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Research article

The coupling system of Kirchhoff and Euler-Bernoulli plates with logarithmic source terms: Strong damping versus weak damping of variable-exponent type

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Abstract: In this paper, we study the asymptotic behavior of solutions of the dissipative coupled system where we have interactions between a Kirchhoff plate and a Euler-Bernoulli plate. We investigate the interaction between the internal strong damping acting in the Kirchhoff equation and internal weak damping of variable-exponent type acting in the Euler-Bernoulli equation. By using the potential well, the energy method (multiplier method) combined with the logarithmic Sobolev inequality, we prove the global existence and derive the stability results. We show that the solutions of this system decay to zero sometimes exponentially and other times polynomially. We find explicit decay rates that depend on the weak damping of the variable-exponent type. This outcome extends earlier results in the literature.

Keywords: plate equations; variable-exponent; logarithmic nonlinearity; systems; potential well; multiplier method; stability
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1. Introduction

Plate problems have been broadly explored by mathematicians and other scientists. These types of problems have a lot of applications in different areas of science and engineering, such as elasticity, material engineering, mechanical engineering, nuclear physics and optics. In linear elasticity theory, one of the equations widely used in the construction of engineering equipment is based on the plate equation:

$$\rho u_{tt} - \gamma \Delta u_{tt} + \beta \Delta^2 u + L_u = 0, \text{ in } \Omega, \ t > 0, \tag{1.1}$$

where $\gamma > 0$, and L_u denotes some dissipative mechanism. This plate equation corresponds to the model formulated by G. Kirchhoff. In the absence of the rotational inertia $\gamma = 0$, the model is known as the Euler-Bernoulli plate. These two models have different characteristics: While one is hyperbolic, the other is elliptic. When these equations are dissipative by the same mechanism, the asymptotic behavior of their solutions differs. For example, if $L_u = u_t$, under appropriate boundary conditions, the Euler-Bernoulli model decays exponentially while the Kirchhoff model does not. An exhaustive study of the asymptotic behavior of these models with different dissipative mechanisms can be found in J. E. Lagnese's book [1]. We start off by reviewing some works related to the quasi-linear wave equation and plate equation. Cavalcanti et al. [2] considered the following equation:

$$|u_t|^\rho u_{tt} - \Delta u_{tt} - \Delta u - \gamma \Delta u_t = 0, \text{ in } \Omega, \ t > 0, \tag{1.2}$$

and proved the global existence of weak solutions and uniform decay rates of the energy in the presence of a strong damping of the form $-\Delta u_t$ acting as the domain and assuming that the relaxation function decays exponentially. Messaoudi and Tatar [3] studied (1.2), but without a strong damping ($\gamma = 0$). They showed that the memory term is enough to stabilize the solution. Han and Wang [4], for ($\gamma = 0$), investigated the general decay result of the energy of (1.2) with nonlinear damping. In [5], Liu investigated (1.2) with weakly nonlinear time-dependent dissipation and source terms, and he established explicit and general energy decay rate results without imposing any restrictive growth assumption on the damping term at the origin. For the quasi-linear plate equations, we mention the work of Al-Gharabli et al. [6] where they studied the well-posedness and asymptotic stability for a quasi-linear viscoelastic plate equation with a logarithmic nonlinearity. Recently, Al-Mahdi [7] studied the same problem as in Al-Gharabli et al. [6], but with infinite memory. With the imposition of a minimal condition on the relaxation function, he obtained an explicit and general decay rate result for the energy. In [8], Kakumani and Yadav considered a plate equation with infinite memory, nonlinear damping, and logarithmic source. They proved the explicit and general decay rate of the solution.

For the damped wave equation, Chen and Xu [9] considered the following wave equation with the logarithmic source term:

$$u_{tt} - \Delta u + \Delta^2 u - \omega\left(\Delta u_{tt} + \Delta u_t\right) + |u_t|^{r-1} u_t = u \ln |u|, \text{ in } \Omega, \ t > 0, \tag{1.3}$$

where $\Omega \subset \mathcal{R}^n (n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $\omega \in \{0, 1\}$, and $r \geq 1$. Based on the potential well method, they constructed several conditions to prove the global existence or infinite time blow-up with subcritical initial energy. They also used the scaling technique to extend these results to the critical initial energy. Moreover, they surrounded the blow-up at arbitrarily high initial energy. Lian et al. [10] considered the following fourth-order nonlinear wave equations:

$$u_{tt} - \Delta u + \Delta^2 u + \sum_{i=1}^{n} \sigma_i\left(u_{x_i}\right) - \Delta u_t + |u_t|^{r-1} u_t = f(u), \text{ in } \Omega, \ t > 0, \tag{1.4}$$

where $\Omega \subset \mathcal{R}^n (n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, and $r \geq 1$. The nonlinear function $f(u)$ and the function $\sigma_i (i = 1, ..., n)$ satisfy some specific conditions. They proved the local solution by using the fix point theory. Then, by constructing the potential well structure frame, they established the global existence, asymptotic behavior and blow-up of solutions for the subcritical initial energy and critical initial energy, respectively. In addition, they proved the blow-up in a finite time of solutions for the arbitrarily positive initial energy case. For the single plate equation with nonlinear damping and a logarithmic source term, Gongwei [11] considered the following:

$$u_{tt} + \Delta^2 u + |u_t|^{m-2} u_t = |u|^{p-2} u \ln |u|^k, \text{ in } \Omega, \ t > 0, \tag{1.5}$$

where $\Omega \subset \mathcal{R}^n (n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$ and k is a positive real number. The constant exponents $m \geq 2$ and p satisfies:

$$2 < p < \frac{2(n-2)}{n-4} \text{ if } n \geq 5, \text{ and } 2 < p < +\infty, \text{ if } n \leq 4. \tag{1.6}$$

He established the local existence, global existence, and decay estimate of the solution at subcritical initial energy. He also proved that the solution with negative initial energy experiences a blow-up in a finite time under suitable conditions. Moreover, he proved the blow-up in a finite time of solution at the arbitrarily high initial energy when $m = 2$. For the wave equation with weak and strong damping terms and the logarithmic source term, Lian and Xu [12] considered the following:

$$u_{tt} - \Delta u - \omega (\Delta u_{tt} + \Delta u_t) + \mu u_t = u \ln |u|, \text{ in } \Omega, \ t > 0, \tag{1.7}$$

where $\Omega \subset \mathcal{R}^n (n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $\omega \geq 0$, and $\mu > -\omega\lambda_1$ where λ_1 being the first eigenvalue of the operator $-\Delta$ under homogeneous Dirichlet boundary conditions. By using the contraction mapping principle and the potential well, they proved the local existence, global existence, energy decay and, infinite time blow-up of the solution with three different levels of initial energy. For the Kirchhoff plate equation, Liua et al. [13] considered the following viscoelastic Kirchhoff-like plate equation:

$$u_{tt} - \Delta u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)ds - \Delta_p u + u_t - \Delta u_t = |u|^{q-2} u, \text{ in } \Omega, \ t > 0, \tag{1.8}$$

where $\Omega \subset \mathcal{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, $\Delta_p u = div\left(|\nabla u|^{q-2}\nabla u\right)$, and the kernel g and the growth exponents p, q satisfy some specific conditions. The authors proved the local existence and uniqueness of the solution by linearization and the contraction mapping principle. Then, they established the global existence of solution with subcritical and critical initial energy by applying the potential well theory. Moreover, they proved the asymptotic behavior of the global solution with positive initial energy strictly below the depth of potential well. We also refer the reader to the recent work in [14] for more existence, stability, and blow-up results of semilinear hyperbolic equations.

