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Research article

The exact transcendental entire solutions of complex equations with three quadratic terms

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Abstract: In this paper, we study the entire solutions of two quadratic functional equations in the complex plane. One consists of three basic terms, $f(z)$, $f'(z)$ and $f(z+c)$, and the other one consists of $f(z)$, $f'(z)$ and $f(qz)$. These two equations can be transformed into functional equations of Fermat-type. We prove that if these two equations admit finite order transcendental entire solutions, then the solutions of these two equations are both exponential functions, and their exponents are one degree polynomials, whose coefficients of the first degree term are closely related to the coefficients of the functional equation. Moreover, examples are given to show that the theorems are true. The feature of this paper is that the Fermat-type equations contain three quadratic terms, while the equations that have been studied in the previous articles in this field contain only two quadratic terms. The addition of $f(qz)$ will make the proof methods in this paper very different from those in the existing literature. The proof becomes more difficult, and the number of cases that need to be discussed becomes much larger. In addition, when dealing with the analytical property of f, we also use a different method from the previous literature.

Keywords: entire function; differential; difference; q-difference; Fermat-type equation
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1. Introduction

We all know Fermat's last theorem: For an integer $n > 2$, the equation $x^n + y^n = z^n$ for x, y, z has no positive integer solution. It took 356 years from when it was proposed in 1637, to 1993 when Wiles conquered it. This equation can also be extended to functional equations. In complex analysis, the researchers began to focus on meromorphic solutions of the equation $f^n(z) + g^n(z) = h^n(z)$. As far as we know, Montel [22] was the first scholar to study this problem, and later Gross and Baker carried out the follow-up research work [2, 9, 10]. Then, applying Nevanlinna theory to the study of this kind of functional equation became a hot topic. For example, Yang et al. [31] studied the transcendental meromorphic solution of the functional equation $f(z)^2 + f'(z)^2 = 1$ and found that the solution to this equation must have the form of $f(z) = \frac{1}{2}\left(pe^{\lambda z} + \frac{1}{p}e^{-\lambda z}\right)$, where p, λ are nonzero constants. With the establishment of the Nevanlinna theory with the difference of meromorphic function [8, 11], more attention was paid to Fermat-type functional equation with the difference of meromorphic function. Liu et al. [15–17] studied the finite order transcendental entire solutions of Fermat-type difference equations $f(z+c)^2 + f(z)^2 = 1$ and $f(z+c)^2 + f'(z)^2 = 1$, and they obtained that the solutions of these two equations are sine functions. Subsequently, more attention has been paid to this area of research [14, 18–21, 23–25, 28–30, 33, 34].

In 2016, Liu and Yang [18] studied the existence of solutions to quadratic trinomial functional equations, as well as the entire function and its derivatives and differences, and they converted the equations in the following two theorems into Fermat- type equations by transformation.

Theorem 1.1. *([18, Theorem 1.4]) If $\alpha \neq 0, \pm 1$, then the finite order transcendental entire solution of equation*

$$f(z)^2 + 2\alpha f(z)f(z+c) + f(z+c)^2 = 1$$

is of order one.

Theorem 1.2. *([18, Theorem 1.6]) If $\alpha \neq 0, \pm 1$, then the equation*

$$f(z)^2 + 2\alpha f(z)f'(z) + f'(z)^2 = 1$$

has no transcendental meromorphic solutions.

On the other hand, Han and Lü [12] studied the existence of solutions to the Fermat-type equation when the right side was an exponential function. Here we only list the $n = 2$ case in their results.

Theorem 1.3. *([12, Theorem 1.1]) The meromorphic solutions of f of the following differential equation*

$$f(z)^2 + f'(z)^2 = e^{\alpha z+\beta}$$

are

$$f(z) = e^{\frac{\beta}{2}}\sin(z+b)$$

if $\alpha = 0$, and

$$f(z) = de^{\frac{\alpha z+\beta}{2}}$$

if $\alpha \neq 0$ with $d^2(1 + (\frac{\alpha}{2})^2) = 1$.

In the same article, they also studied the case of replacing $f'(z)$ with $f(z+c)$ in the above equation, and found that the solution of equation

$$f(z)^2 + f(z+c)^2 = e^{\alpha z+\beta}$$

is $f(z) = de^{\frac{\alpha z+\beta}{2}}$ with $d^2(1 + e^{ac}) = 1$.

Combining these conclusions above, Luo et al. [20] studied the following three equations with finite order transcendental entire solutions. These three equations are

$$f(z+c)^2 + 2\alpha f(z)f(z+c) + f(z)^2 = e^{g(z)}, \tag{1.1}$$

$$f(z+c)^2 + 2\alpha f'(z)f(z+c) + f'(z)^2 = \mathrm{e}^{g(z)} \tag{1.2}$$

and

$$f(z)^2 + 2\alpha f(z)f'(z) + f'(z)^2 = \mathrm{e}^{g(z)}, \tag{1.3}$$

where $\alpha \neq 0, \pm 1$, c are constants and $g(z)$ is a polynomial. If all these equations admit finite order transcendental entire solutions, $g(z)$ must be a polynomial with the degree of one, and the solutions f of these equations are all exponential functions or the sum of two exponential functions whose exponents are polynomials of the degree of one, as seen in Theorems 2.1–2.3 [20]. Each of these three equations contains only two terms of $f(z), f'(z)$ or $f(z+c)$, so can we consider a quadratic equation that contains all three of these terms?

Inspired by this, we shall study the problem of finite order transcendental entire solutions of functional equations involving the quadratic of f, its derivative and its difference. In fact, we studied the finite order transcendental entire solution for (1.4) below.

Theorem 1.4. *Suppose that $\alpha \neq 0, \pm 1$, $\beta \neq 0$, $\gamma \neq 0$ and $c \neq 0$ are four constants such that $\alpha^2+\beta^2+\gamma^2 \neq 1 + 2\alpha\beta\gamma$, and $g(z)$ is a nonconstant polynomial. If the complex equation*

$$[f'(z)]^2 + [f(z+c)]^2 + f^2(z) + 2\alpha f'(z)f(z+c) + 2\beta f(z)f(z+c) + 2\gamma f(z)f'(z) = \mathrm{e}^{g(z)} \tag{1.4}$$

admits a transcendental entire solution $f(z)$ of finite order, then for

$$\delta = \frac{1-\alpha^2-\beta^2-\gamma^2+2\alpha\beta\gamma}{1-\alpha^2},$$

the solution has two forms:

(1)

$$f(z) = \frac{d\mathrm{e}^{\frac{az+b}{2}}}{2i\sqrt{\delta}},$$

where $a \neq 0$ and b is an arbitrary constant. Moreover, $g(z) = az + b$, and d is a constant with

$$1 + \frac{a^2}{4} + \mathrm{e}^{ac} + (a\alpha + 2\beta)\mathrm{e}^{ac/2} + a\gamma = -\frac{4\delta}{d^2}.$$

(2)

$$f(z) = \frac{\mathrm{e}^{a_1z+b_1} - \mathrm{e}^{a_2z+b_2}}{2i\sqrt{\delta}},$$

where a_1, a_2 $(a_1 \neq a_2)$ are nonzero constants satisfying (1.5), and b_1, b_2 are arbitrary constants, $g(z) = (a_1 + a_2)z + b_1 + b_2$.

$$\begin{cases} a_1^2 + 2\gamma a_1 + \mathrm{e}^{2a_1c} + (2\alpha a_1 + 2\beta)\cdot \mathrm{e}^{a_1c} + 1 = 0; \\ a_2^2 + 2\gamma a_2 + \mathrm{e}^{2a_2c} + (2\alpha a_2 + 2\beta)\cdot \mathrm{e}^{a_2c} + 1 = 0; \\ [\mathrm{e}^{a_1c} - \mathrm{e}^{a_2c} + \alpha(a_1-a_2)]^2 + (1-\alpha^2)(a_1-a_2)^2 + 4\delta = 0. \end{cases} \tag{1.5}$$

Let's give two examples to show that Theorem 1.4 is true.

Example 1.5. Suppose $\alpha = 1/3, \beta = \gamma = 1$ and $c = 2$ in (1.4), then $\delta = -1/2$. Set $a = 2, b = 2$, then by the relationship of a and d, we have

$$d = \frac{\sqrt{2}}{\sqrt{4 + \frac{8}{3}e^2 + e^4}}.$$

We can verify that the entire function

$$f(z) = \frac{\pm 1}{\sqrt{4 + \frac{8}{3}e^2 + e^4}} \cdot e^{z+1}$$

is a solution of

$$[f'(z)]^2 + [f(z+2)]^2 + f^2(z) + \frac{2}{3}f'(z)f(z+2) + 2f(z)f(z+2) + 2f(z)f'(z) = e^{2z+2}.$$

Example 1.6. Suppose

$$\alpha = \frac{3\pi i}{4}, \beta = 1 + \frac{3\pi^2}{2}, \gamma = -\frac{5\pi i}{4}$$

and $c = 1$ in (1.4), then $\delta = \pi^2$. $a_1 = \pi i, a_2 = 3\pi i$, b_1, b_2 are arbitrary constants. We can verify that the entire function

$$f(z) = \frac{e^{\pi i z + b_1} - e^{3\pi i z + b_2}}{\pm 2\pi i}$$

is a solution of

$$[f'(z)]^2 + [f(z+1)]^2 + f^2(z) + \frac{3\pi i}{2}f'(z)f(z+1) + (2 + 3\pi^2)f(z)f(z+1) - \frac{5\pi i}{2}f(z)f'(z) = e^{4\pi i z + b_1 + b_2}.$$

From Theorem 1.4 we have the following corollary.

Corollary 1.7. *Under the assumption of Theorem 1.4, if the degree of polynomial $g(z)$ is greater than one, then* (1.4) *does not have a transcendental solution with finite order.*

If $q(\neq 0, 1)$ is a constant, then $f(qz)$ is called the q-difference of meromorphic function $f(z)$. The q-difference is also an important research content in the value distribution theory, and the research on it can be traced back to the early 20th century [5, 13].

