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Research article

Global gradient estimates in directional homogenization

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Abstract: In this research, we investigate a higher regularity result in periodic directional homogenization for divergence-form elliptic systems with discontinuous coefficients in a bounded nonsmooth domain. The coefficients are assumed to have small bounded mean oscillation (BMO) seminorms and the domain has the δ-Reifenberg property. Under these assumptions we derive global uniform Calderón-Zygmund estimates by proving that the gradient of the weak solution is as integrable as the given nonhomogeneous term.

Keywords: directional homogenization; uniform estimate; Calderón-Zygmund theory; Reifenberg domain; BMO coefficient
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1. Introduction

In this study, we consider elliptic systems in directional homogenization of the following form:

$$\begin{cases} D_\alpha\left(A_{ij}^{\alpha\beta,\epsilon}(x)D_\beta u_\epsilon^j(x)\right) = D_\alpha f_\alpha^i(x) & \text{in} \quad \Omega \\ u_\epsilon^i(x) = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1.1}$$

for $1 \le \alpha,\beta \le n$ and $1 \le i, j \le m$ with $m \ge 1$, where the nonhomogeneous term $F = \{f_\alpha^i\}$ is given by a matrix-valued function. Here, Ω is a bounded domain in $\mathbb{R}^n$ with $n \ge 2$ and $0 < \epsilon \le 1$. Especially, in order to treat directional homogenization we define the coefficients $A^\epsilon = \{A_{ij}^{\alpha\beta,\epsilon}\}$ for $0 < \epsilon \le 1$ from $A = \{A_{ij}^{\alpha\beta}\}$, $A_{ij}^{\alpha\beta} : \mathbb{R}^n \to \mathbb{R}$, to be as follows:

$$A_{ij}^{\alpha\beta}(x) = A_{ij}^{\alpha\beta,1}(x) \quad \text{and} \quad A_{ij}^{\alpha\beta,\epsilon}(x', x'') = A_{ij}^{\alpha\beta}\left(x', \frac{x''}{\epsilon}\right) \tag{1.2}$$

where $x = (x', x'')$, $x' = (x_1, \cdots, x_l) \in \mathbb{R}^l$ and $x'' = (x_{l+1}, \cdots, x_n) \in \mathbb{R}^{n-l}$ with $0 \le l \le n$. In addition, we assume the following periodicity condition on $\left\{A_{ij}^{\alpha\beta}(x)\right\}$:

$$A_{ij}^{\alpha\beta}(x', x'' + z'') = A_{ij}^{\alpha\beta}(x', x'') \quad ((x', x'') \in \mathbb{R}^l \times \mathbb{R}^{n-l}, z'' \in \mathbb{Z}^{n-l}). \tag{1.3}$$

The coefficients are assumed to have uniform ellipticity and uniform boundedness. In other words, we assume that there exist positive constants ν and L such that

$$\nu|\xi|^2 \le A_{ij}^{\alpha\beta}(x)\xi_\alpha^i \xi_\beta^j \quad \text{and} \quad \|A\|_{L^\infty(\mathbb{R}^n, \mathbb{R}^{mn\times mn})} \le L, \tag{1.4}$$

for every matrix $\xi \in \mathbb{R}^{mn}$ and for almost every $x \in \mathbb{R}^n$. We note that since if $l = n$, we do not need to treat homogenization and if $l = 0$, our problem is periodic homogenization, throughout this research in order to consider directional homogenzition we assume that $1 \le l \le n - 1$.

In this paper, we consider the weak solution $u_\epsilon = (u_\epsilon^1, \cdots, u_\epsilon^m) \in H_0^1(\Omega, \mathbb{R}^m)$ to (1.1) which satisfies

$$\int_\Omega A_{ij}^{\alpha\beta,\epsilon} D_\beta u_\epsilon^j D_\alpha \phi^i dx = \int_\Omega f_\alpha^i D_\alpha \phi^i dx, \quad \forall \phi = (\phi^1, \ldots, \phi^m) \in H_0^1(\Omega, \mathbb{R}^m). \tag{1.5}$$

Here we note that if $F \in L^2(\Omega, \mathbb{R}^{mn})$, the weak solution $u_\epsilon \in H_0^1(\Omega, \mathbb{R}^m)$ exists and satisfies the estimate

$$\|Du_\epsilon\|_{L^2(\Omega)} \le c\|F\|_{L^2(\Omega)}, \tag{1.6}$$

where the constant c does not depend on ϵ, by the Lax-Milgram lemma.

Now, we introduce some basic facts for directional homogenization; see for details in [2, 16]. The matrix of correctors $\chi = \left\{\chi_\alpha^{ij}(x', x'')\right\}$, with $1 \le i, j \le m$ and $l + 1 \le \alpha \le n$, is the weak solution to the following cell problem:

$$\begin{cases} -D_\alpha\left(A_{ij}^{\alpha\beta}(x', x'')D_\beta \chi_\gamma^{jk}(x', x'')\right) = D_\alpha A_{ik}^{\alpha\gamma}(x', x'') \\ \int_{[0,1]^{n-l}} \chi_\gamma^{jk}(x', x'')dx'' = 0 \\ \chi_\gamma^{jk}(x', x'') \text{ is } \mathbb{Z}^{n-l} \text{ periodic,} \end{cases} \tag{1.7}$$

which satisfies the following estimate :

$$\|D_{x''}\chi(x', x'')\|_{L^2([0,1]^{n-l})} \le c(\nu, L, m, n, l). \tag{1.8}$$

Let

$$A_{ij}^{\alpha\beta,0}(x') = \int_{[0,1]^{n-l}} \left(A_{ij}^{\alpha\beta}(x', x'') + A_{ik}^{\alpha\gamma}(x', x'')D_\gamma \chi_\beta^{kj}(x', x'')\right) dx''. \tag{1.9}$$

Then the linear elliptic system given by

$$\begin{cases} D_\alpha\left(A_{ij}^{\alpha\beta,0}(x')D_\beta u_0^j(x)\right) = D_\alpha f_\alpha^i(x) & \text{in} \quad \Omega \\ u_0^i(x) = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1.10}$$

is the homogenized problem of (1.1), whose weak solution u_0 of (1.10) is the weak limit of the weak solution u_ϵ in $H_0^1(\Omega, \mathbb{R}^m)$ as $\epsilon \to 0$.

Regularity theories for elliptic equations in homogenization are widely studied for partial differential equations; see [1, 3–5, 11, 15–18, 20, 23, 27] and the references therein. Among these, under the settings, our goal is to obtain global uniform Calderón-Zygmund estimates, that is, we would like to prove that if $F \in L^p(\Omega, \mathbb{R}^{mn})$, then the L^p norm of Du_ϵ is controlled by the L^p norm of F and is independent of ϵ. The authors proved the Calderón-Zygmund theory for (1.1) under the condition of periodic homogenization in [3]. Recently, the authors of [15, 16] gave several different interior regularity results for directional homogenization. Given these viewpoints, here, we consider the global estimates in directional homogenization.

