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Research article

Entire solutions of two certain types of quadratic trinomial q-difference differential equations

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Abstract: The main purpose of this paper is to find the explicit forms for entire solutions of two certain types of Fermat-type q-difference differential equations. Some previous results are generalized and examples are constructed to show that the results are accurate.

Keywords: entire solutions; q-difference differential equations; Nevanlinna theory
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1. Introduction and main results

The classical Fermat's last theorem that equation $x^n + y^n = 1$ has no non-trivial rational solutions, when $n \geq 3$, had been proved by Wiles in [1]. Considering x, y in $x^n + y^n = 1$ as elements in function fields, we arrive at looking equations that may be called Fermat type functional equations

$$f(z)^n + g(z)^n = 1. \tag{1.1}$$

In 1966, Gross [2] proved the Fermat type functional equation (1.1) has no transcendental meromorphic solutions when $n \geq 4$. If $n = 2$, then Eq (1.1) has the entire solutions $f(z) = \sin(h(z))$ and $g(z) = \cos(h(z))$, where $h(z)$ is any entire function, and no other solutions exist [3]. Baker [4] and Yang [5] also obtained some related results on Fermat type functional equation.

In recent years, the analogue of Fermat type equations inspired numerous investigations. Particularly, some authors have gotten a number of interesting results by considering that $g(z)$ has a special relationship with $f(z)$ [6, 7]. For example, Liu et al. [6] considered the difference equation

$$f(z)^2 + f(z + c)^2 = 1, \tag{1.2}$$

and obtained the following result:

Theorem 1.1. *(see [6], Theorem 1.1) The transcendental entire solutions with finite order of Eq (1.2) must satisfy* $f(z) = \sin(Az + B)$*, where* B *is a constant and* $A = \dfrac{(4k+1)\pi}{2c}$*,* k *is an integer.*

Later on, considering a generalization of Eq (1.2) as

$$f(z)^2 + P(z)^2 f(z+c)^2 = Q(z), \tag{1.3}$$

where $P(z), Q(z)$ are non-zero polynomials, Liu and Yang obtained a result (see [8], Theorem 2.1), which is an improvement of Theorem A. Closely related to difference expressions are q-difference expressions, where the usual shift $f(z+c)$ of a meromorphic function will be replaced by the q-shift $f(qz)$. Liu and Cao [9] considered the entire solutions of Fermat type q-difference equations

$$f(z)^2 + P(z)^2 f(qz)^2 = Q(z), \tag{1.4}$$

where $P(z), Q(z)$ are non-zero polynomials and $|q| = 1$. They showed the following theorem:

Theorem 1.2. *(see [9], Theorem 2.6) If Eq (1.4) admits a transcendental entire solution of finite order, then* $P(z)$ *must be a constant* P*. This solution can be written as*

$$f(z) = \frac{Q_1(z)e^{p(z)} + Q_2(z)e^{-p(z)}}{2}$$

satisfying one of the following conditions:

(1) q *satisfies* $p(qz) = p(z)$ *and* $Q_1(z) - iPQ_1(qz) \equiv 0, Q_2(z) + iPQ_2(qz) \equiv 0$*,* $P^4 Q(q^2 z) = Q(z)$*;*
(2) q *satisfies* $p(qz) + p(z) = 2a_0$*, and* $iPQ_1(qz)e^{2a_0} \equiv -Q_2(z)$*,* $iPQ_2(qz) \equiv Q_1(z)e^{2a_0}$*,* $P^4 Q(q^2 z) = Q(z)$*,* $e^{8a_0} = 1$*, where* $Q(z) = Q_1(z)Q_2(z)$ *and* $p(z)$ *is a non-constant polynomial.*

Liu and Yang [7] in 2016 studied the existence and the forms of solutions of some quadratic trinomial functional equations and obtained some precise properties on the meromorphic solutions of the following equations

$$f(z)^2 + 2\alpha f(z)f'(z) + f'(z)^2 = 1 \tag{1.5}$$

and

$$f(z)^2 + 2\alpha f(z)f(z+c) + f(z+c)^2 = 1. \tag{1.6}$$

If $\alpha \neq \pm 1, 0$, then Eq (1.5) has no transcendental meromorphic solutions (see [7], Theorem 1.3) and the finite order transcendental entire functions of Eq (1.6) must be of order equal to one (see [7], Theorem 1.4).

Recently, Luo et al. [10] investigated the transcendental entire solutions with finite order of the quadratic trinomial difference equation

$$f(z+c)^2 + 2\alpha f(z)f(z+c) + f(z)^2 = e^{g(z)}, \tag{1.7}$$

and differential difference equation

$$f(z+c)^2 + 2\alpha f(z+c)f'(z) + f'(z)^2 = e^{g(z)}, \tag{1.8}$$

where $\alpha^2 (\neq 0, 1), c$ are constants and $g(z)$ is a polynomial.