In recent decades, with the establishment of Nevanlinna theory related to it [3], the research on q-difference has been vigorously developed, and this theory has been applied to many q-difference equations to get a lot of results [4, 6, 7, 16, 26, 27]. Therefore, we considered to replace $f(z+c)$ in (1.4) by $f(qz)$ as to get a q-difference functional equation, and then studied the finite order transcendental entire solution of this equation. Through the complicated discussion and calculation of different cases, we came to the following conclusion.

Theorem 1.8. *Suppose that $\alpha \neq 0, \pm 1$, $\beta \neq 0$, $\gamma \neq 0, \pm 1$ and $q \neq 0, 1$ are four constants such that $\alpha^2 + \beta^2 + \gamma^2 \neq 1 + 2\alpha\beta\gamma$, and $g(z)$ is a nonconstant polynomial. If the complex equation*

$$[f'(z)]^2 + [f(qz)]^2 + f^2(z) + 2\alpha f'(z)f(qz) + 2\beta f(z)f(qz) + 2\gamma f(z)f'(z) = e^{g(z)} \tag{1.6}$$

admits a transcendental entire solution $f(z)$ of finite order, then

$$f(z) = \pm e^{\frac{az+b}{2}}$$

and $g(z) = aqz + b$, b is an arbitrary constant, $a \neq 0$ and $\gamma^2 \neq 1$ such that

$$\begin{cases} \frac{a^2}{4} + \gamma a + 1 = 0, \\ \alpha a + 2\beta = 0. \end{cases} \tag{1.7}$$

Here is an example to test the truth of the Theorem 1.8.

Example 1.9. Suppose $\alpha = 1/2, \beta = -1, \gamma = -5/4$ and q is any constant except $0, 1$ in (1.6), then we can verify that the entire function $f(z) = \pm e^{2z+1}$ is a solution of

$$[f'(z)]^2 + [f(qz)]^2 + f^2(z) + f'(z)f(qz) - 2f(z)f(qz) - \frac{5}{2}f(z)f'(z) = e^{4qz+2}.$$

A corollary also can be obtained from Theorem 1.8.

Corollary 1.10. *Under the assumption of Theorem 1.8, if the degree of polynomial $g(z)$ is greater than one, then* (1.6) *does not admit transcendental entire solution with finite order.*

Remark 1.11. Equations (1.4) and (1.6) can be transformed into three term quadratic equations by linear transformation. The purpose of restrictions $\alpha^2 \neq 1$ and $\alpha^2 + \beta^2 + \gamma^2 \neq 1 + 2\alpha\beta\gamma$ in Theorems 1.4 and 1.8 is to not allow these three term quadratic equations to degenerate into quadratic equations with two or one terms, which have been studied in previous literatures. This can be seen easily from (3.2) in the proof below.

Remark 1.12. From the proof of Theorems 1.4 and 1.8 and the above three examples, we can find that if the two equations have finite order transcendental entire solutions, then the solutions of both equations are exponential functions and their exponents are polynomials with the degree of one. For (1.4), after the solution was substituted into the equation, the terms of the equation contained the common factor $e^{g(z)}$. After dividing both sides of the equation by $e^{g(z)}$, the relationship between the coefficients of the equation and the coefficients of the exponent was obtained. For (1.6), when one substitutes the solution into it, the term $e^{g(z)}$ in the right side of the equation is equal to $[f(qz)]^2$ in the left side, which can be subtracted from both sides of the equation. The signs of the other two mixed terms containing $f(qz)$ are opposite to each other, so these two mixed terms were canceled out.

2. Preliminary lemmas

The following lemma played a key role in the proofs of this paper. It is about the factorization of an entire function. In particular, if $f(z)$ was a finite order entire function without zero, then $f(z) = e^{h(z)}$ where $h(z)$ was a nonconstant polynomial, as seen in Theorems 1.42 and 1.44 [32].

Lemma 2.1. *(Hadamard's factorization theorem) [32, Theorem 2.5] Let f be an entire function of finite order $\lambda(f)$ with zeros $\{a_1, a_2, \cdots\} \subset \mathbb{C} \setminus \{0\}$ and a k-fold zero at the origin. Then,*

$$f(z) = z^k P(z) e^{Q(z)}$$

where $P(z)$ is the canonical product of f formed with the non-null zeros of f,

$$P(z)=\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots+\frac{1}{h}\left(\frac{z}{a_n}\right)^h},$$

and h is the smallest integer for which this series converges, called the genus of the canonical product. $Q(z)$ is a polynomial of degree $\leq \lambda(f)$ and $h \leq \lambda$.

The second lemma belongs to Borel. It's about the combination of entire functions, and we'll use it repeatedly in the proofs in Sections 3 and 4. When using it, the key is to verify the second condition below.

Lemma 2.2. *[32, Theorem 1.52] If $f_j(z)(j=1,2,\cdots,n)$ and $g_j(z)(j=1,2,\cdots,n)(n\geq 2)$ are entire functions satisfying*

(1) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$,

(2) the orders of f_j are less than that of $e^{g_h(z)-g_k(z)}$ for $1\leq j\leq n, 1\leq h<k\leq n$.

Then, $f_j\equiv 0, (j=1,2,\cdots,n)$.

3. Proof of Theorem 1.4

According to the linear algebra, any quadratic form can be reduced to the standard form by a non-degenerate linear transformation. So, setting

$$\begin{cases} f(z)=w, \\ f'(z)=u-\alpha v+\frac{\alpha\beta-\gamma}{1-\alpha^2}w, \\ f(z+c)=v-\frac{\beta-\alpha\gamma}{1-\alpha^2}w \end{cases} \tag{3.1}$$

and substituting it into (1.4), we obtain that

$$u^2+(1-\alpha^2)v^2+\frac{1-\alpha^2-\beta^2-\gamma^2+2\alpha\beta\gamma}{1-\alpha^2}w^2=e^{g(z)}. \tag{3.2}$$

For simplicity and convenience, let's denote

$$\delta := \frac{1-\alpha^2-\beta^2-\gamma^2+2\alpha\beta\gamma}{1-\alpha^2},$$

then (3.2) can convert into

$$\left(\frac{\sqrt{u^2+(1-\alpha^2)v^2}}{e^{\frac{g(z)}{2}}}\right)^2+\left(\frac{\sqrt{\delta}w}{e^{\frac{g(z)}{2}}}\right)^2=1. \tag{3.3}$$

Consequently, we have

$$\left(\frac{\sqrt{u^2+(1-\alpha^2)v^2}}{e^{\frac{g(z)}{2}}}+i\cdot\frac{\sqrt{\delta}w}{e^{\frac{g(z)}{2}}}\right)\cdot\left(\frac{\sqrt{u^2+(1-\alpha^2)v^2}}{e^{\frac{g(z)}{2}}}-i\cdot\frac{\sqrt{\delta}w}{e^{\frac{g(z)}{2}}}\right)=1. \tag{3.4}$$

By Hadamard's factorization theorem, if the multiplicities of any zeros of the entire function $u^2 + (1-\alpha^2)v^2$ is even number, then $\sqrt{u^2+(1-\alpha^2)v^2}$ is also an entire function. The following (3.5) holds in the complex plane, where $p(z)$ is a polynomial. If $u^2+(1-\alpha^2)v^2$ have some zeros with odd number multiplicities, then $\sqrt{u^2+(1-\alpha^2)v^2}$ has branch points in the complex plane. Branches are obtained by connecting finite branch points and infinity points appropriately by line segments. These segments are called branch cuts, and $\sqrt{u^2+(1-\alpha^2)v^2}$ is analytic and univalent in every branch [1,35]. Since the two analytical factors on the left side of (3.4) have no zeros in each branch, there exists an analytical function $p(z)$ such that the equations

$$\begin{cases} \frac{\sqrt{u^2+(1-\alpha^2)v^2}}{e^{\frac{g(z)}{2}}} + i\cdot\frac{\sqrt{\delta}w}{e^{\frac{g(z)}{2}}} = e^{p(z)}, \\ \frac{\sqrt{u^2+(1-\alpha^2)v^2}}{e^{\frac{g(z)}{2}}} - i\cdot\frac{\sqrt{\delta}w}{e^{\frac{g(z)}{2}}} = e^{-p(z)} \end{cases} \tag{3.5}$$

hold in every branch. Denote

$$\lambda_1(z) := p(z) + \frac{g(z)}{2}, \quad \lambda_2(z) := -p(z) + \frac{g(z)}{2},$$

then,

$$w = \frac{e^{\lambda_1(z)} - e^{\lambda_2(z)}}{2i\sqrt{\delta}} \tag{3.6}$$

hold in every branch. Moreover, noting that $w = f(z)$ is an entire function with finite order, the righthand side of (3.6) can be extended to the whole complex plane. Therefore, one can supplement the definition of function $p(z)$ at the points on branch cuts by the limiting values, and it is still called $p(z)$ after supplementary definition. Thus, $p(z)$ is analytic on the whole complex plane, so it's an entire function. Because $e^{p(z)}$ is of finite order, $p(z)$ is actually a polynomial, so we get

$$u^2 + (1-\alpha^2)v^2 = \left(\frac{e^{\lambda_1(z)} + e^{\lambda_2(z)}}{2}\right)^2. \tag{3.7}$$

Noting that $f(z) = w$, we have

$$\begin{cases} f(z) = \frac{e^{\lambda_1(z)} - e^{\lambda_2(z)}}{2i\sqrt{\delta}}, \\ f'(z) = \frac{\lambda_1'(z)e^{\lambda_1(z)} - \lambda_2'(z)e^{\lambda_2(z)}}{2i\sqrt{\delta}}, \\ f(z+c) = \frac{e^{\lambda_1(z+c)} - e^{\lambda_2(z+c)}}{2i\sqrt{\delta}}. \end{cases} \tag{3.8}$$

