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Research article

Solution of fractional differential equation by fixed point results in orthogonal $\mathcal{F}$-metric spaces

Mohammed H. Alharbi and Jamshaid Ahmad*

Department of Mathematics, College of Science, University of Jeddah, Jeddah 21589, Saudi Arabia

* **Correspondence:** Email: jkhan@uj.edu.sa.

Abstract: In this paper, we solve the existence and uniqueness of a solution for a fractional differential equation by introducing some new fixed point results for rational (α, β, ψ)-contractions in the framework of orthogonal $\mathcal{F}$-metric spaces. We derive some well-known results in literature as consequences of our leading result.

Keywords: fixed point; orthogonal $\mathcal{F}$-metric space; orthogonal cyclic (α, β)-admissible; rational (α, β, ψ)-contraction; fractional differential equations
Mathematics Subject Classification: 46S40, 47H10, 54H25

1. Introduction

The analysis of metric spaces has played a crucial role in different fields of pure and applied sciences. We can find countless effective and stunning applications of this notion in different fields of science (see [1, 2]). Many mathematicians extrapolated, generalized and expanded the idea of metric spaces to different spaces. Branciari [3] gave the idea of a generalized metric by putting more general inequality in the place of natural triangle inequality of a metric space. This inequality is called a rectangular inequality which consists of four terms instead of three terms. Such a metric space is famous as the Branciari metric space in literature. Bakhtin [4] initiated the concept of b-metric space in 1989, which was properly described by Czerwik [5] in 1993. The b-metric is not continuous unlike the classical metric in the topology created by it . Gordji et al. [6] gave the idea of orthogonality in the notion of metric spaces and established some fixed point results for contraction mappings in orthogonal metric spaces. In 2018, Jleli et al. [7] initiated the concept of $\mathcal{F}$-metric space as a generalizations of above all metric spaces. Recently, Kanwal et al. [8] united the concepts of $\mathcal{F}$-MS and orthogonal set to initiate the conception of orthogonal $\mathcal{F}$-metric space. They presented some common fixed point theorems in the framework of orthogonal $\mathcal{F}$-MS.

On the other hand, Stefan Banach [9] was the pioneer researcher in this theory, having given the conception of contraction in the framework of metric spaces and established a fixed point theorem. It has been tremendously appropriate in many areas such as optimization problems, differential equations, economics and in various other fields. A lot of research work in this field have been given to the improvement and generalization of Banach contraction principle in different ways. In 2011, Samet et al. [10] gave the ideas of (α, ψ)-contraction and α-admissibility to generalize the Banach fixed point theorem. Later on, Ramezani et al. [11] combined the concepts of orthogonality and α-admissibility to introduce orthogonal α-admissible mapping. Subsequetly, Alizadeh [12] introduced the notion of cyclic (α, β)-admissibility to prove some fixed point results.

In the present research work, we give rational (α, β, ψ)-contractions in the framework of orthogonal $\mathcal{F}$-MS and establish some fixed point results. We solve the existence and uniqueness of a fractional differential equation by our leading theorem.

2. Preliminaries

We start this section with the definition of metric space in this way.

Let $\mathcal{P} \neq \emptyset$. A metric is a function $\mathfrak{u} : \mathcal{P} \times \mathcal{P} \to [0, +\infty)$ satisfying:

(i) $\mathfrak{u}(\mathfrak{y}, \varsigma) = 0$ if and only if $\mathfrak{y} = \varsigma$,

(ii) $\mathfrak{u}(\mathfrak{y}, \varsigma) = \mathfrak{u}(\varsigma, \mathfrak{y})$,

(iii) $\mathfrak{u}(\mathfrak{y}, \varsigma) \leq \mathfrak{u}(\mathfrak{y}, \omega) + \mathfrak{u}(\omega, \varsigma)$,

for all $\mathfrak{y}, \omega, \varsigma \in \mathcal{P}$. If $\mathfrak{u}$ be a metric, then $(\mathcal{P}, \mathfrak{u})$ is said to be a metric space.

In 1980, Fisher [13] established a fixed point theorem for mapping satisfying

$$\mathfrak{u}(\mathfrak{R}\mathfrak{y}, \mathfrak{R}\varsigma) \leq \varrho_1 \mathfrak{u}(\mathfrak{y}, \varsigma) + \varrho_1 \frac{\mathfrak{u}(\mathfrak{y}, \mathfrak{R}\mathfrak{y})\mathfrak{u}(\varsigma, \mathfrak{R}\varsigma)}{1 + \mathfrak{u}(\mathfrak{y}, \varsigma)},$$

for all $\mathfrak{y}, \varsigma \in \mathcal{P}$, where $\varrho_1, \varrho_2 \in [0, \frac{1}{2})$ with $\varrho_1 + \varrho_2 < 1$.

Czerwik [5] presented the conception of b-metric by altering the triangular inequality of metric space in this manner: for all $\mathfrak{y}, \omega, \varsigma \in \mathcal{P}$ and for some $b \geq 1$,

$$\mathfrak{u}(\mathfrak{y}, \varsigma) \leq b \left[\mathfrak{u}(\mathfrak{y}, \omega) + \mathfrak{u}(\omega, \varsigma)\right].$$

Gordji et al. [6] gave the idea of the orthogonal set (O-set) and reinforced the concept of metric space by introducing orthogonal metric space in 2017.

Definition 1. Let $\mathcal{P} \neq \emptyset$. $\mathcal{P}$ is said to be an O-set if there exists some binary relation $\perp \subseteq \mathcal{P} \times \mathcal{P}$ fulfilling the following axiom

$$\text{there exists } \mathfrak{y}_0 \text{ such that } \varsigma \perp \mathfrak{y}_0 \text{ or } \mathfrak{y}_0 \perp \varsigma \text{ for all } \varsigma \in \mathcal{P}.$$

Furthermore, $\mathfrak{y}_0$ is said to be an orthogonal point. An O-set is represented as $(\mathcal{P},\perp)$.

Definition 2. ([6]) A sequence $\{\mathfrak{y}_n\}$ in O-set $(\mathcal{P},\perp)$ is called an orthogonal sequence if $\mathfrak{y}_n \perp \mathfrak{y}_{n+1}$ or $\mathfrak{y}_{n+1} \perp \mathfrak{y}_n$, for all $n \in \mathbb{N}$. We represent an orthogonal sequence by O-sequence.

Definition 3. ([6]) Let $(\mathcal{P},\perp)$ be an O-set. A mapping $\mathfrak{R} : \mathcal{P} \to \mathcal{P}$ is said to be $\perp$-preserving if $\mathfrak{y} \perp \varsigma$ implies $\mathfrak{R}\mathfrak{y} \perp \mathfrak{R}\varsigma$.

On the other hand, Jleli et al. [7] initiated the concept of an $\mathcal{F}$-metric space ($\mathcal{F}$-MS) as follows.

Let $\mathcal{F}$ be the class of functions $\xi : (0, +\infty) \to \mathbb{R}$ satisfying

($\mathcal{F}_1$) $0 < \mathfrak{y}_1 < \mathfrak{y}_2 \Rightarrow \xi(\mathfrak{y}_1) \leq \xi(\mathfrak{y}_2)$,

($\mathcal{F}_2$) for all $\{\mathfrak{y}_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to\infty} \mathfrak{y}_n = 0 \iff \lim_{n\to\infty} \xi(\mathfrak{y}_n) = -\infty$.

Definition 4. ([7]) Let $\mathcal{P} \neq \emptyset$ and $\mathfrak{u} : \mathcal{P}\times\mathcal{P} \to [0, +\infty)$. Suppose that there exists $(\xi, \alpha) \in \mathcal{F} \times [0, +\infty)$ such that

(D_1) $(\mathfrak{y}, \varsigma) \in \mathcal{P} \times \mathcal{P}$, $\mathfrak{u}(\mathfrak{y}, \varsigma) = 0$ if and only if $\mathfrak{y} = \varsigma$;

(D_2) $\mathfrak{u}(\mathfrak{y}, \varsigma) = \mathfrak{u}(\varsigma, \mathfrak{y})$, for all $\mathfrak{y}, \varsigma \in \mathcal{P}$;

(D_3) for all $(\mathfrak{y}, \varsigma) \in \mathcal{P} \times \mathcal{P}$, and $(\mathfrak{y}_i)_{i=1}^N \subset \mathcal{P}$, with $(\mathfrak{y}_1, \mathfrak{y}_N) = (\mathfrak{y}, \varsigma)$, we have

$$\mathfrak{u}(\mathfrak{y}, \varsigma) > 0 \Rightarrow \xi(\mathfrak{u}(\mathfrak{y}, \varsigma)) \leq \xi(\sum_{i=1}^{N-1} \mathfrak{u}(\mathfrak{y}_i, \mathfrak{y}_{i+1})) + \alpha,$$

for all $N \geq 2$. Then $(\mathcal{P}, \mathfrak{u})$ is said to be an $\mathcal{F}$-MS.

