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Research article

Existence and nonexistence of positive solutions to a class of nonlocal discrete Kirchhoff type equations

Yuhua Long[1,2,*]

[1] School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China

[2] Guangzhou Center for Applied Mathematics, Guangzhou University, Guangzhou 510006, China

* **Correspondence:** Email: sxlongyuhua@gzhu.edu.cn.

Abstract: In this paper, we investigate the existence and nonexistence of positive solutions to a class of nonlocal partial difference equations via a variant version of the mountain pass theorem. The conditions in our obtained results release the classical (AR) condition in some sense.

Keywords: difference equation; positive solution; mountain pass theorem; existence; nonexistence
Mathematics Subject Classification: 39A10, 34B15, 35B38

1. Introduction

Consider existence and nonexistence of positive solutions to the following type of nonlocal discrete Kirchhoff equation:

$$\begin{aligned}&-[a+b(\sum_{j=1}^{n}\sum_{i=1}^{m+1}|\Delta_1 x(i-1,j)|^2+\sum_{i=1}^{m}\sum_{j=1}^{n+1}|\Delta_2 x(i,j-1)|^2)]\cdot\left(\Delta_1^2 x(i-1,j)+\Delta_2^2 x(i,j-1)\right)\\ &=f((i,j),x(i,j)),\quad \forall (i,j)\in[1,m]\times[1,n],\end{aligned}\tag{1.1}$$

subject to Dirichlet boundary conditions

$$x(i,0)=x(i,n+1)=0,\quad i\in[0,m+1],\qquad x(0,j)=x(m+1,j)=0,\quad j\in[0,n+1],\tag{1.2}$$

where, given constants $a,b>0$ and $m,n>0$ are integers. For integers $\hbar\le\wp$, let $[\hbar,\wp]=\{\hbar,\hbar+1,\cdots,\wp\}$ denote a discrete segment. Forward difference operators $\Delta_1 x(i,j)=x(i+1,j)-x(i,j)$, $\Delta_2 x(i,j)=x(i,j+1)-x(i,j)$ and $\Delta^2 x(i,j)=\Delta(\Delta x(i,j))$. $\mathbf{R}_+$ denotes the set of all nonnegative real numbers and the nonlinearity $f((i,j),x)$ fulfills:

(H_1) $f : [0, m+1] \times [0, n+1] \times \mathbf{R} \to \mathbf{R}_+$ is continuous in x. If $x \le 0$, then $f((i,j),x) \equiv 0$ for all $(i,j) \in [0, m+1] \times [0, n+1]$;
(H_2) for $(i,j) \in [1,m] \times [1,n]$, $\frac{f((i,j),x)}{x^3}$ is nondecreasing with respect to $x \ge 0$.

Notice that (1.1) with Dirichlet boundary conditions (1.2) is usually taken in regard to the discrete analogue of the following Kirchhoff type problem:

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2\right)\triangle u = f(x,u), & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \Omega. \end{cases} \tag{1.3}$$

Owing to taking into account the effects of the changes in the length of a string during vibrations, (1.3) is an extension of the classical d'Alembert's wave equations [1]. Kirchhoff type equation (1.3) concerns not only the non-Newton mechanics, but also the physical laws of the universe, population dynamics models, the problem of plasma and so on. Consequently, it has captured keen research interest and there are many papers that have emerged. For example, Perera and Zhang [2] achieved a nontrivial solution by combining a critical group with the Yang index. Both in [3] and [4], the authors displayed results on multiple solutions including sign-changing solutions. Without the Ambrosetti-Rabinowitz condition, ground state solutions to the N-Kirchhoff equation was studied in [5]. For more interesting results, we refer the reader to [6, 7] and references therein.

It is well known that difference equation models are established in lots of areas, for instance, mechanical engineering, neural networks, biology, computer science and so on. For example, using difference equations, a two-patch SIR disease model was established in [8]. The authors studied the interaction between wild and sterile mosquitoes by a difference equation model in [9]. Because of wide applications, difference equations have been investigated extensively and many results have been achieved. Here we mention a few. Yu, Guo and Zuo [10] considered periodic solutions of second order self-adjoint difference equations, Zhou and Ling [11] presented results on positive solutions to a discrete two-point boundary value problem, and Kuang and Guo [12] dealt with heteroclinic solutions for p-Laplacian difference equations with a parameter and Nastasi, Tersian and Vetro [13] gave results on the existence of at least two non-zero homoclinic solutions without using Ambrosetti-Rabinowitz type-conditions.