For homogenization problems, we want to derive estimates which are independent of $0 < \epsilon \leq 1$. Since our desired result includes the case $\epsilon = 1$ for (1.1) when there is no homogenization, our research relies on the conditions that the L^p regularity theory for the gradient is established; see [6–9, 14, 22]. Thus, we prove the global Calderón-Zygmund theory for (1.1) subject to periodic directional homogenization under the conditions of Definitions 2.2 and 2.3 described in Section 2. In fact, in view of the condition for the coefficients of the regularity theory for (1.1) with $\epsilon = 1$ in this literature, we may consider some weaker assuptions than Definition 2.2 but in that case, we can only obtain some local results instead of the global regularity; see Remark 2.8.

To prove our result, we use a perturbation argument based on localization which includes scaling, translation and rotation. In fact, even though in (1.1) with (1.2) the direction of homogenization is globally fixed so that the direction is changed under rotation, the condition of the coefficients in Definition 2.2 below is invariant under rotation. This makes our method suitable for directional homogenization. Also, for directional homogenization, when we solve our problem, there are some differences between the x'-direction which is not involved with homogenization and the x''-direction for homogenization. The former gives macroscopic properties in the x'-direction and the latter represents microscopic oscillation in the x''-direction. With this observation, we can apply estimates from the case without homogenization; see [6, 7] for x' and results corresponding to periodic homogenization in [3] for x'' in our proof; see details in Lemma 3.3.

This paper is organized as follows. In Section 2, we introduce some notations and definitions and announce our result as Theorem 2.6. In Section 3, we show our key lemma, Lemma 3.3, and then finally give the proof of the result.

2. Assumptions and main result

We start this section with some notations and definitions.

Notations 2.1. *(1) An open ball in $\mathbb{R}^n$ with a center y with radius $r > 0$ is defined to be*

$$B_r(y) = \{x \in \mathbb{R}^n : |x - y| < r\}.$$

If the center is the origin, we denote $B_r(0)$ by B_r. Similarly, for $y = (y', y'') \in \mathbb{R}^l \times \mathbb{R}^{n-l}$, an open ball in $\mathbb{R}^l$ with the center y' with radius $r > 0$ is defined to be

$$B'_r(y') = \{x' \in \mathbb{R}^l : |x' - y'| < r\},$$

an open ball in $\mathbb{R}^{n-l}$ with the center y'' with radius $r > 0$ is defined to be

$$B''_r(y'') = \{x'' \in \mathbb{R}^{n-l} : |x'' - y''| < r\},$$

and if the center is the origin, $y = (0', 0'')$, then we denote $B'_r(0') \subset \mathbb{R}^l$ by B'_r and $B''_r(0'') \subset \mathbb{R}^{n-l}$ by B''_r.

(2) The integral average of $g \in L^1(U)$ over the bounded domain U in $\mathbb{R}^n$ is denoted by

$$\overline{g}_U = \fint_U g(x)dx = \frac{1}{|U|}\int_U g(x)dx.$$

When $U = U' \times U'' \subset \mathbb{R}^l \times \mathbb{R}^{n-l}$, we denote the integral average over $U' \subset \mathbb{R}^l$ by

$$\overline{g}_{U'}(x'') = \fint_{U'} g(x)dx' = \frac{1}{|U'|}\int_{U'} g(x)dx'$$

and we denote the integral average over $U'' \subset \mathbb{R}^{n-l}$ by

$$\overline{g}_{U''}(x') = \fint_{U''} g(x)dx'' = \frac{1}{|U''|}\int_{U''} g(x)dx''.$$

(3) $B_\rho^+ = B_\rho \cap \{x_n > 0\}$, $T_\rho = B_\rho \cap \{x_n = 0\}$.

(4) $\Omega_\rho(y) = B_\rho(y) \cap \Omega$, $\Omega_\rho = \Omega_\rho(0)$, $\partial_w\Omega_\rho(y) = B_\rho(y) \cap \partial\Omega$, $\partial_w\Omega_\rho = \partial_w\Omega_\rho(0)$.

For our global regularity result we assume that the coefficient A enjoys the small bounded mean oscillation (BMO) condition which is a generalization of the vanishing mean oscillation (VMO) condition. The following is the precise definition that is to be used throughout this paper.

Definition 2.2. *We say that $A_{ij}^{\alpha\beta}$ is (δ, R)-vanishing if*

$$\sup_{0<r\le R}\sup_{y\in\mathbb{R}^n}\fint_{B_r(y)}\left|A_{ij}^{\alpha\beta}(x) - \overline{A_{ij}^{\alpha\beta}}_{B_r(y)}\right|^2 dx \le \delta^2.$$

Additionally, we consider the domain Ω is a Reifenberg domain; see [26], which is an extension of Lipschitz domains with small Lipschitz constants. The definition is as follows:

Definition 2.3. *Let Ω be a bounded domain in $\mathbb{R}^n$. We say that Ω is (δ, R)-Reifenberg flat if for every $x \in \partial\Omega$ and every $r \in (0, R]$, there exists a coordinate system $\{y_1, \ldots, y_n\}$ that is dependent on r and x so that $x = 0$ in this coordinate system and*

$$B_r \cap \{y_n > \delta r\} \subset B_r \cap \Omega \subset B_r \cap \{y_n > -\delta r\}. \tag{2.1}$$

Similar to Definition 2.2, we can define $A_{ij}^{\alpha\beta}$ as (δ, R)-*vanishing with respect to* x' if

$$\sup_{0<r\le R}\sup_{(y',y'')\in\mathbb{R}^l\times\mathbb{R}^{n-l}}\fint_{B_r'(y')\times B_r''(y'')}\left|A_{ij}^{\alpha\beta}(x', x'') - \overline{A_{ij}^{\alpha\beta}}_{B_r'(y')}(x'')\right|^2 dx \le \delta^2 \tag{2.2}$$

and $A_{ij}^{\alpha\beta}$ is (δ, R)-*vanishing with respect to* x'' if

$$\sup_{0<r\le R}\sup_{(y',y'')\in\mathbb{R}^l\times\mathbb{R}^{n-l}}\fint_{B_r'(y')\times B_r''(y'')}\left|A_{ij}^{\alpha\beta}(x', x'') - \overline{A_{ij}^{\alpha\beta}}_{B_r''(y'')}(x')\right|^2 dx \le \delta^2. \tag{2.3}$$