From (3.1), we know that

$$u = f'(z) + \alpha f(z+c) + \gamma f(z)$$

and

$$v = f(z+c) + \frac{\beta - \alpha\gamma}{1-\alpha^2} f(z).$$

Substituting the above u, v into (3.7) we get

$$\begin{aligned}
&[f'(z)]^2 + [f(z+c)]^2 + \frac{\beta^2+\gamma^2-2\alpha\beta\gamma}{1-\alpha^2} f^2(z) \\
&+2\alpha f'(z)f(z+c) + 2\beta f(z+c)f(z) + 2\gamma f'(z)f(z) \\
&= \frac{e^{2\lambda_1(z)} + e^{2\lambda_2(z)} + 2e^{\lambda_1(z)+\lambda_2(z)}}{4}.
\end{aligned} \tag{3.9}$$

For simplicity and convenience, we give the following notation:

$$\lambda_1 := \lambda_1(z), \lambda_2 := \lambda_2(z), \overline{\lambda_1} := \lambda_1(z+c), \overline{\lambda_2} := \lambda_2(z+c).$$

Substituting (3.8) into (3.9) we obtain

$$\begin{aligned}
&\frac{\lambda_1'^2 e^{2\lambda_1} + \lambda_2'^2 e^{2\lambda_2} - 2\lambda_1'\lambda_2' e^{\lambda_1+\lambda_2}}{-4\delta} + \frac{e^{2\overline{\lambda_1}} + e^{2\overline{\lambda_2}} - 2e^{\overline{\lambda_1}+\overline{\lambda_2}}}{-4\delta} \\
&+\frac{\beta^2+\gamma^2-2\alpha\beta\gamma}{1-\alpha^2} \cdot \frac{e^{2\lambda_1} + e^{2\lambda_2} - 2e^{\lambda_1+\lambda_2}}{-4\delta} \\
&+2\alpha \cdot \frac{\lambda_1' e^{\lambda_1+\overline{\lambda_1}} - \lambda_1' e^{\lambda_1+\overline{\lambda_2}} - \lambda_2' e^{\overline{\lambda_1}+\lambda_2} + \lambda_2' e^{\lambda_2+\overline{\lambda_2}}}{-4\delta} \\
&+2\beta \cdot \frac{e^{\lambda_1+\overline{\lambda_1}} - e^{\lambda_1+\overline{\lambda_2}} - e^{\overline{\lambda_1}+\lambda_2} + e^{\lambda_2+\overline{\lambda_2}}}{-4\delta} \\
&+2\gamma \cdot \frac{\lambda_1' e^{2\lambda_1} - \lambda_1' e^{\lambda_1+\lambda_2} - \lambda_2' e^{\lambda_1+\lambda_2} + \lambda_2' e^{2\lambda_2}}{-4\delta} \\
&= \frac{e^{2\lambda_1} + e^{2\lambda_2} + 2e^{\lambda_1+\lambda_2}}{4}.
\end{aligned} \tag{3.10}$$

The transcendental terms appearing in the above equation have exponents

$$2\lambda_1,\ 2\lambda_2,\ \lambda_1+\lambda_2,\ 2\overline{\lambda_1},\ 2\overline{\lambda_2},\ \overline{\lambda_1}+\overline{\lambda_2},\ \lambda_1+\overline{\lambda_1},\ \lambda_1+\overline{\lambda_2},\ \overline{\lambda_1}+\lambda_2 \text{ and } \lambda_2+\overline{\lambda_2}.$$

In order to apply Lemma 2.2 to (3.10), we checked whether the pairwise difference between these exponents was constant. If $\lambda_1 \equiv \lambda_2$, then $f(z) \equiv 0$ this was impossible, so $\lambda_1 \not\equiv \lambda_2$. The following two cases are discussed.

Case 1. If $\lambda_1 - \lambda_2$ is a nonzero constant, then $p(z)$ is a constant, denoted by p in the following. Consequently, for $p \neq k\pi i (k \in \mathbb{Z})$,

$$f(z) = w = \frac{(e^p - e^{-p})e^{g(z)/2}}{2i\sqrt{\delta}}. \tag{3.11}$$

Then,

$$f'(z) = \frac{(e^p - e^{-p})e^{g(z)/2}}{2i\sqrt{\delta}} \cdot \frac{g'(z)}{2} \tag{3.12}$$

and

$$f(z+c) = \frac{(e^p - e^{-p})e^{g(z+c)/2}}{2i\sqrt{\delta}}. \tag{3.13}$$

Substituting (3.11)–(3.13) into (1.4), the terms in the left side of (1.4) can be expressed respectively as

$$\begin{cases} [f'(z)]^2 = \frac{d^2 e^{g(z)}}{-4\delta} \cdot \frac{(g'(z))^2}{4}, \\ [f(z+c)]^2 = \frac{d^2 e^{g(z+c)}}{-4\delta}, \\ f^2(z) = \frac{d^2 e^{g(z)}}{-4\delta}, \\ 2\alpha f'(z) f(z+c) = 2\alpha \cdot \frac{d^2 e^{\frac{g(z)+g(z+c)}{2}}}{-4\delta} \cdot \frac{g'(z)}{2}, \\ 2\beta f(z) f(z+c) = 2\beta \cdot \frac{d^2 e^{\frac{g(z)+g(z+c)}{2}}}{-4\delta}, \\ 2\gamma f(z) f'(z) = 2\gamma \cdot \frac{d^2 e^{g(z)}}{-4\delta} \cdot \frac{g'(z)}{2}, \end{cases} \tag{3.14}$$

where $d := e^p - e^{-p}$. If the degree of polynomial $g(z)$ is greater than one, the three exponents $g(z), g(z+c)$ and $\frac{g(z)+g(z+c)}{2}$ are pairwise distinct. By Lemma 2.2, we obtained that after combining like terms, the coefficients of these three exponential terms $e^{g(z)}, e^{g(z+c)}$ and $e^{\frac{g(z)+g(z+c)}{2}}$ are zero. In particular, $\frac{d^2}{-4\delta} = 0$ since it's the coefficient of the sole term $e^{g(z+c)}$. This is impossible, because that means $f \equiv 0$, so the degree of $g(z)$ is one. Therefore, suppose $g(z) = az + b$, $a(\neq 0), b$ are constants. Substitute it into (3.14), then into (1.4), and eliminate $e^{g(z)}$ from both sides of this equation. Then, we get

$$1 + \frac{a^2}{4} + e^{ac} + (a\alpha + 2\beta)e^{ac/2} + a\gamma = -\frac{4\delta}{d^2}. \tag{3.15}$$

This means if the constants α, β, γ, c in the original (1.4) are known, then the solution is

$$f(z) = \frac{d e^{\frac{az+b}{2}}}{2i\sqrt{\delta}},$$

where constants a, d should satisfy the relationship of (3.15), and b is an arbitrary constant.

Case 2. If $\lambda_1 - \lambda_2$ is not a constant, then $p(z)$ is not a constant; instead, it is a nonconstant polynomial. For (3.10) we multiply -4δ on both sides, combine like terms and move all the terms to the left side of this equation, then the right side is just zero. Thus, the coefficients of the distinct transcendental terms can be listed in Table 1.

Table 1. Transcendental terms and corresponding coefficients.

Transcendental terms	Corresponding coefficients
$e^{2\lambda_1}$	$\lambda_1'^2 + 2\gamma\lambda_1' + 1$
$e^{2\lambda_2}$	$\lambda_2'^2 + 2\gamma\lambda_2' + 1$
$e^{\lambda_1+\lambda_2}$	$-2\lambda_1'\lambda_2' - 2\gamma(\lambda_1' + \lambda_2') + 4\delta - 2$
$e^{2\overline{\lambda_1}}$	1
$e^{2\overline{\lambda_2}}$	1
$e^{\overline{\lambda_1}+\overline{\lambda_2}}$	-2
$e^{\lambda_1+\overline{\lambda_1}}$	$2\alpha\lambda_1' + 2\beta$
$e^{\lambda_1+\overline{\lambda_2}}$	$-2\alpha\lambda_1' - 2\beta$
$e^{\overline{\lambda_1}+\lambda_2}$	$-2\alpha\lambda_2' - 2\beta$
$e^{\lambda_2+\overline{\lambda_2}}$	$2\alpha\lambda_2' + 2\beta$

Because the difference between any two of $2\lambda_1, 2\lambda_2, \lambda_1+\lambda_2$ is not constant, the term containing $e^{2\lambda_1}$ cannot combine with terms containing $e^{2\lambda_2}$ or $e^{\lambda_1+\lambda_2}$.

Suppose that $\deg(\lambda_1) = m > 1$ and $\deg(\lambda_2) = n > 1$. If the term containing $e^{2\lambda_1}$ cannot combine with any other transcendental terms, then its coefficient has to be zero for any $z \in \mathbb{C}$ by Lemma 2.2. This is impossible, since its coefficients are nonconstant polynomials. The only term in the coefficient that may cancel out with $\lambda_1'^2$ is the term that contains λ_2'. They must have the same degree, so we have $2(m-1) = n-1$. By the same arguments, we have $m-1 = 2(n-1)$ by considering $\lambda_2'^2$ with λ_1'. Then, we get a contradiction, and it yields that $\deg(\lambda_1), \deg(\lambda_2)$ are both at most one.