Example 1. ([7]) Let $\mathcal{P} = \mathbb{R}$. Then $\mathfrak{u} : \mathcal{P} \times \mathcal{P} \to [0, +\infty)$ is an $\mathcal{F}$-metric defined by

$$\mathfrak{u}(\mathfrak{y}, \varsigma) = \begin{cases} (\mathfrak{y} - \varsigma)^2 & \text{if } (\mathfrak{y}, \varsigma) \in [0, 3] \times [0, 3] \\ |\mathfrak{y} - \varsigma| & \text{if } (\mathfrak{y}, \varsigma) \notin [0, 3] \times [0, 3] \end{cases}$$

with $\xi(t) = \ln(t)$ and $\alpha = \ln(3)$.

Definition 5. ([7]) Let $(\mathcal{P}, \mathfrak{u})$ be a $\mathcal{F}$-MS.

(i) A sequence $\{\mathfrak{y}_n\} \subseteq \mathcal{P}$ is called an $\mathcal{F}$-convergent if

$$\lim_{n\to\infty} \mathfrak{u}(\mathfrak{y}_n, \mathfrak{y}) = 0.$$

(ii) A sequence $\{\mathfrak{y}_n\}$ is an $\mathcal{F}$-Cauchy, if

$$\lim_{n,m\to\infty} \mathfrak{u}(\mathfrak{y}_n, \mathfrak{y}_m) = 0.$$

In due course, Kanwal et al. [8] united the concepts of $\mathcal{F}$-MS and O-set and introduced the notion of orthogonal $\mathcal{F}$-MS in such wise.

Definition 6. ([8]) Let $(\mathcal{P}, \perp)$ be an O-set and $\mathfrak{u} : \mathcal{P} \times \mathcal{P} \to [0, +\infty)$ be an $\mathcal{F}$-metric, then $(\mathcal{P}, \perp, \mathfrak{u})$ is said to be an $O\mathcal{F}$-MS.

Example 2. ([8]) Let $\mathcal{P} = [0, 1]$. Define an $\mathcal{F}$-metric $\mathfrak{u}$ given as

$$\mathfrak{u}(\mathfrak{y}, \varsigma) = \begin{cases} e^{(|\mathfrak{y}-\varsigma|)}, & \text{if } \mathfrak{y} \neq \varsigma, \\ 0, & \text{if } \mathfrak{y} = \varsigma, \end{cases}$$

for all $\mathfrak{y}, \varsigma \in \mathcal{P}$, $\xi(t) = -\frac{1}{t}$, $t > 0$ and $\alpha = 1$. Define $\mathfrak{y}\perp\varsigma$ if and only if $\mathfrak{y}\varsigma \leq \mathfrak{y}$ or $\mathfrak{y}\varsigma \leq \varsigma$. Then, for all $\mathfrak{y} \in \mathcal{P}$, $0\perp\varsigma$, so $(\mathcal{P},\perp)$ is an O-set. Then, $(\mathcal{P}, \mathfrak{u}, \perp)$ is an $O\mathcal{F}$-MS.

Definition 7. ([8]) Let $(\mathcal{P}, \mathfrak{u}, \perp)$ be an orthogonal $\mathcal{F}$-MS and $\mathfrak{R} : \mathcal{P} \to \mathcal{P}$. Then $\mathfrak{R}$ is professed to be $\perp$-continuous at $\mathfrak{y} \in \mathcal{P}$ if for each O-sequence $\{\mathfrak{y}_n\}$ in $\mathcal{P}$ if $\mathfrak{y}_n \to \mathfrak{y}$, then $\mathfrak{R}\mathfrak{y}_n \to \mathfrak{R}\mathfrak{y}$. Also the mapping $\mathfrak{R}$ is $\perp$-continuous on $\mathcal{P}$ if the mapping $\mathfrak{R}$ is $\perp$-continuous in each $\mathfrak{y} \in \mathcal{P}$.

Definition 8. ([8]) Let $(\mathcal{P}, \mathfrak{u}, \perp)$ be an $O\mathcal{F}$-MS. Then $(\mathcal{P}, \mathfrak{u}, \perp)$ is said to be a complete $O\mathcal{F}$-MS, if every Cauchy O-sequence is $\mathcal{F}$-convergent in $\mathcal{P}$.

Samet et al. [10] gave the idea of α-admissible mapping as follows:

Definition 9. A mapping $\mathfrak{R} : \mathcal{P} \to \mathcal{P}$ is called an α-admissible mapping if

$$\alpha(\mathfrak{y}, \varsigma) \geq 1 \quad \text{implies} \quad \alpha(\mathfrak{R}\mathfrak{y}, \mathfrak{R}\varsigma) \geq 1.$$

Ramezani [11] introduced the concept of orthogonal α-admissible mapping in such wise.

Definition 10. A mapping $\mathfrak{R} : \mathcal{P} \to \mathcal{P}$ is called orthogonal α-admissible mapping if

$$\mathfrak{y} \perp \varsigma \ \text{ and } \ \alpha(\mathfrak{y}, \varsigma) \geq 1 \quad \text{implies } \ \alpha(\mathfrak{R}\mathfrak{y}, \mathfrak{R}\varsigma) \geq 1.$$

Alizadeh [12] gave the notion of cyclic (α,β)-admissibility of self mapping $\mathfrak{R} : \mathcal{P} \to \mathcal{P}$.

Definition 11. A mapping $\mathfrak{R} : \mathcal{P} \to \mathcal{P}$ is called a cyclic (α,β)-admissible mapping if there exist two functions $\alpha, \beta : \mathcal{P} \to \mathbb{R}_0^+$ such that

(i) $\alpha(\mathfrak{y}) \geq 1$ implies $\beta(\mathfrak{R}\mathfrak{y}) \geq 1$, for some $\mathfrak{y} \in \mathcal{P}$,
(ii) $\beta(\mathfrak{y}) \geq 1$ implies $\alpha(\mathfrak{R}\mathfrak{y}) \geq 1$, for some $\mathfrak{y} \in \mathcal{P}$.

We extend the above notion to orthogonal cyclic (α,β)-admissibility of self mapping $\mathfrak{R} : \mathcal{P} \to \mathcal{P}$.

Definition 12. A mapping $\mathfrak{R} : \mathcal{P} \to \mathcal{P}$ is called an orthogonal cyclic (α, β)-admissible mapping if there exist two functions $\alpha, \beta : \mathcal{P} \to \mathbb{R}_0^+$ such that

(i) $\alpha(\mathfrak{y}) \geq 1$ implies $\beta(\mathfrak{R}\mathfrak{y}) \geq 1$, for all $\mathfrak{y} \in \mathcal{P}$ with $\mathfrak{y} \perp \mathfrak{R}\mathfrak{y}$,
(ii) $\beta(\mathfrak{y}) \geq 1$ implies $\alpha(\mathfrak{R}\mathfrak{y}) \geq 1$, for all $\mathfrak{y} \in \mathcal{P}$ with $\mathfrak{y} \perp \mathfrak{R}\mathfrak{y}$.

For more details in this direction, we refer the readers to [14–19].

We also give a property (W) for orthogonal cyclic (α, β)-admissible mapping as follows:

Property (W): $\alpha(\mathfrak{y}) \geq 1$ and $\beta(\mathfrak{y}) \geq 1$ for any $\mathfrak{y}, \varsigma \in \{\mathfrak{y}^* \in \mathcal{P}\colon \mathfrak{y}^* = \mathfrak{R}\mathfrak{y}^*\}$ and $\mathfrak{y} \perp \varsigma$.

3. Results and discussions

Let Ψ be the set of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, where ψ^n is the n-th iterate of ψ. These functions are known as comparison functions. Moreover, $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$.

We define the notion of rational (α, β, ψ)-contraction as follows:

Definition 13. Let $(\mathcal{P}, \mathfrak{u}, \perp)$ be an $O\mathcal{F}$-MS. A mapping $\mathfrak{R} : \mathcal{P} \to \mathcal{P}$ is said to be a rational (α, β, ψ)-contraction if there exist $\psi \in \Psi$, $\alpha, \beta : \mathcal{P} \longrightarrow \mathbb{R}_0^+$ such that

$$\text{for all } \mathfrak{y}, \varsigma \in \mathcal{P}, \mathfrak{y} \perp \varsigma, \ \ \alpha(\mathfrak{y})\beta(\varsigma) \geq 1 \text{ implies } \mathfrak{u}(\mathfrak{R}\mathfrak{y}, \mathfrak{R}\varsigma) \leq \psi\left[\mathcal{M}(\mathfrak{y}, \varsigma)\right], \tag{3.1}$$

where

$$\mathcal{M}(\mathfrak{y}, \varsigma) = \max\left\{ \begin{array}{c} \mathfrak{u}(\mathfrak{y}, \varsigma), \frac{\mathfrak{u}(\mathfrak{y}, \mathfrak{R}\mathfrak{y})\mathfrak{u}(\varsigma, \mathfrak{R}\varsigma)}{1+\mathfrak{u}(\mathfrak{y}, \varsigma)}, \\ \frac{\mathfrak{u}(\mathfrak{y}, \mathfrak{R}\varsigma)\mathfrak{u}(\varsigma, \mathfrak{R}\mathfrak{y})}{1+\mathfrak{u}(\mathfrak{y}, \varsigma)} \end{array} \right\}.$$