Remark 2.4. *We can see that $A_{ij}^{\alpha\beta}$ being (δ, R)-vanishing is equivalent to the condition that $A_{ij}^{\alpha\beta}$ is (δ, R)-vanishing with respect to x' and x'' in the following sense. From direct computations by using the properties of averages, if $A_{ij}^{\alpha\beta}$ is (δ, R)-vanishing with respect to x' and x'', then $A_{ij}^{\alpha\beta}$ is $(c_1\delta, R)$-vanishign; conversely if $A_{ij}^{\alpha\beta}$ is (δ, R)-vanishing, then $A_{ij}^{\alpha\beta}$ is $\left(c_2\delta, \frac{1}{\sqrt{2}}R\right)$-vanishing with respect to x' and x'' where $c_1 = \sqrt{\frac{2|B_r'\times B_r''|}{|B_r|}}$ and $c_2 = \sqrt{\frac{|B_{2r}|}{|B_r'\times B_r''|}}$. From this equivalence, we consider the (δ, R)-vanishing condition instead of the (δ, R)-vanishing conditions with respect to x' and x'' throughout this paper since the (δ, R)-vanishing condition has a rotational invariant property.*

Remark 2.5. *We give some comments on Definitions 2.2 and 2.3. First of all, by the scaling invariant property of our problem* (1.1)*, the value of R in the definitions of both coefficients and domains can be* 1 *or any other constants greater than* 1*. For this reason $R \geq 1$ is to be selected for our purpose. In addition, the constant δ to be determined is also invariant under this scaling.*

For (2.1)*, with Ω being a Reifenberg flat domain, it is known that $\delta \leq \delta_*$ for some constant $\delta_* = \delta_*(n)$. We note that $\delta_* = \delta_*(n) < 2^{-n-1} \leq \frac{1}{8}$ for $n \geq 2$; see [28]. From this, we can assume that $\delta < \frac{1}{8}$ throughout this paper. Moreover, even though the Reifenberg flatness condition given by* (2.1) *does not mean any smoothness on the boundary, this gives the following measure density condition:*

$$\frac{|B_r(y)|}{|B_r(y) \cap \Omega|} \leq \left(\frac{2}{1-\delta}\right)^n \leq \left(\frac{16}{7}\right)^n \tag{2.4}$$

for every $y \in \partial\Omega$ and $r \in (0, R]$. This will be used in our L^2 approach since (2.4) *implies the p-capacity condition with $p = 2$; see [25, Section 2.2.3], which makes us apply the higher integrability result in [21] to our method.*

Now let us state the global estimate of this paper.

Theorem 2.6. *Suppose that $F \in L^p(\Omega, \mathbb{R}^{mn})$ for some $2 < p < \infty$. Then there exists a small positive constant $\delta_0 = \delta_0(\nu, L, m, n, p)$ such that if $A_{ij}^{\alpha\beta}$ is $(\delta, 336)$-vanishing and Ω is $(\delta, 336)$-Reifenberg flat with $\delta \leq \delta_0$, then for the weak solution $u_\epsilon \in H_0^1(\Omega, \mathbb{R}^m)$ to* (1.1)*, we have*

$$Du_\epsilon \in L^p(\Omega, \mathbb{R}^{mn}) \tag{2.5}$$

with the estimate

$$\|Du_\epsilon\|_{L^p(\Omega)} \leq c\|F\|_{L^p(\Omega)}, \tag{2.6}$$

where the positive constant $c = c(|\Omega|, \nu, L, m, n, p)$ is independent of ϵ.

Remark 2.7. *Since* (1.6) *comes from the Lax-Milgram lemma, Theorem 2.6 holds for $p = 2$ without any assumptions. After the estimate* (2.6) *is obtained for $2 < p < \infty$, the estimate* (2.6) *for the case when $1 < p < 2$ follows by a duality argument if the weak solution $u_\epsilon \in H_0^1(\Omega, \mathbb{R}^m)$ to* (1.1) *satisfies that $Du_\epsilon \in L^p(\Omega, \mathbb{R}^{mn})$ for some $1 < p < 2$.*

Remark 2.8. *In view of the regularity results [6, 9, 14, 22], just for interior estimates or local boundary estimates we can give weaker conditions than those of Theorem 2.6. The weaker conditions are that $A_{ij}^{\alpha\beta}$ is (δ, R)-vanishing of codimension* 1 *in [4] for interior cases and $(A_{ij}^{\alpha\beta}, \Omega)$ is (δ, R)-vanishing of codimension* 1 *in [6, 19] for boundary cases. Both conditions allow $A_{ij}^{\alpha\beta}$ to be merely measurable* *for one variable, while they have small BMO seminorms for the other variables in some appropriate coordinates. Second, Ω is to be a (δ, R) Reifenberg flat domain.*

In fact, for the interior case, our argument in this paper can be applied to the interior version for Theorem 2.6 when the coefficients $A_{ij}^{\alpha\beta}$ are (δ, R)-vanishing of codimension 1. Especially, for the interior case with $l = n - 1$, in view of [6, 12] the interior estimate corresponding to Theorem 2.6 is obtained without considering homogenization since we can regard the direction of homogenization as just a measurable direction.

On the other hand, for the boundary case we cannot consider in general $(A_{ij}^{\alpha\beta}, \Omega)$ to be (δ, R)-vanishing of codimension 1 because of consistency between the coefficients of homogenization and the domain even though the global regularity result holds under this condition when there is no homogenization; see [6]. From this observation, with the same idea in this research, we can obtain a local boundary estimate at the point $x_0 \in \partial\Omega$ whose normal direction is only related to x'' in the sense of Definition 2.3 under the condition that $(A_{ij}^{\alpha\beta}, \Omega)$ is (δ, R)-vanishing of codimension 1.

3. Global gradient estimate

To establish our global gradient estimate, we first introduce some tools for the proof of Theorem 2.6. Our method is based on the Hardy-Littlewood maximal function.

First, let us recall the Hardy-Littlewood maximal function and its basic properties. If we suppose that g is a locally integrable function on $\mathbb{R}^n$, then the Hardy-Littlewood maximal function is given by

$$(\mathcal{M}g)(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |g(y)| dy.$$

If g is defined only on a bounded subset of $\mathbb{R}^n$, then we define

$$\mathcal{M}g = \mathcal{M}\overline{g},$$

where $\overline{g}$ is the zero extension of g from the bounded set to $\mathbb{R}^n$. This maximal function satisfies the conditions of the weak 1-1 estimate and the strong p-p estimates. Also, we define the restricted maximal function

$$\mathcal{M}_U g = \mathcal{M}(g \mathbb{1}_U)$$

where $\mathbb{1}_U$ is the characteristic function of $U \subset \mathbb{R}^n$.

Our goal in this article is to show the L^p integrability of Du_ϵ. For this, we would like to use a sum of certain estimates for super-level sets. The next lemma gives a relation between the integration and summation of super-level sets.