Theorem 1.3. *(see [10], Theorem 2.1) Let $\alpha^2 \neq 0, 1$, $c(\neq 0) \in \mathbb{C}$ and g be a polynomial. If the difference equation (1.7) admits a transcendental entire solution $f(z)$ of finite order, then $g(z)$ must be of the form $g(z) = az + b$, where $a, b \in \mathbb{C}$.*

In the above results, Nevanlinna theory of meromorphic functions [11, 12] and its difference counterparts [13, 14] play a critical role. For related results, we refer the reader to [15–23] and the references therein.

Motivated by the above equations and results, we investigate the existence and forms of entire solutions of the following two quadratic trinomial q-difference differential equations

$$f(qz)^2 + 2\alpha f(z)f(qz) + f(z)^2 = e^{g(z)}, \tag{1.9}$$

where $\alpha^2 \neq 0, 1$ and $q \neq 0, \pm 1$ are complex numbers, and $g(z)$ is a polynomial.

$$f(qz)^2 + 2\alpha f'(z)f(qz) + f'(z)^2 = e^{g(z)}, \tag{1.10}$$

where $\alpha^2 \neq 0, 1$ and $q \neq 0, 1$ are complex numbers, and $g(z)$ is a polynomial.

Below, for convenience, let

$$A_1 = \frac{1}{2\sqrt{1+\alpha}} + \frac{1}{2i\sqrt{1-\alpha}} \text{ and } A_2 = \frac{1}{2\sqrt{1+\alpha}} - \frac{1}{2i\sqrt{1-\alpha}}. \tag{1.11}$$

Theorem 1.4. *If Eq (1.9) admits a transcendental entire solution $f(z)$ with finite order, then $g(z)$ must satisfy* $\deg(g(z)) > 2$ *and* $q^{\deg(g(z))} = 1$. *Furthermore,*

$$f(z) = \pm\frac{\sqrt{2}}{2(\sqrt{1+\alpha})}e^{\frac{g(z)}{2}}.$$

We give an example to show that the result of Theorem 1.4 is precise as follows:

Example 1.1. $f(z) = \pm\frac{\sqrt{6}}{6}e^{\frac{z^3}{2}}$ *is a transcendental entire solution of*

$$f\left((-\frac{1}{2} + \frac{\sqrt{3}}{2}i)z\right)^2 + 4f(z)f\left((-\frac{1}{2} + \frac{\sqrt{3}}{2}i)z\right) + f(z)^2 = e^{z^3}.$$

Here, $g(z) = z^3$, $q = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $\alpha = 2$, $A_1 = \frac{\sqrt{3}-3}{6}$ *and* $A_2 = \frac{\sqrt{3}+3}{6}$.

Corollary 1.1. *If* $\deg(g(z)) \leq 2$, *then Eq (1.9) has no transcendental entire solution of $f(z)$ with finite order.*

Corollary 1.2. *If $|q| \neq 1$, then Eq (1.9) has no transcendental entire solution of $f(z)$ with finite order.*

Theorem 1.5. *If Eq (1.10) admits a transcendental entire solution $f(z)$ with finite order, then $g(z) \equiv \beta$, $q = -1$ and*

$$f(z) = \frac{\sqrt{2}}{2t}(A_1 e^{tz+y_1} - A_2 e^{-tz+y_2}),$$

where $t, y_1, y_2, \beta \in \mathbb{C}$ satisfying $\beta = y_1 + y_2$ and $t = \pm i$.

We give an example to show that the result of Theorem 1.5 is precise as follows:

Example 1.2. $f(z) = \frac{\sqrt{2}}{2i}(\frac{\sqrt{3}-3}{6}e^{iz+\ln i} - \frac{\sqrt{3}+3}{6}e^{-iz})$ *is a transcendental entire solution of*

$$f(-z)^2 + 4f'(z)f(-z) + f'(z)^2 = e^{\ln i}.$$

Here, $g(z) \equiv \ln i$, $q = -1$, $\alpha = 2$, $A_1 = \frac{\sqrt{3}-3}{6}$ *and* $A_2 = \frac{\sqrt{3}+3}{6}$.

