Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(11): 27328–27346.
DOI: 10.3934/math.20231398
Received: 14 July 2023
Revised: 06 September 2023
Accepted: 08 September 2023
Published: 26 September 2023

Research article

On a class of weakly Landsberg metrics composed by a Riemannian metric and a conformal 1-form

Fangmin Dong and Benling Li*

School of Mathematics and Statistics, Ningbo University, Ningbo, Zhejiang Province 315211, China

* **Correspondence:** Email: libenling@nbu.edu.cn; Tel: +8615825558107.

Abstract: Without the quadratic restriction, there are many non-Riemannian geometric quantities in Finsler geometry. Among these geometric quantities, Berwald curvature, Landsberg curvature and mean Landsberg curvature are related directly to the famous "unicorn problem" in Finsler geometry. In this paper, Finsler metrics with vanishing weakly Landsberg curvature (i.e., weakly Landsberg metrics) are studied. For the general (α,β)-metrics, which are composed by a Riemannian metric α and a 1-form β, we found that if the expression of the metric function doesn't depend on the dimension n, then any weakly Landsberg (α,β)-metric with a conformal 1-form must be a Landsberg metric. In the two-dimensional case, the weakly Landsberg case is equivalent to the Landsberg case. Further, we classified two-dimensional Berwald general (α,β)-metrics with a conformal 1-form.

Keywords: Finsler metric; weakly Landsberg metric; Landsberg metric; Berwald metric; general (α,β)-metric
Mathematics Subject Classification: 53B40, 53C60

1. Introduction

Without the quadratic restriction, Finsler geometry has many non-Riemannian properties compared to Riemannian geometry. There are many non-Riemannian quantities in Finsler geometry, such as the Berwald curvature, the Landsberg curvature, the mean Landsberg curvature and more. A Finsler metric is called a *Berwald metric* if its Berwald curvature vanishes, it is called a *Landsberg metric* if its Landsberg curvature vanishes, or it is called a *weakly Landsberg metric* if its mean Landsberg curvature vanishes. It is a fact that all Berwald metrics are Landsberg metrics, and all Landsberg metrics are weakly Landsberg metrics. M. Matsumoto showed a list of reduction theorems of certain Landsberg metrics to Berwald metrics [10]. Afterward, a natural problem arose in Finsler geometry:

The question of whether or not any Landsberg metric is a Berwald metric.

This problem, presented by D. Bao, is called the "unicorn problem" because it is one of the most important unsolved problems in Finsler geometry. There are some structures of Berwald metrics studied by Z. I. Szabó [18–20]. While some geometers got other results [2,5,7,11,16,17,21,24,26]. However, the "unicorn problem" is still open. Meanwhile, there is another related problem in Finsler geometry:

The question of whether or not any weakly Landsberg metric is a Landsberg metric.

Clearly, in the two-dimensional case, any weakly Landsberg metric is a Landsberg metric. Some rigidity problems are related to weakly Landsberg metrics [3, 6, 15]. T. Aikou studied the weakly Landsberg spaces, which he referred to as the generalized Landsberg [1]. B. Li and Z. Shen studied weakly Landsberg (α,β)-metrics and found that there exists weakly Landsberg metrics, which are not Landsberg metrics [9]. Thus, it is natural to study more general metrics than ever before. In this paper, we mainly study the general (α,β)-metric that was first introduced by C. Yu and H. Zhu [22]. A *general* (α,β)*-metric* is a Finsler metric expressed in the following form,

$$F = \alpha\phi(b^2, s), \quad s = \frac{\beta}{\alpha},$$

where $\phi(b^2, s)$ is a C^∞ function, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form, $b := \|\beta_x\|_\alpha$. Specially, F is the *spherically symmetric metric* when $\alpha = |y|$ is the Euclidean metric and $\beta = \langle x, y\rangle$ is the Euclidean inner production. It is easy to see that β is closed and conformal to α for all spherically symmetric metrics, i.e.,

$$b_{i|j} = ca_{ij}, \tag{1.1}$$

where $c = c(x)$ is a C^∞ scalar function and the subscript "|" denotes the covariant derivative with respect to the Riemannian metric α. When $c = 0$, then β is said to be parallel to α. Some papers studied general (α,β)-metrics under this condition [8,22,23,25] and many interesting properties were found. Thus, it is meaningful to study the problems when β is closed and conformal to α.

In this paper, we first study weakly Landsberg (α,β)-metrics satisfying (1.1). We make an assumption that the expression of $\phi = \phi(b^2, s)$ does not depend on n, and this assumption is used to prove the following theorem. After getting the formula of the mean Landsberg curvature of these metrics, the following theorem is obtained.

Theorem 1.1. *Let $F = \alpha\phi(b^2, s)$ be a general (α,β)-metric on an n-dimensional manifold ($n \geq 2$). Suppose that the expression of the function ϕ does not depend on the dimension n and β is closed and conformal to α satisfying (1.1) with $c \neq 0$, then F is a weakly Landsberg metric if, and only if, it is a Landsberg metric.*

In the above theorem, the expression of ϕ does not depend on n, which means that ϕ is only a function of b^2 and s. For instance, it does depend on n if $\phi = n + s$, while it does not if $\phi = 1 + s$. Of course, F is still related to the dimension n when we express $F = \alpha\phi$ in α, β and b^2. F will change with the change of dimension, but the variable n will not appear in the expression of $\phi = \phi(b^2, s)$. Then the expression of X and H in (2.13) are just functions of b^2 and s. The dimension n will not be a variable in their expressions. Then the expression of Berwald curvature and Landsberg curvature do not depend on n in this case. For example, all of the spherically symmetric Finsler metrics satisfy this condition. This condition is used in getting (3.23) and (3.24) by (3.22) in proving Theorem 1.1 in Section 3.

When ϕ depends on n, we guess Theorem 1.1 is also correct, although we haven't found a workable way to prove it. Here $c = c(x) \neq 0$ is natural because F must be trivial Berwaldian if $c = 0$ (i.e., β is parallel to α).

S. Zhou and B. Li proved that the almost regular Landsberg general (α,β)-metric must be Berwaldian when the dimension $n \geq 3$ [25]. However, the two-dimensional case has not been solved. In Theorem 1.2, we give the equivalent conditions of Berwald metric in the two-dimensional case, then the expression of ϕ can also be given.

Theorem 1.2. *Let $F = \alpha\phi(b^2, s)$ be an almost regular general (α,β)-metric on a 2-dimensional manifold. Suppose that β is closed and conformal to α satisfying (1.1) with $c \neq 0$. Thus, F is a Berwald metric if, and only if, ϕ is given by*

$$\phi = c_7 e^{\int_0^s A(b^2,t)dt}, \tag{1.2}$$

where

$$A(b^2, t) = \frac{b^2 t c_1 \sqrt{b^2 - t^2} + 2c_2(b^2 + t^2) + b^2 c_3(2b^2 - t^2) + t^2 c_4 + 2b^2 t c_6 \sqrt{b^2 - t^2}}{\sqrt{b^2 - t^2}\Big\{b^2 t^2 c_1 - (2tc_2 - b^2 t c_3 + tc_4)\sqrt{b^2 - t^2} - 2b^2 c_6(b^2 - t^2) + b^2\Big\}} \tag{1.3}$$

and $c_i = c_i(b^2), (i = 1, 2, ..., 7)$ are C^∞ functions of b^2 satisfying the following condition,

$$\begin{aligned}
&\int_0^s \frac{\partial A(b^2, t)}{\partial b^2}\,\mathrm{d}t \\
&= \frac{1}{2b^4 s c_7}\Big\{\Big[2s(3b^2 - 4s^2)c_2\sqrt{b^2 - s^2} + 2sc_4(b^2 - s^2)^{3/2} + b^4 s^2 c_5(b^2 - s^2) \\
&+ 2b^4 c_6(b^2 - s^2) - b^4\Big]c_7 A(b^2, s) + \Big[2b^4 s c_1 + 2(b^2 + 4s^2)c_2\sqrt{b^2 - s^2} \\
&+ 2b^4 c_3\sqrt{b^2 - s^2} + 2s^2 c_4\sqrt{b^2 - s^2} + b^4 s^3 c_5 + 2b^4 s c_6\Big]c_7 - 2b^4 s\frac{\mathrm{d}(c_7)}{\mathrm{d}(b^2)}\Big\}.
\end{aligned} \tag{1.4}$$

Two explicit examples of two-dimensional Berwald metrics are given in Section 4 (Examples 4.1 and 4.2).