Theorem 1. *Let $(\mathcal{P}, \mathfrak{u}, \perp)$ be an O-complete $O\mathcal{F}$-MS, $\mathfrak{R} : \mathcal{P} \to \mathcal{P}$ is $\perp$-preserving, orthogonal cyclic (α, β, ψ)-admissible and a rational (α, β, ψ)-contraction. Assume that the following assertions hold:*

(i) There exists $\mathfrak{y}_0 \in \mathcal{P}$ such that $\alpha(\mathfrak{y}_0) \geq 1$ and $\beta(\mathfrak{y}_0) \geq 1$;
(ii) $\mathfrak{R} : \mathcal{P} \to \mathcal{P}$ is $\perp$-continuous or;
(iii) If $\{\mathfrak{y}_n\}$ is an othogonal sequence in $\mathcal{P}$ such that $\mathfrak{y}_n \to \mathfrak{y}^*$ and $\beta(\mathfrak{y}_n) \geq 1$, $\forall\, n \in \mathbb{N}$, then $\beta(\mathfrak{y}^*) \geq 1$;
Then, there exists $\mathfrak{y}^* \in \mathcal{P}$ such that $\mathfrak{R}\mathfrak{y}^* = \mathfrak{y}^*$.
(iv) If the mapping $\mathcal{P}$ satisfies the (W), then the fixed point $\mathfrak{y}^*$ is unique.

Proof. Let $\epsilon > 0$ be fixed and $(\xi, \alpha) \in \mathcal{F} \times [0, +\infty)$ such that (D_3) holds. By $(\mathcal{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \Longrightarrow \xi(t) < \xi(t) - \alpha. \tag{3.2}$$

Since $\mathcal{P}$ is O-set and $\mathfrak{R} : \mathcal{P} \to \mathcal{P}$ so there exists $\mathfrak{y}_0 \in \mathcal{P}$ such that $\mathfrak{y}_0 \perp \mathfrak{R}\mathfrak{y}_0$. By the assumption (i), we have $\alpha(\mathfrak{y}_0) \geq 1$ and $\beta(\mathfrak{y}_0) \geq 1$. Now, we define the sequence $\{\mathfrak{y}_n\}$ as

$$\mathfrak{y}_1 = \mathfrak{R}\mathfrak{y}_0, \cdots, \mathfrak{y}_{n+1} = \mathfrak{R}\mathfrak{y}_n = \mathfrak{R}^{n+1}\mathfrak{y}_0,$$

for all $n \geq 0$. Since $\mathfrak{y}_0 \perp \mathfrak{R}\mathfrak{y}_0 = \mathfrak{y}_1$, $\alpha(\mathfrak{y}_0) \geq 1$, $\beta(\mathfrak{y}_0) \geq 1$ and $\mathfrak{R}$ is orthogonal cyclic (α, β)-admissible, so we have

$$\alpha(\mathfrak{y}_1) = \alpha(\mathfrak{R}\mathfrak{y}_0) \geq 1 \text{ and } \beta(\mathfrak{y}_1) = \beta(\mathfrak{R}\mathfrak{y}_0) \geq 1.$$

By continuing this process, we get $\mathfrak{y}_{n-1} \perp \mathfrak{y}_n, \alpha(\mathfrak{y}_n) \geq 1$ and $\beta(\mathfrak{y}_n) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$. If $\mathfrak{y}_n = \mathfrak{y}_{n+1}$, for any $n \in \mathbb{N} \cup \{0\}$, then clearly $\mathfrak{y}_n$ is a fixed point of $\mathfrak{R}$. Suppose that $\mathfrak{y}_n \neq \mathfrak{y}_{n+1}$, for all $n \in \mathbb{N} \cup \{0\}$. Hence, we assume that

$$\mathfrak{u}(\mathfrak{R}\mathfrak{y}_{n-1}, \mathfrak{R}\mathfrak{y}_n) = \mathfrak{u}(\mathfrak{y}_n, \mathfrak{y}_{n+1}) > 0,$$

for all $n \in \mathbb{N} \cup \{0\}$. Since $\mathfrak{y}_{n-1} \perp \mathfrak{y}_n$ and $\mathfrak{R}$ is $\perp$-preserving, so we get

$$\mathfrak{y}_n = \mathfrak{R}\mathfrak{y}_{n-1} \perp \mathfrak{R}\mathfrak{y}_n = \mathfrak{y}_{n+1}$$

or

$$\mathfrak{y}_{n+1} = \mathfrak{R}\mathfrak{y}_n \perp \mathfrak{R}\mathfrak{y}_{n-1} = \mathfrak{y}_n,$$

for all $n \in \mathbb{N} \cup \{0\}$. It implies that $\{\mathfrak{y}_n\}$ is an O-sequence. Since $\alpha(\mathfrak{y}_n) \geq 1$ and $\beta(\mathfrak{y}_n) \geq 1$, $\forall\, n \in \mathbb{N} \cup \{0\}$, so

$$\alpha(\mathfrak{y}_{n-1})\beta(\mathfrak{y}_n) \geq 1, \tag{3.3}$$

for all $n \in \mathbb{N} \cup \{0\}$. From (3.1) and (3.3), we get

$$\begin{aligned} \mathfrak{u}(\mathfrak{y}_n, \mathfrak{y}_{n+1}) &= \mathfrak{u}(\mathfrak{R}\mathfrak{y}_{n-1}, \mathfrak{R}\mathfrak{y}_n) \\ &\leq \psi\left[\mathcal{M}(\mathfrak{y}_{n-1}, \mathfrak{y}_n)\right], \end{aligned} \tag{3.4}$$

where

$$\mathcal{M}(\mathfrak{y}_{n-1}, \mathfrak{y}_n) = \max\left\{ \begin{array}{c} \mathfrak{u}(\mathfrak{y}_{n-1}, \mathfrak{y}_n), \frac{\mathfrak{u}(\mathfrak{y}_{n-1}, \mathfrak{R}\mathfrak{y}_{n-1})\mathfrak{u}(\mathfrak{y}_n, \mathfrak{R}\mathfrak{y}_n)}{1+\mathfrak{u}(\mathfrak{y}_{n-1}, \mathfrak{y}_n)}, \\ \frac{\mathfrak{u}(\mathfrak{y}_{n-1}, \mathfrak{R}\mathfrak{y}_n)\mathfrak{u}(\mathfrak{y}_n, \mathfrak{R}\mathfrak{y}_{n-1})}{1+\mathfrak{u}(\mathfrak{y}_{n-1}, \mathfrak{y}_n)} \end{array} \right\}$$

$$\begin{aligned}
&= \max\left\{ \begin{array}{c} \mathfrak{u}(\mathfrak{y}_{n-1},\mathfrak{y}_n), \frac{\mathfrak{u}(\mathfrak{y}_{n-1},\mathfrak{y}_n)\mathfrak{u}(\mathfrak{y}_n,\mathfrak{y}_{n+1})}{1+\mathfrak{u}(\mathfrak{y}_{n-1},\mathfrak{y}_n)}, \\ \frac{\mathfrak{u}(\mathfrak{y}_{n-1},\mathfrak{y}_{n+1})\mathfrak{u}(\mathfrak{y}_n,\mathfrak{y}_n)}{1+\mathfrak{u}(\mathfrak{y}_{n-1},\mathfrak{y}_n)} \end{array} \right\} \\
&= \max\left\{ \mathfrak{u}(\mathfrak{y}_{n-1},\mathfrak{y}_n), \frac{\mathfrak{u}(\mathfrak{y}_{n-1},\mathfrak{y}_n)\mathfrak{u}(\mathfrak{y}_n,\mathfrak{y}_{n+1})}{1+\mathfrak{u}(\mathfrak{y}_{n-1},\mathfrak{y}_n)} \right\} \\
&\leq \max\left\{ \mathfrak{u}(\mathfrak{y}_{n-1},\mathfrak{y}_n), \mathfrak{u}(\mathfrak{y}_n,\mathfrak{y}_{n+1}) \right\}.
\end{aligned}$$