For the stability of coupled quasi-linear systems, we referred to [15] and [16]. In [17], Hajjej considered the following coupled system of quasi-linear viscoelastic Kirchhoff plate equations:

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u_{tt} + \Delta^2 u + \int_0^t g_1(t-s)\Delta^2 u(s)ds + f_1(u,z) = 0, & \text{in } \Omega, \ t > 0, \\ |z_t|^\rho z_{tt} - \Delta z_{tt} + \Delta^2 z + \int_0^t g_2(t-s)\Delta^2 z(s)ds + f_2(u,z) = 0, & \text{in } \Omega, \ t > 0. \end{cases} \tag{1.9}$$

He established the existence of local weak solutions by the Faedo-Galerkin approach and, by using the perturbed energy method, he proved a general decay rate of the energy for a wide class of relaxation functions.

In what follows, we will present some previously studied results that motivated this work. For example, for the decoupled Kirchhoff equation , we point out the work of Oquendo and Astudillo [18] where the authors studied the asymptotic behavior of the solutions of the equation:

$$u_{tt} - \gamma\Delta u_{tt} + \beta\Delta^2 u + \int_0^\infty g(s)\Delta^{2\theta} u(t-s)ds = 0, \text{ in } \Omega,\ t > 0, \tag{1.10}$$

where the kernel decreases exponentially. Using the semigroup theory, they have shown that the solutions decay exponentially when $\theta = 1$ and decay polynomially when $\theta < 1$. They also showed that these decay rates are optimal. For a wider class of kernels, recently Al-Mahdi [19] considered this equation for $\theta = 1$ with g satisfying some convexity inequalities. Using multiplier methods, it was proved that the solutions decay in a general way depending on the decay of the kernel. In relation to stability results for problems with short memory, we noticed that the solutions present asymptotic behavior similar to those with long memory [20–24]. We refer to the work of Rivera and Naso [25] for (1.10) with $\gamma = 0$. Regarding indirectly dissipative coupled systems, it is well known that the study of this kind of systems started with Russell [26]. He introduced a general framework for evolution systems with indirect damping mechanisms. Later, Alabau et al. [27] studied a general framework for the stabilization of weakly coupled wave equations dissipative indirectly by frictional dampings. She showed that the solutions do not decay exponentially, but explicit polynomial decay rates were obtained. Recently, studies on asymptotic behavior of wave-plate interactions were developed by Tebou et al. [28]. He studied the stability for two systems where the dissipation acted only in one equation as follows:

$$\begin{cases} u_{tt} - \gamma\Delta u_{tt} + \Delta^2 u + \alpha z + u_t = 0, & \text{in } \Omega,\ t > 0, \\ z_{tt} - \Delta z + \alpha u = 0, & \text{in } \Omega,\ t > 0 \end{cases} \tag{1.11}$$

and

$$\begin{cases} u_{tt} - \gamma\Delta u_{tt} + \Delta^2 u + \alpha z = 0, & \text{in } \Omega,\ t > 0, \\ z_{tt} - \Delta z + \alpha u + z_t = 0, & \text{in } \Omega,\ t > 0, \end{cases} \tag{1.12}$$

It was proved that the solutions of both systems have a polynomial decay. Concerning viscoelastic systems, we cited the work of Guesmia [29, 30], [31] and the references therein. Recently, Tyszka et al. [32] considered the following coupled Kirchhoff and Euler-Bernoulli plates:

$$\begin{cases} \rho_1 u_{tt} - \gamma\Delta u_{tt} + \beta_1\Delta^2 u + \int_0^\infty g_1(s)\Delta^{2\theta_1} u(t-s)ds + \alpha(u-z) = 0, & \text{in } \Omega,\ t > 0, \\ \rho_2 z_{tt} + \beta_2\Delta^2 z + \int_0^\infty g_2(s)\Delta^{2\theta_2} z(t-s)ds + \alpha(u-z) = 0, & \text{in } \Omega,\ t > 0, \end{cases} \tag{1.13}$$

and they established explicit decay rates that depend on the fractional exponents of the memory. They concluded that the memory effects in the Euler-Bernoulli equation dissipate the system more slowly than memory effects in the Kirchhoff equation.

1.1. Our problem

Motivated by all the above works, in this paper, we are interested in the asymptotic behavior of the coupled system of Kirchhoff and the Euler-Bernoulli models. These models are governed by the following equations:

$$\begin{cases}\rho_1 u_{tt} - \gamma\Delta u_{tt} + \beta_1\Delta^2 u + u + \alpha(u-z)^2 - \delta_1\Delta u_t = \kappa u \ln|u|, & \text{in } \Omega,\ t>0,\\ \rho_2 z_{tt} + \beta_2\Delta^2 z + z - \alpha(u-z)^2 + \delta_2|z_t|^{(\nu(x)-2)}z_t = \kappa z\ln|z|, & \text{in } \Omega,\ t>0,\\ u(\cdot,t) = z(\cdot,t) = \frac{\partial u}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, & \text{on } \partial\Omega, t\geq 0,\\ (u(0),z(0)) = (u_0,z_0), (u_t(0),z_t(0)) = (u_1,z_1), & \text{in } \Omega,\end{cases} \tag{P}$$

where γ is the rotational inertia coefficient, for $(i = 1,2)$ ρ_i and β_i are the mass densities and the flexural rigidity coefficients, respectively. The constants α is the coupling coefficient and we assume all the considered coefficients in the system are positive. Here, Ω is a bounded and regular domain of $\mathbb{R}^2$, with the smooth boundary $\partial\Omega$. The vector ν is the unit outer normal to $\partial\Omega$, $\omega(x)$ and $\nu(x)$ are the variable-exponents and the constant κ is a small positive real number satisfying some specific conditions. The initial data u_0, z_0, u_1 and z_1 lie in appropriate Hilbert space. The symbol Δ is the Laplacian operator.

Model (P) describes the interaction of Kirchhoff and the Euler-Bernoulli plates and the extensional vibrations of thin rods [33]. Each one of these two plates are clamped along the boundary $\partial\Omega$. The analysis of stability issues for plate models is more challenging due to free boundary conditions and the presence of variable-exponents nonlinearity and the logarithmic source terms. Moreover, in our case, the source term competes with the dissipation induced by the variable-exponents dampings. The strong damping term in the Kirchhoff equation $(-\delta\Delta u_t)$ is introduced to treat the problems arising from the rotational inertia term $(-\gamma\Delta u_{tt})$ in the same Kirchhoff equation.

As would be expected, nonlinearities enabled the detection of some obscure events. The distribution form was altered by a logarithmic expression; it minimized sample skewness and, in some situations, data skewness. Since most of the behaviors of some models in real-life applications are nonlinear, the nonlinearity can be used to explain why torsional oscillation occurs. If we used logarithmic nonlinearity, the oscillation's amplitude would also decrease.

In addition, the considered system (P) has nonlinear dissipations induced by the variable-exponents dampings. Equations with nonstandard growth conditions occur in the mathematical modeling of various physical phenomena, such as the flows of electro-rheological fluids or fluids with temperature-dependent viscosity, nonlinear viscoelasticity, processes of filtration through a porous media and the image processing [34]. Therefore, it will be very interesting to study this interaction.

1.2. Our objectives

We studied the asymptotic behavior of solutions of the dissipative coupled system (P) where we had interaction between a Kirchhoff plate and an Euler-Bernoulli plate. The Kirchhoff equation was dissipated by a strong damping mechanism, while the Euler-Bernoulli equation was dissipated by a nonlinear weak damping mechanism of variable-exponent type. We investigated the interactions between Kirchhoff and Euler-Bernoulli plates and the level of the effectiveness of the damping mechanism on the two equations and, we studied the competition between the nonlinear source terms and the damping mechanisms.