2. Preliminaries

In this section, some definitions and lemmas needed are introduced. There are most of the notations [4, 14]. A *Finsler metric* on a manifold M is a function $F : TM \to [0, +\infty)$, which satisfies

(i) F is C^∞ on $TM\backslash\{0\}$;
(ii) $F(x, \lambda y) = \lambda F(x, y)$, for any $\lambda > 0$;
(iii) $g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive definitely, for any $y \neq 0$.

Specially, F is called a *Riemannian metric* if $g_{ij} = g_{ij}(x)$. F is called a Minkowskian metric if $g_{ij} = g_{ij}(y)$. A class of computable Finsler metrics called (α,β)-metric is defined by

$$F = \alpha\phi(s), \quad s = \frac{\beta}{\alpha}, \tag{2.1}$$

where ϕ is a C^∞ function, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form. When $\phi = 1$, $F = \alpha$ is a Riemannian metric. When $\phi = 1 + s$, $F = \alpha + \beta$ is a Randers metric [13]. Further, when $\phi = \phi(b^2, s)$, another class of Finsler metric called general (α,β)-metric can be given as follows

$$F = \alpha\phi(b^2, s), \ s = \frac{\beta}{\alpha}. \tag{2.2}$$

When $\alpha = |y|$ is the Euclidean metric and $\beta = \langle x, y\rangle$ is the Euclidean inner production of x and y, $F = |y|\phi(|x|, \frac{\langle x,y\rangle}{|y|})$ is called a *spherically symmetric metric*. There are some results on these metrics [12,23,24]. In this case, (1.1) is always satisfied and $c = 1$.

By a direct computation, the g_{ij} of F in (2.2) can be obtained.

$$g_{ij} = \rho a_{ij} + \bar{\rho} b_i b_j + \tilde{\rho}\left(b_i\alpha_{yj} + b_j\alpha_{yi}\right) - s\tilde{\rho}\alpha_{yi}\alpha_{yj}, \tag{2.3}$$

where

$$\rho = \phi\ (\phi - s\phi_2)\,,\ \bar{\rho} = \phi\,\phi_{22} + (\phi_2)^2,\ \tilde{\rho} = (\phi - s\phi_2)\phi_2 - s\phi\,\phi_{22}. \tag{2.4}$$

In this paper, the subscript "1" means partial derivative with respect to b^2, the subscript "2" means partial derivative with respect to s, such as $\phi_1 = \frac{\partial\phi}{\partial(b^2)}$, $\phi_2 = \frac{\partial\phi}{\partial s}$.

The positivity of general (α,β)-metric can be ensured by the following lemma.

Lemma 2.1. *Let M be an n-dimensional manifold [22]. $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ is a Finsler metric on M for any Riemannian metric α and 1-form β with $\|\beta\|_\alpha < b_0$ if and only if $\phi = \phi(b^2, s)$ is a positive C^∞ function satisfying*

$$\phi - s\phi_2 > 0,\ \ \phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0, \tag{2.5}$$

when $n \geq 3$ or

$$\phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0,$$

when $n = 2$, where s and b are arbitrary numbers with $|s| \leq b < b_0$.

In order to obtain geodesic coefficients and mean Landsberg curvature of F, the first step is to compute the inverse of (g_{ij}).

$$g^{ij} = \rho^{-1}\{a^{ij} + \eta b^i b^j + \bar{\eta}\alpha^{-1}(b^i y^j + b^j y^i) + \tilde{\eta}\alpha^{-2} y^i y^j\}, \tag{2.6}$$

where $a^{ij} = (a_{ij})^{-1}$, $b^i = a^{ij} b_j$,

$$\eta = -\frac{\phi_{22}}{\phi - s\phi_2 + (b^2 - s^2)\phi_{22}}, \quad \bar{\eta} = -\frac{(\phi - s\phi_2)\phi_2 - s\phi\phi_{22}}{\phi[\phi - s\phi_2 + (b^2 - s^2)\phi_{22}]},$$

$$\tilde{\eta} = \frac{[s\phi + (b^2 - s^2)\phi_2][(\phi - s\phi_2)\phi_2 - s\phi\phi_{22}]}{\phi^2[\phi - s\phi_2 + (b^2 - s^2)\phi_{22}]}.$$

In this paper, the subscript "|" denotes the covariant derivative with respect to the Riemannian metric α, such as $b_{i|j}$. In order to simplify the computation, we use the following expression

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}),\ s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}),\ r_{00} = r_{ij}y^i y^j,\ s^i{}_0 = a^{ij}s_{jk}y^k, \tag{2.7}$$

$$r_i = b^j r_{ji},\ s_i = b^j s_{ji},\ r_0 = r_i y^i,\ s_0 = s_i y^i,\ r^i = a^{ij} r_j,\ s^i = a^{ij} s_j,\ r = b^i r_i. \tag{2.8}$$

The *spray* $G = y^i\frac{\partial}{\partial x^i} - 2G^i\frac{\partial}{\partial y^i}$ of Finsler metric F is a vector field on TM, where $G^i = G^i(x, y)$ are called *geodesic coefficients (spray coefficients)* and defined by

$$G^i = \frac{1}{4}g^{il}\{[F^2]_{x^m y^l}y^m - [F^2]_{x^l}\}, \tag{2.9}$$

where $(g^{ij}) = (g_{ij})^{-1}$.

The following lemma gives G^i of general (α,β)-metrics.

Lemma 2.2. *The geodesic coefficients G^i of a general (α,β)-metric $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ [22] are given as*

$$G^i = G^i_\alpha + Py^i + Q^i, \tag{2.10}$$

where

$$P = \Big\{\Theta(-2\alpha\Lambda s_0 + r_{00} + 2\alpha^2 Rr) + \alpha\Omega(r_0 + s_0)\Big\}\alpha^{-1},$$

$$Q^i = \alpha\Lambda s^i{}_0 - \alpha^2 R(r^i + s^i) + \Big\{\Psi(-2\alpha\Lambda s_0 + r_{00} + 2\alpha^2 Rr) + \alpha\Pi(r_0 + s_0)\Big\}b^i,$$

$$\Lambda = \frac{\phi_2}{\phi - s\phi_2}, \quad \Theta = \frac{(\phi - s\phi_2)\phi_2 - s\phi\phi_{22}}{2\phi[\phi - s\phi_2 + (b^2 - s^2)\phi_{22}]}, \quad \Psi = \frac{\phi_{22}}{2[\phi - s\phi_2 + (b^2 - s^2)\phi_{22}]},$$

$$R = \frac{\phi_1}{\phi - s\phi_2}, \quad \Pi = \frac{(\phi - s\phi_2)\phi_{12} - s\phi_1\phi_{22}}{(\phi - s\phi_2)[\phi - s\phi_2 + (b^2 - s^2)\phi_{22}]}, \quad \Omega = \frac{2\phi_1}{\phi} - \frac{s\phi + (b^2 - s^2)\phi_2}{\phi}\Pi.$$

Here when (1.1) is satisfied, the expression of G^i can be simplified. By (1.1),

$$r_{ij} = ca_{ij}, \quad s_{ij} = 0, \quad r_{00} = c\alpha^2, \quad s^i_0 = 0, \tag{2.11}$$

$$r_i = cb_i, \quad s_i = 0, \quad r_0 = c\beta, \quad s_0 = 0, \quad r^i = cb^i, \quad s^i = 0, \quad r = cb^2. \tag{2.12}$$

Then the geodesic coefficients G^i can be rewritten as

$$G^i = G^i_\alpha + c\alpha Xy^i + c\alpha^2 Hb^i, \tag{2.13}$$

where

$$X = \frac{\phi_2 + 2s\phi_1}{2\phi} - H\frac{s\phi + (b^2 - s^2)\phi_2}{\phi} \tag{2.14}$$

and

$$H = \frac{\phi_{22} - 2(\phi_1 - s\phi_{12})}{2[\phi - s\phi_2 + (b^2 - s^2)\phi_{22}]}. \tag{2.15}$$