Corollary 1.3. *If* $\deg(g(z)) \geq 1$, *then Eq (1.10) has no transcendental entire solution of* $f(z)$ *with finite order.*

Corollary 1.4. *If* $q \neq 0, \pm 1$, *then Eq (1.10) has no transcendental entire solution of* $f(z)$ *with finite order.*

2. Some lemmas

Lemma 2.1. *[12] Let* $f_j(z)$, $j = 1, 2, 3$ *be meromorphic functions and* $f_1(z)$ *is not a constant. If*

$$\sum_{j=1}^{3} f_j(z) \equiv 1,$$

and

$$\sum_{j=1}^{3} N\left(r, \frac{1}{f_j}\right) + 2\sum_{j=1}^{3} \overline{N}(r, f_j) < (\lambda + o(1))T(r), \ r \in I,$$

where $\lambda < 1$, $T(r) = \max_{1\leq j\leq 3}\{T(r, f_j)\}$ *and* I *represents a set of* $r \in (0, \infty)$ *with infinite linear measure. Then,* $f_2 \equiv 1$ *or* $f_3 \equiv 1$.

Lemma 2.2. *[12] If* $f_j(z)$, $g_j(z)(1 \leq j \leq n, n \geq 2)$ *are entire functions satisfying*

(1) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;
(2) The orders of f_j *are less than that of* $e^{g_h(z)-g_k(z)}$ *for* $1 \leq j \leq n$, $1 \leq h < k \leq n$.

Then $f_j(z) \equiv 0$ *for* $1 \leq j \leq n$.

Lemma 2.3. *[12] Let* $p(z)$ *be a nonzero polynomial with degree* n. *If* $p(qz) - p(z)$ *is a constant, then* $q^n = 1$ *and* $p(qz) \equiv p(z)$. *If* $p(qz) + p(z)$ *is a constant, then* $q^n = -1$ *and* $p(qz) + p(z) = 2a_0$, *where* a_0 *is the constant term of* $p(z)$.

3. Proof of Theorem 1.4

Let $f(z)$ be a transcendental entire solution with finite order of Eq (1.9). Denote

$$f(z) = \frac{1}{\sqrt{2}}(\mu + \nu) \text{ and } f(qz) = \frac{1}{\sqrt{2}}(\mu - \nu),$$

where μ, ν are entire functions. It can be deduced from Eq (1.9) that

$$(1 + \alpha)\mu^2 + (1 - \alpha)\nu^2 = e^{g(z)}. \tag{3.1}$$

From Eq (3.1), we have

$$\left(\frac{\sqrt{1+\alpha}\mu}{e^{\frac{g(z)}{2}}}\right)^2+\left(\frac{\sqrt{1-\alpha}\nu}{e^{\frac{g(z)}{2}}}\right)^2=1.$$

The above equation leads to

$$\left(\frac{\sqrt{1+\alpha}\mu}{e^{\frac{g(z)}{2}}}+i\frac{\sqrt{1-\alpha}\nu}{e^{\frac{g(z)}{2}}}\right)\left(\frac{\sqrt{1+\alpha}\mu}{e^{\frac{g(z)}{2}}}-i\frac{\sqrt{1-\alpha}\nu}{e^{\frac{g(z)}{2}}}\right)=1. \tag{3.2}$$

We observe that both $\frac{\sqrt{1+\alpha}\mu}{e^{\frac{g(z)}{2}}}+i\frac{\sqrt{1-\alpha}\nu}{e^{\frac{g(z)}{2}}}$ and $\frac{\sqrt{1+\alpha}\mu}{e^{\frac{g(z)}{2}}}-i\frac{\sqrt{1-\alpha}\nu}{e^{\frac{g(z)}{2}}}$ have no zeros. Combining Eq (3.2) with the Hadamard factorization theorem, there exists a polynomial $p(z)$ such that

$$\frac{\sqrt{1+\alpha}\mu}{e^{\frac{g(z)}{2}}}+i\frac{\sqrt{1-\alpha}\nu}{e^{\frac{g(z)}{2}}}=e^{p(z)} \text{ and } \frac{\sqrt{1+\alpha}\mu}{e^{\frac{g(z)}{2}}}-i\frac{\sqrt{1-\alpha}\nu}{e^{\frac{g(z)}{2}}}=e^{-p(z)}. \tag{3.3}$$

Set

$$\gamma_1(z)=p(z)+\frac{g(z)}{2} \text{ and } \gamma_2(z)=-p(z)+\frac{g(z)}{2}. \tag{3.4}$$

It follows from Eq (3.3) that

$$\mu=\frac{e^{\gamma_1(z)}+e^{\gamma_2(z)}}{2\sqrt{1+\alpha}} \text{ and } \nu=\frac{e^{\gamma_1(z)}-e^{\gamma_2(z)}}{2i\sqrt{1-\alpha}}.$$

