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Research article

Exploration of indispensable Banach-space valued functions

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Abstract: In the paper, we present a necessary and sufficient condition for the existence of a sequence of measurable functions with finite values, which converge to any given essential bounded function in the topology of essential supremum in a Banach space. A new convergence method is proposed, which allows for the discovery of an essential bounded function F that is valued in a Banach space. Generally speaking, there exists a Banach-valued essential bounded function F which F_n can't converge to F in the topology of essential supremum for any sequence of finite-valued measurable function.

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1. Introduction

The property of $L^p(\Omega, \mathscr{F}, \mu; \mathcal{B})$ will be discussed, here $(\Omega, \mathscr{F}, \mu)$ is a σ-finite measure space, and $\mathcal{B}$ is a real Banach space. For $1 \le p < \infty$, $L^p(\Omega, \mathscr{F}, \mu; \mathcal{B})$ is a linear space of all $\mathcal{B}$-valued Bochner L^p integral function with the norm given by the formula

$$\|F\|_{L^p(\Omega,\mathscr{F},\mu;\mathcal{B})} \equiv \left(\int_\Omega \|F(\omega)\|_{\mathcal{B}}^p \, d\mu(\omega) \right)^{\frac{1}{p}}.$$

If $p = \infty$, $L^\infty(\Omega, \mathscr{F}, \mu; \mathcal{B})$ is a linear space of all $\mathcal{B}$-valued essential bounded function with norm defined by letting

$$\|F\|_{L^\infty(\Omega,\mathscr{F},\mu;\mathcal{B})} \equiv \inf_{\substack{E \in \mathscr{F} \\ \mu(E)=0}} \left(\sup_{\omega \in E^c} \|F(\omega)\|_{\mathcal{B}} \right).$$

If $\mathcal{B} = \mathbb{R}$, when $p \in [1, \infty]$, it is known that there exists a sequence of finite-valued simple measurable function $\{F_n, n \geq 1\}$ such that

$$\lim_{n\to\infty} \|F - F_n\|_{L^p(\Omega,\mathscr{F},\mu;\mathbb{R})} = 0,$$

for any $F \in L^p(\Omega, \mathscr{F}, \mu; \mathbb{R})$ (see [1, 2]). If $\mathcal{B}$ is a general Banach space, $p \in [1, \infty)$, there exists a sequence of finite-valued simple measurable function F_n such that

$$\lim_{n\to\infty} \|F - F_n\|_{L^p(\Omega,\mathscr{F},\mu;\mathcal{B})} = 0,$$

for any $F \in L^p(\Omega, \mathscr{F}, \mu; \mathcal{B})$, and there exists a sequence of countable-valued simple measurable function F_n such that

$$\lim_{n\to\infty} \|F - F_n\|_{L^\infty(\Omega,\mathscr{F},\mu;\mathcal{B})} = 0,$$

for any $F \in L^\infty(\Omega, \mathscr{F}, \mu; \mathcal{B})$ (see [3]).

The difference between infinite dimension Banach space $\mathcal{B}$ and $\mathbb{R}$ is that the closed ball of $\mathbb{R}$ is compact set and the closed ball of $\mathcal{B}$ is non-compact set (see [4]), which makes the property of $L^\infty(\Omega, \mathscr{F}, \mu; \mathcal{B})$ is very different form $L^\infty(\Omega, \mathscr{F}, \mu; \mathbb{R})$.

Convergence methods of Banach-valued function were defined in serval ways. For example, Zheng and Cui [5] investigated that $l^\infty(X)$- evaluation uniform convergence of operator series can be described completed by the essential bounded subset of $l^\infty(X)$. Here X is a Banach space,

$$l^\infty(X) \equiv \left\{(x_j) : x_j \in X, \sup_{j\in\mathbb{N}} \left\|x_j\right\| < \infty\right\},$$

and $l^\infty(X)$ equip the norm of

$$\left\|x_j\right\|_\infty \equiv \sup_{j\in\mathbb{N}} \left\|x_j\right\|.$$

León-Saavedra considered unconditionally convergence of a series $\sum_i x_i$ in a Banach space. [6] showed that a series is unconditionally convergent if and only if the series is weakly subseries convergent with respect to a regular linear summability method. Furthermore, this paper unifies several versions of the Orlicz-Pettis theorem that incorporate summability methods. [7] give a another version of the Orlicz-Pettis theorem within the frame of the strong ρ-Cesàro convergence. [8] unified several results which characterize when a series is weakly unconditionally Cauchy (wuc) in terms of the completeness of a convergence space associated with the wuc series. [9] gave a new characterization of weakly unconditionally Cauchy series and unconditionally convergent series through the strong ρ-Cesàro summability is obtained.

In this work, we will present a necessary and sufficient condition for the existence of F_n in $L^\infty(\Omega, \mathscr{F}, \mu; \mathcal{B})$ for $F \in L^\infty(\Omega, \mathscr{F}, \mu; \mathcal{B})$, by constructing a sequence of finite-valued measurable functions that converge to F in some sense. A counterexample is also discussed to demonstrate that there exists $F \in L^\infty(\Omega, \mathscr{F}, \mu; \mathcal{B})$ for which F_n cannot converge to F in the norm topology of $L^\infty(\Omega, \mathscr{F}, \mu; \mathcal{B})$ for any sequence of finite-valued measurable functions F_n.