Non-Riemannian quantities play an important role in Finsler geometry, such as the Berwald curvature, the mean Berwald curvature, the Landsberg curvature and the mean Landsberg curvature. The *Berwald curvature* of F is a tensor defined by

$$B : TM \otimes TM \otimes TM \longrightarrow TM,$$

where

$$B = B^i_{jkl}\frac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k \otimes dx^l \tag{2.16}$$

and

$$B^i_{jkl} = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}. \tag{2.17}$$

A Finsler metric is called a *Berwald metric* if and only if $B^i_{jkl} = 0$. Obviously, G^i are quadratic in y if and only if F is a Berwald metric. By the definition of Berwald curvature, S. Zhou and B. Li obtained the expression of B^i_{jkl} [25]. We found there were some typing errors of B^i_{jkl}, and the expression of B^i_{jkl} under condition (1.1) can be given by

$$\begin{aligned} B^i_{jkl} = &\frac{c}{\alpha^4}\Big\{(H_2 - sH_{22})[h_{jk}h_l + h_{jl}h_k + h_{kl}h_j] + H_{222}h_jh_kh_l\Big\}b^i \\ &- \frac{c}{\alpha^5}\Big\{(X - sX_2)[h_{jk}y_l + h_{jl}y_k + h_{kl}y_j] + X_{22}[h_jh_ky_l + h_jh_ly_k + h_kh_ly_j] \\ &+ sX_{22}[h_{jk}h_l + h_{jl}h_k + h_{kl}h_j] - X_{222}h_jh_kh_l\Big\}y^i + \frac{c}{\alpha^3}[(X - sX_2)h_{kl} + X_{22}h_kh_l]\delta^i_j \\ &+ \frac{c}{\alpha^3}[(X - sX_2)h_{jl} + X_{22}h_jh_l]\delta^i_k + \frac{c}{\alpha^3}[(X - sX_2)h_{jk} + X_{22}h_jh_k]\delta^i_l, \end{aligned} \tag{2.18}$$

where

$$h_j = \alpha b_j - sy_j, \quad h_{jk} = \alpha^2 a_{jk} - y_jy_k. \tag{2.19}$$

The *mean Berwald curvature* of F is a tensor which can be defined by

$$E : TM \otimes TM \longrightarrow R,$$

where

$$E_{jk} = \frac{1}{2}B^m_{mjk} = \frac{1}{2}\frac{\partial^3 G^m}{\partial y^m \partial y^j \partial y^k}. \tag{2.20}$$

A Finsler metric is called a *weakly Berwald metric* if and only if $B^m_{mjk} = 0$.

The *Landsberg curvature* of F is a tensor defined by

$$L : TM \otimes TM \otimes TM \longrightarrow TM,$$

where

$$L = L_{jkl}dx^j \otimes dx^k \otimes dx^l \tag{2.21}$$

and

$$L_{jkl} = -\frac{1}{2}FF_{y^i}B^i_{jkl}. \tag{2.22}$$

A Finsler metric is called a *Landsberg metric* if and only if $L_{jkl} = 0$.

Based on (2.18), the expression of L_{jkl} is given [25] as follows

$$L_{jkl} = -\frac{\rho}{6\alpha^5}\Big\{h_jh_kC_l + h_jh_lC_k + h_kh_lC_j + 3E_jh_{kl} + 3E_kh_{jl} + 3E_lh_{jk}\Big\}, \tag{2.23}$$

where

$$C_j = c\alpha^2\Big\{[b^2\Lambda + s]H_{222} + [1 + s\Lambda]X_{222} + 3\Lambda X_{22}\Big\}h_j, \tag{2.24}$$

$$E_j = c\alpha^2\Big\{[b^2\Lambda + s](H_2 - sH_{22}) - [1 + s\Lambda]sX_{22} + \Lambda[X - sX_2]\Big\}h_j. \tag{2.25}$$

The following lemma gives the equivalent conditions of Berwald metric in two-dimensional case.

Lemma 2.3. *Let F be a two-dimensional Finsler metric. F is Berwaldian if, and only if, it is Landsbergian and weakly Berwaldian [14].*

The definitions of mean Landsberg curvature and weakly Landsberg metric are given in next section.

3. Weakly Landsberg general (α,β)-metric

In this section, the expression of J_j is given and Theorem 1.1 is proved. The *mean Landsberg curvature* is a tensor defined by

$$J : TM \longrightarrow R,$$

where

$$J = J_j dx^j \tag{3.1}$$

and

$$J_j = g^{kl} L_{jkl}. \tag{3.2}$$

A Finsler metric is called a *weakly Landsberg metric* if and only if $J_j = 0$. Here we give the formula of J_j directly by (2.6), (2.23) and (3.2).

Lemma 3.1. *Let $F = \alpha\phi(b^2, s)$ be a general (α,β)-metric on an n-dimensional manifold, then the mean Landsberg curvature $J = J_j dx^j$ is given by*

$$J_j = -\frac{1}{2}\Big\{\frac{\Big(b^2 - s^2\Big)\Big[(b^2 - s^2)\eta + 1\Big]C + \Big[3(b^2 - s^2)\eta + n + 1\Big]E}{\alpha^3}\Big\}h_j, \tag{3.3}$$

where

$$C = c\alpha^2\Big\{[b^2\Lambda + s]H_{222} + [1 + s\Lambda]X_{222} + 3\Lambda X_{22}\Big\}, \tag{3.4}$$

$$E = c\alpha^2\Big\{[b^2\Lambda + s](H_2 - sH_{22}) - [1 + s\Lambda]sX_{22} + \Lambda[X - sX_2]\Big\}. \tag{3.5}$$

Proof. By direct contractions, we get

$$h_{kl}y^k = 0, \quad h_k y^k = 0, \quad E_k y^k = 0, \quad C_k y^k = 0, \tag{3.6}$$

$$C_k b^k = \alpha(b^2 - s^2)C, \quad E_k b^k = \alpha(b^2 - s^2)E, \quad h_{jk}b^k = \alpha h_j, \quad h_k b^k = \alpha(b^2 - s^2), \tag{3.7}$$

$$a^{kl}h_k h_l = \alpha^2(b^2 - s^2), \quad a^{kl}h_{kl} = \alpha^2(n-1), \quad a^{kl}h_{jk}h_l = \alpha^2 h_j. \tag{3.8}$$

Substituting (2.6) and (2.23) into (3.2) yields

$$\begin{aligned} J_j =& \frac{-1}{6\alpha^5}\Big\{a^{kl} + \eta b^k b^l + \frac{\bar{\eta}}{\alpha}(b^k y^l + b^l y^k) + \frac{\tilde{\eta}}{\alpha^2}y^k y^l\Big\}\Big\{h_j h_k C_l + h_j h_l C_k + h_k h_l C_j \\ &+ 3E_j h_{kl} + 3E_k h_{jl} + 3E_l h_{jk}\Big\} \\ =& \frac{-1}{6\alpha^5}\Big\{a^{kl} + \eta b^k b^l\Big\}\Big\{h_j h_k C_l + h_j h_l C_k + h_k h_l C_j + 3E_j h_{kl} + 3E_k h_{jl} + 3E_l h_{jk}\Big\} \\ =& \frac{-1}{6\alpha^5}\Big\{3a^{kl}h_k h_l h_j C + 3\eta b^k b^l h_j h_k h_l C + 3a^{kl}[h_j h_{kl} + h_k h_{jl} + h_l h_{jk}]E \\ &+ 3\eta b^k b^l [h_j h_{kl} + h_k h_{jl} + h_l h_{jk}]E\Big\} \\ =& \frac{-1}{2\alpha^5}\Big\{[a^{kl}h_k h_l + \eta b^k b^l h_k h_l]h_j C + [a^{kl} + \eta b^k b^l][h_j h_{kl} + h_k h_{jl} + h_l h_{jk}]E\Big\} \\ =& \frac{-1}{2\alpha^3}\Big\{(b^2 - s^2)[(b^2 - s^2)\eta + 1]C + [3(b^2 - s^2)\eta + n + 1]E\Big\}h_j. \end{aligned} \tag{3.9}$$

The second equality in the above equation is given by (3.6), and the other equalities are given by (3.7) and (3.8).