Therefore, we can assume that $\lambda_1 = a_1 z + b_1$ and $\lambda_2 = a_2 z + b_2$ where $a_1 \neq a_2$ are constants, and b_1, b_2 are arbitrary constants. The transcendental terms and the corresponding coefficients in Table 1 can convert into those in Table 2. Since the three transcendental terms $e^{2\lambda_1}$, $e^{2\lambda_2}$ and $e^{\lambda_1+\lambda_2}$ cannot be combined with each other, we get the following system of (3.16) with respect to the coefficients by Lemma 2.2.

$$\begin{cases} a_1^2 + 2\gamma a_1 + e^{2a_1 c} + (2\alpha a_1 + 2\beta)\cdot e^{a_1 c} + 1 = 0, \\ a_2^2 + 2\gamma a_2 + e^{2a_2 c} + (2\alpha a_2 + 2\beta)\cdot e^{a_2 c} + 1 = 0, \\ -2a_1 a_2 - 2\gamma(a_1+a_2) - 2e^{(a_1+a_2)c} - (2\alpha a_1 + 2\beta)\cdot e^{a_2 c} \\ -(2\alpha a_2 + 2\beta)\cdot e^{a_1 c} + 4\delta - 2 = 0. \end{cases} \tag{3.16}$$

Table 2. The transcendental term after the change and the corresponding coefficients.

Before	After	Corresponding coefficients
$e^{2\lambda_1}$	$e^{2\lambda_1}$	$a_1^2 + 2\gamma a_1 + 1$
$e^{2\overline{\lambda_1}}$	$e^{2\lambda_1}$	$e^{2a_1 c}$
$e^{\lambda_1+\overline{\lambda_1}}$	$e^{2\lambda_1}$	$(2\alpha a_1 + 2\beta)\cdot e^{a_1 c}$
$e^{2\lambda_2}$	$e^{2\lambda_2}$	$a_2^2 + 2\gamma a_2 + 1$
$e^{2\overline{\lambda_2}}$	$e^{2\lambda_2}$	$e^{2a_2 c}$
$e^{\lambda_2+\overline{\lambda_2}}$	$e^{2\lambda_2}$	$(2\alpha a_2 + 2\beta)\cdot e^{a_2 c}$
$e^{\lambda_1+\lambda_2}$	$e^{\lambda_1+\lambda_2}$	$-2a_1 a_2 - 2\gamma(a_1+a_2) + 4\delta - 2$
$e^{\overline{\lambda_1}+\overline{\lambda_2}}$	$e^{\lambda_1+\lambda_2}$	$-2e^{(a_1+a_2)c}$
$e^{\lambda_1+\overline{\lambda_2}}$	$e^{\lambda_1+\lambda_2}$	$(-2\alpha a_1 - 2\beta)\cdot e^{a_2 c}$
$e^{\overline{\lambda_1}+\lambda_2}$	$e^{\lambda_1+\lambda_2}$	$(-2\alpha a_2 - 2\beta)\cdot e^{a_1 c}$

Adding the three equations in (3.16) together, they convert into

$$\begin{cases} a_1^2 + 2\gamma a_1 + e^{2a_1 c} + (2\alpha a_1 + 2\beta)\cdot e^{a_1 c} + 1 = 0, \\ a_2^2 + 2\gamma a_2 + e^{2a_2 c} + (2\alpha a_2 + 2\beta)\cdot e^{a_2 c} + 1 = 0, \\ [e^{a_1 c} - e^{a_2 c} + \alpha(a_1 - a_2)]^2 + (1-\alpha^2)(a_1-a_2)^2 + 4\delta = 0. \end{cases} \tag{3.17}$$

Therefore, the original (1.4) has solutions of the form

$$f(z) = \frac{e^{a_1 z + b_1} - e^{a_2 z + b_2}}{2i\sqrt{\delta}},$$

where a_1, a_2 $(a_1 \neq a_2)$ are nonzero constants satisfying (3.17), and b_1, b_2 are arbitrary constants.

4. Proof of Theorem 1.8

For an exponential polynomial $f(z)$ with finite order, the exponents for each exponential terms of $f'(z)$ are the same as those of $f(z)$, but the exponents of $f(qz)$ are not the same as the exponents of $f(z)$ for $q \neq 0, 1$. The term $e^{g(z)}$ in the right side of (1.6) with coefficient one must combine with one of these two kinds of exponential terms, transcendental terms in $f(z)$ or $f(qz)$, by Lemma 2.2. The following are divided into two cases for discussion.

Case 1. If $e^{g(z)}$ can combine with the exponential terms in $f(z)$, then replacing $f(z+c)$ by $f(qz)$ in Section 3 and using the same methods in it we get

$$\begin{cases} f(z) = \frac{e^{\lambda_1(z)} - e^{\lambda_2(z)}}{2i\sqrt{\delta}}, \\ f'(z) = \frac{\lambda_1'(z)e^{\lambda_1(z)} - \lambda_2'(z)e^{\lambda_2(z)}}{2i\sqrt{\delta}}, \\ f(qz) = \frac{e^{\lambda_1(qz)} - e^{\lambda_2(qz)}}{2i\sqrt{\delta}}, \end{cases} \tag{4.1}$$

where $\lambda_1(z), \lambda_2(z), \delta$ are the same as in Section 3, and we also have

$$u = f'(z) + \alpha f(qz) + \gamma f(z), \quad v = f(qz) + \frac{\beta - \alpha\gamma}{1-\alpha^2} f(z).$$

Substituting the above u, v into

$$u^2 + (1-\alpha^2)v^2 = \left(\frac{e^{\lambda_1(z)} + e^{\lambda_2(z)}}{2}\right)^2, \tag{4.2}$$

it yields

$$\begin{aligned} [f'(z)]^2 + [f(qz)]^2 &+ \frac{\beta^2+\gamma^2-2\alpha\beta\gamma}{1-\alpha^2} f^2(z) + 2\alpha f'(z)f(qz) + 2\beta f(qz)f(z) + 2\gamma f'(z)f(z) \\ &= \frac{e^{2\lambda_1(z)} + e^{2\lambda_2(z)} + 2e^{\lambda_1(z)+\lambda_2(z)}}{4}. \end{aligned} \tag{4.3}$$

For simplicity and convenience, we denote

$$\lambda_1 := \lambda_1(z), \lambda_2 := \lambda_2(z), \widetilde{\lambda_1} := \lambda_1(qz), \widetilde{\lambda_2} := \lambda_2(qz).$$

Substituting (4.1) into (4.3) we get

$$\begin{aligned} &\frac{\lambda_1'^2 e^{2\lambda_1} + \lambda_2'^2 e^{2\lambda_2} - 2\lambda_1'\lambda_2' e^{\lambda_1+\lambda_2}}{-4\delta} + \frac{e^{2\widetilde{\lambda_1}} + e^{2\widetilde{\lambda_2}} - 2e^{\widetilde{\lambda_1}+\widetilde{\lambda_2}}}{-4\delta} \\ &+ \frac{\beta^2+\gamma^2-2\alpha\beta\gamma}{1-\alpha^2} \cdot \frac{e^{2\lambda_1} + e^{2\lambda_2} - 2e^{\lambda_1+\lambda_2}}{-4\delta} \\ &+ 2\alpha \cdot \frac{\lambda_1' e^{\lambda_1+\widetilde{\lambda_1}} - \lambda_1' e^{\lambda_1+\widetilde{\lambda_2}} - \lambda_2' e^{\widetilde{\lambda_1}+\lambda_2} + \lambda_2' e^{\lambda_2+\widetilde{\lambda_2}}}{-4\delta} \\ &+ 2\beta \cdot \frac{e^{\lambda_1+\widetilde{\lambda_1}} - e^{\lambda_1+\widetilde{\lambda_2}} - e^{\widetilde{\lambda_1}+\lambda_2} + e^{\lambda_2+\widetilde{\lambda_2}}}{-4\delta} \end{aligned} \tag{4.4}$$

$$+2\gamma\cdot\frac{\lambda_1'e^{2\lambda_1}-\lambda_1'e^{\lambda_1+\lambda_2}-\lambda_2'e^{\lambda_1+\lambda_2}+\lambda_2'e^{2\lambda_2}}{-4\delta}$$
$$=\frac{e^{2\lambda_1}+e^{2\lambda_2}+2e^{\lambda_1+\lambda_2}}{4}.$$

The transcendental terms appearing in (4.4) have exponents

$$2\lambda_1,\ 2\lambda_2,\ \lambda_1+\lambda_2,\ 2\widetilde{\lambda_1},\ 2\widetilde{\lambda_2},\ \widetilde{\lambda_1}+\widetilde{\lambda_2},\ \lambda_1+\widetilde{\lambda_1},\ \lambda_1+\widetilde{\lambda_2},\ \widetilde{\lambda_1}+\lambda_2 \text{ and } \lambda_2+\widetilde{\lambda_2}.$$

In order to apply Lemma 2.2 to (4.4), we checked whether the pairwise difference between these exponents was constant. If $\lambda_1\equiv\lambda_2$, then $f(z)\equiv 0$. This is impossible, so $\lambda_1\not\equiv\lambda_2$.

Subcase 1.1. If $\lambda_1(z)-\lambda_2(z)$ is a nonzero constant, then $p(z)$ is a nonzero constant, denoted by p in the following for simplicity. Consequently,

$$f(z)=w=\frac{(e^{p}-e^{-p})e^{g(z)/2}}{2i\sqrt{\delta}}. \tag{4.5}$$

Then,

$$f'(z)=\frac{(e^{p}-e^{-p})e^{g(z)/2}}{2i\sqrt{\delta}}\cdot\frac{g'(z)}{2} \tag{4.6}$$

and

$$f(qz)=\frac{(e^{p}-e^{-p})e^{g(qz)/2}}{2i\sqrt{\delta}}. \tag{4.7}$$

Substituting (4.5)–(4.7) into (1.6), the terms in the left side of this equation can be expressed as

$$\begin{cases}[f'(z)]^2=\frac{d^2e^{g(z)}}{-4\delta}\cdot\frac{(g'(z))^2}{4},\\ [f(qz)]^2=\frac{d^2e^{g(qz)}}{-4\delta},\\ f^2(z)=\frac{d^2e^{g(z)}}{-4\delta},\\ 2\alpha f'(z)f(qz)=2\alpha\cdot\frac{d^2e^{\frac{g(z)+g(qz)}{2}}}{-4\delta}\cdot\frac{g'(z)}{2},\\ 2\beta f(z)f(qz)=2\beta\cdot\frac{d^2e^{\frac{g(z)+g(qz)}{2}}}{-4\delta},\\ 2\gamma f(z)f'(z)=2\gamma\cdot\frac{d^2e^{g(z)}}{-4\delta}\cdot\frac{g'(z)}{2},\end{cases} \tag{4.8}$$

where $d:=e^{p}-e^{-p}$.