2. Preliminaries

The following definitions are about Banach-valued measurable function.

Definition 2.1. [10] If $(\Omega, \mathscr{F})$ is a measurable space, $\mathcal{B}$ is a Banach space, $\Omega_1, \cdots, \Omega_n \in \mathscr{F}$ are pairwise disjoined nonempty sets, $x_1, \cdots, x_n \in \mathcal{B}$, then the map

$$F(\omega) = \sum_{i=1}^{n} x_i \mathbb{I}_{\Omega_i}(\omega),$$

is called finite-valued simple function. And the map

$$F(\omega) = \sum_{i=1}^{\infty} x_i \mathbb{I}_{\Omega_i}(\omega),$$

is called countable-valued simple function. A map $F : \Omega \to \mathcal{B}$ is called measurable if $\forall A \in \mathscr{B}(\mathcal{B}), F^{-1}(B) \in \mathscr{F}$. F is called strongly measurable if there is a sequence of finite-valued simple function F_n such that $\forall \omega \in \Omega$,

$$\lim_{n\to\infty} \|F(\omega) - F_n(\omega)\|_{\mathcal{B}} = 0.$$

Definition 2.2. [11] Let $F : \Omega \to \mathcal{B}$ be a map, for all $f \in \mathcal{B}^*$, the function $f(F(\omega))$ is measurable function on $(\Omega, \mathscr{F}, \mu)$, then F is called weak measurable function on $(\Omega, \mathscr{F}, \mu)$.

The following theorem describes the relationship weak and strong measurable.

Theorem 2.1. *(Pettis) [11] Let $F : \Omega \to \mathcal{B}$ be a map, the following assertions are equivalent:*

(1) F is strongly measurable.
(2) F is weakly measurable and $F(\Omega)$ is almost separable.

By Theorem 2.1, if $\mathcal{B}$ is separable space, then F is strongly measurable if and only if it's weakly measurable.

Then the definition of Bochner L^P-space is given as follows.

Definition 2.3. [3, 10] Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, and let $F : \Omega \to \mathcal{B}$ be a finite-valued simple function with a form of

$$F(\omega) = \sum_{i=1}^{n} x_i \mathbb{I}_{\Omega_i}(\omega).$$

If $\sum_{i=1}^{n} \mu(\Omega_i) < \infty$, then the Bochner integral of F is defined by

$$\int_\Omega F(\omega) d\mu(\omega) = \sum_{i=1}^{n} x_i \mu(\Omega_i).$$

And let $F : \Omega \to \mathcal{B}$ be a strongly measurable function. If there exists a $p \in [1, \infty)$ such that

$$\int_\Omega \|F(\omega)\|_{\mathcal{B}}^p \, d\mu(\omega) < \infty,$$

then F is called L^p-integrable on $(\Omega, \mathscr{F}, \mu)$. The linear space of all L^p-integrable function with the following seminorm

$$\|F\|_{L^p(\Omega,\mathscr{F},\mu;\mathcal{B})} \equiv \left(\int_\Omega \|F(\omega)\|_{\mathcal{B}}^p \, d\mu(\omega) \right)^{\frac{1}{p}},$$

is denoted by $L^p(\Omega, \mathscr{F}, \mu; \mathcal{B})$. If the function

$$\omega \mapsto \|F(\omega)\|_{\mathcal{B}}$$

is essential bounded, then F is called essential bounded. The linear space of all essential bounded function with the following seminorm

$$\|F\|_{L^\infty(\Omega,\mathscr{F},\mu;\mathcal{B})} \equiv \text{ess sup}\left\{\|F(\omega)\|_{\mathcal{B}} : \omega \in \Omega\right\},$$

is denoted by $L^\infty(\Omega, \mathscr{F}, \mu; \mathcal{B})$.

The following theorems show that the collection of finite-valued function is dense in $L^p(\Omega, \mathscr{F}, \mu; \mathcal{B})$ if $p \in [1, \infty)$, and the collection of countable-valued function is dense in $L^\infty(\Omega, \mathscr{F}, \mu; \mathcal{B})$.

Theorem 2.2. *[3] Let $(\Omega, \mathscr{F}, \mu)$ be a measurable space, $F : \Omega \to \mathcal{B}$ is a strongly measurable function, $p \in [1, \infty)$, then the following statements are the same in meaning:*

(1) $F \in L^p(\Omega, \mathscr{F}, \mu; \mathcal{B})$.
(2) There exists a sequence of finite-valued simple function F_n such that

$$\lim_{n\to\infty} \int_\Omega \|F_n(\omega) - F(\omega)\|_{\mathcal{B}}^p \, d\mu(\omega) = 0.$$

Theorem 2.3. *[3] Let $(\Omega, \mathscr{F}, \mu)$ be a measurable space, $F : \Omega \to \mathcal{B}$ be a strongly measurable function, then the following statements are synonymous:*

(1) $F \in L^\infty(\Omega, \mathscr{F}, \mu; \mathcal{B})$.
(2) There exists a sequence of countable-valued simple function F_n such that

$$\lim_{n\to\infty} \inf_{\substack{E\in\mathscr{F}\\ \mu(E)=0}} \left(\sup_{\omega\in E^c} \|F_n(\omega) - F(\omega)\|_{\mathcal{B}} \right) = 0.$$

3. Main result

Theorem 3.1. *If $(\Omega, \mathscr{F}, \mu)$ is a measure space, and $\mathcal{B}$ is a real Banach space, $F \in L^\infty(\Omega, \mathscr{F}, \mu; \mathcal{B})$, then the following assertions are equivalent:*

(1) There exists a sequence of finite-valued simple function F_n such that

$$\lim_{n\to\infty} \|F - F_n\|_{L^\infty(\Omega,\mathscr{F},\mu;\mathcal{B})} = 0.$$

(2) There exists a measurable set $\tilde{\Omega} \in \mathscr{F}$ such that $\mu(\tilde{\Omega}) = 0$ and $F(\tilde{\Omega}^c)$ is a sequential compact set.