To this end, we started using the potential well technique to prove the global existence of the solutions of the system (P). Then, we applied the energy method (multiplier method) combined with logarithmic Sobolev inequality to establish the stability results. We showed that the solutions of the system (P) decay to zero sometimes exponentially and other times polynomially based on the value of the exponents of the weaker damping. We derived explicit decay rates that depend on the variable-exponents of the dissipative mechanisms.

2. Preliminaries

In this section, we present some materials needed in the proof of our results. We used the standard Lebesgue space $L^2(\Omega)$ and the Sobolev space $H_0^2(\Omega)$ with their usual scalar products and norms. Throughout this paper, c is used to denote a generic positive constant, and we shall assume the following hypotheses:

(A_1) : The variable exponent $\nu : \overline{\Omega} \to [1, \infty)$ is a continuous function such that

$$\nu_1 := \text{essinf}_{x\in\Omega}\nu(x), \quad \nu_2 := \text{esssup}_{x\in\Omega}\nu(x)$$

and $1 < \nu_1 \leq \nu(x) \leq \nu_2 < \infty$. Moreover, the variable function ν satisfies the log-Hölder continuity condition; that is, for any δ with $0 < \delta < 1$, there exists a constant $A > 0$ such that,

$$|\nu(x) - \nu(y)| \leq -\frac{A}{\log|x-y|}, \text{ for all } x, y \in \Omega, \text{ with } |x-y| < \delta. \tag{2.1}$$

(A_2) : The constant κ in (P) satisfies $0 < \kappa < \kappa_0$, where κ_0 is the unique solution of the equation $f(\kappa_0) = 0$ such that

$$f(s) = \sqrt{\frac{2\beta\pi}{c_p s}} - e^{-\frac{3}{2}-\frac{1}{s}}$$

is a continuous and decreasing function on $(0, \infty)$, with

$$\lim_{s\to 0^+} f(s) = \infty \text{ and } \lim_{s\to\infty} f(s) = -e^{-\frac{3}{2}}.$$

Here, $\beta = \min\{\beta_1, \beta_2\}$ and c_p is the smallest positive number satisfying

$$\|\nabla u\|_2^2 \leq c_p\|\Delta u\|_2^2, \quad \forall u \in H_0^2(\Omega), \tag{2.2}$$

and $\|.\|_2 = \|.\|_{L^2(\Omega)}$.

Remark 2.1. The Assumption (A_2) is needed only for the local existence.

The system (P) has a unique solution:

$$u \in L^\infty(\mathbb{R}^+; H^4(\Omega) \cap H_0^2(\Omega)) \cap W^{1,\infty}(\mathbb{R}^+; H_0^2(\Omega)) \cap W^{2,\infty}(\mathbb{R}^+; L^2(\Omega)).$$

We state, without proof, the following standard existence and regularity result. It can be proved by using the Faedo-Galerkin method and Banach fixed point theorem, as well as by following the procedure by M. Cavalcanti [2] and the recent paper by Al-Mahdi et al. [35].

Proposition 2.1. *Let* $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$ *be given. Assume that* $(A_1) - (A_2)$ *hold, then problem* (P) *has a unique global (weak) solution*

$$u, z \in C(\mathbb{R}^+; H_0^2(\Omega)) \cap C^1(\mathbb{R}^+; L^2(\Omega)).$$

$$u_t \in L^\infty((0,T); L^2(\Omega)), z_t \in L^\infty((0,T); L^2(\Omega)) \cap L^{\nu(.)}(\Omega \times (0,T)).$$

Moreover, if

$$(u_0, u_1) \in (H^4(\Omega) \cap H_0^2(\Omega)) \times H_0^2(\Omega),$$

then the solution satisfies

$$u \in L^\infty(H^4(\Omega) \cap H_0^2(\Omega)) \cap W^{1,\infty}(\mathbb{R}^+; H_0^2(\Omega)) \cap W^{2,\infty}(\mathbb{R}^+; L^2(\Omega)).$$

Lemma 2.1. *[36, 37] (Logarithmic Sobolev inequality) Let u be any function in* $H_0^1(\Omega)$ *and* $a > 0$ *be any number. Then,*

$$\int_\Omega u^2 \ln |u| dx \leq \frac{1}{2} \|u\|_2^2 \ln \|u\|_2^2 + \frac{a^2}{2\pi} \|\nabla u\|_2^2 - (1 + \ln a) \|u\|_2^2. \tag{2.3}$$

Corollary 2.1. *Let u be any function in* $H_0^2(\Omega)$ *and a be any positive real number. Then*

$$\int_\Omega u^2 \ln |u| dx \leq \frac{1}{2} \|u\|_2^2 \ln \|u\|_2^2 + \frac{c_p a^2}{2\pi} \|\Delta u\|_2^2 - (1 + \ln a) \|u\|_2^2. \tag{2.4}$$

Proof. The proof of Corollary (2.1) can be established by using (2.2) and Corollary (2.3). □

We define the energy functional $E(t)$ associated to system (P) as follows:

$$\begin{aligned} E(t) := & \frac{1}{2}\Big[\rho_1 \|u_t\|_2^2 + \rho_2 \|z_t\|_2^2\Big] + \frac{1}{2}\Big[\beta_1 \|\Delta u\|_2^2 + \beta_2 \|\Delta z\|_2^2\Big] + \frac{\kappa + 2}{4}\Big[\|u\|_2^2 + \|z\|_2^2\Big] \\ & + \frac{\gamma}{2} \|\nabla u_t\|_2^2 + \frac{\alpha}{3} \int_\Omega (u - z)^3 dx - \frac{\kappa}{2} \int_\Omega u^2 \ln |u| dx - \frac{\kappa}{2} \int_\Omega z^2 \ln |z| dx. \end{aligned} \tag{2.5}$$

By multiplying the two equations in (P) by u_t and z_t, respectively, integrating over Ω, using integration by parts and using the fact that

$$\frac{d}{dt} \frac{\kappa}{2} \int_\Omega u^2 \ln |u| dx = \kappa \int_\Omega u u_t \ln |u| dx + \frac{d}{dt} \frac{\kappa}{4} \int_\Omega u^2 dx, \tag{2.6}$$

we added the results together, and get

$$\frac{d}{dt} E(t) = -\delta_1 \int_\Omega |\nabla u_t|^2 dx - \delta_2 \int_\Omega |z_t|^{\nu(\cdot)} dx \leq 0. \tag{2.7}$$

3. Global existence

In this section, we state and prove a global existence result using the potential wells corresponding to the Logarithmic nonlinearity. For this purpose, we define the following functionals

$$J(u,z)=\frac{1}{2}\Big[\|\beta_1\Delta u\|_2^2+\beta_2\|\Delta z\|_2^2\Big]-\frac{\kappa}{2}\int_\Omega u^2\ln|u|dx-\frac{\kappa}{2}\int_\Omega z^2\ln|z|dx+\frac{\kappa+2}{4}\Big[\|u\|_2^2+\|z\|_2^2\Big]. \quad (3.1)$$

$$I(u,z)=\Big[\beta_1\|\Delta u\|_2^2+\beta_2\|\Delta z\|_2^2\Big]-\kappa\int_\Omega u^2\ln|u|dx-\kappa\int_\Omega z^2\ln|z|dx+\Big[\|u\|_2^2+\|z\|_2^2\Big]. \quad (3.2)$$

Remark 3.1. (1) From the above definitions, it is clear that

$$J(u,z)=\frac{1}{2}I(u,z)+\frac{\kappa}{4}\Big[\|u\|_2^2+\|z\|_2^2\Big], \quad (3.3)$$

$$E(t)=\frac{1}{2}\Big(\rho_1\|u_t\|_2^2+\rho_2\|z_t\|_2^2\Big)+\frac{\gamma}{2}\|\nabla u_t\|_2^2+J(u,z)+\frac{\alpha}{3}\int_\Omega(u-z)^3dx. \quad (3.4)$$

(2) According to the Logarithmic Sobolev inequality, $J(u,z)$ and $I(u,z)$ are well defined.