If polynomial $g(z)$ contains at least two nonconstant terms, without loss of generality, we set

$$g(z)=a_nz^n+\cdots+a_mz^m+\cdots+a_0, n>m,$$

then,

$$g(qz)=a_n(qz)^n+\cdots+a_m(qz)^m+\cdots+a_0$$

and

$$g(qz)-g(z)=a_n(q^n-1)z^n+\cdots+a_m(q^m-1)z^m+\cdots.$$

If $g(qz) - g(z)$ is a constant and one has $q = 1$, then this contradicts the assumption and $g(qz) - g(z)$ is not a constant. By the same argument, $g(qz) - \frac{g(z)+g(qz)}{2}$ is also not a constant. Therefore, the three exponential terms, $e^{g(z)}, e^{g(qz)}$ and $e^{\frac{g(z)+g(qz)}{2}}$, are pairwise distinct, even if we don't consider their constant coefficients. Substituting (4.8) into (1.6) and applying Lemma 2.2 to the obtained equation, we get that the coefficients of the three exponential terms are zero. In particular, $\frac{d^2}{-4\delta} = 0$ since it's the coefficient of the sole term $e^{g(qz)}$. This is impossible, because that means that $f \equiv 0$.

So, $g(z)$ is the form of $g(z) = a_n z^n + b$, where $a_n(\neq 0), b$ are constants. Substitute this into (4.8) and we obtain that

$$\begin{cases} [f'(z)]^2 = \frac{d^2 e^{a_n z^n + b}}{-4\delta} \cdot \frac{(n a_n z^{n-1})^2}{4}, \\ [f(qz)]^2 = \frac{d^2 e^{a_n q^n z^n + b}}{-4\delta}, \\ f^2(z) = \frac{d^2 e^{a_n z^n + b}}{-4\delta}, \\ 2\alpha f'(z) f(qz) = 2\alpha \cdot \frac{n a_n z^{n-1}}{2} \cdot \frac{d^2 e^{\frac{a_n(1+q^n)z^n}{2} + b}}{-4\delta}, \\ 2\beta f(z) f(qz) = 2\beta \cdot \frac{d^2 e^{\frac{a_n(1+q^n)z^n}{2} + b}}{-4\delta}, \\ 2\gamma f(z) f'(z) = 2\gamma \cdot \frac{d^2 e^{a_n z^n + b}}{-4\delta} \cdot \frac{n a_n z^{n-1}}{2}, \end{cases} \tag{4.9}$$

then take the above expressions into (1.6). If $q^n \neq 1$, the expression of $[f(qz)]^2$ has a zero coefficient by Lemma 2.2, that is $\frac{d^2}{-4\delta} = 0$, which is impossible, so $q^n = 1$. Then, for all $z \in \mathbb{C}$ we have

$$\frac{(n a_n z^{n-1})^2}{4} + 2 + (\alpha + \beta) n a_n z^{n-1} + 2\beta \equiv -\frac{4\delta}{d^2}$$

by eliminating $e^{g(z)}$ from both sides of (1.6), so n has to be one, and $q = q^n = 1$, which contradicts the assumption.

Subcase 1.2. If $\lambda_1 - \lambda_2$ is not a constant, then $p(z)$ is not a constant; instead, it is a nonconstant polynomial. We multiply -4δ, combine like terms in (4.4), and move all the terms to the left side of the equation, then the right side is just zero. Thus, the coefficients of the distinct transcendental terms are listed in Table 3.

Table 3. Transcendental terms and corresponding coefficients.

Transcendental terms	Corresponding coefficients
$e^{2\lambda_1}$	$\lambda_1'^2 + 2\gamma\lambda_1' + 1$
$e^{2\lambda_2}$	$\lambda_2'^2 + 2\gamma\lambda_2' + 1$
$e^{\lambda_1+\lambda_2}$	$-2\lambda_1'\lambda_2' - 2\gamma(\lambda_1' + \lambda_2') + 4\delta - 2$
$e^{2\widetilde{\lambda_1}}$	1
$e^{2\widetilde{\lambda_2}}$	1
$e^{\widetilde{\lambda_1}+\widetilde{\lambda_2}}$	-2
$e^{\lambda_1+\widetilde{\lambda_1}}$	$2\alpha\lambda_1' + 2\beta$
$e^{\lambda_1+\widetilde{\lambda_2}}$	$-2\alpha\lambda_1' - 2\beta$
$e^{\widetilde{\lambda_1}+\lambda_2}$	$-2\alpha\lambda_2' - 2\beta$
$e^{\lambda_2+\widetilde{\lambda_2}}$	$2\alpha\lambda_2' + 2\beta$

By the same method in Case 2 of Section 3 (proof of Theorem 1.4), the degree of $\lambda_1(z)$ and $\lambda_2(z)$ both are at most one. Set $\lambda_1(z) = a_1z + b_1$ and $\lambda_2(z) = a_2z + b_2$, where $a_1 \neq a_2$, b_1, b_2 are arbitrary constants, then $\widetilde{\lambda_1(z)} = a_1qz + b_1$ and $\widetilde{\lambda_2(z)} = a_2qz + b_2$. Substituting these into Table 3, we get the results in Table 4.

Table 4. Transcendental terms and corresponding coefficients.

No.	Before	After	Corresponding coefficients
①	$e^{2\lambda_1}$	$e^{2a_1z+2b_1}$	$a_1^2 + 2\gamma a_1 + 1$
②	$e^{2\lambda_2}$	$e^{2a_2z+2b_2}$	$a_2^2 + 2\gamma a_2 + 1$
③	$e^{\lambda_1+\lambda_2}$	$e^{(a_1+a_2)z+b_1+b_2}$	$-2a_1a_2 - 2\gamma(a_1 + a_2) + 4\delta - 2$
④	$e^{2\widetilde{\lambda_1}}$	$e^{2a_1qz+2b_1}$	1
⑤	$e^{2\widetilde{\lambda_2}}$	$e^{2a_2qz+2b_2}$	1
⑥	$e^{\widetilde{\lambda_1}+\widetilde{\lambda_2}}$	$e^{(a_1+a_2)qz+b_1+b_2}$	-2
⑦	$e^{\lambda_1+\widetilde{\lambda_1}}$	$e^{a_1(1+q)z+2b_1}$	$2\alpha a_1 + 2\beta$
⑧	$e^{\lambda_1+\widetilde{\lambda_2}}$	$e^{(a_1+a_2q)z+b_1+b_2}$	$-2\alpha a_1 - 2\beta$
⑨	$e^{\widetilde{\lambda_1}+\lambda_2}$	$e^{(a_1q+a_2)z+b_1+b_2}$	$-2\alpha a_2 - 2\beta$
⑩	$e^{\lambda_2+\widetilde{\lambda_2}}$	$e^{a_2(1+q)z+2b_2}$	$2\alpha a_2 + 2\beta$

The coefficient of term ④ in Table 4 is one and it must combine with some like terms by Lemma 2.2. $2a_1q$ in term ④ may be equal to $2a_2$ in term ②, $a_1 + a_2$ in term ③, $a_1 + a_2q$ in term ⑧ and $a_2 + a_2q$ in term ⑩ since $q \neq 1, a_1 \neq a_2$. Then, we also considered that terms ⑤ and ⑥ must combine with some other like terms, respectively, since their coefficients are both nonzero, so there are many cases that have to be discussed. Through discussion for all possible cases, it is impossible to have a finite order entire solution (see the Appendix).

Case 2. If $e^{g(z)}$ can combine with the exponential terms in $f(qz)$, then

$$\begin{cases} u = f'(z) + \alpha f(qz) + \gamma f(z), \\ v = f(z) + \frac{\beta-\alpha\gamma}{1-\gamma^2} f(qz), \\ w = f(qz), \end{cases} \tag{4.10}$$

and (1.6) can convert into

$$u^2 + (1 - \gamma^2)v^2 + \delta' w^2 = e^{g(z)}, \tag{4.11}$$

where

$$\delta' := \frac{1 - \alpha^2 - \beta^2 - \gamma^2 + 2\alpha\beta\gamma}{1 - \gamma^2}.$$

By the same method in Section 3, we have

$$\begin{cases} f(qz) = \frac{e^{\lambda_1(z)} - e^{\lambda_2(z)}}{2i\sqrt{\delta'}}, \\ f(z) = \frac{e^{\lambda_1(z/q)} - e^{\lambda_2(z/q)}}{2i\sqrt{\delta'}}, \\ f'(z) = \frac{\lambda_1' e^{\lambda_1(z/q)} - \lambda_2' e^{\lambda_2(z/q)}}{2qi\sqrt{\delta'}}, \end{cases} \tag{4.12}$$

where

$$\lambda_1(z) = p(z) + g(z)/2, \ \ \lambda_2(z) = -p(z) + g(z)/2,$$

$p(z)$ is a polynomial. Here, $\lambda_1(z/q)$, $\lambda_2(z/q)$ are composite functions with respect to z, and according to the chain rule for derivatives, λ_1' and λ_2' represent the derivative of the outer function. By the same method in Case 1, we can divide into two subcases.