By the above results, (3.3) can be given.

The equivalent conditions of Landsberg metric are given [25]; in this section, the following conditions will be used in our proof of Theorem 1.1.

Lemma 3.2. *Let $\phi = \phi(b^2, s)$ be a positive C^∞ function [25], then ϕ is a Landsberg metric if, and only if, the function $\phi = \phi(b^2, s)$ satisfies*

$$X - sX_2 = \frac{c_1}{\sqrt{b^2 - s^2}}, \tag{3.10}$$

$$H_2 - sH_{22} = -\frac{c_1}{(b^2 - s^2)^{\frac{3}{2}}}, \tag{3.11}$$

where $c_1 = c_1(b^2)$ is a C^∞ function of b^2.

For the convenience of calculation, we must first prove the following lemma, which will be used in Theorem 1.1.

Lemma 3.3. *Let $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ be a non-Riemannian general (α,β)-metric on an n-dimensional manifold $n \geq 2$, then ϕ is a C^∞ function satisfies*

$$s\phi + (b^2 - s^2)\phi_2 \neq 0, \tag{3.12}$$

$$(\phi - s\phi_2)\phi_2 - s\phi\,\phi_{22} \neq 0. \tag{3.13}$$

Proof. If $s\phi + (b^2 - s^2)\phi_2 = 0$. Integrating the above equation with respect to s yields

$$\phi = \bar{c}_1 \sqrt{b^2 - s^2},$$

where $\bar{c}_1 = \bar{c}_1(b^2)$ is C^∞ function of b^2. Unquestionably, $F = \alpha\phi$ is a Riemannian metric. If $(\phi - s\phi_2)\phi_2 - s\phi\,\phi_{22} = 0$, then integrating the above equation with respect to s yields

$$\phi = \sqrt{\bar{c}_2 s^2 + 2\bar{c}_3},$$

where $\bar{c}_2 = \bar{c}_2(b^2), \bar{c}_3 = \bar{c}_3(b^2)$ are C^∞ functions of b^2. Since $F = \alpha\phi$ is a Riemannian metric, this is a contradiction to the conditions.

Now by Lemmas 3.1–3.3, Theorem 1.1 can be proved.

Proof of Theorem 1.1.

"Necessity" Let

$$\bar{X} = X - sX_2, \quad \bar{H} = H_2 - sH_{22}. \tag{3.14}$$

Differentiating the first equation of (3.14) with respect to s yields

$$\bar{X}_2 = -sX_{22}. \tag{3.15}$$

Differentiating (3.15) with respect to s yields

$$\bar{X}_{22} = -sX_{222} - X_{22}. \tag{3.16}$$

Differentiating the second equation of (3.14) with respect to s yields

$$\bar{H}_2 = -sH_{222}. \tag{3.17}$$

By rewriting (3.14)–(3.17), we get

$$sX_{222} = -\bar{X}_{22} - X_{22}, \quad sX_{22} = -\bar{X}_2, \quad sX_2 = X - \bar{X}, \tag{3.18}$$

$$sH_{222} = -\bar{H}_2, \quad sH_{22} = H_2 - \bar{H}. \tag{3.19}$$

To get the expression of J_j by substituting the above identities into (3.4) and by (3.4) $\times\, s^2$, we get

$$s^2C = -c\alpha^2[(2s\Lambda - 1)\bar{X}_2 + s(1 + s\Lambda)\bar{X}_{22} + s(b^2\Lambda + s)\bar{H}_2]. \tag{3.20}$$

The expression of the right hand of Eq (3.4) is a fraction by substituting identities (3.18) and (3.19) into it, where s is the denominator. According to the rationality of the equation, the denominator is $s \neq 0$, but by definition, $s = 0$ is reasonable. In fact, the s in the denominator can be eliminated by the numerator. In order to avoid confusion, here we consider (3.4) $\times s^2$, and the result is not affected. Substituting (3.18) and (3.19) into (3.5) yields

$$E = c\alpha^2[\Lambda\bar{X} + (1 + s\Lambda)\bar{X}_2 + (b^2\Lambda + s)\bar{H}]. \tag{3.21}$$

In order to give the proof more conveniently, (3.3) can be rewritten as

$$J_j = -\frac{1}{2}\frac{Eh_j}{\alpha^3}n - \frac{1}{2}\Big\{\frac{\left(b^2 - s^2\right)\left[(b^2 - s^2)\eta + 1\right]C + \left[3(b^2 - s^2)\eta + 1\right]E}{\alpha^3}\Big\}h_j. \tag{3.22}$$

Based on the assumption that ϕ does not depend on the dimension n, there is no n in the expressions of C, E and η. By (3.22), $J_j = 0$ is equivalent to

$$-\frac{1}{2}\frac{Eh_j}{\alpha^3} = 0 \tag{3.23}$$

and

$$-\frac{1}{2}\Big\{\frac{\left(b^2 - s^2\right)\left[(b^2 - s^2)\eta + 1\right]C + \left[3(b^2 - s^2)\eta + 1\right]E}{\alpha^3}\Big\}h_j = 0. \tag{3.24}$$

It is a fact that α is a Riemannian metric, so $\alpha \neq 0$. By (2.19), if $h_j = 0$, α^2 can be divided exactly by βy_j, it is impossible, thus $h_j \neq 0$. By (3.21), (3.23) is equivalent to

$$c[\Lambda\bar{X} + (1 + s\Lambda)\bar{X}_2 + (b^2\Lambda + s)\bar{H}] = 0. \tag{3.25}$$

If $c = 0$, by (2.23) we obtain $L_{jkl} = 0$ and Theorem 1.1 is proved.

If $c \neq 0$. By Lemma 3.3, we get

$$b^2\Lambda + s = \frac{s\phi + (b^2 - s^2)\phi_2}{\phi - s\phi_2} \neq 0. \tag{3.26}$$

Then $\bar{H}$ can be solved from (3.25),

$$\bar{H} = -\frac{\Lambda\bar{X} + (1 + s\Lambda)\bar{X}_2}{b^2\Lambda + s}. \tag{3.27}$$

Differentiating (3.27) with respect to s yields

$$\bar{H}_2 = -\frac{\Lambda_2 \bar{X} + (s\Lambda_2 + 2\Lambda)\bar{X}_2 + (1 + s\Lambda)\bar{X}_{22}}{b^2\Lambda + s} + \frac{\left(b^2\Lambda_2 + 1\right)\left[\Lambda\bar{X} + (1 + s\Lambda)\bar{X}_2\right]}{(b^2\Lambda + s)^2}. \tag{3.28}$$

By the same reason before (3.25), (3.24) is equivalent to

$$\left(b^2 - s^2\right)\left[(b^2 - s^2)\eta + 1\right]C + \left[3(b^2 - s^2)\eta + 1\right]E = 0. \tag{3.29}$$

Plugging (3.20) and (3.21) into (3.29)$\times s^2$ yields

$$\begin{aligned} &c\alpha^2\Big\{[3(b^2 - s^2)\eta + 1]s^2\Lambda\bar{X} - \left(b^2 - s^2\right)[(b^2 - s^2)\eta + 1](1 + s\Lambda)s\bar{X}_{22} \\ &- [\left(b^2 - s^2\right)\left(2\Lambda b^2 s - 5\Lambda s^3 - b^2 - 2s^2\right)\eta + 2\Lambda b^2 s - 3\Lambda s^3 - b^2]\bar{X}_2 \\ &+ [3(b^2 - s^2)\eta + 1]\left(b^2\Lambda + s\right)s^2\bar{H} - \left(b^2 - s^2\right)[(b^2 - s^2)\eta + 1]\left(b^2\Lambda + s\right)s\bar{H}_2\Big\} = 0. \end{aligned} \tag{3.30}$$

Plugging (3.27) and (3.28) into (3.30) yields

$$-\frac{c\alpha^2(\Lambda - s\Lambda_2)\left(b^2 - s^2\right)\left[(b^2 - s^2)\eta + 1\right]\left[s\bar{X} - (b^2 - s^2)\bar{X}_2\right]}{b^2\Lambda + s} = 0. \tag{3.31}$$