We define the potential well (stable set) as

$$\mathcal{W}=\{(u,z)\in H_0^2(\Omega)\times H_0^2(\Omega), I(u,z)>0\}\cup\{(0,0)\}.$$

The potential well depth is defined by

$$0<d=\inf_{(u,z)}\{\sup_{p\geq 0}J(pu,pz):(u,z)\in H_0^2(\Omega)\times H_0^2(\Omega),\|\Delta u\|_2\neq 0 \text{ and } \|\Delta z\|_2\neq 0\}, \quad (3.5)$$

and the well-known Nehari-manifold

$$\mathcal{N}=\{(u,z):(u,z)\in H_0^2(\Omega)\times H_0^2(\Omega): I(u,z)=0,\|\Delta u\|_2\neq 0 \text{ and } \|\Delta z\|_2\neq 0\}. \quad (3.6)$$

Proceeding as in [38, 39], one has

$$0<d=\inf_{(u,z)\in\mathcal{N}}J(u,z). \quad (3.7)$$

Lemma 3.1. *For any $(u,z)\in H_0^2(\Omega)\times H_0^2(\Omega)$, $\|u\|_2\neq 0$ and $\|z\|_2\neq 0$. If $\phi(p):=J(pu,pz)$, then we have*

$$I(pu,pz)=p\phi'(p)\begin{cases}>0, & 0\leq p<p^*,\\ =0, & p=p^*,\\ <0, & p^*<p<+\infty,\end{cases}$$

where

$$p^*=\exp\left(\frac{\beta_1\|\Delta u\|_2^2+\beta_2\|\Delta z\|_2^2-\int_\Omega u^2\ln|u|^\kappa dx-\int_\Omega z^2\ln|z|^\kappa dx}{\kappa(\|u\|_2^2+\|z\|_2^2)}\right)\varrho,$$

where ϱ (will be defined in the proof) is a positive constant that depends on the value of the positive constant κ.

Proof.

$$\phi(p) = J(pu, pz) = \frac{1}{2}p^2\left(\beta_1\|\Delta u\|_2^2 + \beta_2\|\Delta z\|_2^2\right) - \frac{1}{2}p^2\left(\int_\Omega u^2 \ln|u|^\kappa dx + \int_\Omega z^2 \ln|z|^\kappa dx\right)$$
$$+ p^2\left(\frac{\kappa+2}{4} - \frac{\kappa}{2}\ln|p|\right)\left(\|u\|_2^2 + \|z\|_2^2\right).$$

Taking common factors reduces to

$$\phi(p) = \frac{1}{2}p^2\left[\beta_1\|\Delta u\|_2^2 + \beta_2\|\Delta z\|_2^2 - \int_\Omega u^2 \ln|u|^\kappa dx - \int_\Omega z^2 \ln|z|^\kappa dx\right]$$
$$+ \underbrace{\frac{1}{2}p^2\left[\left(\frac{\kappa+2}{2} - \kappa\ln|p|\right)\left(\|u\|_2^2 + \|z\|_2^2\right)\right]}_{F}.$$

Since $\|u\|_2 \neq 0$ and $\|z\|_2 \neq 0$, then, we have

$$I(pu, pz) = p\frac{dJ(pu, pz)}{dp} = p\phi'(p)$$
$$= p^2\left(\beta_1\|\Delta u\|_2^2 + \beta_2\|\Delta z\|_2^2\right) - p^2\left(\int_\Omega u^2 \ln|u|^\kappa dx + \int_\Omega z^2 \ln|z|^\kappa dx\right)$$
$$+ p^2(1 - \kappa\ln|p|)\left(\|u\|_2^2 + \|z\|_2^2\right).$$

The above derivative calculated using the following derivative

$$\frac{dF}{dp} = \left[\frac{\kappa p}{2} + p - \kappa p\ln|p| - \frac{\kappa p}{2}\right]\left(\|u\|_2^2 + \|z\|_2^2\right) = (p - \kappa p\ln|p|)\left(\|u\|_2^2 + \|z\|_2^2\right).$$

Now, solving $\phi'(p) = 0$ with dividing both sides by $p^2\kappa\left(\|u\|_2^2 + \|z\|_2^2\right)$, we get

$$p^* = \exp\left(\frac{\beta_1\|\Delta u\|_2^2 + \beta_2\|\Delta z\|_2^2 - \int_\Omega u^2 \ln|u|^\kappa dx - \int_\Omega z^2 \ln|z|^\kappa dx}{\kappa(\|u\|_2^2 + \|z\|_2^2)}\right)\varrho,$$

and $\varrho = e^{\frac{1}{\kappa}}$. Since $\|u\|_2 \neq 0$ and $\|z\|_2 \neq 0$, then we can prove that $\lim_{p\to 0}\phi(p) = 0$, and $\lim_{p\to+\infty}\phi(p) = -\infty$. Thus, we can find $p > 0$ (small enough) such that $\phi(p_0)$. This means that $J(pu)$ is increasing on $0 < p \leq p^*$ and decreasing on $p^* \leq p < \infty$ and takes the maximum at $p = p^*$. In other words, there exists a unique $p^* \in (0, \infty)$ such that $I(p^*u) = 0$ and so, we have the desired result. □

Lemma 3.2. *Let* $(u, z) \in H_0^2(\Omega) \times H_0^2(\Omega)$ *and if* $0 < \|u\|_2, \|z\|_2 \leq e^{\frac{(\kappa+1)}{\kappa}}\sqrt{\frac{2\pi}{c_p\beta\kappa}}$. *Therefore,* $I(u, z) \geq 0$, *where* $\beta = \min\{\beta_1, \beta_2\}$.

Proof. Using the Logarithmic Sobolev inequality (2.1), for any $a > 0$, we have

$$I(u, z) = \beta_1\|\Delta u\|_2^2 + \beta_2\|\Delta z\|_2^2 - \int_\Omega u^2 \ln|u|^\kappa dx - \int_\Omega z^2 \ln|z|^\kappa dx + \left[\|u\|_2^2 + \|z\|_2^2\right]$$
$$\geq \left(\beta_1 - \frac{c_p\kappa a^2}{2\pi}\right)\|\Delta u\|_2^2 + \left(\beta_2 - \frac{c_p\kappa a^2}{2\pi}\right)\|\Delta z\|_2^2 + \kappa(1 + \ln a)\|u\|_2^2 \tag{3.8}$$
$$- \frac{\kappa}{2}\|u\|_2^2 \ln\|u\|_2^2 + \kappa(1 + \ln a)\|z\|_2^2 - \frac{\kappa}{2}\|z\|_2^2 \ln\|z\|_2^2 + \left[\|u\|_2^2 + \|z\|_2^2\right].$$