This leads to

$$\begin{aligned} f(z)=\frac{1}{\sqrt{2}}(\mu+\nu)&=\frac{1}{\sqrt{2}}\left(\frac{e^{\gamma_1(z)}+e^{\gamma_2(z)}}{2\sqrt{1+\alpha}}+\frac{e^{\gamma_1(z)}-e^{\gamma_2(z)}}{2i\sqrt{1-\alpha}}\right)\\ &=\frac{1}{\sqrt{2}}(A_1e^{\gamma_1(z)}+A_2e^{\gamma_2(z)}) \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} f(qz)=\frac{1}{\sqrt{2}}(\mu-\nu)&=\frac{1}{\sqrt{2}}\left(\frac{e^{\gamma_1(z)}+e^{\gamma_2(z)}}{2\sqrt{1+\alpha}}-\frac{e^{\gamma_1(z)}-e^{\gamma_2(z)}}{2i\sqrt{1-\alpha}}\right)\\ &=\frac{1}{\sqrt{2}}(A_2e^{\gamma_1(z)}+A_1e^{\gamma_2(z)}), \end{aligned} \tag{3.6}$$

where A_1 and A_2 are defined as Eq (1.11).

It follows from Eq (3.5) that

$$f(qz)=\frac{1}{\sqrt{2}}(A_1e^{\gamma_1(qz)}+A_2e^{\gamma_2(qz)}). \tag{3.7}$$

Since $\alpha^2\neq 0,1$, we have that both A_1 and A_2 are nonzero constants. Combining with Eqs (3.6) and (3.7), we have

$$e^{\gamma_1(z)-\gamma_2(qz)}+\frac{A_1}{A_2}e^{\gamma_2(z)-\gamma_2(qz)}-\frac{A_1}{A_2}e^{\gamma_1(qz)-\gamma_2(qz)}=1. \tag{3.8}$$

Case 1. $\gamma_1(z)-\gamma_2(qz)$ is a non-constant polynomial. Using Lemma 2.1 in Eq (3.8), we have

$$\frac{A_1}{A_2}e^{\gamma_2(z)-\gamma_2(qz)}\equiv 1 \text{ or } -\frac{A_1}{A_2}e^{\gamma_1(qz)-\gamma_2(qz)}\equiv 1.$$

If $\frac{A_1}{A_2}e^{\gamma_2(z)-\gamma_2(qz)} \equiv 1$, then $\gamma_2(z)-\gamma_2(qz)$ is a constant. By Lemma 2.3, $\gamma_2(z)-\gamma_2(qz) \equiv 0$. Thus, we have $\frac{A_1}{A_2} = 1$, which contradicts with $\alpha \neq 0, 1$.

If $-\frac{A_1}{A_2}e^{\gamma_1(qz)-\gamma_2(qz)} \equiv 1$, then it follows from Eq (3.8) that $e^{\gamma_1(z)-\gamma_2(z)} = -\frac{A_1}{A_2}$. In view of Eq (3.4), we get that

$$-\frac{A_1}{A_2}e^{2p(qz)} \equiv 1 \text{ and } e^{2p(z)} = -\frac{A_1}{A_2}.$$

It is easy to get that $p(z)$ is a constant and $\frac{A_2}{A_1} = \frac{A_1}{A_2}$. This leads to $A_1^2 = A_2^2$, which contradicts with $\alpha^2 \neq 0, 1$.

Case 2. $\gamma_1(z)-\gamma_2(qz)$ is a constant. Let $\kappa = \gamma_1(z)-\gamma_2(qz), \kappa \in \mathbb{C}$. Then, $\gamma_2(qz) = \gamma_1(z)-\kappa$. In view of Eq (3.4), $2p(z) = \gamma_1(z)-\gamma_2(z)$. Equation (3.8) reduces to

$$\frac{A_2}{A_1}(e^{\kappa}-1) + e^{\kappa}e^{-2p(z)} = e^{2p(qz)}. \tag{3.9}$$

Case 2.1. $\kappa = \gamma_1(z)-\gamma_2(qz) \equiv 0$. From Eq (3.9) we have $e^{2(p(z)+p(qz))} = 1$, which gives that $p(z)+p(qz) \equiv 0$. It follows from Eq (3.4) that

$$0 \equiv p(z)+p(qz) = \frac{1}{2}(\gamma_1(z)-\gamma_2(z)+\gamma_1(qz)-\gamma_2(qz)) = \frac{1}{2}(-\gamma_2(z)+\gamma_1(qz)).$$

Further, we have $\gamma_1(z) \equiv \gamma_1(q^2z)$ and $\gamma_2(z) \equiv \gamma_2(q^2z)$. Recall that $f(z)$ is transcendental, then from Eq (3.5) we have that $\gamma_1(z)$ and $\gamma_2(z)$ cannot be constant at the same time. By the assumption that $q \neq 0, \pm 1$, we get a contradiction.