As pointed out in [14], partial difference equations, involving two or more discrete variables, have been used in recent investigations related to digital control systems, image processing, neural networks, population models and social behaviors. Recently, many authors turned their interest towards study of them. For example, Long and Zhang [15, 16] achieved multiple solutions of second order partial difference equations via Morse theory. Meanwhile, results on periodic solutions of partial difference equations via critical point theorems were presented in [17–19].

The nonlocal discrete Kirchhoff type equation (1.1), a basic nonlinear partial difference equation, not only contains bivariate sequences with two independent integer variables, but also involves the discrete Kirchhoff term

$$b(\sum_{j=1}^{n}\sum_{i=1}^{m+1}|\Delta_1 x(i-1,j)|^2 + \sum_{i=1}^{m}\sum_{j=1}^{n+1}|\Delta_2 x(i,j-1)|^2)(\Delta_1^2 x(i-1,j) + \Delta_2^2 x(i,j-1)).$$

Thus, it is more difficult and interesting to study. Recently, based on critical point theory and variational methods, the authors [20] obtained the existence of at least three solutions. We move our attention to (1.1) and obtain some results. For example, we obtained sign-changing solutions in [21] and displayed results on infinitely many solutions in [22, 23]. Also, in [24], we studied nontrivial solutions via Morse theory. Meanwhile, it is well known that positive solutions play an important role in research, there seems few results concerned with positive solutions of (1.1). Moreover, above mentioned results indicate that critical point theory is a strong candidate for study of (1.1). Consequently, in this paper, we manage to deal with the existence and nonexistence of positive solutions of (1.1) by employing variational methods together with a variant version of the mountain pass theorem, which can be found in [25].

We arrange this paper as follows. In Section 2, we provide preliminaries and display our main results. We prove our main results at length in Section 3.

2. Preliminaries and main results

Let the set of all bivariate sequences be denoted by

$$S = \{x = \{x(i, j)\} : x(i, j) \in \mathbf{R}, (i, j) \in \mathbf{Z} \times \mathbf{Z}\}.$$

For any $x, y \in S$, $\imath, \jmath \in \mathbf{R}$, define $\imath x + \jmath y = \{\imath x(i, j) + \jmath y(i, j)\}$. Then, S is a vector space. Define the subset X, an mn-dimensional Hilbert space, of S as

$$\begin{aligned} X = \{x \in S : x(i, 0) = x(i, n+1) = 0 \quad \text{for} \quad i \in [0, m+1], \\ x(0, j) = x(m+1, j) = 0 \quad \text{for} \quad j \in [0, n+1]\}. \end{aligned}$$

For any $x, y \in X$, endow with the inner product $\langle \cdot, \cdot \rangle$ on E as

$$\langle x, y \rangle = \sum_{j=1}^{n} \sum_{i=1}^{m+1} (\Delta_1 x(i-1, j) \cdot \Delta_1 y(i-1, j)) + \sum_{i=1}^{m} \sum_{j=1}^{n+1} (\Delta_2 x(i, j-1) \cdot \Delta_2 y(i, j-1)), \tag{2.1}$$

which implies that the norm $\|\cdot\|$ induced by (2.1) is

$$\|x\| = \sqrt{\langle x, x \rangle} = \left(\sum_{j=1}^{n} \sum_{i=1}^{m+1} |\Delta_1 x(i-1, j)|^2 + \sum_{i=1}^{m} \sum_{j=1}^{n+1} |\Delta_2 x(i, j-1)|^2 \right)^{1/2}, \quad \forall x \in X.$$

For later use, we denote an mn-dimensional Hilbert space E, which is equipped with usual norm $|\cdot|$ and inner product $(\cdot, \cdot)$, respectively. Then, E is isomorphic to X. Throughout this paper, $x \in X$ is regarded as an extension of $x \in E$ when it is necessary. In what follows, for $1 \leq s < +\infty$, let

$$L^s \triangleq \left\{ x \in S : \|x\|_{L^s} = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |x(i, j)|^s \right)^{\frac{1}{s}} < +\infty \right\}$$

and

$$\|x\|_{L^\infty} = \sup_{(i,j) \in [1,m] \times [1,n]} |x(i, j)| < +\infty.$$

Then,

$$\|x\|_{L^s} \leq \eta_s \|x\|, \qquad \forall x \in X, \tag{2.2}$$

where η_s is the best constant for the embedded of X in L^s.