Because $\frac{\mathfrak{u}(\mathfrak{y}_{n-1},\mathfrak{y}_n)}{1+\mathfrak{u}(\mathfrak{y}_{n-1},\mathfrak{y}_n)} < 1$. Now, if $\max\{\mathfrak{u}(\mathfrak{y}_{n-1},\mathfrak{y}_n), \mathfrak{u}(\mathfrak{y}_n,\mathfrak{y}_{n+1})\} = \mathfrak{u}(\mathfrak{y}_n,\mathfrak{y}_{n+1})$, then by (3.4), we have

$$\mathfrak{u}(\mathfrak{y}_n,\mathfrak{y}_{n+1}) \leq \psi\left[\mathfrak{u}(\mathfrak{y}_n,\mathfrak{y}_{n+1})\right] < \mathfrak{u}(\mathfrak{y}_n,\mathfrak{y}_{n+1})$$

a contradiction. Thus $\max\{\mathfrak{u}(\mathfrak{y}_{n-1},\mathfrak{y}_n), \mathfrak{u}(\mathfrak{y}_n,\mathfrak{y}_{n+1})\} = \mathfrak{u}(\mathfrak{y}_{n-1},\mathfrak{y}_n)$. Hence, by (3.4), we get

$$\mathfrak{u}(\mathfrak{y}_n,\mathfrak{y}_{n+1}) \leq \psi\left[\mathfrak{u}(\mathfrak{y}_{n-1},\mathfrak{y}_n)\right],$$

for all $n \in \mathbb{N} \cup \{0\}$. Inductively, we obtain

$$\mathfrak{u}(\mathfrak{y}_n,\mathfrak{y}_{n+1}) \leq \psi^n\left[\mathfrak{u}(\mathfrak{y}_0,\mathfrak{y}_1)\right],$$

for all $n \in \mathbb{N} \cup \{0\}$. As $\psi \in \Psi$, so there exists some n_0 such that

$$0 < \sum_{n \geq n_0}^{+\infty} \psi^n\left[\mathfrak{u}(\mathfrak{y}_0,\mathfrak{y}_1)\right] < \delta.$$

Hence, by (3.2) and $(\mathcal{F}_1)$, we have

$$\xi\left(\sum_{i=n}^{m-1} \psi^i\left[\mathfrak{u}(\mathfrak{y}_0,\mathfrak{y}_1)\right]\right) \leq \xi\left(\sum_{i=n_0}^{+\infty} \psi^i\left[\mathfrak{u}(\mathfrak{y}_0,\mathfrak{y}_1)\right]\right) < \xi(\epsilon) - a, \tag{3.5}$$

where $m > n \geq n_0$. Thus by (D_3) and (3.5) for $\mathfrak{u}(\mathfrak{y}_n,\mathfrak{y}_m) > 0, m > n \geq n_0$, we have

$$\begin{aligned}
\xi\left(\mathfrak{u}(\mathfrak{y}_n,\mathfrak{y}_m)\right) &\leq \xi\left(\sum_{i=n}^{m-1} \mathfrak{u}(\mathfrak{y}_i,\mathfrak{y}_{i+1})\right) + a \\
&\leq \xi\left(\sum_{i=n}^{m-1} \psi^i\left[\mathfrak{u}(\mathfrak{y}_0,\mathfrak{y}_1)\right]\right) + a \\
&< \xi(\epsilon),
\end{aligned}$$

which, from $(\mathcal{F}_1)$, gives that

$$\mathfrak{u}(\mathfrak{y}_n,\mathfrak{y}_m) < \epsilon,$$

for all $m > n \geq n_0$. Therefore, $\{\mathfrak{y}_n\}$ is a Cauchy O-sequence in $(\mathcal{P}, \perp, \mathfrak{u})$. As $(\mathcal{P}, \perp, \mathfrak{u})$ is O-complete, so there exists $\mathfrak{y}^* \in \mathcal{P}$ such that, $\lim_{n\to\infty} \mathfrak{y}_n \to \mathfrak{y}^*$. Now, we show that $\mathfrak{y}^* = \mathfrak{R}\mathfrak{y}^*$. Since $\mathfrak{R}$ is $\perp$-continuous by the assumption (ii), so we have $\mathfrak{R}\mathfrak{y}_n \to \mathfrak{R}\mathfrak{y}^*$ as $n \to \infty$. Thus

$$\mathfrak{R}\mathfrak{y}^* = \lim_{n\to\infty} \mathfrak{R}\mathfrak{y}_n = \lim_{n\to\infty} \mathfrak{y}_{n+1} = \mathfrak{y}^*.$$

Next, we suppose that (iii) holds. We suppose on the contrary that $\mathfrak{y}^*$ is not the fixed point of $\mathfrak{R}$. Then $\mathfrak{u}(\mathfrak{R}\mathfrak{y}_n, \mathfrak{y}^*) \neq 0$. Now since $\{\mathfrak{y}_n\}$ is an O-sequence in $(\mathcal{P}, \perp, \mathfrak{u})$ such that $\lim_{n\to\infty} \mathfrak{y}_n \to \mathfrak{y}^*$, so by the assumption(iii), we have $\beta(\mathfrak{y}^*) \geq 1$. Hence, $\alpha(\mathfrak{y}_n)\beta(\mathfrak{y}^*) \geq 1$. Now by (3.1), we have

$$\begin{aligned}
\xi\left(\mathfrak{u}(\mathfrak{R}\mathfrak{y}^*, \mathfrak{y}^*)\right) &\leq \xi\left(\mathfrak{u}(\mathfrak{R}\mathfrak{y}^*, \mathfrak{R}\mathfrak{y}_n) + \mathfrak{u}(\mathfrak{R}\mathfrak{y}_n, \mathfrak{y}^*)\right) + \alpha \\
&\leq \xi\left(\begin{array}{c} \psi\left(\max\left\{\begin{array}{c} \mathfrak{u}(\mathfrak{y}^*, \mathfrak{y}_n), \frac{\mathfrak{u}(\mathfrak{y}^*, \mathfrak{R}\mathfrak{y}^*)\mathfrak{u}(\mathfrak{y}_n, \mathfrak{R}\mathfrak{y}_n)}{1+\mathfrak{u}(\mathfrak{y}^*, \mathfrak{y}_n)}, \\ \frac{\mathfrak{u}(\mathfrak{y}^*, \mathfrak{R}\mathfrak{y}_n)\mathfrak{u}(\mathfrak{y}_n, \mathfrak{R}\mathfrak{y}^*)}{1+\mathfrak{u}(\mathfrak{y}^*, \mathfrak{y}_n)} \end{array}\right\}\right) \\ +\mathfrak{u}(\mathfrak{R}\mathfrak{y}_n, \mathfrak{y}^*) \end{array}\right) + \alpha \\
&= \xi\left(\begin{array}{c} \psi\left(\max\left\{\begin{array}{c} \mathfrak{u}(\mathfrak{y}^*, \mathfrak{y}_n), \frac{\mathfrak{u}(\mathfrak{y}^*, \mathfrak{R}\mathfrak{y}^*)\mathfrak{u}(\mathfrak{y}_n, \mathfrak{y}_{n+1})}{1+\mathfrak{u}(\mathfrak{y}^*, \mathfrak{y}_n)}, \\ \frac{\mathfrak{u}(\mathfrak{y}^*, \mathfrak{y}_{n+1})\mathfrak{u}(\mathfrak{y}_n, \mathfrak{R}\mathfrak{y}^*)}{1+\mathfrak{u}(\mathfrak{y}^*, \mathfrak{y}_n)} \end{array}\right\}\right) \\ +\mathfrak{u}(\mathfrak{R}\mathfrak{y}_n, \mathfrak{y}^*) \end{array}\right) + \alpha.
\end{aligned}$$

Letting $n \to \infty$, in the above inequality and using the fact that

$$\lim_{n\to\infty} \xi\left(\psi\left(\max\left\{\begin{array}{c} \mathfrak{u}(\mathfrak{y}^*, \mathfrak{y}_n), \frac{\mathfrak{u}(\mathfrak{y}^*, \mathfrak{R}\mathfrak{y}^*)\mathfrak{u}(\mathfrak{y}_n, \mathfrak{y}_{n+1})}{1+\mathfrak{u}(\mathfrak{y}^*, \mathfrak{y}_n)}, \\ \frac{\mathfrak{u}(\mathfrak{y}^*, \mathfrak{y}_{n+1})\mathfrak{u}(\mathfrak{y}_n, \mathfrak{R}\mathfrak{y}^*)}{1+\mathfrak{u}(\mathfrak{y}^*, \mathfrak{y}_n)} \end{array}\right\}\right) + \mathfrak{u}(\mathfrak{R}\mathfrak{y}_n, \mathfrak{y}^*)\right) + \alpha,$$

we have $\xi\left(\mathfrak{u}(\mathfrak{R}\mathfrak{y}^*, \mathfrak{y}^*)\right) = -\infty$. Hence, by $(\mathcal{F}_2)$, we have $\mathfrak{u}(\mathfrak{R}\mathfrak{y}^*, \mathfrak{y}^*) = 0$, which is a contradiction. Thus $\mathfrak{R}\mathfrak{y}^* = \mathfrak{y}^*$. □