Case 2.2. $\kappa = \gamma_1(z)-\gamma_2(qz) \not\equiv 0$. Using the Nevanlinna second fundamental theorem for $e^{2p(qz)}$, we have

$$\begin{aligned} T(r, e^{2p(qz)}) &\leq \overline{N}(r, e^{2p(qz)}) + \overline{N}\left(r, \frac{1}{e^{2p(qz)}}\right) + \overline{N}\left(r, \frac{1}{e^{2p(qz)} - \frac{A_2}{A_1}(e^{\kappa}-1)}\right) + S(r, e^{2p(qz)}) \\ &\leq \overline{N}\left(r, \frac{1}{e^{2p(z)}}\right) + S(r, e^{2p(qz)}) = S(r, e^{2p(qz)}), \end{aligned}$$

which shows that $p(qz)$ is a constant.

We claim that $g(z)$ is a polynomial. If $g(z)$ is a constant, then by combining with $p(qz)$ as a constant and Eq (3.4), we have both $\gamma_1(z)$ and $\gamma_2(z)$ are constants. From Eq (3.5), we have $f(z)$ is a constant, which contradicts with $f(z)$ is transcendental.

Thus, $\deg(g(z)) \geq 1$. Set $p(z) \equiv \eta$, where $\eta \in \mathbb{C}$. Then, it follows from Eqs (3.4) and (3.8) that

$$\left(e^{2\eta} + \frac{A_1}{A_2}\right)e^{\frac{g(z)-g(qz)}{2}} = 1 + \frac{A_1}{A_2}e^{2\eta}. \tag{3.10}$$

If $g(z)-g(qz)$ is a non-constant polynomial, then by using Lemma 2.2 in Eq (3.10), we have

$$\begin{cases} e^{2\eta} + \dfrac{A_1}{A_2} = 0, \\ 1 + \dfrac{A_1}{A_2}e^{2\eta} = 0. \end{cases}$$

It gives $A_1^2 = A_2^2$, which contradicts with $\alpha^2 \neq 0, 1$. Thus, $g(z) - g(qz)$ is a constant.

Further, by Lemma 2.3, we obtain $g(z) - g(qz) \equiv 0$ and $q^{\deg(g(z))} = 1$. Since $q \neq \pm 1$, then $\deg(g(z)) \neq 1, 2$. Combining with $\deg(g(z)) \geq 1$, we have $\deg(g(z)) > 2$. Moreover, Eq (3.10) reduces to

$$e^{2\eta} + \frac{A_1}{A_2} = 1 + \frac{A_1}{A_2} e^{2\eta}.$$

Thus, we have $\frac{A_1}{A_2} - 1 = \left(\frac{A_1}{A_2} - 1\right) e^{2\eta}$. Since $A_1 \neq A_2$, then $\frac{A_1}{A_2} - 1 \neq 0$. Hence, we have $e^{2\eta} = 1$. It gives $e^{\eta} = \pm 1$, i.e., $e^{p(z)} \equiv \pm 1$.

From Eqs (3.4) and (3.5), we have

$$f(z) = \frac{\sqrt{2}(A_1 e^{p(z)} + A_2 e^{-p(z)})}{2} e^{\frac{g(z)}{2}} = \frac{\pm\sqrt{2}(A_1 + A_2)}{2} e^{\frac{g(z)}{2}}.$$

And together with Eq (1.11), we obtain

$$f(z) = \pm \frac{\sqrt{2}}{2(\sqrt{1+\alpha})} e^{\frac{g(z)}{2}}.$$

We completed the proof of Theorem 1.4.

4. Proof of Theorem 1.5

Let $f(z)$ be a transcendental entire solution with finite order of Eq (1.10). Using the same argument as in the proof of Theorem 1.4, we have

$$f'(z) = \frac{1}{\sqrt{2}}(A_1 e^{\gamma_1(z)} + A_2 e^{\gamma_2(z)}) \tag{4.1}$$

and

$$f(qz) = \frac{1}{\sqrt{2}}(A_2 e^{\gamma_1(z)} + A_1 e^{\gamma_2(z)}). \tag{4.2}$$

In view of Eqs (4.1) and (4.2), it follows that

$$f'(qz) = \frac{1}{\sqrt{2}}(A_1 e^{\gamma_1(qz)} + A_2 e^{\gamma_2(qz)}) = \frac{1}{\sqrt{2}q}(A_2 \gamma_1'(z) e^{\gamma_1(z)} + A_1 \gamma_2'(z) e^{\gamma_2(z)}).$$

This leads to

$$\frac{\gamma_1'(z)}{q} e^{\gamma_1(z)-\gamma_2(qz)} + \frac{A_1}{qA_2}\gamma_2'(z) e^{\gamma_2(z)-\gamma_2(qz)} - \frac{A_1}{A_2} e^{\gamma_1(qz)-\gamma_2(qz)} = 1. \tag{4.3}$$

Case 1. $\gamma_1(qz) - \gamma_2(qz)$ is a constant. From Eq (3.4), we have $\gamma_1(qz) - \gamma_2(qz) = 2p(qz)$. Thus, $p(z)$ is a constant. Let $\iota \equiv e^{p(z)}$, where $\iota \in \mathbb{C}\backslash\{0\}$.