Lemma 3.1. *[10] Assume that g is a nonnegative, measurable function defined on the bounded domain $\Omega \subset \mathbb{R}^n$, and let $\theta > 0$ and $\lambda > 1$ be constants. Then for $0 < q < \infty$, we have*

$$g \in L^q(\Omega) \quad \Longleftrightarrow \quad S = \sum_{k \geq 1} \lambda^{qk} \left| \left\{ x \in \Omega : g(x) > \theta\lambda^k \right\} \right| < \infty$$

and

$$\frac{1}{c} S \leq \|g\|_{L^q(\Omega)}^q \leq c(|\Omega| + S). \tag{3.1}$$

The positive constant c depends only on θ, λ and q.

The following lemma is the Vitali-type covering lemma for our proof. Here, we note that because of the scaling invariant property of δ and R in Definition 2.3 for Reifenberg flat domains, we only need to consider $R = 1$ in the next lemma.

Lemma 3.2. *[7, 29] Assume that C and D are measurable sets with $C \subset D \subset \Omega$ and Ω being $(\delta, 1)$-Reifenberg flat. Also, assume that there exists a small $\eta > 0$ such that*

$$|C| < \eta |B_1| \tag{3.2}$$

and that for each $x \in \Omega$ and $r \in (0, 1]$ with $|C \cap B_r(x)| > \eta |B_r(x)|$, we have

$$B_r(x) \cap \Omega \subset D. \tag{3.3}$$

Then

$$|C| \leq \left(\frac{10}{1-\delta}\right)^n \eta |D|. \tag{3.4}$$

The next one is the main lemma in our argument. This shows the second condition of Lemma 3.2 under our settings. In the following argument, we would like to refer [13, 24] to help the readers.

Lemma 3.3. *Let $2 < p < \infty$. Suppose that $u_\epsilon \in H_0^1(\Omega, \mathbb{R}^m)$ is the weak solution to (1.1). Then there exists a universal constant $\eta = \eta(\nu, L, m, n, p)$ so that a small $\delta = \delta(\nu, L, m, n, p) > 0$ is selected such that if $A_{ij}^{\alpha\beta}$ is $(\delta, 336)$-vanishing, Ω is $(\delta, 336)$-Reifenberg flat, and for all $y \in \Omega$ and every $0 < \rho \leq 1$, $B_\rho(y)$ satisfies*

$$\left|\left\{x \in \Omega : \mathcal{M}(|Du_\epsilon|^2) > N^2\right\} \cap B_\rho(y)\right| > \eta \left|B_\rho(y)\right|, \tag{3.5}$$

where

$$\left(\frac{80}{7}\right)^n N^p \eta = \frac{1}{2}, \tag{3.6}$$

then the following holds:

$$\Omega \cap B_\rho(y) \subset \left\{x \in \Omega : \mathcal{M}(|Du_\epsilon|^2) > 1\right\} \cup \left\{x \in \Omega : \mathcal{M}(|F|^2) > \delta^2\right\}. \tag{3.7}$$

Proof. We prove this by contradiction. We assume that (3.5) holds but (3.7) is false. Then there is a point $y_1 \in \Omega \cap B_\rho(y)$ such that

$$\frac{1}{|B_r(y_1)|}\int_{\Omega_r(y_1)} |Du_\epsilon|^2 dx \leq 1 \quad \text{and} \quad \frac{1}{|B_r(y_1)|}\int_{\Omega_r(y_1)} |F|^2 dx \leq \delta^2 \tag{3.8}$$

for all $r > 0$. Under these conditions, there are two cases that we need to consider. One is $B_{14\rho}(y) \subset \Omega$, which is an interior case, and the other is $B_{14\rho}(y) \not\subset \Omega$, which is a boundary case. Since the proof for the interior case is eventually the same as that of the boundary case, here, we prove this lemma for the boundary case.

Now we consider the case that $B_{14\rho}(y) \not\subset \Omega$. Then we assume that since Ω is $(\delta, 336)$-Reifenberg flat there exists an appropriate coordinate system, after suitable rotation and translation, so that

$$B_{14\rho}(y) \cap \Omega \subset B_{28\rho} \cap \Omega \tag{3.9}$$

and

$$B^+_{336\rho} \subset \Omega_{336\rho} \subset B_{336\rho} \cap \{x_n > -672\delta\}. \tag{3.10}$$

Then from (3.8) we see that

$$\frac{1}{|B_{336\rho}|}\int_{\Omega_{336\rho}} |Du_\epsilon|^2 dx \le \frac{|B_{672\rho}(y_1)|}{|B_{336\rho}|}\frac{1}{|B_{672\rho}|}\int_{\Omega_{672\rho}(y_1)} |Du_\epsilon|^2 dx \le 2^n \tag{3.11}$$

and

$$\frac{1}{|B_{336\rho}|}\int_{\Omega_{336\rho}} |F|^2 dx \le \frac{|B_{672\rho}(y_1)|}{|B_{336\rho}|}\frac{1}{|B_{672\rho}|}\int_{\Omega_{672\rho}(y_1)} |F|^2 dx \le 2^n\delta^2. \tag{3.12}$$

Now we consider the following rescaled maps:

$$\tilde{u}_\epsilon(z) = \frac{u_\epsilon(28\rho z)}{28\rho\sqrt{2^n}}, \quad \tilde{F}(z) = \frac{F(28\rho z)}{\sqrt{2^n}}, \quad \tilde{A}^{\alpha\beta,\epsilon}_{ij}(z) = A^{\alpha\beta,\epsilon}_{ij}(28\rho z) \tag{3.13}$$

for $z \in \tilde{\Omega}_{12} = \dfrac{1}{28\rho}\Omega_{336\rho}$. Then we see that $\tilde{u}_\epsilon$ is a weak solution to the following:

$$\begin{cases} D_\alpha\left(\tilde{A}^{\alpha\beta,\epsilon}_{ij}(z)D_\beta\tilde{u}^j_\epsilon(z)\right) = D_\alpha\tilde{f}^i_\alpha(z) & \text{in} \quad \tilde{\Omega}_{12} \\ \tilde{u}^i_\epsilon(z) = 0 & \text{on} \quad \partial_w\tilde{\Omega}_{12} \end{cases} \tag{3.14}$$

with

$$B^+_{12} \subset \tilde{\Omega}_{12} \subset B_{12} \cap \{z_n > -24\delta\}, \tag{3.15}$$

$$\frac{1}{|B_{12}|}\int_{\tilde{\Omega}_{12}} |D\tilde{u}_\epsilon|^2 dz \le 1, \quad \frac{1}{|B_{12}|}\int_{\tilde{\Omega}_{12}} |\tilde{F}|^2 dz \le \delta^2 \tag{3.16}$$

and $\tilde{A}$ is $(\delta, 12)$-vanishing.