Subcase 2.1. If $\lambda_1(z) - \lambda_2(z)$ is a constant, then (4.12) can be rewritten as

$$\begin{cases} f(z) = \frac{d\mathrm{e}^{\frac{g(z/q)}{2}}}{2i\sqrt{\delta'}}, \\ f'(z) = \frac{d\mathrm{e}^{\frac{g(z/q)}{2}}}{2i\sqrt{\delta'}} \cdot \frac{g'}{2q}, \\ f(qz) = \frac{d\mathrm{e}^{\frac{g(z)}{2}}}{2i\sqrt{\delta'}}, \end{cases} \tag{4.13}$$

where $d = \mathrm{e}^p - \mathrm{e}^{-p}$. Here, $g(z/q)$ is a composite function with respect to z, and according to the chain rule for derivatives, g' here represents the derivative of the outer function. Substituting (4.13) into each term in the right side of (1.6), we have

$$\begin{cases} [f'(z)]^2 = \frac{d^2\mathrm{e}^{g(z/q)}}{-4\delta'} \cdot \frac{(g')^2}{4q^2}, \\ [f(qz)]^2 = \frac{d^2\mathrm{e}^{g(z)}}{-4\delta'}, \\ f^2(z) = \frac{d^2\mathrm{e}^{g(z/q)}}{-4\delta'}, \\ 2\alpha f'(z)f(qz) = 2\alpha \cdot \frac{d^2\mathrm{e}^{\frac{g(z)+g(z/q)}{2}}}{-4\delta'} \cdot \frac{g'}{2q}, \\ 2\beta f(z)f(qz) = 2\beta \cdot \frac{d^2\mathrm{e}^{\frac{g(z)+g(z/q)}{2}}}{-4\delta'}, \\ 2\gamma f(z)f'(z) = 2\gamma \cdot \frac{d^2\mathrm{e}^{g(z/q)}}{-4\delta'} \cdot \frac{g'}{2q}. \end{cases} \tag{4.14}$$

If polynomial $g(z)$ contains at least two nonconstant terms, without loss of generality we set

$$g(z) = a_n z^n + \cdots + a_m z^m + \cdots + a_0, \quad n > m,$$

then

$$g(z/q) = a_n(z/q)^n + \cdots + a_m(z/q)^m + \cdots + a_0$$

and

$$g(z/q) - g(z) = a_n\left(\frac{1}{q^n} - 1\right)z^n + \cdots + a_m\left(\frac{1}{q^m} - 1\right)z^m + \cdots .$$

If $g(z/q) - g(z)$ is a constant and one has $q = 1$, this contradicts the assumption and $g(z/q) - g(z)$ is not a constant. By the same argument, $g(z/q) - \frac{g(z)+g(z/q)}{2}$ is also not a constant. Therefore, the three exponential terms, $\mathrm{e}^{g(z)}$, $\mathrm{e}^{g(qz)}$ and $\mathrm{e}^{\frac{g(z)+g(qz)}{2}}$, are distinct and can't combine like terms. Substituting (4.14) into (1.6) and applying Lemma 2.2 to the obtained equation, we get that the coefficients of the three exponential terms are zeroes after combining like terms:

$$\begin{cases} \frac{d^2}{-4\delta'} - 1 \equiv 0, \\ \frac{d^2}{-4\delta'} \cdot \frac{(g')^2}{4q^2} + \frac{d^2}{-4\delta'} + 2\gamma \cdot \frac{d^2}{-4\delta'} \cdot \frac{g'}{2q} \equiv 0, \\ 2\alpha \cdot \frac{d^2}{-4\delta'} \cdot \frac{g'}{2q} + 2\beta \cdot \frac{d^2}{-4\delta'} \equiv 0. \end{cases} \tag{4.15}$$

Since the three equations hold for all $z \in \mathbb{C}$, $g(z/q)$ has a degree of one, and so does $g(z)$.

Therefore, we can set $g(z/q) = az + b$, then $g(z) = aqz + b$, where $a \neq 0$ and b is an arbitrary constant. Noting that g' represents the derivative of the outer function, so $g' = aq$, then the above equations convert into

$$\begin{cases} \frac{d^2}{-4\delta'} = 1, \\ \frac{a^2}{4} + 1 + 2\gamma a \equiv 0, \\ \alpha a + 2\beta \equiv 0. \end{cases} \tag{4.16}$$

Thus, it yields $\alpha^2 + \beta^2 - 2\alpha\beta\gamma = 0$ and $\gamma^2 \neq 1$.

In other words, (1.6) admits a solution in this case with the form $f(z) = e^{\frac{az+b}{2}}$ and $g(z) = aqz + b$ such that

$$\begin{cases} \frac{a^2}{4} + 1 + 2\gamma a \equiv 0, \\ \alpha a + 2\beta \equiv 0, \end{cases} \tag{4.17}$$

then $\alpha^2 + \beta^2 - 2\alpha\beta\gamma = 0$ and $\gamma^2 \neq 1$.

Subcase 2.2. If $\lambda_1(z) - \lambda_2(z)$ is nonconstant, then from (4.11) we have

$$u^2 + (1-\gamma^2)v^2 = \left(\frac{e^{\lambda_1(z)} + e^{\lambda_2(z)}}{2}\right)^2. \tag{4.18}$$

Substituting (4.10) into (4.18) we deduce that

$$\begin{aligned} &[f'(z)]^2 + [f(z)]^2 + \frac{\alpha^2 + \beta^2 - 2\alpha\beta\gamma}{1-\gamma^2}[f(qz)]^2 \\ &+2\alpha f'(z)f(qz) + 2\beta f(qz)f(z) + 2\gamma f'(z)f(z) \\ &= \frac{e^{2\lambda_1(z)} + e^{2\lambda_2(z)} + 2e^{\lambda_1(z)+\lambda_2(z)}}{4}. \end{aligned} \tag{4.19}$$

For simplicity and convenience, we denote

$$\lambda_1 := \lambda_1(z), \lambda_2 := \lambda_2(z), \widehat{\lambda_1} := \lambda_1(z/q), \widehat{\lambda_2} := \lambda_2(z/q).$$

Substituting (4.12) into (4.19) we obtain

$$\begin{aligned} &\frac{1}{q^2} \cdot \frac{\lambda_1'^2 e^{2\widehat{\lambda_1}} + \lambda_2'^2 e^{2\widehat{\lambda_2}} - 2\lambda_1{}'\lambda_2{}' e^{\widehat{\lambda_1}+\widehat{\lambda_2}}}{-4\delta'} + \frac{e^{2\widehat{\lambda_1}} + e^{2\widehat{\lambda_2}} - 2e^{\widehat{\lambda_1}+\widehat{\lambda_2}}}{-4\delta'} \\ &+\frac{\alpha^2 + \beta^2 - 2\alpha\beta\gamma}{1-\gamma^2} \cdot \frac{e^{2\lambda_1} + e^{2\lambda_2} - 2e^{\lambda_1+\lambda_2}}{-4\delta'} \\ &+2\alpha \cdot \frac{1}{q} \cdot \frac{\lambda_1' e^{\lambda_1+\widehat{\lambda_1}} - \lambda_1' e^{\widehat{\lambda_1}+\lambda_2} - \lambda_2' e^{\lambda_1+\widehat{\lambda_2}} + \lambda_2' e^{\lambda_2+\widehat{\lambda_2}}}{-4\delta'} \\ &+2\beta \cdot \frac{e^{\lambda_1+\widehat{\lambda_1}} - e^{\lambda_1+\widehat{\lambda_2}} - e^{\widehat{\lambda_1}+\lambda_2} + e^{\lambda_2+\widehat{\lambda_2}}}{-4\delta'} \end{aligned}$$

$$+2\gamma\cdot\frac{1}{q}\cdot\frac{\lambda_1' e^{2\widehat{\lambda_1}}-\lambda_1' e^{\widehat{\lambda_1}+\widehat{\lambda_2}}-\lambda_2' e^{\widehat{\lambda_1}+\widehat{\lambda_2}}+\lambda_2' e^{2\widehat{\lambda_2}}}{-4\delta'}$$
$$=\frac{e^{2\lambda_1}+e^{2\lambda_2}+2e^{\lambda_1+\lambda_2}}{4}. \tag{4.20}$$

Let's multiply both sides of (4.20) by $-4\delta'$ and move all terms to the left side of the equation. After combining the terms of the same kind, we get the results in Table 5.

Table 5. Transcendental terms and corresponding coefficients.

Transcendental terms	Corresponding coefficients
$e^{2\lambda_1}$	1
$e^{2\lambda_2}$	1
$e^{\lambda_1+\lambda_2}$	$4\delta'-2$
$e^{2\widehat{\lambda_1}}$	$\frac{1}{q^2}\cdot\lambda_1'^2+\frac{2\gamma}{q}\cdot\lambda_1'+1$
$e^{2\widehat{\lambda_2}}$	$\frac{1}{q^2}\cdot\lambda_2'^2+\frac{2\gamma}{q}\cdot\lambda_2'+1$
$e^{\widehat{\lambda_1}+\widehat{\lambda_2}}$	$-\frac{2}{q^2}\lambda_1'\lambda_2'-\frac{2\gamma}{q}(\lambda_1'+\lambda_2')-2$
$e^{\lambda_1+\widehat{\lambda_1}}$	$\frac{2\alpha}{q}\lambda_1'+2\beta$
$e^{\lambda_1+\widehat{\lambda_2}}$	$-\frac{2\alpha}{q}\lambda_2'-2\beta$
$e^{\widehat{\lambda_1}+\lambda_2}$	$-\frac{2\alpha}{q}\lambda_1'-2\beta$
$e^{\lambda_2+\widehat{\lambda_2}}$	$\frac{2\alpha}{q}\lambda_2'+2\beta$

Using the method in Subcase 1.2, we can also obtain that there are no suitable finite order transcendental entire solutions for (1.6) in this case. The details are omitted here.