Proof. If (1) holds, suppose

$$\|F - F_n\|_{L^\infty(\Omega,\mathscr{F},\mu;\mathcal{B})} < \frac{1}{2n},$$

and

$$F_n = \sum_{i=1}^{K_n} x_{in} \mathbb{I}_{E_{in}},$$

where $\{E_{in}\}_{i=1}^{K_n}$ are pairwise disjoined and $\cup_{i=1}^{K_n} E_{in} = \Omega$. By the definition of essential bounded, there exists $\tilde{E}_n \in \mathscr{F}$ such that $\mu(\tilde{E}_n) = 0$ and

$$\sup_{\omega \in \tilde{E}_n^c} \|F_n(\omega) - F(\omega)\|_{\mathcal{B}} < \frac{1}{n}.$$

Considering

$$\tilde{\Omega} \equiv \bigcup_{n \in \mathbb{N}_+} \left(\bigcup_{i=1}^{K_n} E_{in} \cap \tilde{E}_n^c \right)^c.$$

Then $\mu(\tilde{\Omega}) = 0$. Let

$$\omega \in \tilde{\Omega}^c \subset \bigcup_{i=1}^{K_n} E_{in} \cap \tilde{E}_n^c,$$

then there exist $i = 1, \cdots, K_n$ such that

$$\|x_{in} - F(\omega)\|_{\mathcal{B}} \leq \sup_{\omega \in \tilde{E}_n^c} \|F_n(\omega) - F(\omega)\|_{\mathcal{B}} < \frac{1}{n}.$$

Therefore, $\{x_{in}\}_{i=1}^{K_n}$ is a $1/n-$ web of $F(\tilde{\Omega}^c)$. By the arbitrary of n, $F(\tilde{\Omega}^c)$ is a sequential compact set.

If condition (2) is satisfied, then $\forall n \in \mathbb{N}_+$, there exists a finite $1/n-$ web $\{x_{in}\}_{i=1}^{K_n}$ of $F(\tilde{\Omega}^c)$. Let

$$E_{in} \equiv \left\{ \omega \in \tilde{\Omega}^c : \|x_{in} - F(\omega)\|_{\mathcal{B}} < \frac{1}{n} \right\}.$$

Let $\tilde{E}_{1n} = E_{1n}$, and for $i > 1$, defined by

$$\tilde{E}_{in} \equiv E_{in} \backslash \left(\bigcup_{j=1}^{i-1} E_{jn} \right).$$

Now, let's define a finite-valued function

$$F_n = \sum_{i=1}^{K_n} x_{in} \mathbb{I}_{\tilde{E}_{in}};$$

then

$$\|F - F_n\|_{L^\infty(\Omega, \mathscr{F}, \mu; \mathcal{B})} \leq \sup_{\omega \in \tilde{\Omega}^c} \|F_n(\omega) - F(\omega)\|_{\mathcal{B}} < \frac{1}{n}.$$

By the arbitrary of n, (1) holds. □

From now on, suppose $\mathcal{B}$ is real Banach space which dual space $\mathcal{B}^*$ is separable, and $(\Omega, \mathscr{F}, \mu)$ is complete measure space. Then $\mathcal{B}$ is separable. Let $\{f_n\}_{n \in \mathbb{N}_+}$ be countably dense subset of $\mathcal{B}^*$. We define a new convergence.

Definition 3.1. Let $F_k, k = 1, 2, \cdots$ be a sequence of $\mathcal{B}$-valued strongly measurable function on $(\Omega, \mathscr{F}, \mu)$, we say F_k *weakly converge* to a $\mathcal{B}$-valued function F almost uniformly if there exists $E \in \mathscr{F}$ such that $\mu(E) = 0$ and for all weak neighborhood of origin W, there exists $N \in \mathbb{N}_+$ such that $\forall k > N$,

$$F(\omega) - F_k(\omega) \in W, \quad \forall \omega \in E^c.$$

We say F_i is a almost uniformly weak Cauchy sequence if there exists $E \in \mathscr{F}$ such that $\mu(E) = 0$ and for all weak neighborhood of origin W, there exists $N \in \mathbb{N}_+$ such that $\forall i, j > N$,

$$F_i(\omega) - F_j(\omega) \in W, \quad \forall \omega \in E^c.$$

Theorem 3.2. *(1) F_k weakly converge to F almost uniformly if and only if there exists $E \in \mathscr{F}$ such that $\mu(E) = 0$, and for all $f \in \mathcal{B}^*$, then*

$$\lim_{k\to\infty} \sup_{\omega\in E^c} |f(F_k(\omega)) - f(F(\omega))| = 0.$$

(2) F_k is a almost uniformly weak Cauchy sequence if and only if there exists $E \in \mathscr{F}$ such that $\mu(E) = 0$, and for all $f \in \mathcal{B}^*$, then

$$\lim_{k\to\infty} \sup_{p\in\mathbb{N}_+} \sup_{\omega\in E^c} |f(F_{k+p}(\omega)) - f(F_n(\omega))| = 0.$$