Under these settings, it suffices to consider the following case:

$$\tilde{A}^{\alpha\beta,\epsilon}_{ij}(z) = \tilde{A}^{\alpha\beta,\epsilon}_{ij}(z', z'') = \tilde{A}^{\alpha\beta}_{ij}\left(z', \frac{z''}{\epsilon}\right).$$

This is because the $(\delta, 12)$-vanishing condition, which is a small BMO condition, is invariant under rotation for the coefficients, even though the direction of homogenization for our problem (1.1) is changed under rotation.

Next, we let $\tilde{w}_\epsilon \in H^1(\tilde{\Omega}_{11}, \mathbb{R}^m)$ be the weak solution to the following:

$$\begin{cases} D_\alpha\left(\tilde{A}^{\alpha\beta,\epsilon}_{ij}(z)D_\beta\tilde{w}^j_\epsilon(z)\right) = 0 & \text{in} \quad \tilde{\Omega}_{11} \\ \tilde{w}^i_\epsilon(z) = \tilde{u}^i_\epsilon(z) & \text{on} \quad \partial\tilde{\Omega}_{11}. \end{cases} \tag{3.17}$$

Then since $\tilde{u}_\epsilon - \tilde{w}_\epsilon \in H^1_0(\tilde{\Omega}_{11}, \mathbb{R}^m)$ is the weak solution to

$$\begin{cases} D_\alpha\left(\tilde{A}^{\alpha\beta,\epsilon}_{ij}(z)D_\beta\left(\tilde{u}^j_\epsilon(z) - \tilde{w}^j_\epsilon(z)\right)\right) = D_\alpha\tilde{f}^i_\alpha(z) & \text{in} \quad \tilde{\Omega}_{11}, \\ \tilde{u}^i_\epsilon(z) - \tilde{w}^i_\epsilon(z) = 0 & \text{on} \quad \partial\tilde{\Omega}_{11}, \end{cases} \tag{3.18}$$

a standard L^2 estimate follows from (3.18) and (3.16)

$$\frac{1}{|B_{11}|}\int_{\tilde{\Omega}_{11}} |D\tilde{u}_\epsilon - D\tilde{w}_\epsilon|^2 dz \le \frac{c}{|B_{11}|}\int_{\tilde{\Omega}_{11}} |\tilde{F}|^2 dz \le c\delta^2 \tag{3.19}$$

for some positive constant $c = c(\nu, L, m, n)$. In addition, since our domain satisfies the measure density condition which implies the p-capacity condition with $p = 2$ according to Remark 2.5, from (3.16) and (3.17) there exists positive constants σ_1 and $c = c(\nu, L, m, n)$ such that

$$\left(\int_{\tilde{\Omega}_{10}} |D\tilde{w}_\epsilon|^{2+\sigma_1} dz\right)^{\frac{1}{2+\sigma_1}} \leq c. \tag{3.20}$$

For the sake of our perturbation argument, by using the notation given by (2.2), we now let $\tilde{h}_\epsilon \in H^1(\tilde{\Omega}_5, \mathbb{R}^m)$ be the weak solution to

$$\begin{cases} D_\alpha\left(\overline{\tilde{A}^{\alpha\beta,\epsilon}_{ij}}_{B'_5}(z'') D_\beta \tilde{h}^j_\epsilon(z)\right) &= 0 \quad \text{in} \quad \tilde{\Omega}_5 \\ \tilde{h}^i_\epsilon(z) &= \tilde{w}^i_\epsilon(z) \quad \text{on} \quad \partial\tilde{\Omega}_5. \end{cases} \tag{3.21}$$

Then $\tilde{w}_\epsilon - \tilde{h}_\epsilon \in H^1_0(\tilde{\Omega}_5, \mathbb{R}^m)$ is the weak solution to

$$\begin{cases} D_\alpha\left(\overline{\tilde{A}^{\alpha\beta,\epsilon}_{ij}}_{B'_5}(z'') D_\beta(\tilde{w}^j_\epsilon(z) - \tilde{h}^j_\epsilon(z))\right) = -D_\alpha\left(\left(\tilde{A}^{\alpha\beta,\epsilon}_{ij}(z) - \overline{\tilde{A}^{\alpha\beta,\epsilon}_{ij}}_{B'_5}(z'')\right) D_\beta \tilde{w}^j_\epsilon(z)\right) & \text{in } \tilde{\Omega}_5 \\ \tilde{w}^i_\epsilon(z) - \tilde{h}^i_\epsilon(z) = 0 & \text{on } \partial\tilde{\Omega}_5. \end{cases} \tag{3.22}$$

From this, (1.4), Remark 2.4 and (3.20), we compute

$$\begin{aligned} \frac{1}{|B_5|}\int_{\tilde{\Omega}_5} |D\tilde{w}_\epsilon - D\tilde{h}_\epsilon|^2 dz &\leq \frac{c}{|B_5|}\int_{\tilde{\Omega}_5} \left|\tilde{A}^{\alpha\beta,\epsilon}_{ij}(z) - \overline{\tilde{A}^{\alpha\beta,\epsilon}_{ij}}_{B'_5}(z'')\right|^2 |D\tilde{w}_\epsilon|^2 dz \\ &\leq c\left(\frac{1}{|B'_5 \times B''_5|}\int_{B'_5\times B''_5\cap\tilde{\Omega}_{10}} \left|\tilde{A}^{\alpha\beta,\epsilon}_{ij}(z) - \overline{\tilde{A}^{\alpha\beta,\epsilon}_{ij}}_{B'_5}(z'')\right|^2 dz\right)^{\frac{\sigma_1}{2+\sigma_1}} \\ &\leq c\left(\frac{1}{|B_{10}|}\int_{\tilde{\Omega}_{10}} \left|\tilde{A}^{\alpha\beta,\epsilon}_{ij}(z) - \overline{\tilde{A}^{\alpha\beta,\epsilon}_{ij}}_{B'_5}(z'')\right|^2 dz\right)^{\frac{\sigma_1}{2+\sigma_1}} \end{aligned}$$

for some constant $c = c(\nu, L, m, n)$ and hence

$$\frac{1}{|B_5|}\int_{\tilde{\Omega}_5} |D\tilde{w}_\epsilon - D\tilde{h}_\epsilon|^2 dz \leq c\delta^{\frac{\sigma_1}{2+\sigma_1}}. \tag{3.23}$$