5. Conclusions

In this paper we proved two theorems (Theorems 1.4 and 1.8), studied the finite order entire solutions of (1.4) and (1.6), respectively and found the concrete forms of solutions of these two equations, both of which were exponential functions. Examples 1.5 and 1.6 verified the two cases of solutions of the equation in Theorem 1.4, and Example 1.9 verified the truth of Theorem 1.8. The equations studied in this paper can be transformed into the Fermat-type equation with three quadratic terms by linear transformation, which improves the previous Fermat-type equations with only two quadratic terms, so it is very novel.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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Appendix

We divided Table 4 into the following four cases with 10 subcases, but in each case there is no finite order transcendental entire solution. The classification is based on the fact that terms ④,⑤ and ⑥ in Table 4 must be combined with other terms, since their coefficients are nonzero. Otherwise, they contradict with Lemma 2.2.

Case A1. Term ④ can combine with term ② and ⑨ in Table 4, that is, $a_1 q = a_2$. Since term ⑤ should also combine with other terms, we split into some subcases.

Subcase A1.1. Term ⑤ may combine with term ① in Table 4. Thus, we get the results in Table A1.

Table A1. $q = \frac{a_2}{a_1} = -1, a_2 = -a_1$.

No.	Transcendental terms	Corresponding coefficients
①	$e^{2a_1 z}$	$e^{2b_1}\left(a_1^2 + 2\gamma a_1 + 1\right)$
②	$e^{-2a_1 z}$	$e^{2b_2}\left(a_1^2 - 2\gamma a_1 + 1\right)$
③	e^0	$e^{b_1+b_2}\left(2a_1^2 + 4\delta - 2\right)$
④	$e^{-2a_1 z}$	e^{2b_1}
⑤	$e^{2a_1 z}$	e^{2b_2}
⑥	e^0	$-2e^{b_1+b_2}$
⑦	e^0	$e^{2b_1}(2\alpha a_1 + 2\beta)$
⑧	$e^{2a_1 z}$	$e^{b_1+b_2}(-2\alpha a_1 - 2\beta)$
⑨	$e^{-2a_1 z}$	$e^{b_1+b_2}(2\alpha a_1 - 2\beta)$
⑩	e^0	$e^{2b_2}(-2\alpha a_1 + 2\beta)$

After combining terms of the same kind in Table A1, according to Lemma 2.2, we know that the coefficients must always be zero, and the following equations were obtained

$$\begin{cases} e^{2b_1}\left(a_1^2 + 2\gamma a_1 + 1\right) + e^{2b_2} + e^{b_1+b_2}(-2\alpha a_1 - 2\beta) = 0, \\ e^{2b_2}\left(a_1^2 - 2\gamma a_1 + 1\right) + e^{2b_1} + e^{b_1+b_2}(2\alpha a_1 - 2\beta) = 0, \\ e^{b_1+b_2}\left(a_1^2 + 2\delta - 2\right) + e^{2b_1}(\alpha a_1 + \beta) + e^{2b_2}(-\alpha a_1 + \beta) = 0. \end{cases}$$

Thus, we have

$$\begin{cases} e^{b_1} = \pm e^{b_2}, \\ \alpha = \pm\gamma, \\ \beta = \pm 1, \end{cases}$$

it yields $\delta = 0$, which is impossible.

Subcase A1.2. Term ⑤ may combine with term ③ in Table 4. Then, we obtained the results in Table A2 as follows.

Table A2. $q = \frac{a_2}{a_1} = -1/2, a_2 = -a_1/2.$

No.	Transcendental terms	Corresponding coefficients
①	$e^{2a_1 z}$	$e^{2b_1}\left(a_1^2 + 2\gamma a_1 + 1\right)$
②	$e^{-a_1 z}$	$e^{2b_2}\left(\frac{a_1^2}{4} - \gamma a_1 + 1\right)$
③	$e^{\frac{a_1}{2}\cdot z}$	$e^{b_1+b_2}\left(a_1^2 - \gamma a_1 + 4\delta - 2\right)$
④	$e^{-a_1 z}$	e^{2b_1}
⑤	$e^{\frac{a_1}{2}\cdot z}$	e^{2b_2}
⑥	$e^{-\frac{a_1}{4}\cdot z}$	$-2e^{b_1+b_2}$
⑦	$e^{\frac{a_1}{2}\cdot z}$	$e^{2b_1}(2\alpha a_1 + 2\beta) = 0$
⑧	$e^{\frac{5a_1}{4}\cdot z}$	$e^{b_1+b_2}(-2\alpha a_1 - 2\beta) = 0$
⑨	$e^{-a_1 z}$	$e^{b_1+b_2}(\alpha a_1 - 2\beta) = e^{b_1+b_2}(-3\beta)$
⑩	$e^{-\frac{a_1}{4}\cdot z}$	$e^{2b_2}(-\alpha a_1 + 2\beta)$

Combining terms of the same kind and according to Lemma 2.2, we know that the coefficients must always be zero, and the following equations are obtained

$$\begin{cases} e^{2b_1}\left(a_1^2 + 2\gamma a_1 + 1\right) = 0, \\ e^{2b_2}\left(\frac{a_1^2}{4} - \gamma a_1 + 1\right) + e^{2b_1} + e^{b_1+b_2}(\alpha a_1 - 2\beta) = 0, \\ e^{b_1+b_2}\left(a_1^2 - \gamma a_1 + 4\delta - 2\right) + e^{2b_2} + e^{2b_1}(2\alpha a_1 + 2\beta) = 0, \\ -2e^{b_1+b_2} + e^{2b_2}(-\alpha a_1 + 2\beta) = 0, \\ e^{b_1+b_2}(-2\alpha a_1 - 2\beta) = 0. \end{cases}$$

For the above equation system, there is no suitable solution a_1.

Case A2. Term ④ can combine with term ③ in Table 4, that is, $2a_1 q = a_1 + a_2$. Since term ⑤ should also combine with other terms, we split it into three subcases.

Subcase A2.1. Term ⑤ may combine with term ① in Table 4. We get the results in Table A3.

From Table A3, after combining terms of the same kind, according to Lemma 2.2, we know that the coefficients must always be zero, and the following equations are obtained

$$\begin{cases} e^{2b_1}\left(a_1^2 + 2\gamma a_1 + 1\right) + e^{2b_2} + e^{b_1+b_2}(-2\alpha a_1 - 2\beta) = 0, \\ e^{2b_2}\left(4a_1^2 - 4\gamma a_1 + 1\right) = 0, \\ e^{b_1+b_2}\left(4a_1^2 + 2\gamma a_1 + 4\delta - 2\right) + e^{2b_1} + e^{2b_2}(-4\alpha a_1 + 2\beta) = 0, \\ -2e^{b_1+b_2} + e^{2b_1}(2\alpha a_1 + 2\beta) = 0, \\ e^{b_1+b_2}(4\alpha a_1 - 2\beta) = 0. \end{cases}$$

For the above equation system, there is no suitable solution a_1.

Table A3. $q = \frac{a_1+a_2}{2a_1} = -1/2, a_2 = -2a_1$.

No.	Transcendental terms	Corresponding coefficients
①	$e^{2a_1 z}$	$e^{2b_1}\left(a_1^2 + 2\gamma a_1 + 1\right)$
②	$e^{-4a_1 z}$	$e^{2b_2}\left(4a_1^2 - 4\gamma a_1 + 1\right)$
③	$e^{-a_1 z}$	$e^{b_1+b_2}\left(4a_1^2 + 2\gamma a_1 + 4\delta - 2\right)$
④	$e^{-a_1 z}$	e^{2b_1}
⑤	$e^{2a_1 z}$	e^{2b_2}
⑥	$e^{\frac{a_1}{2}\cdot z}$	$-2e^{b_1+b_2}$
⑦	$e^{\frac{a_1}{2}\cdot z}$	$e^{2b_1}\left(2\alpha a_1 + 2\beta\right)$
⑧	$e^{2a_1 z}$	$e^{b_1+b_2}\left(-2\alpha a_1 - 2\beta\right)$
⑨	$e^{-\frac{5a_1}{2}z}$	$e^{b_1+b_2}\left(4\alpha a_1 - 2\beta\right)$
⑩	$e^{-a_1 z}$	$e^{2b_2}\left(-4\alpha a_1 + 2\beta\right)$

Subcase A2.2. Term ⑤ may combine with term ⑦ in Table 4. Then, we get the results in Table A4.

Table A4. $q = \frac{a_1+a_2}{2a_1} = -\frac{1}{4}, a_2 = -\frac{3}{2}a_1$.

No.	Transcendental terms	Corresponding coefficients
①	$e^{2a_1 z}$	$e^{2b_1}\left(a_1^2 + 2\gamma a_1 + 1\right)$
②	$e^{-3a_1 z}$	$e^{2b_2}\left(\frac{9}{4}a_1^2 - 3\gamma a_1 + 1\right)$
③	$e^{-\frac{1}{2}a_1 z}$	$e^{b_1+b_2}\left(3a_1^2 + \gamma a_1 + 4\delta - 2\right)$
④	$e^{-\frac{1}{2}a_1 z}$	e^{2b_1}
⑤	$e^{\frac{3}{4}a_1 z}$	e^{2b_2}
⑥	$e^{\frac{1}{8}a_1 z}$	$-2e^{b_1+b_2}$
⑦	$e^{\frac{3}{4}a_1\cdot z}$	$e^{2b_1}\left(2\alpha a_1 + 2\beta\right)$
⑧	$e^{\frac{11}{8}a_1 z}$	$e^{b_1+b_2}\left(-2\alpha a_1 - 2\beta\right)$
⑨	$e^{-\frac{7a_1}{4}z}$	$e^{b_1+b_2}\left(3\alpha a_1 - 2\beta\right)$
⑩	$e^{-\frac{9}{8}a_1 z}$	$e^{2b_2}\left(-3\alpha a_1 + 2\beta\right)$

In Table A4, the term ⑥ cannot combine with other transcendental terms; it's impossible.

Subcase A2.3. Term ⑤ may combine with term ⑨ in Table 4. Then, we deduce the results in Table A5.