By Lemma 2.1,

$$(b^2 - s^2)\eta + 1 = \frac{\phi - s\phi_2}{\phi - s\phi_2 + (b^2 - s^2)\phi_{22}} \neq 0. \tag{3.32}$$

By Lemma 3.3,

$$\Lambda - s\Lambda_2 = \frac{(\phi - s\phi_2)\phi_2 - s\phi\phi_{22}}{(\phi - s\phi_2)^2} \neq 0. \tag{3.33}$$

Meanwhile, by the definition of b and s, we get

$$b^2 - s^2 \neq 0. \tag{3.34}$$

Hence, (3.31) is equivalent to

$$c[s\bar{X} - (b^2 - s^2)\bar{X}_2] = 0. \tag{3.35}$$

Because $c \neq 0$, (3.35) is equivalent to

$$s\bar{X} - (b^2 - s^2)\bar{X}_2 = 0.$$

Integrating the above equation with respect to s yields

$$\bar{X} = \frac{c_1}{\sqrt{b^2 - s^2}}, \tag{3.36}$$

where $c_1 = c_1(b^2)$ is a C^∞ function of b^2. Differentiating (3.36) with respect to s yields

$$\bar{X}_2 = \frac{sc_1}{(b^2 - s^2)^{\frac{3}{2}}}. \tag{3.37}$$

Plugging (3.36) and (3.37) into (3.27) yields

$$\bar{H} = -\frac{c_1}{(b^2 - s^2)^{\frac{3}{2}}}. \tag{3.38}$$

By Lemma 3.2, F is a Landsberg metric.

"Sufficiency" By definition, all Landsberg metrics are weakly Landsberg metrics.

As spherically symmetric metrics compose a special class in general (α,β)-metrics and satisfy (1.1). Thus, the following Corollary 3.4 can be obtained as a special case of Theorem 1.1.

Corollary 3.4. *Let $F = |y|\phi(|x|, \frac{\langle x,y\rangle}{|y|})$ be a spherically symmetric metric on an n-dimensional manifold ($n \geq 2$). Suppose n vanishes in the expression of ϕ. Then F is a weakly Landsberg metric if, and only if, it is a Landsberg metric.*

Proof. When F is a spherically symmetric metric, then

$$r_{ij} = \delta_{ij}, \quad s_{ij} = 0, \quad r_{00} = |y|^2, \quad s^i_0 = 0,$$

$$r_i = x_i, \quad s_i = 0, \quad r_0 = \langle x, y\rangle, \quad s_0 = 0, \quad r^i = x^i, \quad s^i = 0.$$

Obviously, Theorem 1.1 is satisfied in spherically symmetric case.

4. Berwald general (α,β)-metric

S. Zhou and B. Li studied the Berwald general (α,β)-metric on an n-dimensional manifold $n \geq 3$ and gave the classification by obtaining the equivalent equations of the Berwald metric when $n \geq 3$ [25]. In this section, by Lemma 2.3 and based on the equivalent conditions of the Berwald metric, we get Lemmas 4.2 and 4.3. From these, we get two equivalent equations in Proposition 4.4, then Theorem 1.2 can be proved.

The following lemma [25] is obtained, which is needed in the proof of Lemma 4.2.

Lemma 4.1. *Let $F = \alpha\phi(b^2, s)$ be a non-Riemannian general (α,β)-metric on an n-dimensional manifold $n \geq 2$ [25]. Suppose $I = I(b^2, s)$ and $N = N(b^2, s)$ are arbitrary C^∞ functions, then the following facts hold:*

(i) $(n \geq 3)$ $h_j h_k h_l I + h_{jk} h_l N + h_{jl} h_k N + h_{kl} h_j N = 0$ *if and only if* $I = 0$ *and* $N = 0$*;*

(ii) $(n = 2)$ $h_j h_k h_l I + h_{jk} h_l N + h_{jl} h_k N + h_{kl} h_j N = 0$ *if and only if* $(b^2 - s^2)I + 3N = 0$.

Based on Lemma 4.1, in order to prove Proposition 4.4, the Landsberg curvature of the general (α,β)-metric can be rewritten in two-dimensional case as follows.

Lemma 4.2. *Let $F = \alpha\phi(b^2, s)$ be a non-Riemannian general (α,β)-metric on a two-dimensional manifold. The Landsberg curvature L_{jkl} can be given as*

$$L_{jkl} = -\frac{\phi}{2\alpha^3}\Big\{h_j h_k h_l \bar{I} + h_{jk} h_l \bar{N} + h_{jl} h_k \bar{N} + h_{kl} h_j \bar{N}\Big\}, \tag{4.1}$$

where

$$\bar{I} = (\phi - s\phi_2)\widehat{I} + \phi_2 \widetilde{I}, \quad \bar{N} = (\phi - s\phi_2)\widehat{N} + \phi_2 \widetilde{N}.$$

Proof. Contracting (2.18) with b_i and y_i respectively yields

$$B^i_{jkl} b_i = \frac{1}{\alpha^4}\Big\{h_j h_k h_l \widetilde{I} + h_{jk} h_l \widetilde{N} + h_{jl} h_k \widetilde{N} + h_{kl} h_j \widetilde{N}\Big\} \tag{4.2}$$

and

$$B^i_{jkl} y_i = \frac{1}{\alpha^3}\Big\{h_j h_k h_l \widehat{I} + h_{jk} h_l \widehat{N} + h_{jl} h_k \widehat{N} + h_{kl} h_j \widehat{N}\Big\}, \tag{4.3}$$

where

$$\widetilde{I} = b^2 H_{222} + s X_{222} + 3X_{22}, \quad \widetilde{N} = X - sX_2 - s^2 X_{22} + b^2(H_2 - sH_{22}), \tag{4.4}$$

$$\widehat{I} = X_{222} + sH_{222}, \quad \widehat{N} = s(H_2 - sH_{22}) - sX_{22}. \tag{4.5}$$

From $F = \alpha\phi(b^2, s)$, we get

$$F_{y^i} = (\frac{\phi - s\phi_2}{\alpha}) y_i + \phi_2 b_i. \tag{4.6}$$

By (2.22),

$$L_{jkl} = -\frac{1}{2}\phi[(\phi - s\phi_2) y_i + \alpha\phi_2 b_i] B^i_{jkl} = -\frac{1}{2}\phi(\phi - s\phi_2) B^i_{jkl} y_i - \frac{1}{2}\alpha\phi\phi_2 B^i_{jkl} b_i. \tag{4.7}$$

Plugging (4.2) and (4.3) into (4.7) yields

$$L_{jkl} = -\frac{\phi}{2\alpha^3}\Big\{h_j h_k h_l \bar{I} + h_{jk} h_l \bar{N} + h_{jl} h_k \bar{N} + h_{kl} h_j \bar{N}\Big\},$$

where

$$\bar{I} = (\phi - s\phi_2)\widehat{I} + \phi_2 \widetilde{I}, \quad \bar{N} = (\phi - s\phi_2)\widehat{N} + \phi_2 \widetilde{N}.$$

By Lemma 2.3, the mean Berwald curvature is essential in our proof. In the following lemma, we present it in a two-dimensional case.