Now we note that since our desired δ has an upper bound by Remark 2.5 we have from (3.23) that such $\tilde{h}_\epsilon$ satisfies

$$\frac{1}{|B_5|}\int_{\tilde{\Omega}_5} |D\tilde{h}_\epsilon|^2 dz \leq c$$

for some constant $c = c(\nu, L, m, n)$ and similar to $\tilde{w}_\epsilon$

$$\left(\int_{\tilde{\Omega}_4} |D\tilde{h}_\epsilon|^{2+\sigma_1} dz\right)^{\frac{1}{2+\sigma_1}} \leq c, \tag{3.24}$$

for some constant $c = c(\nu, L, m, n)$. Thus according to [18, Lemma 3.4] for any $\kappa > 0$ there exists a small $\delta = \delta(\nu, L, m, n)$, which depends only on the given structure conditions, such that there exists a weak solution $\tilde{v}_\epsilon \in H^1(B^+_4, \mathbb{R}^m)$ to

$$\begin{cases} D_\alpha\left(\overline{\tilde{A}^{\alpha\beta,\epsilon}_{ij}}_{B'_5}(z'') D_\beta \tilde{v}^j_\epsilon(z)\right) &= 0 \quad \text{in} \quad B^+_4 \\ \tilde{v}^i_\epsilon(z) &= 0 \quad \text{on} \quad T_4 \end{cases} \tag{3.25}$$

satisfying

$$\frac{1}{|B_4|}\int_{B_4^+}|D\tilde{v}_\epsilon(z)|^2dz \le c \tag{3.26}$$

for some constant $c = c(\nu, L, m, n)$ and

$$\frac{1}{|B_4|}\int_{B_4^+}|D\tilde{h}_\epsilon - D\tilde{v}_\epsilon|^2dz \le \kappa^2. \tag{3.27}$$

Moreover, since $\overline{\tilde{A}_{ij}^{\alpha\beta,\epsilon}}_{B_5'}(z'')$ is independent of z' and $\overline{\tilde{A}_{ij}^{\alpha\beta,\epsilon}}_{B_5'}(z'') = \overline{\tilde{A}_{ij}^{\alpha\beta}}_{B_5'}\left(\frac{z''}{\epsilon}\right)$, we can extend our coefficient $\overline{\tilde{A}_{ij}^{\alpha\beta,\epsilon}}_{B_5'}(z'')$ for the z''-variable to the z-variable, that is, $\overline{\tilde{A}_{ij}^{\alpha\beta,\epsilon}}_{B_5'}(z) = \overline{\tilde{A}_{ij}^{\alpha\beta}}_{B_5'}\left(\frac{z}{\epsilon}\right)$. For this reason, we can apply the result [3, 27] for periodic homogenization to (3.25); then, we obtain for any $2 < q < \infty$, that there exists $\delta = \delta(\nu, L, m, n, q)$ such that

$$D\tilde{v}_\epsilon \in L^q(B_3^+)$$

with the estimate

$$\left(\fint_{B_3^+}|D\tilde{v}_\epsilon|^q dz\right)^{\frac{1}{q}} \le c\left(\fint_{B_4^+}|D\tilde{v}_\epsilon|^2 dz\right)^{\frac{1}{2}}.$$

for some constant $c = c(\nu, L, m, n, q)$ independent of ϵ. Especially, by taking $q = p+1$ we have that there exists $\delta = \delta(\nu, L, m, n, p)$ so that

$$\left(\fint_{B_3^+}|D\tilde{v}_\epsilon|^{p+1} dz\right)^{\frac{1}{p+1}} \le c\left(\fint_{B_4^+}|D\tilde{v}_\epsilon|^2 dz\right)^{\frac{1}{2}} \tag{3.28}$$

where the constant $c = c(\nu, L, m, n, p)$ is independent of ϵ.

Considering u_ϵ as the zero extension outside of the domain Ω, we assert that if $N_1 \ge 1$, then

$$\{z \in \tilde{\Omega}_1 : \mathcal{M}(|D\tilde{u}_\epsilon|^2) > N_1^2\} \subset \{z \in \tilde{\Omega}_1 : \mathcal{M}_{B_3}(|D\tilde{u}_\epsilon|^2) > N_1^2\}. \tag{3.29}$$

For this, we denote $z_1 = \frac{y_1}{28\rho}$ and let $z_0 \in \{z \in \tilde{\Omega}_1 : \mathcal{M}_{B_3}(|D\tilde{u}_\epsilon|^2) \le N_1^2\}$. If $r \le 2$, we have that $B_r(z_0) \subset B_3$ and hence

$$\frac{1}{|B_r|}\int_{B_r(z_0)}|D\tilde{u}_\epsilon|^2dz \le \mathcal{M}_{B_3}(|D\tilde{u}_\epsilon|^2)(z_0) \le N_1^2.$$

If $r > 2$, since $z_1 \in B_r(z_0) \subset B_{2r}(z_1)$, we obtain the following from (3.8) and (3.13)

$$\frac{1}{|B_r|}\int_{B_r(z_0)}|D\tilde{u}_\epsilon|^2dz \le \frac{|B_{2r}|}{|B_r|}\frac{1}{|B_{2r}|}\int_{B_{2r}(z_1)}|D\tilde{u}_\epsilon|^2dz \le 1 \le N_1^2.$$

Thus, we prove (3.29) by showing that $z_0 \in \{z \in \tilde{\Omega}_1 : \mathcal{M}(|D\tilde{u}_\epsilon|^2) \le N_1^2\}$.

Now, we let $\tilde{V}_\epsilon$ be the zero extension of $\tilde{v}_\epsilon$ from B_4^+ to B_4, and we let

$$N^2 = N_1^2 2^n$$

where $N_1 \geq 1$ is to be determined. Then from (3.29) we compute the following:

$$
\begin{aligned}
&\frac{1}{|B_\rho|}|\{x \in \Omega : \mathcal{M}(|Du_\epsilon|^2) > N^2\} \cap B_\rho(y)| \\
&\quad \leq \frac{1}{|B_\rho|}|\{x \in \Omega_{28\rho} : \mathcal{M}(|Du_\epsilon|^2) > N^2\}| \\
&\quad = \frac{28^n}{|B_1|}|\{z \in \tilde{\Omega}_1 : \mathcal{M}(|D\tilde{u}_\epsilon|^2) > N_1^2\}| \\
&\quad \leq \frac{28^n}{|B_1|}|\{z \in \tilde{\Omega}_1 : \mathcal{M}_{B_3}(|D\tilde{u}_\epsilon|^2) > N_1^2\}| \\
&\quad \leq \frac{28^n}{|B_1|}|\{z \in \tilde{\Omega}_1 : \mathcal{M}_{B_3}(4|D\tilde{u}_\epsilon - D\tilde{w}_\epsilon|^2 + 4|D\tilde{w}_\epsilon - D\tilde{h}_\epsilon|^2 + 4|D\tilde{h}_\epsilon - D\tilde{V}_\epsilon|^2 + 4|D\tilde{V}_\epsilon|^2) > N_1^2\}| \\
&\quad \leq \frac{28^n}{|B_1|}\left|\left\{z \in \tilde{\Omega}_1 : \mathcal{M}_{B_3}(|D\tilde{u}_\epsilon - D\tilde{w}_\epsilon|^2) > \frac{N_1^2}{16}\right\}\right| \\
&\qquad + \frac{28^n}{|B_1|}\left|\left\{z \in \tilde{\Omega}_1 : \mathcal{M}_{B_3}(|D\tilde{w}_\epsilon - D\tilde{h}_\epsilon|^2) > \frac{N_1^2}{16}\right\}\right| \\
&\qquad + \frac{28^n}{|B_1|}\left|\left\{z \in \tilde{\Omega}_1 : \mathcal{M}_{B_3}(|D\tilde{h}_\epsilon - D\tilde{V}_\epsilon|^2) > \frac{N_1^2}{16}\right\}\right| \\
&\qquad + \frac{28^n}{|B_1|}\left|\left\{z \in \tilde{\Omega}_1 : \mathcal{M}_{B_3}(|D\tilde{V}_\epsilon|^2) > \frac{N_1^2}{16}\right\}\right| \\
&\quad =: I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