Now, we suppose that $\mathfrak{y}' = \mathfrak{R}\mathfrak{y}'$ is another fixed point of $\mathfrak{R}$ such that $\mathfrak{y}' \neq \mathfrak{y}^*$. Since the mapping $\mathcal{P}$ satisfies the property (W), so by the assumption (iv), we have $\mathfrak{y}^* \perp \mathfrak{y}'$ or $\mathfrak{y}' \perp \mathfrak{y}^*$ and $\alpha(\mathfrak{y}^*) \geq 1$ and $\beta\left(\mathfrak{y}'\right) \geq 1$. Then as $\alpha(\mathfrak{y}^*)\beta\left(\mathfrak{y}'\right) \geq 1$, so by (3.1), we have

$$\begin{aligned}
\mathfrak{u}(\mathfrak{y}^*, \mathfrak{y}') &= \mathfrak{u}(\mathfrak{R}\mathfrak{y}^*, \mathfrak{R}\mathfrak{y}') \leq \psi\left[\mathcal{M}(\mathfrak{y}^*, \mathfrak{y}')\right] \\
&= \psi\left[\max\left\{\begin{array}{c} \mathfrak{u}(\mathfrak{y}^*, \mathfrak{y}'), \frac{\mathfrak{u}(\mathfrak{y}^*, \mathfrak{R}\mathfrak{y}^*)\mathfrak{u}(\mathfrak{y}', \mathfrak{R}\mathfrak{y}')}{1+\mathfrak{u}(\mathfrak{y}^*, \mathfrak{y}')}, \\ \frac{\mathfrak{u}(\mathfrak{y}^*, \mathfrak{R}\mathfrak{y}')\mathfrak{u}(\mathfrak{y}', \mathfrak{R}\mathfrak{y}^*)}{1+\mathfrak{u}(\mathfrak{y}^*, \mathfrak{y}')} \end{array}\right\}\right] \\
&= \psi\left[\max\left\{\begin{array}{c} \mathfrak{u}(\mathfrak{y}^*, \mathfrak{y}'), \frac{\mathfrak{u}(\mathfrak{y}^*, \mathfrak{y}^*)\mathfrak{u}(\mathfrak{y}', \mathfrak{y}')}{1+\mathfrak{u}(\mathfrak{y}^*, \mathfrak{y}')}, \\ \frac{\mathfrak{u}(\mathfrak{y}^*, \mathfrak{y}')\mathfrak{u}(\mathfrak{y}', \mathfrak{y}^*)}{1+\mathfrak{u}(\mathfrak{y}^*, \mathfrak{y}')} \end{array}\right\}\right] \\
&= \psi\left[\max\left\{\mathfrak{u}(\mathfrak{y}^*, \mathfrak{y}'), \frac{\mathfrak{u}(\mathfrak{y}^*, \mathfrak{y}')\mathfrak{u}(\mathfrak{y}', \mathfrak{y}^*)}{1 + \mathfrak{u}(\mathfrak{y}^*, \mathfrak{y}')}\right\}\right] \\
&\leq \psi\left(\mathfrak{u}(\mathfrak{y}^*, \mathfrak{y}')\right) < \mathfrak{u}(\mathfrak{y}^*, \mathfrak{y}'),
\end{aligned}$$

which is a contradiction. Hence, $\mathfrak{y}' = \mathfrak{y}^*$. Thus the fixed point is unique.

Corollary 1. *Let $(\mathcal{P}, \mathfrak{u}, \perp)$ be an O-complete $\mathcal{OF}$-MS, $\mathfrak{R} : \mathcal{P} \to \mathcal{P}$ is $\perp$-preserving and orthogonal cyclic (α,β)-admissible. Suppose that there exist $\psi \in \Psi$, $\alpha, \beta : \mathcal{P} \longrightarrow \mathbb{R}_0^+$ and $l > 0$, such that*

$$\left(\mathfrak{u}(\mathfrak{R}\mathfrak{y}, \mathfrak{R}\varsigma) + l\right)^{\alpha(\mathfrak{y})\beta(\varsigma)} \leq \left(\psi\left[\mathcal{M}(\mathfrak{y}, \varsigma)\right] + l\right), \tag{3.6}$$

for all $\mathfrak{y}, \varsigma \in \mathcal{P}$, $\mathfrak{y} \perp \varsigma$, where

$$\mathcal{M}(\mathfrak{y}, \varsigma) = \max\left\{\begin{array}{c} \mathfrak{u}(\mathfrak{y}, \varsigma), \frac{\mathfrak{u}(\mathfrak{y}, \mathfrak{R}\mathfrak{y})\mathfrak{u}(\varsigma, \mathfrak{R}\varsigma)}{1+\mathfrak{u}(\mathfrak{y}, \varsigma)}, \\ \frac{\mathfrak{u}(\mathfrak{y}, \mathfrak{R}\varsigma)\mathfrak{u}(\varsigma, \mathfrak{R}\mathfrak{y})}{1+\mathfrak{u}(\mathfrak{y}, \varsigma)} \end{array}\right\}.$$

Assume that the following assertions hold:
(i) There exists $\mathfrak{y}_0 \in \mathcal{P}$ such that $\alpha(\mathfrak{y}_0) \geq 1$ and $\beta(\mathfrak{y}_0) \geq 1$;
(ii) $\mathfrak{R} : \mathcal{P} \to \mathcal{P}$ is $\perp$-continuous or;
(iii) If $\{\mathfrak{y}_n\}$ is an orthogonal sequence in $\mathcal{P}$ such that $\mathfrak{y}_n \to \mathfrak{y}^*$ and $\beta(\mathfrak{y}_n) \geq 1$, for all $n \in \mathbb{N}$; then $\beta(\mathfrak{y}^*) \geq 1$.
Then there exists $\mathfrak{y}^* \in \mathcal{P}$ such that $\mathfrak{R}\mathfrak{y}^* = \mathfrak{y}^*$.
(iv) If the mapping $\mathcal{P}$ satisfies the property (W), then the fixed point $\mathfrak{y}^*$ is unique.

Proof. Let $\alpha(\mathfrak{y})\beta(\varsigma) \geq 1$ for $\mathfrak{y}, \varsigma \in \mathcal{P}$. Then, from (3.6), we get

$$(\mathfrak{u}(\mathfrak{R}\mathfrak{y}, \mathfrak{R}\varsigma) + l) \leq (\mathfrak{u}(\mathfrak{R}\mathfrak{y}, \mathfrak{R}\varsigma) + l)^{\alpha(\mathfrak{y})\beta(\varsigma)} \leq (\psi[\mathcal{M}(\mathfrak{y}, \varsigma)] + l).$$

Then we obtain,

$$\mathfrak{u}(\mathfrak{R}\mathfrak{y}, \mathfrak{R}\varsigma) \leq \psi[\mathcal{M}(\mathfrak{y}, \varsigma)]$$

where

$$\mathcal{M}(\mathfrak{y}, \varsigma) = \max\left\{ \begin{array}{c} \mathfrak{u}(\mathfrak{y}, \varsigma), \frac{\mathfrak{u}(\mathfrak{y}, \mathfrak{R}\mathfrak{y})\mathfrak{u}(\varsigma, \mathfrak{R}\varsigma)}{1+\mathfrak{u}(\mathfrak{y}, \varsigma)}, \\ \frac{\mathfrak{u}(\mathfrak{y}, \mathfrak{R}\varsigma)\mathfrak{u}(\varsigma, \mathfrak{R}\mathfrak{y})}{1+\mathfrak{u}(\mathfrak{y}, \varsigma)} \end{array} \right\}.$$

Hence, the conditions of Theorem 1 are satisfied and $\mathfrak{R}$ has a unique fixed point. □

Corollary 2. *Let $(\mathcal{P}, \mathfrak{u}, \perp)$ be an O-complete orthogonal $\mathcal{F}$-MS, $\mathfrak{R} : \mathcal{P} \to \mathcal{P}$ is $\perp$-preserving and orthogonal cyclic (α,β)-admissible mapping. Suppose that there exist $\psi \in \Psi$ and $\alpha, \beta : \mathcal{P} \longrightarrow \mathbb{R}_0^+$, such that*

$$(\alpha(\mathfrak{y})\beta(\varsigma) + 1)^{\mathfrak{u}(\mathfrak{R}\mathfrak{y}, \mathfrak{R}\varsigma)} \leq 2^{\psi[\mathcal{M}(\mathfrak{y}, \varsigma)]}, \tag{3.7}$$

for all $\mathfrak{y}, \varsigma \in \mathcal{P}$ with $\mathfrak{y} \perp \varsigma$, where

$$\mathcal{M}(\mathfrak{y}, \varsigma) = \max\left\{ \begin{array}{c} \mathfrak{u}(\mathfrak{y}, \varsigma), \frac{\mathfrak{u}(\mathfrak{y}, \mathfrak{R}\mathfrak{y})\mathfrak{u}(\varsigma, \mathfrak{R}\varsigma)}{1+\mathfrak{u}(\mathfrak{y}, \varsigma)}, \\ \frac{\mathfrak{u}(\mathfrak{y}, \mathfrak{R}\varsigma)\mathfrak{u}(\varsigma, \mathfrak{R}\mathfrak{y})}{1+\mathfrak{u}(\mathfrak{y}, \varsigma)} \end{array} \right\}.$$

Assume that the following assertions hold:
(i) There exists $\mathfrak{y}_0 \in \mathcal{P}$ such that $\alpha(\mathfrak{y}_0) \geq 1$ and $\beta(\mathfrak{y}_0) \geq 1$;
(ii) $\mathfrak{R} : \mathcal{P} \to \mathcal{P}$ is $\perp$-continuous or;
(iii) If $\{\mathfrak{y}_n\}$ is an orthogonal sequence in $\mathcal{P}$ such that $\mathfrak{y}_n \to \mathfrak{y}^*$ and $\beta(\mathfrak{y}_n) \geq 1$, for all $n \in \mathbb{N}$, then $\beta(\mathfrak{y}^*) \geq 1$;
Then there exists $\mathfrak{y}^* \in \mathcal{P}$ such that $\mathfrak{R}\mathfrak{y}^* = \mathfrak{y}^*$.
(iv) If the mapping $\mathcal{P}$ satisfies the property (W), then the fixed point $\mathfrak{y}^*$ is unique.