Proof. We have just proven (1), and likewise, (2) can be demonstrated. Suppose there exists $E \in \mathscr{F}$ such that $\mu(E) = 0$, and for all weak neighborhood of origin W, there exists $N \in \mathbb{N}_+$ such that $\forall k > N$,

$$F(\omega) - F_k(\omega) \in W, \quad \forall \omega \in E^c.$$

Let $f \in \mathcal{B}^*$, given $m \in \mathbb{N}_+$, consider the set

$$V_m \equiv \left\{ x \in \mathcal{B} : |f(x)| < \frac{1}{m} \right\}.$$

Then, for $N \in \mathbb{N}_+$ such that $\forall k > N$,

$$F(\omega) - F_k(\omega) \in V_m, \quad \forall \omega \in E^c.$$

That is

$$\sup_{\omega\in E^c} |f(F_k(\omega)) - f(F(\omega))| < \frac{1}{m}.$$

Let $k \to \infty$,

$$\limsup_{k\to\infty} \sup_{\omega\in E^c} |f(F_k(\omega)) - f(F(\omega))| \le \frac{1}{m}.$$

By the arbitrary of m,

$$\lim_{k\to\infty} \sup_{\omega\in E^c} |f(F_k(\omega)) - f(F(\omega))| = 0.$$

Suppose there exists $E \in \mathscr{F}$ such that $\mu(E) = 0$, and for all $f \in \mathcal{B}^*$, we have

$$\lim_{k\to\infty} \sup_{\omega\in E^c} |f(F_k(\omega)) - f(F(\omega))| = 0.$$

Given a weak neighborhood of origin W, by the definition of weak topology, there exists $g_1, \cdots, g_m \in \mathcal{B}^*$ and $\epsilon > 0$ such that

$$V \equiv \{x \in \mathcal{B} : |g_1(x)| < \epsilon, \cdots, |g_m(x)| < \epsilon\} \subset W.$$

Then, for $i = 1, \cdots, m, \exists N_i \in \mathbb{N}_+$ such that

$$\sup_{\omega \in E^c} |g_i(F_k(\omega)) - g_i(F(\omega))| < \epsilon, \quad \forall n > N_i.$$

Let $N = max(N_1, \cdots, N_m)$, then $\forall k > N$, we have

$$F_k(\omega) - F(\omega) \in V \subset W, \quad \forall \omega \in E^c.$$

Therefore, F_k weakly converge to F almost uniformly. □

Theorem 3.3. *Let $F_k, k = 1, 2, \cdots$ be a sequence of $\mathcal{B}$-valued strongly measurable function. If F_k weakly converge to F almost uniformly, then F is strongly measurable.*

Proof. By Theorem 2.1, it is sufficient to prove that F is weakly measurable. If F_k weakly converge to F almost uniformly, then there exists $E \in \mathscr{F}$ such that $\mu(E) = 0$, and $\forall f \in \mathcal{B}^*$,

$$\lim_{k\to\infty} \sup_{\omega \in E^c} |f(F_k(\omega)) - f(F(\omega))| = 0.$$

Therefore, $f(F_k \mathbb{I}_{E^c})$ pointwise converge to $f(F\mathbb{I}_{E^c})$. By the arbitrary of f, $F\mathbb{I}_{E^c}$ is weakly measurable, thus it is strongly measurable. Because $\mu(E) = 0$ and $(\Omega, \mathscr{F}, \mu)$ is complete, $F\mathbb{I}_E$ is strongly measurable. In summary, $F = F\mathbb{I}_{E^c} + F\mathbb{I}_E$ is measurable. □

Theorem 3.4. *If F_k weakly converge to F' and F'' almost uniformly, then $F' = F'', \mu - a.e.$.*

Proof. If F_k weakly converge to F' and F'' almost uniformly, then there exist $E', E'' \in \mathscr{F}$ such that $\mu(E') = \mu(E'') = 0$, and

$$\lim_{k\to\infty} \sup_{\omega \in E'^c} |f(F_k(\omega)) - f(F(\omega))| = \lim_{k\to\infty} \sup_{\omega \in E''^c} |f(F_k(\omega)) - f(F(\omega))| = 0.$$

Then

$$\mu\left(\{\omega \in \Omega : F'(\omega) \neq F''(\omega)\}\right) \leq \mu(E' \cup E'') \leq \mu(E') + \mu(E'') = 0.$$

Therefore, $F' = F'', \mu - a.e.$. □

Theorem 3.5. *Let $F, F_k, k = 1, 2, \cdots \in L^\infty(\Omega, \mathscr{F}, \mu; \mathcal{B})$, then F_k weakly converge to F almost uniformly if and only if*

(1) $\sup_{k \in \mathbb{N}_+} \|F_k\|_{L^\infty(\Omega, \mathscr{F}, \mu; \mathcal{B})} < \infty$.
(2) There exists $E \in \mathscr{F}$ such that $\mu(E) = 0$, and $\forall n \in \mathbb{N}_+$,

$$\lim_{k\to\infty} \sup_{\omega \in E^c} |f_n(F_k(\omega)) - f_n(F(\omega))| = 0.$$