Taking $a \leq \sqrt{\frac{2\beta\pi}{c_p\kappa}}$ in (3.8) where $\beta = \min\{\beta_1, \beta_2\}$, we obtain

$$I(u,z) \geq \left[1 + \kappa\left(1 + \ln\sqrt{\frac{2\beta\pi}{c_p\kappa}}\right) - \frac{\kappa}{2}\ln\|u\|_2^2\right]\|u\|_2^2 + \left[1 + \kappa\left(1 + \ln\sqrt{\frac{2\beta\pi}{c_p\kappa}}\right) - \frac{\kappa}{2}\ln\|z\|_2^2\right]\|z\|_2^2. \tag{3.9}$$

Now, to calculate the potential well depth d, we have

$$\begin{aligned}\sup_{p\geq 0} J(pu, pz)\} = J(p^*u, p^*z) &= \frac{1}{2}I(p^*u, p^*z) + \frac{\kappa(p^*)^2}{4}\Big[\|u\|_2^2 + \|z\|_2^2\Big] \\ &= \frac{\kappa(p^*)^2}{4}\Big[\|u\|_2^2 + \|z\|_2^2\Big].\end{aligned} \tag{3.10}$$

A combination of (3.9) and Lemma (3.1) gives us

$$\begin{aligned}0 = I(p^*u, p^*z) \geq &\left[1 + \kappa\left(1 + \ln\sqrt{\frac{2\beta\pi}{c_p\kappa}}\right) - \frac{\kappa}{2}\ln\|p^*u\|_2^2\right]\|p^*u\|_2^2 \\ &+ \left[1 + \kappa\left(1 + \ln\sqrt{\frac{2\beta\pi}{c_p\kappa}}\right) - \frac{\kappa}{2}\ln\|p^*z\|_2^2\right]\|p^*z\|_2^2.\end{aligned} \tag{3.11}$$

Therefore, we must have

$$p^*\|u\|_2 \geq \sqrt{\frac{2\beta\pi}{c_p\kappa}}e^{\frac{(\kappa+1)}{\kappa}} \Rightarrow (p^*)^2\|u\|_2^2 \geq \frac{2\beta\pi}{c_p\kappa}e^{\frac{2(\kappa+1)}{\kappa}} \tag{3.12}$$

and

$$p^*\|z\|_2 \geq \sqrt{\frac{2\beta\pi}{c_p\kappa}}e^{\frac{(\kappa+1)}{\kappa}} \Rightarrow (p^*)^2\|z\|_2^2 \geq \frac{2\beta\pi}{c_p\kappa}e^{\frac{2(\kappa+1)}{\kappa}}. \tag{3.13}$$

From, (3.12) and (3.13), we find

$$(p^*)^2\Big[\|u\|_2^2 + \|z\|_2^2\Big] \geq \frac{2\beta\pi}{c_p\kappa}e^{\frac{(\kappa+1)}{\kappa}} \Rightarrow \kappa\frac{(p^*)^2}{4}\Big[\|u\|_2^2 + \|z\|_2^2\Big] \geq \frac{\beta\pi}{2c_p}e^{\frac{2(\kappa+1)}{\kappa}}. \tag{3.14}$$

Hence, from (3.10) and (3.14), we conclude that the potential well depth d satisfies

$$d \geq \frac{\beta\pi}{2c_p}e^{\frac{2(\kappa+1)}{\kappa}}, \tag{3.15}$$

where $\beta = \min\{\beta_1, \beta_2\}$. In addition, if $0 < \|u\|_2, \|z\|_2 \leq e^{\frac{(\kappa+1)}{\kappa}}\sqrt{\frac{2\beta\pi}{c_p\kappa}}$, then

$$1 + \kappa\left(1 + \ln\sqrt{\frac{2\beta\pi}{c_p\kappa^2}}\right) - \frac{\kappa}{2}\ln\|u\|_2^2 \geq 0 \text{ and } 1 + \kappa\left(1 + \ln\sqrt{\frac{2\beta\pi}{c_p\kappa}}\right) - \frac{\kappa}{2}\ln\|z\|_2^2 \geq 0,$$

which gives $I(u,z) \geq 0$. □

Lemma 3.3. *Let* $(u_0, u_1), (z_0, z_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *such that* $0 < E(0) < d$ *and* $I(u_0, z_0) > 0$. *Then any solution of* (P), $(u, z) \in \mathcal{W}$.

Proof. Let T be the maximal existence time of a weak solution of (u, z). From (2.7) and (3.4), we have for any $t \in [0, T)$,

$$\begin{aligned} &\frac{1}{2}\left(\rho_1\|u_t\|^2 + \rho_2\|z_t\|^2\right) + \frac{\gamma}{2}\|\nabla u_t\|^2 + \frac{\alpha}{3}\int_\Omega (u-z)^3 dx + J(u,z) \\ &\le \frac{1}{2}\left(\rho_1\|u_1\|^2 + \rho_2\|z_1\|^2\right) + \frac{\gamma}{2}\|\nabla u_1\|^2 + \frac{\alpha}{3}\int_\Omega (u_0-z_0)^3 dx + J(u_0,z_0) < d. \end{aligned} \tag{3.16}$$

Then we claim that $(u(t), z(t)) \in \mathcal{W}$ for all $t \in [0, T)$. If not, then there is a $t_0 \in (0, T)$ such that $I(u(t_0), z(t_0)) < 0$. Using the continuity of $I(u(t), z(t))$ in t, we deduce that there exists a $t_* \in (0, T)$ such that $I(u(t_*), z(t_*)) = 0$. Then, using the definition of d in (3.5) gives

$$d \le J(u(t_*), z(t_*)) \le E(u(t_*), z(t_*)) \le E(0) < d,$$

which is a contradiction. □

4. Technical lemmas

In this section, we state and prove the following lemmas.

Lemma 4.1. *Assume that* $(A_1 - A_2)$ *hold and let* $(u_0, z_0), (u_1, z_1) \in H_0^2(\Omega) \times L^2(\Omega)$. *Then, the functional is defined by*

$$L(t) = NE(t) + \rho_1 \int_\Omega u u_t dx + \rho_2 \int_\Omega z z_t dx + \gamma \int_\Omega \nabla u \cdot \nabla u_t dx \tag{4.1}$$

satisfies, along the solutions of (P),

$$L \sim E, \tag{4.2}$$

and

$$L'(t) \le \begin{cases} -\vartheta E(t) + \rho_2 \int_\Omega z_t^2 dx, & \nu_1 \ge 2; \\ -\vartheta E(t) + \rho_2 \int_\Omega z_t^2 dx - cE'(t) - cE^{-\Theta}(t)E'(t), & 1 < \nu_1 < 2. \end{cases} \tag{4.3}$$

where $\Theta = \frac{2-\nu_1}{\nu_1 - 1} > 0$.