Proof. Let $\alpha(\mathfrak{y})\beta(\varsigma) \geq 1$ for $\mathfrak{y}, \varsigma \in \mathcal{P}$. Then, from (3.7), we get

$$2^{\mathfrak{u}(\mathfrak{R}\mathfrak{y}, \mathfrak{R}\varsigma)} = (1 + 1)^{\mathfrak{u}(\mathfrak{R}\mathfrak{y}, \mathfrak{R}\varsigma)} \leq (\alpha(\mathfrak{y})\beta(\varsigma) + 1)^{\mathfrak{u}(\mathfrak{R}\mathfrak{y}, \mathfrak{R}\varsigma)} \leq 2^{\psi[\mathcal{M}(\mathfrak{y}, \varsigma)]}.$$

Then we obtain,

$$\mathfrak{u}(\mathfrak{R}\mathfrak{y}, \mathfrak{R}\varsigma) \leq \psi[\mathcal{M}(\mathfrak{y}, \varsigma)],$$

where

$$\mathcal{M}(\mathfrak{y},\varsigma) = \max\left\{ \begin{array}{c} \mathfrak{u}(\mathfrak{y},\varsigma), \frac{\mathfrak{u}(\mathfrak{y},\mathfrak{R}\mathfrak{y})\mathfrak{u}(\varsigma,\mathfrak{R}\varsigma)}{1+\mathfrak{u}(\mathfrak{y},\varsigma)}, \\ \frac{\mathfrak{u}(\mathfrak{y},\mathfrak{R}\varsigma)\mathfrak{u}(\varsigma,\mathfrak{R}\mathfrak{y})}{1+\mathfrak{u}(\mathfrak{y},\varsigma)} \end{array} \right\}.$$

Hence, the conditions of Theorem 1 are satisfied and $\mathfrak{R}$ has a unique fixed point. □

Corollary 3. *Let $(\mathcal{P},\mathfrak{u},\perp)$ be an O-complete orthogonal $\mathcal{F}$-MS, $\mathfrak{R}:\mathcal{P}\to\mathcal{P}$ is $\perp$-preserving and orthogonal cyclic (α,β)-admissible mapping. Suppose that there exist $\psi\in\Psi$ and $\alpha,\beta:\mathcal{P}\longrightarrow\mathbb{R}_0^+$, such that*

$$\alpha(\mathfrak{y})\beta(\varsigma)\mathfrak{u}(\mathfrak{R}\mathfrak{y},\mathfrak{R}\varsigma) \leq \psi\left[\mathcal{M}(\mathfrak{y},\varsigma)\right], \tag{3.8}$$

for all $\mathfrak{y},\varsigma\in\mathcal{P}$ with $\mathfrak{y}\perp\varsigma$, where

$$\mathcal{M}(\mathfrak{y},\varsigma) = \max\left\{ \begin{array}{c} \mathfrak{u}(\mathfrak{y},\varsigma), \frac{\mathfrak{u}(\mathfrak{y},\mathfrak{R}\mathfrak{y})\mathfrak{u}(\varsigma,\mathfrak{R}\varsigma)}{1+\mathfrak{u}(\mathfrak{y},\varsigma)}, \\ \frac{\mathfrak{u}(\mathfrak{y},\mathfrak{R}\varsigma)\mathfrak{u}(\varsigma,\mathfrak{R}\mathfrak{y})}{1+\mathfrak{u}(\mathfrak{y},\varsigma)} \end{array} \right\}.$$

Assume that the following conditions hold:
(i) There exists $\mathfrak{y}_0\in\mathcal{P}$ such that $\alpha(\mathfrak{y}_0)\geq 1$ and $\beta(\mathfrak{y}_0)\geq 1$;
(ii) $\mathfrak{R}:\mathcal{P}\to\mathcal{P}$ is $\perp$-continuous or;
(iii) If $\{\mathfrak{y}_n\}$ is an orthogonal sequence in $\mathcal{P}$ such that $\mathfrak{y}_n\to\mathfrak{y}^*$ and $\beta(\mathfrak{y}_n)\geq 1$ for all $n\in\mathbb{N}$, then and $\beta(\mathfrak{y}^*)\geq 1$;
Then there exists $\mathfrak{y}^*\in\mathcal{P}$ such that $\mathfrak{R}\mathfrak{y}^*=\mathfrak{y}^*$.
(iv) If the mapping $\mathcal{P}$ satisfies the property (W), then the fixed point $\mathfrak{y}^*$ is unique.

Proof. Let $\alpha(\mathfrak{y})\beta(\varsigma)\geq 1$ for $\mathfrak{y},\varsigma\in\mathcal{P}$. Then, from (3.8), we get

$$\mathfrak{u}(\mathfrak{R}\mathfrak{y},\mathfrak{R}\varsigma) \leq \alpha(\mathfrak{y})\beta(\varsigma)\mathfrak{u}(\mathfrak{R}\mathfrak{y},\mathfrak{R}\varsigma) \leq \psi\left[\mathcal{M}(\mathfrak{y},\varsigma)\right].$$

Then we obtain,

$$\mathfrak{u}(\mathfrak{R}\mathfrak{y},\mathfrak{R}\varsigma) \leq \psi\left[\mathcal{M}(\mathfrak{y},\varsigma)\right],$$

where

$$\mathcal{M}(\mathfrak{y},\varsigma) = \max\left\{ \begin{array}{c} \mathfrak{u}(\mathfrak{y},\varsigma), \frac{\mathfrak{u}(\mathfrak{y},\mathfrak{R}\mathfrak{y})\mathfrak{u}(\varsigma,\mathfrak{R}\varsigma)}{1+\mathfrak{u}(\mathfrak{y},\varsigma)}, \\ \frac{\mathfrak{u}(\mathfrak{y},\mathfrak{R}\varsigma)\mathfrak{u}(\varsigma,\mathfrak{R}\mathfrak{y})}{1+\mathfrak{u}(\mathfrak{y},\varsigma)} \end{array} \right\}.$$

Hence, the conditions of Theorem 1 are satisfied and $\mathfrak{R}$ has a unique fixed point. □

Corollary 4. *Let $(\mathcal{P},\mathfrak{u},\perp)$ be an O-complete orthogonal $\mathcal{F}$-MS, $\mathfrak{R}:\mathcal{P}\to\mathcal{P}$ is $\perp$-preserving and orthogonal cyclic (α,β)-admissible mapping such that*

$$\alpha(\mathfrak{y})\beta(\varsigma)\geq 1 \text{ implies } \mathfrak{u}(\mathfrak{R}\mathfrak{y},\mathfrak{R}\varsigma)\leq k\mathcal{M}(\mathfrak{y},\varsigma),$$

for all $\mathfrak{y},\varsigma\in\mathcal{P}$ with $\mathfrak{y}\perp\varsigma$, where $0<k<1$ and

$$\mathcal{M}(\mathfrak{y},\varsigma) = \max\left\{ \begin{array}{c} \mathfrak{u}(\mathfrak{y},\varsigma), \frac{\mathfrak{u}(\mathfrak{y},\mathfrak{R}\mathfrak{y})\mathfrak{u}(\varsigma,\mathfrak{R}\varsigma)}{1+\mathfrak{u}(\mathfrak{y},\varsigma)} \\ , \frac{\mathfrak{u}(\mathfrak{y},\mathfrak{R}\varsigma)\mathfrak{u}(\varsigma,\mathfrak{R}\mathfrak{y})}{1+\mathfrak{u}(\mathfrak{y},\varsigma)} \end{array} \right\}.$$

Assume that the following assertions hold:
(i) Yhere exists $\mathfrak{y}_0 \in \mathcal{P}$ such that $\alpha(\mathfrak{y}_0) \geq 1$ and $\beta(\mathfrak{y}_0) \geq 1$;
(ii) $\mathfrak{R} : \mathcal{P} \to \mathcal{P}$ is $\perp$-continuous or;
(iii) If $\{\mathfrak{y}_n\}$ is an orthogonal sequence in $\mathcal{P}$ such that $\mathfrak{y}_n \to \mathfrak{y}^*$ and $\beta(\mathfrak{y}_n) \geq 1$ for all $n \in \mathbb{N}$, then and $\beta(\mathfrak{y}^*) \geq 1$;
Then there exists $\mathfrak{y}^* \in \mathcal{P}$ such that $\mathfrak{R}\mathfrak{y}^* = \mathfrak{y}^*$.
(iv) If the mapping $\mathcal{P}$ satisfies the property (W), then the fixed point $\mathfrak{y}^*$ is unique.

Proof. Taking $\psi(t) = kt$, where $0 < k < 1$ in Theorem 1. □

Now, we prove that our result is a real generalization of main result of Faraji et al. [19].