Furthermore, we have $\deg(g(z)) \geq 1$. Otherwise, from Eq (3.4), we have that both $\gamma_1(z)$ and $\gamma_2(z)$ are constants. It follows from Eq (4.1) that $f'(z)$ is a constant, which conflicts with $f(z)$ being transcendental.

Combining with Eqs (3.4) and (4.3), we get that

$$\left(\frac{\iota^2}{q} + \frac{A_1}{qA_2}\right) \frac{g'(z)}{2} e^{\frac{g(z)-g(qz)}{2}} = 1 + \frac{A_1}{A_2}\iota^2. \tag{4.4}$$

If $g(z)-g(qz)$ is a non-constant polynomial, then by using Lemma 2.2 in Eq (4.4), we get that

$$\begin{cases}\left(\dfrac{\iota^2}{q}+\dfrac{A_1}{qA_2}\right)\dfrac{g'(z)}{2}=0,\\ 1+\dfrac{A_1}{A_2}\iota^2=0.\end{cases} \tag{4.5}$$

The second equation of (4.5) gives that $\iota^2=-\frac{A_2}{A_1}$. Substituting this into the first equation of (4.5), we have

$$\left(\frac{-A_2}{qA_1}+\frac{A_1}{qA_2}\right)\frac{g'(z)}{2}=0.$$

Since $\deg(g(z))\geq 1$ and $q\neq 0,1$, then we have $\frac{-A_2}{A_1}+\frac{A_1}{A_2}=0$. It gives that $A_1^2=A_2^2$, which contradicts with $\alpha^2\neq 0,1$.

If $g(z)-g(qz)$ is a constant, by Lemma 2.3, we have $g(z)-g(qz)\equiv 0$ and $q^{\deg(g(z))}=1$. Since $q\neq 1$, then $\deg(g(z))\neq 1$. Note that $\deg(g(z))\geq 1$, then $\deg(g(z))\geq 2$.

Equation (4.4) reduces to

$$\left(\frac{\iota^2}{q}+\frac{A_1}{qA_2}\right)\frac{g'(z)}{2}=1+\frac{A_1}{A_2}\iota^2.$$

This implies that $\dfrac{\iota^2}{q}+\dfrac{A_1}{qA_2}=0$ and $1+\dfrac{A_1}{A_2}\iota^2=0$. Similar to the above, we also have $A_1^2=A_2^2$, which is a contradiction.

Case 2. $\gamma_1(qz)-\gamma_2(qz)$ is a non-constant polynomial. Since $\gamma_1(qz)-\gamma_2(qz)=2p(qz)$, then we have $p(z)$ is a non-constant polynomial.

Next, we show that $\gamma_1'(z)\not\equiv 0$ and $\gamma_2'(z)\not\equiv 0$. From Eq (4.3), it is easy to get that $\gamma_1'(z)\equiv 0$ and $\gamma_2'(z)\equiv 0$ cannot hold at the same time.

If $\gamma_1'(z)\equiv 0$ and $\gamma_2'(z)\not\equiv 0$, then Eq (4.3) reduces to

$$\frac{A_1}{qA_2}\gamma_2'(z)e^{\gamma_2(z)-\gamma_2(qz)}-\frac{A_1}{A_2}e^{\gamma_1(qz)-\gamma_2(qz)}=1.$$

Using the Nevanlinna second fundamental theorem for $e^{\gamma_1(qz)-\gamma_2(qz)}$, we have that

$$\begin{aligned}T(r,e^{\gamma_1(qz)-\gamma_2(qz)})&\leq \overline{N}(r,e^{\gamma_1(qz)-\gamma_2(qz)})+\overline{N}\left(r,\frac{1}{e^{\gamma_1(qz)-\gamma_2(qz)}}\right)\\&\quad+\overline{N}\left(r,\frac{1}{e^{\gamma_1(qz)-\gamma_2(qz)}+\dfrac{A_2}{A_1}}\right)+S(r,e^{\gamma_1(qz)-\gamma_2(qz)})\\&\leq N\left(r,\frac{1}{\dfrac{A_1}{qA_2}\gamma_2'(z)e^{\gamma_2(z)-\gamma_2(qz)}}\right)+S(r,e^{\gamma_1(qz)-\gamma_2(qz)})\\&=S(r,e^{\gamma_1(qz)-\gamma_2(qz)}),\end{aligned}$$

which is a contradiction.