Lemma 4.3. *Let $F = \alpha\phi(b^2, s)$ be a non-Riemannian general (α,β)-metric on a two-dimensional manifold. The mean Berwald curvature B^m_{mjk} is given by*

$$B^m_{mjk} = \frac{c}{\alpha^3}\Big\{\Big[(b^2 - s^2) H_{222} + 3X_{22}\Big](b^2 - s^2) + 3(b^2 - s^2)(H_2 - sH_{22}) + 3(X - sX_2)\Big\} h_{jk}. \tag{4.8}$$

Proof. By contracting (2.18) for i and l, we get

$$\begin{aligned} B^m_{mjk} = \frac{c}{\alpha^3}\Big\{&\Big[(b^2 - s^2) H_{222} + 2(H_2 - sH_{22}) + (n+1) X_{22}\Big] h_j h_k \\ &+ [(b^2 - s^2)(H_2 - sH_{22}) + (n+1)(X - sX_2)] h_{jk}\Big\}. \end{aligned} \tag{4.9}$$

When $n = 2$, we have

$$\begin{aligned} B^m_{mjk} = \frac{c}{\alpha^3}\Big\{&\Big[(b^2 - s^2) H_{222} + 2(H_2 - sH_{22}) + 3X_{22}\Big] h_j h_k \\ &+ [(b^2 - s^2)(H_2 - sH_{22}) + 3(X - sX_2)] h_{jk}\Big\}, \end{aligned} \tag{4.10}$$

where

$$h_j = \alpha b_j - s y_j, \quad h_{jk} = \alpha^2 a_{jk} - y_j y_k.$$

By the assumption that the dimension is two, $h_i h_j$ can be expressed as follows

$$h_1h_1 = \frac{(b_1y_2 - b_2y_1)^2(y^2)^2}{\alpha^2} = (b^2 - s^2)(y^2)^2 \det a = (b^2 - s^2)h_{11},$$

$$h_2h_2 = \frac{(b_1y_2 - b_2y_1)^2(y^1)^2}{\alpha^2} = (b^2 - s^2)(y^1)^2 \det a = (b^2 - s^2)h_{22},$$

$$h_1h_2 = \frac{-(b_1y_2 - b_2y_1)^2y^1y^2}{\alpha^2} = -(b^2 - s^2)y^1y^2 \det a = (b^2 - s^2)h_{12}.$$

By the above expressions,

$$h_jh_k = (b^2 - s^2)h_{jk}. \tag{4.11}$$

Therefore, (4.10) is equivalent to

$$B^m_{mjk} = \frac{ch_{jk}}{\alpha^3}\Big\{\Big[(b^2 - s^2)H_{222} + 3X_{22}\Big](b^2 - s^2) + 3(b^2 - s^2)(H_2 - sH_{22}) + 3(X - sX_2)\Big\}. \tag{4.12}$$

By Lemmas 2.3, 4.2 and 4.3, two equivalent equations of the Berwald general (α,β)-metric can be obtained as follows.

Proposition 4.4. *Let $F = \alpha\phi(b^2, s)$ be a non-Riemannian general (α,β)-metric on a 2-dimensional manifold, then F is a Berwald metric if, and only if, it satisfies*

$$\begin{aligned}&(b^2 - s^2)\Big\{3\phi_2X_{22} + \phi X_{222} + [s\phi + (b^2 - s^2)\phi_2]H_{222}\Big\}\\&+ 3\Big\{ - s\phi X_{22} + \phi_2(X - sX_2) + [s\phi + (b^2 - s^2)\phi_2](H_2 - sH_{22})\Big\} = 0\end{aligned} \tag{4.13}$$

and

$$\Big[(b^2 - s^2)H_{222} + 3X_{22}\Big](b^2 - s^2) + 3(b^2 - s^2)(H_2 - sH_{22}) + 3(X - sX_2) = 0. \tag{4.14}$$

Proof. By Lemma 4.1 and (4.1) in Lemma 4.2 when $n = 2$, $L_{jkl} = 0$ if, and only if,

$$(b^2 - s^2)\bar{I} + 3\bar{N} = 0. \tag{4.15}$$

Which is equivalent to

$$\begin{aligned}&(b^2 - s^2)\Big\{3\phi_2X_{22} + \phi X_{222} + [s\phi + (b^2 - s^2)\phi_2]H_{222}\Big\}\\&+ 3\Big\{ - s\phi X_{22} + \phi_2(X - sX_2) + [s\phi + (b^2 - s^2)\phi_2](H_2 - sH_{22})\Big\} = 0.\end{aligned} \tag{4.16}$$

By (4.11), $b^2 - s^2 \neq 0$ and $h_j \neq 0$, we get $h_{jk} \neq 0$. By Lemma 4.3, $B^m_{mjk} = 0$ if, and only if,

$$\Big[(b^2 - s^2)H_{222} + 3X_{22}\Big](b^2 - s^2) + 3(b^2 - s^2)(H_2 - sH_{22}) + 3(X - sX_2) = 0. \tag{4.17}$$

By above results and Lemma 2.3, F is a Berwald metric if, and only if, it satisfies (4.13) and (4.14).
By solving (4.13) and (4.14), ϕ can be given.

Proof of Theorem 1.2.
By Proposition 4.4 and considering (4.13) $\times\, (b^2 - s^2)$ − (4.14) $\times\, [s\phi + (b^2 - s^2)\phi_2]$, we get

$$\phi X_{222}(b^2 - s^2)^2 - 6s\,\phi X_{22}(b^2 - s^2) - 3s\,\phi(X - sX_2) = 0. \tag{4.18}$$

By (3.18), the above equation can be rewritten as

$$s\bar{X}_{22}\left(b^2-s^2\right)^2-\left(b^2-s^2\right)\left(b^2+5\,s^2\right)\bar{X}_2+3\,s^3\bar{X}=0. \tag{4.19}$$

Integrating the above equation twice yields

$$\bar{X}=\frac{\gamma_2 s^2+\gamma_3}{(b^2-s^2)^{3/2}}, \tag{4.20}$$

where $\gamma_2=\gamma_2(b^2),\gamma_3=\gamma_3(b^2)$ are C^∞ functions of b^2. By plugging (4.20) into the first equation of (3.14) and solving X, we get

$$X=c_1 s+\frac{c_2}{\sqrt{b^2-s^2}}+c_3\sqrt{b^2-s^2}, \tag{4.21}$$

where $c_1=c_1(b^2), c_2=-\gamma_2(b^2), c_3=\frac{b^2\gamma_2(b^2)+2\gamma_3(b^2)}{b^4}$ are C^∞ functions of b^2. Differentiating (4.21) with respect to s yields

$$X_2=c_1+\frac{c_2 s}{(b^2-s^2)^{3/2}}-\frac{c_3 s}{\sqrt{b^2-s^2}}. \tag{4.22}$$

Differentiating (4.22) with respect to s yields

$$X_{22}=\frac{3c_2 s^2}{(b^2-s^2)^{5/2}}+\frac{c_2-c_3 s^2}{(b^2-s^2)^{3/2}}-\frac{c_3}{\sqrt{b^2-s^2}}. \tag{4.23}$$

Plugging (3.19), (4.21)–(4.23) into (4.14) yields

$$3s\left(b^2-s^2\right)\bar{H}-\left(b^2-s^2\right)^2\bar{H}_2+\frac{6c_2 b^2 s}{(b^2-s^2)^{3/2}}=0. \tag{4.24}$$

By solving the above equation, we get

$$\bar{H}=\frac{c_4}{(b^2-s^2)^{3/2}}+\frac{3c_2 b^2}{(b^2-s^2)^{5/2}}, \tag{4.25}$$

where $c_4=c_4(b^2)$ is a C^∞ function of b^2. By (4.25) and the second equation of (3.14), H can be solved as

$$H=-\frac{(4\,c_2+c_4)\,s^3}{b^4\sqrt{b^2-s^2}}+\frac{(3c_2+c_4)\,s}{b^2\sqrt{b^2-s^2}}+\frac{1}{2}c_5 s^2+c_6, \tag{4.26}$$

where $c_5=c_5(b^2), c_6=c_6(b^2)$ are C^∞ functions of b^2. Differentiating (4.26) with respect to s yields

$$H_2=-\frac{3\,(4\,c_2+c_4)\,s^2}{b^4\sqrt{b^2-s^2}}-\frac{(4\,c_2+c_4)\,s^4}{b^4\,(b^2-s^2)^{3/2}}+\frac{3\,c_2+c_4}{b^2\sqrt{b^2-s^2}}+\frac{(3\,c_2+c_4)\,s^2}{b^2\,(b^2-s^2)^{3/2}}+c_5 s. \tag{4.27}$$

By (2.14) and (2.15), we have

$$2\phi X-\phi_2-2s\phi_1+2H[s\phi+(b^2-s^2)\phi_2]=0 \tag{4.28}$$

and

$$2H[\phi - s\phi_2 + (b^2 - s^2)\phi_{22}] - \phi_{22} + 2(\phi_1 - s\phi_{12}) = 0. \tag{4.29}$$