Proof. Suppose F_k weakly converge to F almost uniformly, since (2) is self-evident, we will focus on demonstrating (1). By the conditions, there exists a $E \in \mathscr{F}$ such that $\mu(E) = 0$ and

$$\lim_{k\to\infty} \sup_{\omega\in E^c} |f(F_k(\omega)) - f(F(\omega))| = 0.$$

In addition, we can suppose $\sup_{\omega\in E^c} \|F(\omega)\| < \infty$ and $\sup_{\omega\in E^c} \|F_k(\omega)\| < \infty (k \in \mathbb{N}_+)$. Fixed $f \in \mathscr{B}^*$, then there exists $k_0 \in \mathbb{N}_+$ such that for $k \geq k_0$,

$$\sup_{\omega\in E^c} |f(F_k(\omega)) - f(F(\omega))| < 1.$$

For $k \geq k_0$,

$$\begin{aligned}\sup_{\omega\in E^c} |f(F_k(\omega))| &\leq \sup_{\omega\in E^c} |f(F_k(\omega)) - f(F(\omega))| + \sup_{\omega\in E^c} |f(F(\omega))| \\ &\leq 1 + \|f\|_{\mathscr{B}^*} \sup_{\omega\in E^c} \|F(\omega)\|_{\mathscr{B}} < \infty.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sup \{|f(F_k(\omega))| : k \in \mathbb{N}_+, \omega \in E^c\} \\ \leq &\max \left\{ \|f\|_{\mathscr{B}^*} \sup_{\omega\in E^c} \|F_1(\omega)\|_{\mathscr{B}}, \cdots, \|f\|_{\mathscr{B}^*} \sup_{\omega\in E^c} \left\|F_{k_0}(\omega)\right\|_{\mathscr{B}}, 1 + \|f\|_{\mathscr{B}^*} \sup_{\omega\in E^c} \|F(\omega)\|_{\mathscr{B}} \right\} < \infty.\end{aligned}$$

By Uniform Boundedness Principle,

$$\sup \{\|F_k(\omega)\|_{\mathscr{B}} : k \in \mathbb{N}_+, \omega \in E^c\} < \infty.$$

Thus, $\sup_{k\in\mathbb{N}_+} \|F_k\|_{L^\infty(\Omega,\mathscr{F},\mu;\mathscr{B})} < \infty$. Now we suppose (1) and (2) are true, then there exists a $E \in \mathscr{F}$ such that $\mu(E) = 0$ and $\forall n \in \mathbb{N}_+$,

$$\lim_{k\to\infty} \sup_{\omega\in E^c} |f_n(F_k(\omega)) - f_n(F(\omega))| = 0.$$

We can assume

$$M \equiv \sup_{\omega\in E^c} \|F(\omega)\|_{\mathscr{B}} + \sup_{k\in\mathbb{N}_+} \sup_{\omega\in E^c} \|F_k(\omega)\|_{\mathscr{B}} < \infty.$$

Fixed $f \in \mathscr{B}^*$, then $\forall \epsilon > 0, \exists n_0 \in \mathbb{N}_+$ such that

$$\left\|f - f_{n_0}\right\|_{\mathscr{B}^*} < \frac{\epsilon}{2M}.$$

Then $\forall k \in \mathbb{N}_+, \forall \omega \in E^c$,

$$\begin{aligned}&|f(F_k(\omega)) - f(F(\omega))| \\ \leq &|f(F_k(\omega)) - f_{n_0}(F_k(\omega))| + |f_{n_0}(F_k(\omega)) - f_{n_0}(F(\omega))| + |f_{n_0}(F(\omega)) - f_{n_0}(F(\omega))| \\ \leq &\left\|f - f_{n_0}\right\|_{\mathscr{B}^*} \|F_k(\omega)\|_{\mathscr{B}} + |f_{n_0}(F_k(\omega)) - f_{n_0}(F(\omega))| + \left\|f - f_{n_0}\right\|_{\mathscr{B}^*} \|F(\omega)\|_{\mathscr{B}} \\ < &\epsilon + |f_{n_0}(F_k(\omega)) - f_{n_0}(F(\omega))|.\end{aligned}$$

By the arbitrary of ω,

$$\sup_{\omega\in E^c} |f(F_k(\omega)) - f(F(\omega))| < \epsilon + \sup_{\omega\in E^c} |f_{n_0}(F_k(\omega)) - f_{n_0}(F(\omega))|.$$

Therefore,

$$\limsup_{k\to\infty} \sup_{\omega\in E^c} |f(F_k(\omega)) - f(F(\omega))| < \epsilon + \limsup_{k\to\infty} \sup_{\omega\in E^c} |f_{n_0}(F_k(\omega)) - f_{n_0}(F(\omega))| \le \epsilon,$$

By the arbitrary of ϵ,

$$\limsup_{k\to\infty} \sup_{\omega\in E^c} |f(F_k(\omega)) - f(F(\omega))| = 0.$$

Finally, by the arbitrary of f, F_k weakly converge to F almost uniformly □

Theorem 3.6. *Suppose F is a $\mathcal{B}$-valued strongly measurable function on $(\Omega, \mathscr{F}, \mu)$, then $F \in L^\infty(\Omega, \mathscr{F}, \mu; \mathcal{B})$ if and only if there exists a sequence of finite-valued simple function F_k such that F_k weakly converge to F almost uniformly.*