For I_1, we use (3.19); then,

$$
I_1 \leq \frac{c}{N_1^2|B_1|}\int_{\tilde{\Omega}_3} |D\tilde{u}_\epsilon - D\tilde{w}_\epsilon|^2 dz \leq \frac{c\delta^2}{N^2} \tag{3.30}
$$

for some constant $c = c(\nu, L, m, n)$.

Applying (3.23), we see that

$$
I_2 \leq \frac{c}{N_1^2|B_1|}\int_{\tilde{\Omega}_3} |D\tilde{w}_\epsilon - D\tilde{h}_\epsilon|^2 dz \leq \frac{c\delta^{\frac{\sigma_1}{2+\sigma_1}}}{N^2} \tag{3.31}
$$

for some constant $c = c(\nu, L, m, n)$.

For any $\kappa > 0$, if $\delta > 0$ is small enough to satisfy (3.24), then (3.24) and (3.27) yield

$$
\begin{aligned}
I_3 &\leq \frac{c}{N_1^2}\left(\frac{1}{|B_1|}\int_{B_3^+} |D\tilde{h}_\epsilon - D\tilde{V}_\epsilon|^2 dz + \frac{1}{|B_1|}\int_{\tilde{\Omega}_3 \setminus B_3^+} |D\tilde{h}_\epsilon|^2 dz\right) \\
&\leq \frac{c}{N^2}\left(\kappa^2 + \left(\int_{\tilde{\Omega}_3} |D\tilde{h}_\epsilon|^{2+\sigma_1} dz\right)^{\frac{2}{2+\sigma_1}}\left(\int_{\tilde{\Omega}_3 \setminus B_3^+} dz\right)^{\frac{\sigma_1}{2+\sigma_1}}\right) \\
&\leq \frac{c}{N^2}\left(\kappa^2 + \delta^{\frac{\sigma_1}{2+\sigma_1}}\right)
\end{aligned} \tag{3.32}
$$

for some constant $c = c(\nu, L, m, n)$.

Finally, for $\delta > 0$ in (3.32), from (3.28), we estimate I_4 as

$$I_4 \leq \frac{c}{N_1^{p+1}|B_1|}\int_{B_3^+}|D\tilde{v}_\epsilon|^{p+1}dz \leq \frac{c}{N^{p+1}}\left(\fint_{B_4^+}|D\tilde{v}_\epsilon|^2 dz\right)^{\frac{p+1}{2}} \leq \frac{c}{N^{p+1}} \tag{3.33}$$

for some constant $c = c(\nu, L, m, n, p)$.

By the estimates from (3.30) to (3.33), we see with (3.6) that

$$\begin{aligned}
&\frac{1}{|B_\rho|}|\{x \in \Omega : \mathcal{M}(|Du_\epsilon|^2) > N^2\} \cap B_\rho(y)| \\
&\quad \leq I_1 + I_2 + I_3 + I_4 \\
&\quad \leq \frac{c}{N^2}\left(\kappa^2 + \delta^2 + \delta^{\frac{\sigma_1}{2+\sigma_1}}\right) + \frac{c}{N^{p+1}} \\
&\quad = \eta\left(c\eta^{\frac{2}{p}-1}\left(\kappa^2 + \delta^2 + \delta^{\frac{\sigma_1}{2+\sigma_1}}\right) + c\eta^{\frac{1}{p}}\right).
\end{aligned}$$

for some constant $c = c(\nu, L, m, n, p)$. Now we choose η satisfying

$$0 < c\eta \leq \frac{1}{3}; \tag{3.34}$$

then N is given by (3.6). Next, we take $\kappa > 0$ so that

$$0 < c\eta^{\frac{2}{p}-1}\kappa \leq \frac{1}{3}. \tag{3.35}$$

Finally, we can select $\delta > 0$ for such $\kappa > 0$ in (3.35) which makes (3.26) and (3.27) possible so that

$$0 < c\eta^{\frac{2}{p}-1}\left(\delta^2 + \delta^{\frac{\sigma_1}{2+\sigma_1}}\right) \leq \frac{1}{3}. \tag{3.36}$$

Therefore from (3.34) to (3.36), we obtain

$$\frac{1}{|B_\rho|}|\{x \in \Omega : \mathcal{M}(|Du_\epsilon|^2) > N^2\} \cap B_\rho(y)| \leq \eta$$

which contradicts (3.5). This completes the proof.

□

Proof of Theorem 2.6. Fix any $p \in (2, \infty)$ and let $u_\epsilon \in H_0^1(\Omega, \mathbb{R}^m)$ be the weak solution to (1.1). We assume that $F \in L^p(\Omega, \mathbb{R}^{mn})$, $A_{ij}^{\alpha\beta}$ is $(\delta, 336)$-vanishing and Ω is $(\delta, 336)$-Reifenberg flat, and we let η and N be constants in Lemma 3.3. Additionally, we let δ_1 be the constant $\delta = \delta(\nu, L, m, n, p)$ in Lemma 3.3 and we denote c by the constants given by the known quantities such as $|\Omega|, \nu, L, m, n$and p.