Differentiating (4.28) with respect to s yields

$$2\phi_2 X + 2\phi X_2 - \phi_{22} - 2\phi_1 - 2s\phi_{12} + 2H_2[s\phi + (b^2 - s^2)\phi_2] + 2H[\phi - s\phi_2 + (b^2 - s^2)\phi_{22}] = 0. \tag{4.30}$$

By (4.30) and (4.29), we get

$$\phi_2 X + \phi X_2 - 2\phi_1 + H_2[s\phi + (b^2 - s^2)\phi_2] = 0. \tag{4.31}$$

By (4.31) $\times$ s $-$ (4.28), we get

$$\{sX + sH_2(b^2 - s^2) - 2H(b^2 - s^2) + 1\}\phi_2 + \{sX_2 - 2X + s^2 H_2 - 2sH\}\phi = 0. \tag{4.32}$$

Plugging (4.21) and (4.22), (4.26) and (4.27) into (4.32) yields

$$\begin{aligned}&\frac{1}{b^2}\Big\{b^2 s^2 c_1 - 2sc_2\sqrt{b^2 - s^2} + b^2 sc_3\sqrt{b^2 - s^2} - sc_4\sqrt{b^2 - s^2} - 2b^2 c_6(b^2 - s^2)\\ &+ b^2\Big\}\phi_2 - \frac{1}{b^2\sqrt{b^2 - s^2}}\Big\{b^2 sc_1\sqrt{b^2 - s^2} + 2c_2(b^2 + s^2) + (2b^2 - s^2)b^2 c_3 + s^2 c_4\\ &+ 2b^2 sc_6\sqrt{b^2 - s^2}\Big\}\phi = 0,\end{aligned} \tag{4.33}$$

which is equivalent to

$$\begin{aligned}&\sqrt{b^2 - s^2}\Big\{b^2 s^2 c_1 - 2sc_2\sqrt{b^2 - s^2} + b^2 sc_3\sqrt{b^2 - s^2} - sc_4\sqrt{b^2 - s^2} - 2b^2 c_6(b^2 - s^2)\\ &+ b^2\Big\}\Big(\ln\phi\Big)_2 - \Big\{b^2 sc_1\sqrt{b^2 - s^2} + 2c_2(b^2 + s^2) + (2b^2 - s^2)b^2 c_3 + s^2 c_4 + 2b^2 sc_6\sqrt{b^2 - s^2}\Big\}\\ &= 0.\end{aligned}$$

Integrating the above equation, (1.2) is obtained, where $c_7 = c_7(b^2)$ is C^∞ function of b^2. To prove the sufficiency, by plugging (4.21), (4.26) and (1.2) into (4.28), we get the condition (1.4). Plugging (4.21) and (4.26) into (2.13) yields

$$\begin{aligned}G^i =& G^i_\alpha + c\alpha X y^i + c\alpha^2 H b^i\\ =& G^i_\alpha + c\alpha\Big[c_1 s + \frac{c_2}{\sqrt{b^2 - s^2}} + c_3\sqrt{b^2 - s^2}\Big]y^i + c\alpha^2\Big[-\frac{(4c_2 + c_4)s^3}{b^4\sqrt{b^2 - s^2}}\\ &+ \frac{(3c_2 + c_4)s}{b^2\sqrt{b^2 - s^2}} + \frac{1}{2}c_5 s^2 + c_6\Big]b^i\\ =& G^i_\alpha + c\Phi^i + cc_2\Gamma^i,\end{aligned} \tag{4.34}$$

where

$$\Phi^i = \Big[c_1\beta + c_3\alpha\sqrt{b^2 - s^2}\Big]y^i + \Big[\frac{(4c_2 + c_4)\alpha\beta}{b^4}\sqrt{b^2 - s^2} + \frac{1}{2}c_5\beta^2 + c_6\alpha^2\Big]b^i, \tag{4.35}$$

$$\Gamma^i = \frac{\alpha(b^2 y^i - \beta b^i)}{b^2\sqrt{b^2 - s^2}}. \tag{4.36}$$

The following steps are to prove G^i are quadratic in y. In fact, in the two-dimensional case, $\sqrt{b^2 - s^2}$ can be simplified. By assumption,

$$
\begin{aligned}
&b^2 - s^2\\
=&b_1\Big(\frac{a_{22}b_1}{\det a} - \frac{a_{12}b_2}{\det a}\Big) + b_2\Big(\frac{a_{11}b_2}{\det a} - \frac{a_{12}b_1}{\det a}\Big) - \frac{[b_1(a^{11}y_1 + a^{12}y_2) + b_2(a^{12}y_1 + a^{22}y_2)]^2}{\alpha^2}\\
=&\frac{(b_2y_1 - b_1y_2)^2}{\alpha^2 \det a}.
\end{aligned}
$$

Thus,

$$
\sqrt{b^2 - s^2} = \frac{|b_2y_1 - b_1y_2|}{\alpha\sqrt{\det a}}, \tag{4.37}
$$

where $\det a = a_{11}a_{22} - (a_{12})^2$ is the determinant of a_{ij}. By (4.35) and (4.37),

$$
\Phi^i = \Big[c_1\beta + \frac{c_3|b_2y_1 - b_1y_2|}{\sqrt{\det a}}\Big]y^i + \Big[\frac{(4c_2 + c_4)|b_2y_1 - b_1y_2|\beta}{b^4\sqrt{\det a}} + \frac{1}{2}c_5\beta^2 + c_6\alpha^2\Big]b^i, \tag{4.38}
$$

which are quadratic in y, and then by (4.36) and (4.37), Γ^1 can be expressed as follows

$$
\begin{aligned}
\Gamma^1 =&\frac{\alpha}{\sqrt{b^2 - s^2}}y^1 - \frac{\alpha\beta}{b^2\sqrt{b^2 - s^2}}b^1\\
=&\frac{\alpha^2\sqrt{\det a}}{|b_2y_1 - b_1y_2|}y^1 - \frac{\alpha^2\beta(\det a)^{3/2}}{[(a_{22}b_1 - a_{12}b_2)b_1 + (a_{11}b_2 - a_{12}b_1)b_2]|b_2y_1 - b_1y_2|}b^1\\
=&\frac{\alpha^2(a^{11}y_1 + a^{12}y_2)\sqrt{\det a}}{|b_2y_1 - b_1y_2|}\\
&- \frac{\alpha^2[b_1(a^{11}y_1 + a^{12}y_2) + b_2(a^{12}y_1 + a^{22}y_2)](a_{22}b_1 - a_{12}b_2)\sqrt{\det a}}{[(a_{22}b_1 - a_{12}b_2)b_1 + (a_{11}b_2 - a_{12}b_1)b_2]|b_2y_1 - b_1y_2|}\\
=&\frac{\alpha^2 b_2[(a^{11}y_1 + a^{12}y_2)(a_{11}b_2 - a_{12}b_1) - (a^{12}y_1 + a^{22}y_2)(a_{22}b_1 - a_{12}b_2)]\sqrt{\det a}}{[(a_{22}b_1 - a_{12}b_2)b_1 + (a_{11}b_2 - a_{12}b_1)b_2]|b_2y_1 - b_1y_2|}\\
=&\frac{\alpha^2 b_2\sqrt{\det a}}{(a_{22}b_1 - a_{12}b_2)b_1 + (a_{11}b_2 - a_{12}b_1)b_2}\\
=&\frac{\alpha^2 b_2}{b^2\sqrt{\det a}}.
\end{aligned}
$$

Γ^2 can be given in the same method

$$
\Gamma^2 = \frac{-\alpha^2 b_1}{b^2\sqrt{\det a}}. \tag{4.39}
$$

Undoubtedly, Γ^i is quadratic in y, and it is proved that G^i is quadratic in y. Therefore, F is a Berwald metric.

As a special class in general (α,β)-metric, for spherically symmetric metric, the following corollary can be easily proved.