Corollary 5. *([19]) Let $(\mathcal{P}, \mathfrak{u}, \perp)$ be a complete $\mathcal{F}$-MS, $\mathfrak{R} : \mathcal{P} \to \mathcal{P}$ is cyclic (α,β)-admissible mapping. Suppose that there exist $\psi \in \Psi$ and $\alpha, \beta : \mathcal{P} \longrightarrow \mathbb{R}_0^+$ such that*

$$\alpha(\mathfrak{y})\beta(\varsigma) \geq 1 \text{ implies } \mathfrak{u}(\mathfrak{R}\mathfrak{y}, \mathfrak{R}\varsigma) \leq \psi\left[\mathcal{M}(\mathfrak{y}, \varsigma)\right], \tag{3.9}$$

where

$$\mathcal{M}(\mathfrak{y}, \varsigma) = \max\left\{ \begin{array}{c} \mathfrak{u}(\mathfrak{y}, \varsigma), \frac{\mathfrak{u}(\mathfrak{y}, \mathfrak{R}\mathfrak{y})\mathfrak{u}(\varsigma, \mathfrak{R}\varsigma)}{1+\mathfrak{u}(\mathfrak{y},\varsigma)}, \\ \frac{\mathfrak{u}(\mathfrak{y}, \mathfrak{R}\varsigma)\mathfrak{u}(\varsigma, \mathfrak{R}\mathfrak{y})}{1+\mathfrak{u}(\mathfrak{y},\varsigma)} \end{array} \right\}$$

for all $\mathfrak{y}, \varsigma \in \mathcal{P}$.
Assume that the following assertions hold:
(i) There exists $\mathfrak{y}_0 \in \mathcal{P}$ such that $\alpha(\mathfrak{y}_0) \geq 1$ and $\beta(\mathfrak{y}_0) \geq 1$;
(ii) $\mathfrak{R} : \mathcal{P} \to \mathcal{P}$ is continuous or;
(iii) If $\{\mathfrak{y}_n\}$ is a sequence in $\mathcal{P}$ such that $\mathfrak{y}_n \to \mathfrak{y}^*$ and $\beta(\mathfrak{y}_n) \geq 1$, for all $n \in \mathbb{N}$, then $\beta(\mathfrak{y}^*) \geq 1$;
Then there exists a unique point $\mathfrak{y}^* \in \mathcal{P}$ such that $\mathfrak{R}\mathfrak{y}^* = \mathfrak{y}^*$.

Proof. Suppose that

$$\mathfrak{y} \perp \varsigma \iff \mathfrak{u}(\mathfrak{R}\mathfrak{y}, \mathfrak{R}\varsigma) \leq \psi\left[\mathcal{M}(\mathfrak{y}, \varsigma)\right]$$

where

$$\mathcal{M}(\mathfrak{y}, \varsigma) = \max\left\{ \begin{array}{c} \mathfrak{u}(\mathfrak{y}, \varsigma), \frac{\mathfrak{u}(\mathfrak{y}, \mathfrak{R}\mathfrak{y})\mathfrak{u}(\varsigma, \mathfrak{R}\varsigma)}{1+\mathfrak{u}(\mathfrak{y},\varsigma)}, \\ \frac{\mathfrak{u}(\mathfrak{y}, \mathfrak{R}\varsigma)\mathfrak{u}(\varsigma, \mathfrak{R}\mathfrak{y})}{1+\mathfrak{u}(\mathfrak{y},\varsigma)} \end{array} \right\}$$

Fix $\mathfrak{y}_0 \in \mathcal{P}$. As $\mathfrak{R}$ satisfies the condition (3.9) for all $\varsigma \in \mathcal{P}$, $\mathfrak{y}_0 \perp \varsigma$. Hence, $(\mathcal{P}, \perp)$ is an O-set. It is obviously that $\mathcal{P}$ is O-complete $\mathcal{F}$-MS and $\mathfrak{R}$ is an $\perp$-contraction, $\perp$-continuous and $\perp$-preserving. By applying Theorem 1, $\mathfrak{R}$ has a unique fixed point in $\mathcal{P}$.

□

4. Applications

Fixed point theory is a very important tool to solve differential and integral equations used to obtain solutions of different mathematical models, dynamical systems, models in economy, game theory, physics, computer science, engineering, neural networks and many others (see [20–30]).

In this section, let us give an application of our fixed point theorem to a nonlinear differential equation of fractional order

$$^{C}D^{\eta}(\mathfrak{y}(t)) = g(t, \mathfrak{y}(t)), \ \ (0 < t < 1, \ 1 < \eta \leq 2) \tag{4.1}$$

along with the integral boundary conditions

$$\mathfrak{y}(0) = 0\,, \ \mathfrak{y}^{/}(0) = I, \ (0 < I < 1),$$

where $^{C}D^{\eta}$ represents the Caputo fractional derivative of order η defined by

$$^{C}D^{\eta}g(t) = \frac{1}{\Gamma(j-\eta)}\int\limits_{0}^{t}(t-s)^{j-\eta-1}\,g^{j}(s)ds,$$

$(j-1 < \eta < j, \ j = [\eta]+1)$ and g is a continuous function. Consider

$$\mathcal{P}=\{\mathfrak{y} : \mathfrak{y} \in C\,([0,1],\mathbb{R})\}$$

with supremum norm $\|\mathfrak{y}\|_{\infty} = \sup_{t\in[0,1]}|\mathfrak{y}(t)|$. Thus, $(\mathcal{P}, \|\mathfrak{y}\|_{\infty})$ is a Banach space. Note that

$$I^{\eta}g(t) = \frac{1}{\Gamma(\eta)}\int\limits_{0}^{t}(t-s)^{\eta-1}\,g(s)ds, \quad \text{with } \eta > 0$$

is Riemann-Liouville fractional integral.

Lemma 1. *([8]) A Banach space $(\mathcal{P}, \|\cdot\|_{\infty})$ equipped with the $\mathcal{F}$-metric d defined by*

$$d(\mathfrak{y},\varsigma) = \|\mathfrak{y}-\varsigma\|_{\infty} = \sup_{t\in[0,1]}|\mathfrak{y}(t)-\varsigma(t)|$$

and orthogonal relation $\mathfrak{y}\perp\varsigma \Leftrightarrow \mathfrak{y}\varsigma \geq 0$, where $\mathfrak{y},\varsigma \in \mathcal{P}$, is an orthogonal $\mathcal{F}$-metric space

Theorem 2. *Suppose that the mapping g is continuous. Assume that the following conditions hold:*

(i) There exists a constant ϑ such that

$$\|g\,(t,\mathfrak{y}) - g\,(t,\varsigma)\| \leq \vartheta\,\|\mathfrak{y}-\varsigma\|\,,$$

for $t \in [0,1]$ and for all $\mathfrak{y},\varsigma \in \mathcal{P}$ such that $\mathfrak{y}(t)\varsigma(t) \geq 0$ and with $\vartheta\varpi < 1$, where

$$\varpi = \frac{1}{\Gamma(\eta+1)} + \frac{2\lambda^{\eta+1}\Gamma(\eta)}{(2-\lambda^{2})\,\Gamma(\eta+1)}.$$

(ii) There exists $\mathfrak{R} : (\mathcal{P},\perp,d) \to (\mathcal{P},\perp,d)$ defined by

$$\begin{aligned}\mathfrak{R}\mathfrak{y}(t) &= \frac{1}{\Gamma(\eta)}\int\limits_{0}^{t}(t-s)^{\eta-1}\,g\,(s,\mathfrak{y}(s))\,ds \\ &\quad + \frac{2t}{(2-\lambda^{2})\,\Gamma(\eta)}\int\limits_{0}^{\lambda}\left(\int\limits_{0}^{s}(s-m)^{\eta-1}\,g\,(m,\mathfrak{y}(m))\,dm\right)ds,\end{aligned}$$

for all $\mathfrak{y},\varsigma \in \mathcal{P}$ such that $\mathfrak{y}(t)\varsigma(t) \geq 0$. Also $\mathfrak{R}$ is orthogonal cyclic (α,β)-admissible.
(iii) There exists $\mathfrak{y}_0(t) \in (\mathcal{P},\perp,d)$ with $\mathfrak{y}_0(t) \geq 0$ for $t \in [0,1]$ such that $\mathfrak{y}_0(t) \perp \mathfrak{R}\mathfrak{y}_0(t)$ and there exist $\alpha,\beta : \mathcal{P} \to \mathbb{R}_0^+$ such that $\alpha(\mathfrak{y}_0(t)) \geq 1$ and $\beta(\mathfrak{y}_0(t)) \geq 1$.
(iv) $\mathfrak{R} : \mathcal{P} \to \mathcal{P}$ is $\perp$-continuous or.
(v) If $\{\mathfrak{y}_n\}$ is an othogonal sequence in $\mathcal{P}$ such that $\mathfrak{y}_n \to \mathfrak{y}^*$ and $\beta(\mathfrak{y}_n) \geq 1$ for all $n \in \mathbb{N}$, then and $\beta(\mathfrak{y}^*) \geq 1$.

Then (4.1) has a solution.