Proof. Suppose $F \in L^\infty(\Omega, \mathscr{F}, \mu; \mathcal{B})$, then there is a $E \in \mathscr{F}$ such that $\mu(E) = 0$, and

$$M \equiv \sup_{\omega\in E^c} \|F(\omega)\|_{\mathcal{B}} < \infty.$$

By Banach-Alaoglu theorem, $F(E^c)$ is weak relatively compact sets. Let $k \in \mathbb{N}_+$, and

$$V_k \equiv \left\{ x \in \mathcal{B} : |f_1(x)| < \frac{1}{k}, \cdots, |f_k(x)| < \frac{1}{k} \right\}.$$

Then there exist $\forall k \in \mathbb{N}_+, \exists \{x_{ik}\}_{i=1}^{N_k} \subset F(E^c)$ such that

$$\{F(\omega) : \omega \in E^c\} \subset \bigcup_{i=1}^{N_k} (x_{ik} + V_k).$$

Let

$$E_{ik} \equiv \{\omega \in E^c : F(\omega) - x_{ik} \in V_k\}.$$

and $\tilde{E}_{1k} = E_{1k}$, for $i > 1$,we can define

$$\tilde{E}_{ik} \equiv E_{ik} \backslash \left(\bigcup_{j=1}^{i-1} E_{jk} \right).$$

We can construct a finite-valued measurable function

$$F_k = \sum_{i=1}^{K_k} x_{ik} \mathbb{I}_{\tilde{E}_{ik}}.$$

Because $F_k(E^c) \subset F(E^c)$,

$$\sup \{\|F_k(\omega)\|_{\mathcal{B}} : \omega \in E^c, k \in \mathbb{N}_+\} \le M < \infty.$$

For all $n \in \mathbb{N}_+$, for $k > n$,

$$\sup_{\omega \in E^c} |f_n(F_k(\omega)) - f_n(F(\omega))| < \frac{1}{k}.$$

Therefore,

$$\lim_{k\to\infty} \sup_{\omega \in E^c} |f_n(F_k(\omega)) - f_n(F(\omega))| = 0.$$

By Theorem 3.5, F_k weakly converge to F almost uniformly.

Consider the possibility that a sequence of finite-valued simple function F_k that weakly converge to F almost uniformly, then there exists a $E \in \mathscr{F}$ such that $\mu(E) = 0$ and $\forall f \in \mathcal{B}^*$,

$$\lim_{n\to\infty} \sup_{\omega \in E^c} |f(F_k(\omega)) - f(F(\omega))| = 0.$$

Then there exists a $k_0 \in \mathbb{N}_+$ such that

$$\sup_{\omega \in E^c} |f(F_{k_0}(\omega)) - f(F(\omega))| < 1.$$

Because F_{k_0} is finite-valued function,

$$\begin{aligned}\sup_{\omega \in E^c} |f(F(\omega))| &\le \sup_{\omega \in E^c} |f(F_{k_0}(\omega)) - f(F(\omega))| + \sup_{\omega \in E^c} |f(F_{k_0}(\omega))| \\ &\le 1 + \|f\|_{\mathcal{B}^*} \sup_{\omega \in E^c} \left\|F_{k_0}(\omega)\right\|_{\mathcal{B}} < \infty.\end{aligned}$$

By Uniform Boundedness Principle,

$$\sup_{\omega \in E^c} \|F(\omega)\|_{\mathcal{B}} < \infty.$$

Therefore, $F \in L^\infty(\Omega, \mathscr{F}, \mu; \mathcal{B})$. □

Now we will proof $L^\infty(\Omega, \mathscr{F}, \mu; \mathcal{B})$ is complete in the sense of weak convergence almost uniformly.

Theorem 3.7. *Let $F_k, k = 1, 2, \cdots \in L^\infty(\Omega, \mathscr{F}, \mu; \mathcal{B})$ be a almost uniformly weak Cauchy sequence, then $\exists F \in L^\infty(\Omega, \mathscr{F}, \mu; \mathcal{B})$ such that F_k weakly converge to F almost uniformly.*

Proof. Suppose $F_k \in L^\infty(\Omega, \mathscr{F}, \mu; \mathcal{B})$ is a almost uniformly weak Cauchy sequence, then there exists a $E \in \mathscr{F}$ such that $\mu(E) = 0$ and for all $k \in \mathbb{N}_+$,

$$\sup_{\omega \in E^c} \|F_k(\omega)\|_{\mathcal{B}} < \infty.$$

And $\forall f \in \mathcal{B}^*$,

$$\lim_{n\to\infty} \sup_{p \in \mathbb{N}_+} \sup_{\omega \in E^c} |f(F_{n+p}(\omega)) - f(F_n(\omega))| = 0.$$

Fixed $f \in \mathcal{B}^*$, $\exists k_0 \in \mathbb{N}_+$ such that $\forall k > k_0$,

$$\sup_{\omega \in E^c} |f(F_k(\omega)) - f(F_{k_0}(\omega))| < 1.$$

For $i = 1, \cdots, k_0$,

$$\sup_{\omega \in E^c} |f(F_i(\omega))| \le \|f\|_{\mathcal{B}^*} \sup_{\omega \in E^c} \|F_i(\omega)\|_{\mathcal{B}}.$$