Similarly, if $\gamma_1'(z) \not\equiv 0$ and $\gamma_2'(z) \equiv 0$, we also get a contradiction.

Then, by using Lemma 2.1 in Eq (4.3), we have

$$\frac{\gamma_1'(z)}{q} e^{\gamma_1(z)-\gamma_2(qz)} \equiv 1 \text{ or } \frac{A_1}{qA_2}\gamma_2'(z) e^{\gamma_2(z)-\gamma_2(qz)} \equiv 1.$$

Case 2.1. If $\frac{A_1}{qA_2}\gamma_2'(z)e^{\gamma_2(z)-\gamma_2(qz)} \equiv 1$, it implies that $\gamma_2'(z)$ is a nonzero constant, and $\gamma_2(z) - \gamma_2(qz)$ is a constant.

By Lemma 2.3, we have $\gamma_2(z) - \gamma_2(qz) \equiv 0$ and $q^{\deg(\gamma_2(z))} = 1$. Since $q \neq 1$, then $\deg(\gamma_2(z)) \neq 1$, which contradicts with $\gamma_2'(z)$ being a nonzero constant.

Case 2.2. If $\frac{\gamma_1'(z)}{q}e^{\gamma_1(z)-\gamma_2(qz)} \equiv 1$, then from Eq (4.3) we have

$$\frac{\gamma_2'(z)}{q} e^{\gamma_2(z)-\gamma_1(qz)} = 1. \tag{4.6}$$

The above two equations give that $\gamma_1(z) - \gamma_2(qz)$ and $\gamma_2(z) - \gamma_1(qz)$ are constants. Moreover, we also have that $\gamma_i'(z)(i = 1, 2)$ are nonzero constants, i.e., $\deg(\gamma_i(z)) = 1(i = 1, 2)$.

Set

$$\eta_1 = \gamma_1(z) - \gamma_2(qz) \text{ and } \eta_2 = \gamma_2(z) - \gamma_1(qz),$$

where $\eta_1, \eta_2 \in \mathbb{C}$.

In view of Eq (3.4), we have

$$\begin{cases} 2p(z) + 2p(qz) = [\gamma_1(z) - \gamma_2(qz)] - [\gamma_2(z) - \gamma_1(qz)] = \eta_1 - \eta_2, \\ g(z) - g(qz) = [\gamma_1(z) - \gamma_2(qz)] + [\gamma_2(z) - \gamma_1(qz)] = \eta_1 + \eta_2. \end{cases} \tag{4.7}$$

By Lemma 2.3, we get that $q^{\deg(p(z))} = -1$ and $q^{\deg(g(z))} = 1$. Since $q \neq 1$, then $\deg(g(z)) \neq 1$.

We now show that $\deg(g(z)) = 0$. If $\deg(g(z)) \geq 2$, by combining with $\deg(\gamma_i(z)) = 1$ and Eq (3.4), then we have $\deg(p(z)) = \deg(g(z))$. Therefore, $q^{\deg(p(z))} = q^{\deg(g(z))} = 1$, which contradicts with $q^{\deg(p(z))} = -1$. Hence, we have $g(z) \equiv \beta$, where $\beta \in \mathbb{C}$.

Recall that $\deg \gamma_i(z) = 1(i = 1, 2)$. It follows from Eq (3.4) that $\gamma_1(z) + \gamma_2(z) = g(z) \equiv \beta$.

Set

$$\gamma_1(z) = tz + y_1 \text{ and } \gamma_2(z) = -tz + y_2, \tag{4.8}$$

where $t \in \mathbb{C}\backslash\{0\}, y_1, y_2 \in \mathbb{C}$ such that $\beta = y_1 + y_2$.

It follows from Eqs (3.4) and (4.8) that $p(z) = tz + \frac{y_1-y_2}{2}$. And together with $q^{\deg(p(z))} = -1$, then we have $q = -1$.

By substituting $q = -1$ and Eq (4.8) into $\frac{\gamma_1'(z)}{q}e^{\gamma_1(z)-\gamma_2(qz)} \equiv 1$ and Eq (4.6), we obtain

$$-te^{y_1-y_2} = 1 \text{ and } te^{y_2-y_1} = 1,$$

respectively. It gives that $t = \pm i$.

Furthermore, substituting Eq (4.8) into Eq (4.1), we have

$$f'(z) = \frac{1}{\sqrt{2}}(A_1 e^{tz+y_1} + A_2 e^{-tz+y_2}).$$

Integration of the above equation gives that

$$f(z) = \frac{\sqrt{2}}{2t}(A_1 e^{tz+y_1} - A_2 e^{-tz+y_2}).$$

We completed the proof of Theorem 1.5.