Proof. It is well known from (see [8, 31]) that $\mathfrak{y} \in \mathcal{P}$ is a solution of (4.1) if $\mathfrak{y} \in \mathcal{P}$ is a solution of the integral equation

$$\begin{aligned}\mathfrak{y}(t) &= \frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1} g(s,\mathfrak{y}(s))\,ds \\ &\quad + \frac{2t}{(2-\lambda^2)\Gamma(\eta)}\int_0^\lambda \left(\int_0^s (s-m)^{\eta-1} g(m,\mathfrak{y}(m))\,dm\right) ds.\end{aligned}$$

Then, problem (4.1) is equivalent to find $\mathfrak{y} \in \mathcal{P}$ which is a fixed point of mapping $\mathfrak{R}$. Suppose that $\perp \subseteq \mathcal{P}\times\mathcal{P}$ be defined by

$$\mathfrak{y}\perp\varsigma \Leftrightarrow \mathfrak{y}(t)\varsigma(t) \geq 0,$$

for all $t \in [0,1]$. Then the mapping $\mathcal{P}$ is orthogonal under $\perp$, since for $\mathfrak{y} \in \mathcal{P}$, there exists $\varsigma(t) = 0$, for all $t \in [0,1]$ such that $\mathfrak{y}(t)\varsigma(t) = 0$. Now define $d : \mathcal{P}\times\mathcal{P} \to [0,+\infty)$ by

$$d(\mathfrak{y},\varsigma) = \|\mathfrak{y}-\varsigma\|_\infty = \sup_{t\in[0,1]} \|\mathfrak{y}(t)-\varsigma(t)\|,$$

for all $\mathfrak{y},\varsigma \in \mathcal{P}$, then $(\mathcal{P},d,\perp)$ is a complete orthogonal $\mathcal{F}$-MS. It is very easy to prove that $\mathfrak{R}$ is $\perp$-continuous. We first prove that $\mathfrak{R}$ is $\perp$-preserving. Suppose $\mathfrak{y}(t)\perp\varsigma(t)$, for all $t \in [0,1]$. Now

$$\begin{aligned}\mathfrak{R}\mathfrak{y}(t) &= \frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1} g(s,\mathfrak{y}(s))\,ds \\ &\quad + \frac{2t}{(2-\lambda^2)\Gamma(\eta)}\int_0^\lambda \left(\int_0^s (s-m)^{\eta-1} g(m,\mathfrak{y}(m))\,dm\right) ds > 0,\end{aligned}$$

which implies that $\mathfrak{R}\mathfrak{y}(t)\perp\mathfrak{R}\varsigma(t)$, i.e., $\mathfrak{R}$ is $\perp$-preserving. Thus, for all $t \in [0,1]$, $\mathfrak{y}(t)\perp\varsigma(t)$, we have

$$\begin{aligned}&\alpha(\mathfrak{y}(t))\beta(\varsigma(t))\left\|\mathfrak{R}\mathfrak{y}(t)-\mathfrak{R}\varsigma(t)\right\| \\ \leq\ & \left\|\mathfrak{R}\mathfrak{y}(t)-\mathfrak{R}\varsigma(t)\right\| = \left\|\begin{array}{c}\frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1} g(s,\mathfrak{y}(s))\,ds \\ +\frac{2t}{(2-\lambda^2)\Gamma(\eta)}\int_0^\lambda \left(\int_0^s (s-m)^{\eta-1} g(m,\mathfrak{y}(m))\,dm\right) ds \\ -\frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1} g(s,\varsigma(s))\,ds \\ -\frac{2t}{(2-\lambda^2)\Gamma(\eta)}\int_0^\lambda \left(\int_0^s (s-m)^{\eta-1} g(m,\varsigma(m))\,dm\right) ds\end{array}\right\|\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\eta)} \int\limits_0^t \|t-s\|^{\eta-1} \|g(s, \mathfrak{y}(s)) - g(s, \varsigma(s))\| \, ds \\
&+ \frac{2t}{(2-\lambda^2)\Gamma(\eta)} \int\limits_0^\lambda \left(\int\limits_0^s (s-m)^{\eta-1} \|g(m, \varsigma(m)) - g(m, \mathfrak{y}(m))\| \, dm \right) ds,
\end{aligned}
$$

which implies

$$
\begin{aligned}
& \alpha(\mathfrak{y}(t)) \beta(\varsigma(t)) \left\| \mathfrak{R}\mathfrak{y}(t) - \mathfrak{R}\varsigma(t) \right\| \\
\leq & \begin{pmatrix} \frac{1}{\Gamma(\eta)} \int\limits_0^t \|t-s\|^{\eta-1} \, ds \\ + \frac{2t}{(2-\lambda^2)\Gamma(\eta)} \int\limits_0^\lambda \left(\int\limits_0^s \|s-m\|^{\eta-1} \, dm \right) ds \end{pmatrix} \vartheta \varpi \|\mathfrak{y} - \varsigma\|_\infty \\
= & \left(\frac{1}{\Gamma(\eta+1)} + \frac{2\lambda^{\eta+1}\Gamma(\eta)}{(2-\lambda^2)\Gamma(\eta+1)} \right) \vartheta \|\mathfrak{y}(s) - \varsigma(s)\|_\infty \\
= & \vartheta \varpi \|\mathfrak{y} - \varsigma\|_\infty \\
\leq & \vartheta \varpi \mathcal{M}(\mathfrak{y}, \varsigma),
\end{aligned}
$$

where

$$
\mathcal{M}(\mathfrak{y}, \varsigma) = \max \left\{ \begin{array}{c} \|\mathfrak{y} - \varsigma\|_\infty, \frac{\|\mathfrak{y} - \mathfrak{R}\mathfrak{y}\|_\infty \|\varsigma - \mathfrak{R}\varsigma\|_\infty}{1 + \|\mathfrak{y} - \varsigma\|_\infty}, \\ \frac{\|\mathfrak{y} - \mathfrak{R}\varsigma\|_\infty \|\varsigma - \mathfrak{R}\mathfrak{y}\|_\infty}{1 + \|\mathfrak{y} - \varsigma\|_\infty} \end{array} \right\}.
$$

Taking $\varrho = \vartheta\varpi < 1$. Define $\alpha, \beta : \mathcal{P} \to \mathbb{R}_0^+$ by

$$
\alpha(\mathfrak{y}) = \begin{cases} 1, & \text{if } \mathfrak{y}(t) > 0 \\ 0, & \text{otherwise.} \end{cases}
$$

and

$$
\beta(\varsigma) = \begin{cases} 1, & \text{if } \varsigma(t) > 0 \\ 0, & \text{otherwise.} \end{cases}
$$

Now, if $\psi(t) = \varrho t$, where $\varrho \in (0, 1)$, then $\psi \in \Psi$. Hence from above, we have

$$
\alpha(\mathfrak{y})\beta(\varsigma)d(\mathfrak{R}\mathfrak{y}, \mathfrak{R}\varsigma) \leq \psi(\mathcal{M}(\mathfrak{y}, \varsigma)),
$$

where

$$
\mathcal{M}(\mathfrak{y}, \varsigma) = \max \left\{ \begin{array}{c} \mathfrak{u}(\mathfrak{y}, \varsigma), \frac{\mathfrak{u}(\mathfrak{y}, \mathfrak{R}\mathfrak{y})\mathfrak{u}(\varsigma, \mathfrak{R}\varsigma)}{1 + \mathfrak{u}(\mathfrak{y}, \varsigma)}, \\ \frac{\mathfrak{u}(\mathfrak{y}, \mathfrak{R}\varsigma)\mathfrak{u}(\varsigma, \mathfrak{R}\mathfrak{y})}{1 + \mathfrak{u}(\mathfrak{y}, \varsigma)} \end{array} \right\},
$$

for all $\mathfrak{y}, \varsigma \in \mathcal{P}$. Hence, all the condition of Corollary 4 are satisfied and thus Eq (4.1) has a unique solution. □

5. Conclusions

In the present research work, we introduced rational (α, β, ψ)-contraction in the framework of $\mathcal{OF}$-MS and established some fixed point results. We derived the main result of Faraji et al. [19] as consequences of our leading result. We solved the existence and uniqueness of fractional differential equation by our leading theorem. Our results are new and significantly contribute to the existing literature in the fixed point theory.

In this direction, obtaining the fixed points of multi-valued mappings and fuzzy mappings for rational (α, β, ψ)-contraction in the framework of $\mathcal{OF}$-MS can be fascinating results for the researchers. Moreover, one can solve fractional differential inclusion problems as applications of these proposed outlines.

Nomenclature

The following abbreviations are used in this manuscript:

$(\mathcal{P}, \mathfrak{u}, \perp)$	Orthogonal $\mathcal{F}$-metric space
$\mathfrak{R}$	Self mapping
$\{\mathfrak{y}_n\}$	Orthogonal sequence
${}^{C}D^{\eta}$	Caputo fractional derivative of order η
$(\mathcal{P}, \|\|\mathfrak{y}\|\|_{\infty})$	Banach space.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of Interest

The authors declare that they have no conflict of interest.

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