Therefore,

$$\sup\{|f(F_k(\omega))| : k \in \mathbb{N}_+, \omega \in E^c\} \leq 1 + \|f\|_{\mathcal{B}^*} \max\left(\sup_{\omega\in E^c}\|F_1(\omega)\|_{\mathcal{B}}, \cdots, \sup_{\omega\in E^c}\left\|F_{k_0}(\omega)\right\|_{\mathcal{B}}\right).$$

By Uniform Boundedness Principle, there exists $M \in (0, \infty)$ such that

$$\{F_k(\omega) : k \in \mathbb{N}_+, \omega \in E^c\} \subset \{x \in \mathcal{B} : \|x\|_{\mathcal{B}} \leq M\}.$$

Fixed $\omega \in E^c$, $\{F_k(\omega)\}_{k\in\mathbb{N}_+}$ is a bounded sequence, which means it's a weak relatively compact sequence. Therefore exists a subsequence $\{F_{k_i}(\omega)\}_{i\in\mathbb{N}_+}$ and $F(\omega) \in \mathcal{B}$ such that $F_{k_i}(\omega)$ weakly converge to $F(\omega)$. Because $\{F_k(\omega)\}_{k\in\mathbb{N}_+}$ is a weak cauchy sequence, $\forall f \in \mathcal{B}^*, \forall\epsilon > 0, \exists K \in \mathbb{N}_+$ such that $\forall m, n \geq K$,

$$|f(F_m(\omega)) - f(F_n(\omega))| < \frac{\epsilon}{2}.$$

Meanwhile, $\exists i_0 \in \mathbb{N}_+$ such that $k_{i_0} > N$ and

$$|f(F_{k_{i_0}}(\omega)) - f(F(\omega))| < \frac{\epsilon}{2},$$

when $k > k_{i_0}$,

$$|f(F_k(\omega)) - f(F(\omega))| \leq |f(F_k(\omega)) - f_{k_{i_0}}(F(\omega))| + |f(F_{k_{i_0}}(\omega)) - f(F(\omega))| < \epsilon.$$

which means $F_k(\omega)$ weakly converge to $F(\omega)$. By Mazur Theorem,

$$F(\omega) \in \bar{co}\{F_k(\omega)\}_{k\in\mathbb{N}_+} \subset \{x \in \mathcal{B} : \|x\|_{\mathcal{B}} \leq M\},$$

where $\bar{co}\{F_k(\omega)\}_{k\in\mathbb{N}_+}$ is convex hull of $\{F_k(\omega)\}_{k\in\mathbb{N}_+}$ in norm topology of $\mathcal{B}$. Therefore, F is essential bounded, and

$$\|F\|_{L^\infty(\Omega,\mathscr{F},\mu;\mathcal{B})} \leq \sup_{\omega\in E^c}\|F(\omega)\|_{\mathcal{B}} \leq M.$$

Finally, F_k weakly converge to F almost uniformly can be show. Fixed $f \in \mathcal{B}^*$, there exists a $\{F_{k_i}\}_{i\in\mathbb{N}}$ such that

$$\sup_{\omega\in E^c}|f(F_{k_i}(\omega)) - f(F_{k_{i-1}}(\omega))| < \frac{1}{2^i}.$$

Then

$$f(F_{k_0}(\omega)) + \sum_{i=1}^{\infty}(f(F_{k_i}(\omega)) - f(F_{k_{i-1}}(\omega))) = f(F(\omega)).$$

Let $j \to \infty$,

$$\sup_{\omega\in E^c}|f(F_{k_j}(\omega)) - f(F(\omega))| \leq \sum_{i=j+1}^{\infty}\sup_{\omega\in E^c}|f(F_{k_i}(\omega)) - f(F_{k_{i-1}}(\omega))| \leq \sum_{i=j+1}^{\infty}\frac{1}{2^i} \to 0.$$

$\forall\epsilon > 0, \exists K \in \mathbb{N}_+$ such that $\forall m, n \geq K$,

$$\sup_{\omega\in E^c}|f(F_m(\omega)) - f(F_n(\omega))| < \frac{\epsilon}{2}.$$

Meanwhile, $\exists i_0 \in \mathbb{N}_+$ such that $k_{i_0} > N$ and

$$\sup_{\omega\in E^c} |f(F_{k_{i_0}}(\omega)) - f(F(\omega))| < \frac{\epsilon}{2}.$$

Therefore, for $k > k_{i_0}$, we have

$$\sup_{\omega\in E^c} |f(F_k(\omega)) - f(F(\omega))| \leq \sup_{\omega\in E^c} |f(F_k(\omega)) - f_{k_{i_0}}(F(\omega))| + \sup_{\omega\in E^c} |f(F_{k_{i_0}}(\omega)) - f(F(\omega))| < \epsilon,$$

which means F_k weakly converge to F almost uniformly. □

Finally, we will provide a counterexample to demonstrate that there exists an $F \in L^\infty(\Omega; \mathcal{B})$ for which F_n cannot converge to F in the norm topology of $L^\infty(\Omega; \mathcal{B})$ for any sequence of finite-valued measurable functions F_n.