Corollary 4.5. *Let $F = |y|\phi(|x|, \frac{\langle x,y\rangle}{|y|})$ be a spherically symmetric metric on a two-dimensional manifold. Then F is a Berwald metric if, and only if, ϕ is given by (1.2), and $c_1 = c_1(b^2), c_2 = c_2(b^2), c_3 = c_3(b^2), c_4 = c_4(b^2), c_6 = c_6(b^2), c_7 = c_7(b^2)$ are C^∞ functions of b^2 satisfy (1.4).*

Proof. As F is a spherically symmetric metric, we get

$$r_{ij} = \delta_{ij}, \quad s_{ij} = 0, \quad r_{00} = |y|^2, \quad s^i_0 = 0,$$

$$b_i = r_i = x_i, \quad s_i = 0, \quad r_0 = \langle x, y\rangle, \quad s_0 = 0, \quad r^i = x^i, \quad s^i = 0.$$

Therefore, Theorem 1.2 is satisfied in spherically symmetric case. □

Next, by choosing suitable $c_1, c_2, c_3, c_4, c_6, c_7$, some explicit examples can be given as follows.

Example 4.1. *Let $c_1 = 0$, $c_2 = 0$, $c_3 = 1$, $c_4 = b^2$, $c_6 = 0$ and $c_7 = c_7(b^2) > 0$ in (1.3), then*

$$A(b^2, t) = \frac{2b^2}{\sqrt{b^2 - t^2}}. \tag{4.40}$$

By (1.2), ϕ can be given as

$$\phi = c_7 e^{2\,b^2 \arctan\left(\frac{s}{\sqrt{b^2-s^2}}\right)}. \tag{4.41}$$

By a direct computation,

$$\phi - s\phi_2 + (b^2 - s^2)\phi_{22} = c_7 e^{2\,b^2 \arctan\left(\frac{s}{\sqrt{b^2-s^2}}\right)}\left[1 + 4b^4\right] > 0.$$

This proves ϕ satisfies Lemma 2.1. Therefore, $F = \alpha\phi$ is a Berwald metric.

Example 4.2. *Let $c_1 = 0$, $c_2 = b^2$, $c_3 = 2$, $c_4 = 0$, $c_6 = 0$ and $c_7 = c_7(b^2) > 0$ in (1.3), then*

$$A(b^2, t) = \frac{6b^2}{\sqrt{b^2 - t^2}}. \tag{4.42}$$

By (1.2), ϕ can be given as

$$\phi = c_7 e^{6\,b^2 \arctan\left(\frac{s}{\sqrt{b^2-s^2}}\right)}. \tag{4.43}$$

It is easy to verify that ϕ satisfies Lemma 2.1. Therefore, $F = \alpha\phi$ is a Berwald metric.

5. Conclusions

In this paper, we firstly get the expression for the mean Landsberg curvature of general (α,β)-metric under the condition that β is closed and conformal to α. When the expression of the function ϕ does not depend on n, the equivalent relationship between Landsberg metric and weakly Landsberg metric is obtained. This is Theorem 1.1. Secondly, we get the expressions of mean Berwald curvature and Landsberg curvature in two-dimensional case. Base when based on Lemma 2.3, the classification of two-dimensional Berwald general (α,β)-metrics under the condition (1.1) with $c \neq 0$. This is Theorem 1.2. Hence, these results can be used in studying (weakly) Landsberg metrics and Berwald metrics in future.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgements

This research is partly supported by the ZPNSFC (Y23A010060), NNSF (2021J110) and K. C. Wong Magna Fund in Ningbo University.

Conflict of interest

There is no conflict of interest.

References

1. T. Aikou, Some remarks on the geometry of tangent bundle of Finsler manifolds, *Tensor*, **52** (1993), 234–242.
2. G. S. Asanov, Finsleroid-Finsler space with Berwald and Landsberg conditions, *Rep. Math. Phys.*, **58** (2006), 275–300. https://doi.org/10.1016/S0034-4877(06)80053-4
3. D. Bao, S. S. Chern, A note on the Gauss-Bonnet theorem for Finsler spaces, *Ann. Math.*, **143** (1996), 233–252. https://doi.org/10.2307/2118643
4. D. Bao, S. S. Chern, Z. Shen, *An introduction to Riemann-Finsler geometry*, New York: Springer, 2000. https://doi.org/10.1007/978-1-4612-1268-3
5. S. Bácsó, M. Matsumoto, Reduction theorems of certain Landsberg spaces to Berwald spaces, *Publ. Math. Debrecen*, **48** (1996), 357–366.
6. D. Bao, Z. Shen, On the volume of unit tangent spheres in a Finsler space, *Results Math.*, **26** (1994), 1–17. https://doi.org/10.1007/BF03322283
7. M. Crampin, On Landsberg spaces and the Landsberg-Berwald problem, *Houston J. Math.*, **37** (2011), 1103–1124.
8. L. Huang, X. Mo, On some explicit constructions of dually flat Finsler metrics, *J. Math. Anal. Appl.*, **405** (2013), 565–573. https://doi.org/10.1016/j.jmaa.2013.04.028
9. B. Li, Z. Shen, On a class of weakly Landsberg metrics, *Sci. China Ser. A*, **50** (2007), 573–589. https://doi.org/10.1007/s11425-007-0021-8
10. M. Matsumoto, Remarks on Berwald and Landsberg spaces, *Contemp. Math.*, 1996.
11. V. S. Matveev, On "All regular Landsberg metrics are always Berwald" by Z. I. Szabó, *Balkan J. Geom. Appl.*, **14** (2008), 50–52.
12. X. Mo, L. Zhou, The curvatures of spherically symmetric Finsler metrics in R^n, 2012, arXiv: 1202.4543. https://doi.org/10.48550/arXiv.1202.4543
13. G. Randers, On an asymmetric metric in the four-space of general relativity, *Phys. Rev.*, **59** (1941), 195. https://doi.org/10.1103/PhysRev.59.195

14. Z. Shen, *Differential geometry of Spray and Finsler spaces*, Dordrecht: Springer, 2001. https://doi.org/10.1007/978-94-015-9727-2

15. Z. Shen, Finsler manifolds with nonpositive flag curvature and constant S-curvature, *Math. Z.*, **249** (2005), 625–639. https://doi.org/10.1007/s00209-004-0725-1

16. Z. Shen, On a class of Landsberg metrics in Finsler geometry, *Can. J. Math.*, **61** (2009), 1357–1374. https://doi.org/10.4153/CJM-2009-064-9

17. Z. Shen, H. Xing, On randers metrics with isotropic S-curvature, *Acta. Math. Sin.-English Ser.*, **24** (2008), 789–796. https://doi.org/10.1007/s10114-007-5194-0

18. Z. I. Szabó, Positive definite Berwald spaces. Structure theorem on Berwald spaces, *Tensor (NS)*, **35** (1981), 25–39.

19. Z. I. Szabó, All regular Landsberg metrics are Berwald, *Ann. Glob. Anal. Geom.*, **34** (2008), 381–386. https://doi.org/10.1007/s10455-008-9115-y

20. Z. I. Szabó, Correction to "All regular Landsberg metrics are Berwald", *Ann. Glob. Anal. Geom.*, **35** (2009), 227–230. https://doi.org/10.1007/s10455-008-9131-y

21. M. Xu, V. S. Matveev, Proof of Laugwitz Conjecture and Landsberg Unicorn Conjecture for Minkowski norms with $SO(k) \times SO(n-k)$-symmetry, *Can. J. Math.*, **74** (2022), 1486–1516. https://doi.org/10.4153/S0008414X21000304

22. C. Yu, H. Zhu, On a new class of Finsler metrics, *Differ. Geom. Appl.*, **29** (2011), 244–254. https://doi.org/10.1016/j.difgeo.2010.12.009

23. L. Zhou, Projective spherically symmetric Finsler metrics with constant flag curvature in R^n, *Geom. Dedicata*, **158** (2012), 353–364. https://doi.org/10.1007/s10711-011-9639-3

24. L. Zhou, The Finsler surface with $K = 0$ and $J = 0$, *Differ. Geom. Appl.*, **35** (2014), 370–380. https://doi.org/10.1016/j.difgeo.2014.02.003

25. S. Zhou, B. Li, On Landsberg general (α,β)-metrics with a conformal 1-form, *Differ. Geom. Appl.*, **59** (2018), 46–65. https://doi.org/10.1016/j.difgeo.2018.04.001

26. M. Zohrehvand, H. Maleki, On general (α,β)-metrics of Landsberg type, *Int. J. Geom. Methods M.*, **13** (2016), 1650085. https://doi.org/10.1142/S0219887816500857

AIMS Press