5. Conclusions

In this paper, we showed that the explicit forms for entire solutions of two certain types of Fermat-type q-difference differential equations. In addition, we have given specific examples to illustrate our results.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors state no conflict of interest.

References

1. A. Wiles, Modular elliptic curves and Fermat's last theorem, *Ann. Math.*, **141** (1995), 443–551.
2. F. Gross, On the equation $f^n + g^n = 1$, *Bull. Amer. Math. Soc.*, **72** (1966), 86–88.
3. F. Gross, On the equation $f^n + g^n = h^n$, *Am. Math. Mon.*, **73** (1966), 1093–1096. https://doi.org/10.2307/2314644
4. I. N. Baker, On a class of meromorphic functions, *Proc. Am. Math. Soc.* **17** (1966), 819–822. https://doi.org/10.2307/2036259
5. C. C. Yang, A generalization of a theorem of p. montel on entire functions, *Proc. Am. Math. Soc.*, **26** (1970), 332–334. https://doi.org/10.2307/2036399
6. K. Liu, T. B. Cao, H. Z. Cao, Entire solutions of Fermat type differential-difference equations, *Arch. Math.*, **99** (2012), 147–155. https://doi.org/10.1007/s00013-012-0408-9
7. K. Liu, L. Z. Yang, A note on meromorphic solutions of Fermat types equations, *An. Stiint. Univ. Al. I. Cuza Lasi. Mat. (N.S.)*, **1** (2016), 317–325.
8. K. Liu, L. Z. Yang, On entire solutions of some differential-difference equations, *Comput. Methods Funct. Theory*,**13** (2013), 433–447. https://doi.org/10.1007/s40315-013-0030-2
9. K. Liu, T. B. Cao, Entire solutions of Fermat type q-difference differential equations, *Electron. J. Differ. Equ.*, **59** (2013), 1–10.

10. J. Luo, H. Y. Xu, F. Hu, Entire solutions for several general quadratic trinomial differential difference equations, *Open Math.*, **19** (2021), 1018–1028. https://doi.org/10.1515/math-2021-0080
11. W. K. Hayman, *Meromorphic functions*, Oxford: Clarendon Press, 1964.
12. C. C. Yang, H. X. Yi, *Uniqueness theory of meromorphic functions*, Dordrecht: Springer, 2003.
13. Y. M. Chiang, S. J. Feng, On the nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane, *Ramanujan J.*, **16** (2008), 105–129. https://doi.org/10.1007/s11139-007-9101-1
14. R. G. Halburd, R. J. Korhonen, Difference analogue of the lemma on the logarithmic derivative with applications to difference equations, *J. Math. Anal. Appl.*, **314** (2006), 477–487. https://doi.org/10.1016/j.jmaa.2005.04.010
15. K. Ishizaki, A note on the functional equation $f^n + g^n + h^n = 1$ and some complex differential equations, *Comput. Methods Funct. Theory*, **2** (2003), 67–85. https://doi.org/10.1007/BF03321010
16. B. Q. Li, Entire solutions of $(u_{z_1})^m + (u_{z_2})^n = e^g$, *Nagoya Math. J.*, **178** (2005), 151–162.
17. B. Q. Li, On certain non-linear differential equations in complex domains, *Arch. Math.*, **91** (2008), 344–353. https://doi.org/10.1007/s00013-008-2648-2
18. K. Liu, Meromorphic functions sharing a set with applications to difference equations, *J. Math. Anal. Appl.*, **359** (2009), 384–393. https://doi.org/10.1016/j.jmaa.2009.05.061
19. M. L. Liu, L. Y. Gao, Transcendental solutions of systems of complex differential-difference equations, *Sci. Sin. Math.*, **49** (2019), 1633. https://doi.org/10.1360/N012018-00061
20. J. F. Tang, L. W. Liao, The transcendental meromorphic solutions of a certain type of nonlinear differential equations, *J. Math. Anal. Appl.*, **334** (2007), 517–527. https://doi.org/10.1016/j.jmaa.2006.12.075
21. H. Y. Xu, Y. Y Jiang, Results on entire and meromorphic solutions for several systems of quadratic trinomial functional equations with two complex variables, *RACSAM*, **116** (2022), 8. https://doi.org/10.1007/s13398-021-01154-9
22. H. Y. Xu, Y. H. Xu, X. L. Liu, On solutions for several systems of complex nonlinear partial differential equations with two variables, *Anal. Math. Phys.*, **13** (2023), 47. https://doi.org/10.1007/s13324-023-00811-z
23. H. Y. Xu, L. Xu, Transcendental entire solutions for several quadratic binomial and trinomial PDEs with constant coefficients, *Anal. Math. Phys.*, **12** (2022), 64. https://doi.org/10.1007/s13324-022-00679-5