Let $\mathcal{H} = L^2([-\pi, \pi], \mathscr{B}([-\pi, \pi]), l)$, where l is Lebesgue measure. It is obvious that $\mathcal{H}$ is real separable Hilbert space and $\mathcal{H}' = \mathcal{H}$ by Riesz Representation Theorem, which means the dual space of $\mathcal{H}$ is separable.

Let $n \in \mathbb{N}$,

$$h_0 \equiv \frac{1}{\sqrt{2\pi}}, \quad h_{2n-1}(x) = \frac{1}{\sqrt{\pi}} \sin nx, \quad h_{2n}(x) = \frac{1}{\sqrt{\pi}} \cos nx, \quad n \in \mathbb{N}_+.$$

Then $\{h_n\}_{n\in\mathbb{N}}$ is the unit orthogonal basis of $\mathcal{H}$, and

$$\|h_n - h_m\|^2_{\mathcal{H}} = 2, \quad \forall n \neq m.$$

A measure space $([0, 1], \mathscr{B}([0, 1]), l)$ is given. Let C be the Cantor set of $[0, 1]$, and the countable connected component of C^c denote by $\{E_n\}_{n\in\mathbb{N}_+}$.

$$F \equiv \sum_{n=1}^{\infty} h_n \mathbb{I}_{E_n}.$$

Then $F \in L^\infty([0, 1], \mathscr{B}([0, 1]), l; \mathcal{H})$, and

$$\|F\|_{L^\infty([0,1],\mathscr{B}([0,1]),l;\mathcal{H})} = 1.$$

However, given any zero measure set E, $F(E^c) = \{h_n\}_{n\in\mathbb{N}}$, which means $F(E^c)$ in not a sequential compact set. Therefore, any sequence of finite-valued measurable function cannot converge to F in norm topology of $L^\infty([0, 1], \mathscr{B}([0, 1]), l; \mathcal{H})$. Let

$$F_k \equiv h_0 \mathbb{I}_C + \sum_{n=1}^{k} h_n \mathbb{I}_{E_n}.$$

Then F_k are finite-valued measurable functions. Given $m \in \mathbb{N}_+$, for $k > m$,

$$\sup_{\omega\in C^c} \langle h_m, F(\omega) - F_k(\omega)\rangle_{\mathcal{H}} = \sup_{\omega\in C^c} \left\langle h_m, \sum_{n=k+1}^{\infty} h_n \mathbb{I}_{E_n}(\omega) \right\rangle_{\mathcal{H}} = 0,$$

and $l(C) = 0$, which means F_k weakly converge to F almost uniformly.

4. Prospect

In this paper, we give a necessary and sufficient condition for the existence of a sequence of finite-valued measurable function which converge to $F \in L^{\infty}(\Omega, \mathscr{F}, \mu; \mathcal{B})$ in topology of essential supremum and give a new convergence which makes any $F \in L^{\infty}(\Omega, \mathscr{F}, \mu; \mathcal{B})$ can find sequence of finite-valued measurable function which converge to F. On this basis, we can raise some valuable problems:

(1) Definition 3.1 and the proof of Theorem 3.2 to Theorem 3.6 depend on the separability of $\mathcal{B}^*$, if we can extended the conclusion to the condition that $\mathcal{B}^*$ is a general Banach space?

(2) If we can use a topology to convergence defined in Definition 3.1. For example, let $f \in \mathscr{B}^*$, we define the seminorm p_f by

$$p_f(F) = \|f(F)\|_{L^{\infty}(\Omega, \mathscr{F}, \mu; \mathbb{R})}.$$

If we can prove that $p_f(F) = 0$ for all f implies $\|F\|_{L^{\infty}(\Omega, \mathscr{F}, \mu; \mathcal{B})} = 0$, then the the family $\mathscr{P} \equiv \left\{p_f : f \in \mathcal{B}^*\right\}$ can determine a new topology to characterize the convergence.

(3) Based on (2) and Theorem 3.2, we guess the following assertions are equivalent:

(a) F_k weakly converge to F almost uniformly.

(b) For all $f \in \mathcal{B}^*$, we have

$$\lim_{n \to \infty} \|f(F_k) - f(F)\|_{L^{\infty}(\Omega, \mathscr{F}, \mu; \mathbb{R})} = 0.$$

(c) There exists $E \in \mathscr{F}$ such that $\mu(E) = 0$, and for all $f \in \mathcal{B}^*$, we have

$$\lim_{n \to \infty} \sup_{\omega \in E^c} |f(F_k(\omega)) - f(F(\omega))| = 0.$$

Here $F, F_k, k = 1, 2, \cdots \in L^{\infty}(\Omega, \mathscr{F}, \mu; \mathcal{B})$.

5. Conclusions

In the work, a necessary and sufficient condition for the existence of a sequence of finite-valued measurable function which converge to any given $F \in L^{\infty}(\Omega, \mathscr{F}, \mu; \mathcal{B})$ is given. A new convergence is defined. In this convergence, any $F \in L^{\infty}(\Omega, \mathscr{F}, \mu; \mathcal{B})$ has a sequence of finite-valued measurable function which converge to F. Finally, a counterexample is also given to show that there exists $F \in L^{\infty}(\Omega, \mathscr{F}, \mu; \mathcal{B})$ for which F_n cannot converge to F in the norm topology of $L^{\infty}(\Omega, \mathscr{F}, \mu; \mathcal{B})$ for any sequence of finite-valued measurable functions F_n.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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