We now suppose that

$$\|F\|_{L^2(\Omega)} \leq \delta_0 \tag{3.37}$$

by the normalization $\dfrac{u_\epsilon}{\frac{1}{\delta_0}\|F\|_{L^2(\Omega)}}$ and $\dfrac{F}{\frac{1}{\delta_0}\|F\|_{L^2(\Omega)}}$ where the constant $\delta_0 = \delta_0(\nu, L, m, n, p)$ is to be determined. Then, we want to show first that

$$\|\mathcal{M}(|Du_\epsilon|^2)\|_{L^{\frac{p}{2}}(\Omega)}^{\frac{p}{2}} \leq c + c\|F\|_{L^p(\Omega)}^p. \tag{3.38}$$

To prove (3.37), we write

$$C = \left\{x \in \Omega : \mathcal{M}(|Du_\epsilon|^2) > N^2\right\}$$

and

$$D = \left\{x \in \Omega : \mathcal{M}(|Du_\epsilon|^2) > 1\right\} \cup \left\{x \in \Omega : \mathcal{M}(|F|^2) > \delta^2\right\}.$$

By the weak 1-1 estimate and the L^2-estimate, we see that if $\|F\|_{L^2(\Omega)} \leq \delta_2$, then we can take $\delta_2 = \delta_2(\nu, L, m, n)$ such that

$$|C| \leq \frac{c}{N^2}\int_\Omega |Du_\epsilon|^2 dx \leq \frac{c}{N^2}\int_\Omega |F|^2 dx \leq \frac{c\delta_2^2}{N^2} < \eta|B_1|. \tag{3.39}$$

Now we let

$$\delta_0 = \min\{\delta_1, \delta_2, \delta_*\}$$

which depends only on ν, L, m, n and p where $\delta_* = \delta_*(n)$ in Remark 2.5.

For any $\delta \leq \delta_0$, this verifies the first condition of Lemma 3.2. Moreover, the second condition of Lemma 3.2 follows from Lemma 3.3. Thus we apply Lemma 3.2 for such δ_0 to see that

$$|C| < \eta_1|D| \leq \eta_1\left|\left\{x \in \Omega : \mathcal{M}(|Du_\epsilon|^2) > 1\right\}\right| + \eta_1\left|\left\{x \in \Omega : \mathcal{M}(|F|^2) > \delta_0^2\right\}\right| \tag{3.40}$$

where

$$\eta_1 = \left(\frac{10}{1-\delta_0}\right)^n \eta \leq \left(\frac{80}{7}\right)^n \eta$$

since $\delta_0 \leq \delta_* < \frac{1}{8}$; see Remark 2.5.

By an iteration from (3.40), we have

$$\begin{aligned}\left|\left\{x \in \Omega : \mathcal{M}(|Du_\epsilon|^2) > N^{2k}\right\}\right| \leq\ & \eta_1^k\left|\left\{x \in \Omega : \mathcal{M}(|Du_\epsilon|^2) > 1\right\}\right| \\ & + \sum_{i=1}^{k}\eta_1^i\left|\left\{x \in \Omega : \mathcal{M}(|F|^2) > \delta_0^2 N^{2(k-i)}\right\}\right|.\end{aligned} \tag{3.41}$$

Applying Lemma 3.1 to (3.41) for

$$g = \mathcal{M}(|Du_\epsilon|^2), \quad \lambda = N^2, \quad \theta = 1, \text{ and } q = \frac{p}{2},$$

we obtain

$$\begin{aligned}\|\mathcal{M}(|Du_\epsilon|^2)\|^{\frac{p}{2}}_{L^{\frac{p}{2}}(\Omega)} &\leq c\left(|\Omega| + \sum_{k\geq 1} N^{2k\frac{p}{2}}\left|\left\{x \in \Omega : \mathcal{M}(|Du_\epsilon|^2) > N^{2k}\right\}\right|\right) \\ &\leq c\left(1 + \sum_{k\geq 1} N^{kp}\eta_1^k\left|\left\{x \in \Omega : \mathcal{M}(|Du_\epsilon|^2) > 1\right\}\right|\right) \\ &\quad + c\sum_{k\geq 1} N^{kp}\sum_{i=1}^{k}\eta_1^i\left|\left\{x \in \Omega : \mathcal{M}(|F|^2) > \delta_0^2 N^{2(k-i)}\right\}\right| \\ &=: S_1 + S_2.\end{aligned}$$

Then S_1 is estimated as follows:

$$S_1 \leq c\left(1 + |\Omega| \sum_{k\geq 1} (N^p \eta_1)^k\right).$$

Also, S_2 is computed as follows:

$$\begin{aligned} S_2 &\leq c \sum_{k\geq 1} N^{kp} \sum_{i=1}^{k} \eta_1^i \left|\left\{x \in \Omega : \mathcal{M}(|F|^2) > \delta_0^2 N^{2(k-i)}\right\}\right| \\ &= c \sum_{i\geq 1} (N^p \eta_1)^i \sum_{k\geq i} (N^p)^{k-i} \left|\left\{x \in \Omega : \mathcal{M}(|F|^2) > \delta_0^2 N^{2(k-i)}\right\}\right| \\ &\leq c \sum_{i\geq 1} (N^p \eta_1)^i \frac{\|F\|^p_{L^p(\Omega)}}{\delta_0^p} \\ &\leq c \sum_{i\geq 1} (N^p \eta_1)^i \|F\|^p_{L^p(\Omega)}. \end{aligned}$$

Therefore, we have

$$\|\mathcal{M}(|Du_\epsilon|^2)\|^{\frac{p}{2}}_{L^{\frac{p}{2}}(\Omega)} \leq S_1 + S_2 \leq c + c\|F\|^p_{L^p(\Omega)} \tag{3.42}$$

since $N^p \eta_1 = N^p \left(\frac{10}{1-\delta_0}\right)^n \eta \leq N^p \left(\frac{80}{7}\right)^n \eta = \frac{1}{2}$ from (3.6). Now, from the estimate (3.42) we renormalize our problem returning to the original function to find that

$$\begin{aligned} \|\mathcal{M}(|Du_\epsilon|^2)\|^{\frac{p}{2}}_{L^{\frac{p}{2}}(\Omega)} &\leq c\|F\|^p_{L^2(\Omega)} + c\|F\|^p_{L^p(\Omega)} \\ &\leq c|\Omega|^{1-\frac{2}{p}}\|F\|^p_{L^p(\Omega)} + c\|F\|^p_{L^p(\Omega)}. \end{aligned}$$

Finally, the strong p-p estimate yields

$$\|Du_\epsilon\|^p_{L^p(\Omega)} = \||Du_\epsilon|^2\|^{\frac{p}{2}}_{L^{\frac{p}{2}}(\Omega)} \leq c\|\mathcal{M}(|Du_\epsilon|^2)\|^{\frac{p}{2}}_{L^{\frac{p}{2}}(\Omega)} \leq c\|F\|^p_{L^p(\Omega)}.$$

This completes the proof.

□

4. Conclusions

The proof of this paper is based on a perturbation argument in the main lemma, Lemma 3.3. Under this argument for directional homegenization, in the proof of Lemma 3.3 we can compare the original coefficients $\tilde{A}^{\alpha\beta,\epsilon}_{ij}(z', z'')$ with $\overline{\tilde{A}^{\alpha\beta,\epsilon}_{ij}}_{B'_5}(z'')$ by using the small BMO condition to derive (3.23) and then we can apply the result in [3] to (3.25) since we can consider (3.25) as homogenization for whole variables. For these reasons, we can derive Theorem 2.6 for directional homogenization.

Use of AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that there is no conflict of interest.

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