Proof. We prove $(4.3)_2$, and the proof of the $(4.3)_1$ is straightforward. To prove $(4.3)_2$, we differentiate

$L(t)$ and use integrations by parts, to get

$$\begin{aligned}
L'(t) = &-N\delta_1 \int_\Omega |\nabla u_t|^2 dx - N\delta_2 \int_\Omega |z_t|^{\nu(\cdot)} dx \\
&+ \rho_1 \int_\Omega u_t^2 dx + \int_\Omega u\Big[-\beta_1\Delta^2 u + \gamma\Delta u_{tt} - u - \alpha(u-z)^2 + \delta_1\Delta u_t + \kappa u \ln|u|\Big]dx \\
&+ \rho_2 \int_\Omega z_t^2 dx + \int_\Omega z\Big[-\beta_2\Delta^2 z - z + \alpha(u-z)^2 - \delta_2|z_t|^{q(\cdot)-2}z_t + \kappa z\ln|z|\Big]dx \\
&+ \gamma \int_\Omega |\nabla u_t|^2 dx + \gamma \int_\Omega \nabla u\cdot\nabla u_{tt} dx \\
= &\int_\Omega \Big(\rho_1|u_t|^2 + \rho_2|z_t|^2\Big)dx - \int_\Omega \Big(\beta_1|\Delta u|^2 + \beta_2|\Delta z|^2\Big)dx \\
&- \int_\Omega u^2 dx - \int_\Omega z^2 dx - \alpha\int_\Omega (u-z)^3 dx - N\delta_2 \int_\Omega |z_t|^{\nu(\cdot)}dx \\
&+ \gamma\int_\Omega |\nabla u_t|^2 dx + \gamma\int_\Omega \nabla u\cdot\nabla u_{tt}dx + \gamma\int_\Omega u\Delta u_{tt}dx + \delta\int_\Omega u\Delta u_t dx \\
&- \delta_2\int_\Omega z|z_t|^{\nu(\cdot)-2}z_t dx + \kappa\int_\Omega u^2\ln|u|dx + \kappa\int_\Omega z^2\ln|z|dx - N\delta_1\int_\Omega |\nabla u_t|^2 dx.
\end{aligned} \tag{4.4}$$

Integration by part leads

$$\begin{aligned}
L'(t) = &\int_\Omega \Big(\rho_1|u_t|^2 + \rho_2|z_t|^2\Big)dx - \int_\Omega \Big(\beta_1|\Delta u|^2 + \beta_2|\Delta z|^2\Big)dx \\
&- \int_\Omega u^2 dx - \int_\Omega z^2 dx - \alpha\int_\Omega (u-z)^3 dx - N\delta_2\int_\Omega |z_t|^{\nu(\cdot)}dx \\
&+ \gamma\int_\Omega |\nabla u_t|^2 dx - \delta\int_\Omega \nabla u\cdot\nabla u_t dx - N\delta_1\int_\Omega |\nabla u_t|^2 dx \\
&- \delta_2\int_\Omega z|z_t|^{\nu(\cdot)-2}z_t dx + \kappa\int_\Omega u^2\ln|u|dx + \kappa\int_\Omega z^2\ln|z|dx.
\end{aligned} \tag{4.5}$$

Since $u \in H_0^2(\Omega)$, we have

$$\int_\Omega u^2 dx \le c_p \int_\Omega |\nabla u|^2 dx \le {c_p}^2 \int_\Omega |\Delta u|^2 dx$$

and

$$\int_\Omega u_t^2 dx \le c_p \int_\Omega |\nabla u_t|^2 dx,$$

where c_p is the Poincare constant. Using this estimate and Young's inequility, Eq (4.5) becomes for a positive constant $\varepsilon > 0$

$$
\begin{aligned}
L'(t) \leq & \rho_2 \int_\Omega |z_t|^2 dx - \left(\beta_1 - \varepsilon c_p\right) \int_\Omega |\Delta u|^2 dx - \beta_2 \int_\Omega |\Delta z|^2 dx \\
& - \int_\Omega u^2 dx - \int_\Omega z^2 dx - N\delta_2 \int_\Omega |z_t|^{\nu(\cdot)} dx \\
& + \kappa \int_\Omega u^2 \ln|u| dx + \kappa \int_\Omega z^2 \ln|z| dx - \alpha \int_\Omega (u-z)^3 dx \\
& - \delta_2 \int_\Omega z|z_t|^{\nu(\cdot)-2} z_t dx - \left(N\delta_1 - \gamma - c_\rho - \frac{\delta^2}{4\varepsilon}\right) \int_\Omega |\nabla u_t|^2 dx.
\end{aligned} \tag{4.6}
$$

Using (2.7) and the Logarthmic Sobolev inequality, (4.6) becomes

$$
\begin{aligned}
L'(t) \leq & \rho_2 \int_\Omega |z_t|^2 dx - \alpha \int_\Omega (u-z)^3 dx \\
& - \left(\beta_1 - \varepsilon c_\rho - \frac{a^2 c_\rho \kappa}{2\pi}\right) \int_\Omega |\Delta u|^2 dx - \left(\beta_2 - \frac{a^2 c_\rho \kappa}{2\pi}\right) \int_\Omega |\Delta z|^2 dx \\
& - \left(1 - \frac{\kappa}{2} \ln \|u\|_2^2 + \kappa(1+\ln a)\right) \|u\|_2^2 - \left(1 - \frac{\kappa}{2} \ln \|z\|_2^2 + \kappa(1+\ln a)\right) \|z\|_2^2 \\
& \underbrace{-\delta_2 \int_\Omega z|z_t|^{\nu(\cdot)-2} z_t dx}_{I_2} - \left(N\delta_1 - \gamma - c_\rho - \frac{\delta^2}{4\varepsilon}\right) \int_\Omega |\nabla u_t|^2 dx.
\end{aligned} \tag{4.7}
$$

Now, we start following [40] for estimating the integrals I_2 in (4.7) as follows:

$$
\begin{cases}
I_2 \leq \lambda_2 {c_\rho}^2 \int_\Omega |\Delta z|^2 dx + \int_\Omega c_{\lambda_2}(x) |z_t|^{\nu(x)} dx \quad \nu_1 \geq 2, \\
I_2 \leq \lambda_2 {c_\rho}^2 \int_\Omega |\Delta z|^2 dx + \int_\Omega c_{\lambda_2}(x) |z_t|^{\nu(x)} dx + \left(\int_\Omega c_{\lambda_2}(x)|z_t|^{\nu(x)} dx\right)^{\nu_1 - 1}, \ 1 < \nu_1 < 2,
\end{cases} \tag{4.8}
$$

where the positive constants λ_2, c come from Young's inequality and c_ρ is the Poincare constant. Inserting the above two estimates in (4.7) we get

$$
\begin{aligned}
L'(t) \leq & \rho_2 \int_\Omega |z_t|^2 dx - \alpha \int_\Omega (u-z)^3 dx - \left(N\delta_1 - \gamma - c_\rho - \frac{\delta^2}{4\varepsilon}\right) \int_\Omega |\nabla u_t|^2 dx \\
& - \left(\beta_1 - \varepsilon c_\rho - \frac{a^2 c_\rho \kappa}{2\pi}\right) \int_\Omega |\Delta u|^2 dx - \left(\beta_2 - \lambda_2 {c_\rho}^2 - \frac{a^2 c_\rho \kappa}{2\pi}\right) \int_\Omega |\Delta z|^2 dx \\
& - \left(1 - \frac{\kappa}{2} \ln \|u\|_2^2 + \kappa(1+\ln a)\right) \|u\|_2^2 - \left(1 - \frac{\kappa}{2} \ln \|z\|_2^2 + \kappa(1+\ln a)\right) \|z\|_2^2 \\
& - [N\delta_2 - c] \int_\Omega |z_t|^{\nu(\cdot)} dx + c\widetilde{\Lambda}_1 \left(\int_\Omega |z_t|^{\nu(x)} dx\right)^{\nu_1 - 1},
\end{aligned} \tag{4.9}
$$

where c is a positive constant that depends on λ_2 and

$$
\widetilde{\Lambda}_1 = \begin{cases} 1, & 1 < \nu_1 < 2; \\ 0, & \nu_1 \geq 2. \end{cases} \tag{4.10}
$$