Table A5. $q = \frac{a_1+a_2}{2a_1} = \frac{1}{4}, a_2 = -\frac{1}{2}a_1.$

No.	Transcendental terms	Corresponding coefficients
①	e^{2a_1z}	$e^{2b_1}\left(a_1^2 + 2\gamma a_1 + 1\right)$
②	e^{-a_1z}	$e^{2b_2}\left(a_2^2 + 2\gamma a_2 + 1\right)$
③	$e^{\frac{1}{2}a_1z}$	$e^{b_1+b_2}\left(-2a_1a_2 - 2\gamma(a_1 + a_2) + 4\delta - 2\right)$
④	$e^{\frac{1}{2}a_1z}$	e^{2b_1}
⑤	$e^{-\frac{1}{4}a_1z}$	e^{2b_2}
⑥	$e^{\frac{1}{8}a_1z}$	$-2e^{b_1+b_2}$
⑦	$e^{\frac{5}{4}a_1z}$	$e^{2b_1}\left(2\alpha a_1 + 2\beta\right)$
⑧	$e^{\frac{7}{8}a_1z}$	$e^{b_1+b_2}\left(-2\alpha a_1 - 2\beta\right)$
⑨	$e^{-\frac{1}{4}a_1z}$	$e^{b_1+b_2}\left(-2\alpha a_2 - 2\beta\right)$
⑩	$e^{-\frac{5}{8}a_1z}$	$e^{2b_2}\left(2\alpha a_2 + 2\beta\right)$

In Table A5, the term ⑥ cannot combine with other transcendental terms; it's impossible.

Case A3. Term ④ can combine with term ⑧ in Table 4, that is, $2a_1q = a_1 + a_2q$. Since term ⑤ should also combine with other terms, we split it into two subcases.

Subcase A3.1. Term ⑤ may combine with term ③ in Table 4. We get the results in Table A6.

Table A6. $q = \frac{a_1}{2a_1-a_2} = \frac{1}{4}, a_2 = -2a_1.$

No.	Transcendental terms	Corresponding coefficients
①	e^{2a_1z}	$e^{2b_1}\left(a_1^2 + 2\gamma a_1 + 1\right)$
②	e^{-4a_1z}	$e^{2b_2}\left(a_2^2 + 2\gamma a_2 + 1\right)$
③	e^{-a_1z}	$e^{b_1+b_2}\left(-2a_1a_2 - 2\gamma(a_1 + a_2) + 4\delta - 2\right)$
④	$e^{\frac{1}{2}a_1z}$	e^{2b_1}
⑤	e^{-a_1z}	e^{2b_2}
⑥	$e^{-\frac{1}{4}a_1z}$	$-2e^{b_1+b_2}$
⑦	$e^{\frac{5}{4}a_1z}$	$e^{2b_1}\left(2\alpha a_1 + 2\beta\right)$
⑧	$e^{\frac{1}{2}a_1z}$	$e^{b_1+b_2}\left(-2\alpha a_1 - 2\beta\right)$
⑨	$e^{-\frac{7}{4}a_1z}$	$e^{b_1+b_2}\left(-2\alpha a_2 - 2\beta\right)$
⑩	$e^{-\frac{5}{2}a_1z}$	$e^{2b_2}\left(2\alpha a_2 + 2\beta\right)$

The term ⑥ cannot combine with other transcendental terms; this is impossible.

Subcase A3.2. Term ⑤ may combine with term ⑨ in Table 4. We get the results in Table A7.

Table A7. $q = \frac{a_1}{2a_1 - a_2} = \frac{1}{3}, a_2 = -a_1.$

No.	Transcendental terms	Corresponding coefficients
①	$e^{2a_1 z}$	$e^{2b_1}\left(a_1^2 + 2\gamma a_1 + 1\right)$
②	$e^{-2a_1 z}$	$e^{2b_2}\left(a_2^2 + 2\gamma a_2 + 1\right)$
③	e^0	$e^{b_1+b_2}\left(-2a_1a_2 - 2\gamma(a_1+a_2) + 4\delta - 2\right)$
④	$e^{\frac{2}{3}a_1 z}$	e^{2b_1}
⑤	$e^{-\frac{2}{3}a_1 z}$	e^{2b_2}
⑥	e^0	$-2e^{b_1+b_2}$
⑦	$e^{\frac{4}{3}a_1 z}$	$e^{2b_1}\left(2\alpha a_1 + 2\beta\right)$
⑧	$e^{\frac{2}{3}a_1 z}$	$e^{b_1+b_2}\left(-2\alpha a_1 - 2\beta\right)$
⑨	$e^{-\frac{2}{3}a_1 z}$	$e^{b_1+b_2}\left(-2\alpha a_2 - 2\beta\right)$
⑩	$e^{-\frac{4}{3}a_1 z}$	$e^{2b_2}\left(2\alpha a_2 + 2\beta\right)$

By ⑦ and ⑩ in Table A7, we have $a_1 = a_2$, which is a contradiction.

Case A4. Term ④ can combine with term ⑩ in Table 4, that is, $2a_1q = a_2 + a_2q$. Since term ⑤ should also combine with other terms, we split it into three subcases.

Subcase A4.1. Term ⑤ may combine with term ① in Table 4. We get the results in Table A8.

Table A8. $q = \frac{a_2}{2a_1 - a_2} = -\frac{1}{2}, a_2 = -2a_1.$

No.	Transcendental terms	Corresponding coefficients
①	$e^{2a_1 z}$	$e^{2b_1}\left(a_1^2 + 2\gamma a_1 + 1\right)$
②	$e^{-4a_1 z}$	$e^{2b_2}\left(4a_1^2 - 4\gamma a_1 + 1\right)$
③	$e^{-a_1 z}$	$e^{b_1+b_2}\left(4a_1^2 + 2\gamma a_1 + 4\delta - 2\right)$
④	$e^{-a_1 z}$	e^{2b_1}
⑤	$e^{2a_1 z}$	e^{2b_2}
⑥	$e^{\frac{1}{2}a_1}$	$-2e^{b_1+b_2}$
⑦	$e^{\frac{1}{2}a_1 z}$	$e^{2b_1}\left(2\alpha a_1 + 2\beta\right)$
⑧	$e^{2a_1 z}$	$e^{b_1+b_2}\left(-2\alpha a_1 - 2\beta\right)$
⑨	$e^{-\frac{5}{2}a_1 z}$	$e^{b_1+b_2}\left(4\alpha a_1 - 2\beta\right)$
⑩	$e^{-a_1 z}$	$e^{2b_2}\left(-4\alpha a_1 + 2\beta\right)$

Subcase A4.2. Term ⑤ may combine with term ③ in Table 4. Then, we have the results in Table A9.

Table A9. $q = \frac{a_2}{2a_1 - a_2} = -\frac{1}{4}, a_2 = -\frac{2}{3}a_1.$

No.	Transcendental terms	Corresponding coefficients
①	$e^{2a_1 z}$	$e^{2b_1}\left(a_1^2 + 2\gamma a_1 + 1\right)$
②	$e^{-\frac{4}{3}a_1 z}$	$e^{2b_2}\left(a_2^2 + 2\gamma a_2 + 1\right)$
③	$e^{\frac{1}{3}a_1 z}$	$e^{b_1+b_2}\left(-2a_1a_2 - 2\gamma(a_1 + a_2) + 4\delta - 2\right)$
④	$e^{-\frac{1}{2}a_1 z}$	e^{2b_1}
⑤	$e^{\frac{1}{3}a_1 z}$	e^{2b_2}
⑥	$e^{-\frac{1}{12}a_1}$	$-2e^{b_1+b_2}$
⑦	$e^{\frac{3}{4}a_1 z}$	$e^{2b_1}\left(2\alpha a_1 + 2\beta\right)$
⑧	$e^{\frac{7}{6}a_1 z}$	$e^{b_1+b_2}\left(-2\alpha a_1 - 2\beta\right)$
⑨	$e^{-\frac{11}{12}a_1 z}$	$e^{b_1+b_2}\left(-2\alpha a_2 - 2\beta\right)$
⑩	$e^{-\frac{1}{2}a_1 z}$	$e^{2b_2}\left(2\alpha a_2 + 2\beta\right)$

The term ⑥ cannot combine with other transcendental terms; it's impossible.

Subcase A4.3. Term ⑤ may combine with term ⑦ in Table 4. Then, we get the results in Table A10.

Table A10. $q = \frac{a_2}{2a_1 - a_2} = -\frac{1}{3}, a_2 = -a_1.$

No.	Transcendental terms	Corresponding coefficients
①	$e^{2a_1 z}$	$e^{2b_1}\left(a_1^2 + 2\gamma a_1 + 1\right)$
②	$e^{-2a_1 z}$	$e^{2b_2}\left(a_2^2 + 2\gamma a_2 + 1\right)$
③	e^{0}	$e^{b_1+b_2}\left(-2a_1a_2 - 2\gamma(a_1 + a_2) + 4\delta - 2\right)$
④	$e^{-\frac{2}{3}a_1 z}$	e^{2b_1}
⑤	$e^{\frac{2}{3}a_1 z}$	e^{2b_2}
⑥	e^{0}	$-2e^{b_1+b_2}$
⑦	$e^{\frac{2}{3}a_1 z}$	$e^{2b_1}\left(2\alpha a_1 + 2\beta\right)$
⑧	$e^{\frac{4}{3}a_1 z}$	$e^{b_1+b_2}\left(-2\alpha a_1 - 2\beta\right)$
⑨	$e^{-\frac{4}{3}a_1 z}$	$e^{b_1+b_2}\left(-2\alpha a_2 - 2\beta\right)$
⑩	$e^{-\frac{2}{3}a_1 z}$	$e^{2b_2}\left(2\alpha a_2 + 2\beta\right)$

From ⑧ and ⑨ in Table A10, we have $a_1 = a_2$; it's impossible.

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