Recall that we selected earlier $a < \sqrt{\frac{2\beta\pi}{c_\rho\kappa}}$ where $\beta = \min\{\beta_1, \beta_2\}$ which makes $\Upsilon_1 := \beta_1 - \frac{a^2 c_\rho \kappa}{2\pi} > 0$ and $\Upsilon_2 := \beta_2 - \frac{a^2 c_\rho \kappa}{2\pi} > 0$. After that, we chose $\varepsilon = \frac{\Upsilon_1}{2c_\rho}$ and $\lambda_2 = \frac{\Upsilon_2}{2c_\rho{}^2}$. Finally, we selected N large enough so that $N\delta_2 - c, N\delta_1 - \gamma - c_\rho - \frac{\delta^2}{4\varepsilon} > 0$. Recalling (2.7) with these choices leads to

$$\begin{aligned} L'(t) \le{}& \rho_2 \int_\Omega |z_t|^2 dx - \alpha \int_\Omega (u-z)^3 dx - cE'(t) \\ &- \frac{\Upsilon_1}{2} \int_\Omega |\Delta u|^2 dx - \frac{\Upsilon_2}{2} \int_\Omega |\Delta z|^2 dx - c \int_\Omega |z_t|^{\nu(\cdot)} dx + c\widetilde{\Lambda}_1 \left(\int_\Omega |z_t|^{\nu(x)} dx \right)^{\nu_1 - 1} \\ &- \left(1 - \frac{\kappa}{2} \ln \|u\|_2^2 + \kappa(1 + \ln a)\right) \|u\|_2^2 - \left(1 - \frac{\kappa}{2} \ln \|z\|_2^2 + \kappa(1 + \ln a)\right) \|z\|_2^2. \end{aligned} \tag{4.11}$$

Now, assume further that $0 < E(0) < \ell\tau < d$, where

$$\ell = \max\{d\} = \frac{\beta\pi}{2c_\rho} e^{2\left(\frac{(\kappa+1)}{\kappa}\right)}, \tag{4.12}$$

and $0 < \tau < 1$ will be carefully selected later (see (4.17)). Combining (2.5), (2.7), and (4.12), we have

$$\ln \|u\|_2^2 < \ln\left(\frac{4}{\kappa} E(t)\right) < \ln\left(\frac{4}{\kappa} E(0)\right) < \ln\left(\frac{4}{\kappa} \ell\tau\right) < \ln\left(\frac{2\tau\beta\pi e^{2+\frac{2}{\kappa}}}{\kappa c_\rho}\right). \tag{4.13}$$

Now, we select a such that

$$\ln \|u\|_2^2 - 2(1 + \ln a) < 0 \Rightarrow \ln \|u\|_2^2 < \ln a^2 + 2 \ln e \Rightarrow \ln \|u\|_2^2 < \ln a^2 e^2. \tag{4.14}$$

This means that we need $\|u\|_2^2 < a^2 e^2$. From, (4.13), we select a such that

$$\frac{2\tau\beta\pi e^{\frac{2}{\kappa}}}{\kappa c_\rho} < a^2, \text{ and recall } a < \sqrt{\frac{2\pi}{c_\rho \kappa}}, \tag{4.15}$$

which means a must satisfy

$$e^{\frac{1}{\kappa}} \sqrt{\frac{2\beta\pi\tau}{\kappa c_\rho}} < a < \sqrt{\frac{2\pi}{c_\rho \kappa}}. \tag{4.16}$$

To estimate τ, it is clear that from (4.16), we have

$$e^{\frac{1}{\kappa}} \sqrt{\frac{2\beta\pi\tau}{\kappa c_\rho}} < \sqrt{\frac{2\pi}{c_\rho \kappa}} \Rightarrow \tau < \frac{1}{\sqrt{\beta} e^{1/\kappa}} < 1, \tag{4.17}$$

where $\beta = \min\{\beta_1, \beta_2\}$. With these choices, we have guaranteed that

$$1 - \frac{\kappa}{2} \ln \|u\|_2^2 + \kappa(1 + \ln a) > 0 \text{ and } 1 - \frac{\kappa}{2} \ln \|z\|_2^2 + \kappa(1 + \ln a) > 0.$$

Then, (4.11) becomes for some positive constant c,

$$L'(t) \le -cE(t) - cE'(t) + \rho_2 \int_\nu z_t^2 dx + c\widetilde{\Lambda}_1 (-E'(t))^{\nu_1 - 1}. \tag{4.18}$$

Using Young's inequality with $\zeta = \frac{1}{\nu_1 - 1}$ and $\zeta^* = \frac{1}{2-\nu_1}$ on this term $E^{\Theta}(t)(-E'(t))^{\nu_1 - 1}$, then for any $\varepsilon > 0$, we have

$$E^{\Theta}(t)(-E'(t))^{\nu_1-1} \leq \varepsilon E^{\frac{\Theta}{2-\nu_1}}(t) + c_\varepsilon(-E'(t)).$$

Multiplying both sides of the last inequality by $E^{-\Theta}$ where $\Theta = \frac{2-\nu_1}{\nu_1-1}$ gives us

$$(-E'(t))^{\nu_1-1} \leq \varepsilon E(t) + c_\varepsilon E^{-\Theta}(t)(-E'(t)).$$

Inserting these estimates in the last term in (4.18), we have

$$L'(t) \leq -(c-\varepsilon)E(t) + \rho_2 \int_\nu z_t^2 dx + c_\varepsilon \widetilde{\Lambda}_1 E^{-\Theta}(t)(-E'(t)). \tag{4.19}$$

Therefore, the estimate (4.3) is established. On the other hand, if needed, we can choose N even larger so that $L \sim E$. □

Lemma 4.2. *Assume that* (A_1) *holds. If* $\nu_1 \geq 2$, *then*

$$\int_\nu z_t^2 dx \leq -cE'(t), \quad \text{if } \nu_2 = 2, \tag{4.20}$$

$$\int_\nu z_t^2 dx \leq -cE'(t) + c\left(-E'(t)\right)^{\frac{2}{\nu_2}}, \quad \text{if } \nu_2 > 2. \tag{4.21}$$

Proof. The proof can be found in [40]. □

5. Decay estimates

We present and prove our results on the decay in this section.

Theorem 5.1. *Under the assumptions* (A_1) *and* (A_2), *the energy functional* (2.5) *satisfies, for some positive constants* λ_1, σ_1 *and for any* $t \geq 0$,

$$E(t) \leq \mu_1 e^{-\lambda_1 t}, \quad \text{if } \nu_1 = \nu_2 = \nu(x) = 2, \tag{5.1}$$

and

$$E(t) \leq \frac{\sigma_1}{(t+1)^{\left(\frac{\nu_2-2}{2}\right)}}, \text{ if } \nu_1 \geq 2 \text{ and } \nu_2 > 2. \tag{5.2}$$

Proof. To prove (5.1), we impose Lemma 4.2 in $(4.3)_1$ and use the equivalence properties $\mathcal{L} \sim E$ to get

$$\mathcal{L}'(t) \leq -c\mathcal{L}(t) + c(-E'(t)).$$

This gives us

$$\mathcal{L}_1'(t) \leq -c\mathcal{L}(t)$$

where $\mathcal{L}_1 = \mathcal{L} + cE \sim E$. Integrating the last estimate over the interval $(0, t)$ and using the equivalence properties $\mathcal{L}_1, \mathcal{L} \sim E$, the proof of (5.1) is completed.

Now, we prove the estimate in (5.2). For this, we impose Lemma 4.2 in $(4.3)_2$ to obtain

$$\mathcal{L}'(t) \leq -c\mathcal{L}(t) + (-E'(t))^{\frac{2}{\nu_2}} + c_\varepsilon \widetilde{\Lambda}_1 E^{-\Theta}(t)(-E'(t)).$$

The following result obtained by multiplying the last equation by E^{α} where $\alpha = \frac{\nu_2-2}{2} > 0$, and noting that $\alpha - \Theta = \frac{\nu_2(\nu_1-2)-2}{2(\nu_1-1)} > 0$,

$$E^{\alpha}\mathcal{L}'(t) \leq -cE^{\alpha}\mathcal{L}(t) + E^{\alpha}\left(-E'(t)\right)^{\frac{2}{\nu_2}} + c\widetilde{\Lambda}_1\left(-E'(t)\right).$$

This reduces to

$$\mathcal{L}_1'(t) \leq -cE^{\alpha}\mathcal{L}(t) + E^{\alpha}\left(-E'(t)\right)^{\frac{2}{\nu_2}},$$

where $\mathcal{L}_2 = E^{\alpha}\mathcal{L} + c\widetilde{\Lambda}_1 E \sim E$. With the use of the Young inequality on the last term, we get for $\varepsilon > 0$

$$\mathcal{L}_1'(t) \leq -cE^{\alpha+1}\mathcal{L}(t) + \varepsilon E^{\frac{\alpha\nu_2}{\nu_2-2}} + c_{\varepsilon}\left(-E'(t)\right).$$

Taking ε small enough, the above estimate becomes:

$$\mathcal{L}_2(t) \leq -cE^{\alpha+1}(t), \qquad \forall t \geq 0, \tag{5.3}$$

where $\mathcal{L}_2 = \mathcal{L}_1 + cE \sim E$. Integration over $(0, t)$ and using $E \sim \mathcal{L}_2$ gives us

$$E(t) < \frac{c_{\nu_2}}{(t+1)^{1/\alpha}}, \ \forall\ t > 0, \tag{5.4}$$

where $\alpha = \frac{\nu_2-2}{2} > 0$. □

Theorem 5.2. *Under the assumptions* (A_1) *and* (A_2)*, the energy functional* (2.5) *satisfies, for a positive constant* C_1 *and* $1 < \nu_1 < 2$*, the following estimate:*

$$E(t) \leq C_1\left(1+t\right)^{\frac{-1}{\Theta}}, \ t > 0, \ \nu_2 = 2, \tag{5.5}$$

where $\Theta = \frac{2-\nu_1}{\nu_1-1} > 0$.

Proof. To prove (5.5), we impose Lemma (4.2) in $(4.3)_2$ to have

$$\mathcal{L}'(t) \leq -cE(t) + c\left(-E'(t)\right) + c\left(-E'(t)\right) + c\widetilde{\Lambda}_1 E^{-\Theta}(t)(-E'(t)), \tag{5.6}$$

where $\Theta = \frac{2-\nu_1}{\nu_1-1} > 0$. Therefore, Eq (5.6) becomes

$$\mathcal{L}_1'(t) \leq -cE(t) + c\widetilde{\Lambda}_1 E^{\Theta}(t)(-E'(t)), \tag{5.7}$$

where $\mathcal{L}_1 = \mathcal{L} + cE \sim E$. The following is obtained by multiplying (5.7) by E^{Θ},

$$E^{\Theta}(t)\mathcal{L}_1'(t) \leq -cE^{\Theta+1}(t) - c\widetilde{\Lambda}_1 E'(t),$$

which leads to

$$\mathcal{L}_2'(t) \leq -cE^{\Theta+1}(t)$$

for $\mathcal{L}_2 = E^{\mu}\mathcal{L}_1 + c\widetilde{\Lambda}_1 E \sim E$. Then, we get the following decay estimate:

$$E(t) \leq C\left[\frac{1}{(t+1)}\right]^{\frac{1}{\Theta}}, \ \forall\ t > 0. \tag{5.8}$$

This completes the proof of (5.5). □

Theorem 5.3. *Under the assumptions* (A_1) *and* (A_2)*, the energy functional* (2.5) *satisfies for a positive constant* C_1*,* $1 < \nu_1 < 2$ *and* $\nu_2 > 2$ *the following estimate:*

$$E(t) < \frac{C_1}{(t+1)^{\left(\frac{2}{\nu_2-2}\right)}}, \quad t > 0. \tag{5.9}$$

Proof. To prove (5.9), we use Lemma 4.2 in $(4.3)_1$ to obtain

$$\mathcal{L}'(t) \leq -cE(t) + c\left(-E'(t)\right)^{\frac{2}{\nu_2}} - c\widetilde{\Lambda}_1 E^{-\Theta}(t)E'(t).$$

Multiplying by E^{α} for $\alpha = \frac{\nu_2-2}{2} > 0$, noting that $\alpha - \Theta > 0$ and Young's inequality, we obtain for a positive constant ε,

$$E^{\alpha}\mathcal{L}'(t) \leq -cE^{\alpha+1}(t) + \varepsilon E^{\frac{\alpha\nu_2}{\nu_2-2}} + c\widetilde{\Lambda}_1\left(-E'(t)\right).$$

Using $\alpha = \frac{\nu_2-2}{2}$ and $E \sim \mathcal{L}$, the above equation reduced to

$$E^{\alpha}\mathcal{L}'(t) \leq -\left(c - c_{\varepsilon}\right)E^{\alpha+1}\mathcal{L}(t) + c\widetilde{\Lambda}_1\left(-E'(t)\right).$$

Taking ε small enough, the above estimate becomes:

$$\mathcal{L}_2(t) \leq -cE^{\alpha+1}(t), \qquad \forall t \geq 0, \tag{5.10}$$

where $\mathcal{L}_2 = E^{\alpha}\mathcal{L} + c\widetilde{\Lambda}_1 E \sim E$.

Integration over $(0,t)$ and using the fact $E \sim \mathcal{L}_2$, gives

$$E(t) < \frac{C_{\nu_2}}{(t+1)^{1/\alpha}}, \ \forall\, t > 0, \tag{5.11}$$

where $\alpha = \frac{\nu_2-2}{2} > 0$. So, the proof of (5.9) is completed. □

6. Concluding remarks

We considered a coupled system of plate equations. We investigated the interaction of Kirchhoff and the Euler-Bernoulli plates. Kirchhoff equation is dissipated by a strong damping mechanism while the Euler-Bernoulli equation is dissipated by a weak damping mechanism of variable-exponent type. We noticed the following:

- The system (P) decays exponentially when $\nu(x) = 2$ and polynomially when $1 < \nu_1 < 2$ or $\nu_1 > 2$.
- We found that decay rates depend on the weak damping of variable-exponent type.
- The strong damping term in the Kirchhoff equation $(-\delta\Delta u_t)$ is introduced to treat the problems arising from the rotational inertia term $(-\gamma\Delta u_{tt})$ in the same Kirchhoff equation and we can obtain the same decay results if we replace this strong damping by a memory damping $\int_0^t g(t-s)\Delta^u(s)ds$ where the memory function g satisfies $g'(s) \leq -g(s)$.
- We can obtain the same decay results if we replace the coupling term $\alpha(u-z)$ by αuz^2 and αzu^2.
- The flexural rigidity coefficients β_i play a role in the analysis and they can control the well depth d either stretching or shrinking while the mass densities coefficients do not play any role.

- The constant κ on the source terms plays a role in the existence and stability. It also affects the well depth d.
- In our system (P), the single term u in the Kirchhoff equation and z in the Euler-Bernoulli equation play important roles in the existence and the stability as well.
- It is an interesting problem if one can investigate the coupling the system (P) where the coupling is on the logarithmic source terms such as if the source terms were $\kappa z \ln|u_t|$ and $\kappa u \ln|z_t|$.
- It is an interesting problem if one can investigate the coupling system (P) where the damping is the logarithmic function such as if the dampings were $-u_t \ln|u_t|$ and $-z_t \ln|z_t|$.

Use of AI tools declaration

The author declares they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that there is no conflict of interest regarding the publication of this paper.

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