For convenience, we give some notations. Denote the well-known discrete Laplacian acting on a function $x(i,j):[0,1+m]\times[0,1+n]$ by $\Xi x(i,j)=\Delta_1^2 x(i-1,j)+\Delta_2^2 x(i,j-1)$. From [26], we get that the distinct Dirichlet eigenvalues of the invertible operator $-\Xi$ on $[1,m]\times[1,n]$ can be expressed as $0<\lambda_1<\lambda_2\leq\lambda_3\leq\cdots\leq\lambda_{mn}$. Specifically,

$$\lambda_1\|x\|_2^2\leq\|x\|^2\leq\lambda_{mn}\|x\|_2^2. \tag{2.3}$$

Consider the following eigenvalue problem:

$$\begin{cases} -\|x\|^2\Xi=\mu x^3(i,j), & [i,j]\in[1,m]\times[1,n] \\ x(i,0)=x(i,n+1)=x(0,j)=x(m+1,j)=0, & i\in[0,m+1],\ j\in[0,n+1]. \end{cases} \tag{2.4}$$

Denote the minimum eigenvalue and the maximum eigenvalue of (2.4) by μ_1 and $\mu_{\max}$, respectively. In the same manner as [22], we get that (2.4) has finitely many eigenvalues which all belong to $[\lambda_1^2, mn\lambda_{mn}^2]$. Clearly, $\mu_1>0$.

Write $F((i,j),x)=\int_0^x f((i,j),\tau)d\tau$. Consider the functional $J:E\to\mathbf{R}$ as the following:

$$\begin{aligned} J(x)=&\frac{a}{2}\left(\sum_{j=1}^{n}\sum_{i=1}^{m+1}|\Delta_1 x(i-1,j)|^2+\sum_{i=1}^{m}\sum_{j=1}^{n+1}|\Delta_2 x(i,j-1)|^2\right)\\ &+\frac{b}{4}\left(\sum_{j=1}^{n}\sum_{i=1}^{m+1}|\Delta_1 x(i-1,j)|^2+\sum_{i=1}^{m}\sum_{j=1}^{n+1}|\Delta_2 x(i,j-1)|^2\right)^2-\sum_{i=1}^{m}\sum_{j=1}^{n}F((i,j),x(i,j))\\ =&\frac{a}{2}\|x\|^2+\frac{b}{4}\|x\|^4-\sum_{i=1}^{m}\sum_{j=1}^{n}F((i,j),x(i,j)), \end{aligned} \tag{2.5}$$

then the continuity of f guarantees that $J\in C^1(E,\mathbf{R})$.

Denote

$$\begin{aligned} \Phi(x)=&\frac{a}{2}\left(\sum_{j=1}^{n}\sum_{i=1}^{m+1}|\Delta_1 x(i-1,j)|^2+\sum_{i=1}^{m}\sum_{j=1}^{n+1}|\Delta_2 x(i,j-1)|^2\right)\\ &+\frac{b}{4}\left(\sum_{j=1}^{n}\sum_{i=1}^{m+1}|\Delta_1 x(i-1,j)|^2+\sum_{i=1}^{m}\sum_{j=1}^{n+1}|\Delta_2 x(i,j-1)|^2\right)^2 \end{aligned}$$

and

$$\Psi(x)=\sum_{i=1}^{m}\sum_{j=1}^{n}F((i,j),x(i,j)),$$

then $J(x)=\Phi(x)-\Psi(x)$. For each $x,z\in E$, we have

$$\langle\Psi'(x),z\rangle=\lim_{\tau\to 0}\frac{\Psi(x+\tau z)-\Psi(x)}{\tau}=\sum_{i=1}^{m}\sum_{j=1}^{n}(f((i,j),x(i,j))\cdot z(i,j)). \tag{2.6}$$

Moreover, using Dirichlet boundary conditions, there holds

$$
\begin{aligned}
\langle \Phi'(x), z\rangle =& \lim_{\tau\to 0}\frac{\Phi(x+\tau z)-\Phi(x)}{\tau}\\
=& a\left(\sum_{j=1}^{n}\sum_{i=1}^{m+1}(\Delta_1 x(i-1,j)\cdot\Delta_1 z(i-1,j))+\sum_{i=1}^{m}\sum_{j=1}^{n+1}(\Delta_2 x(i,j-1)\cdot\Delta_2 z(i-1,j))\right)\\
&+b\left(\sum_{j=1}^{n}\sum_{i=1}^{m+1}|\Delta_1 x(i-1,j)|^2+\sum_{i=1}^{m}\sum_{j=1}^{n+1}|\Delta_2 x(i,j-1)|^2\right)\cdot\\
&\left(\sum_{j=1}^{n}\sum_{i=1}^{m+1}(\Delta_1 x(i-1,j)\cdot\Delta_1 z(i-1,j))+\sum_{i=1}^{m}\sum_{j=1}^{n+1}(\Delta_2 x(i,j-1)\cdot\Delta_2 z(i-1,j))\right)\\
=& a\left(\sum_{j=1}^{n}\sum_{i=1}^{m}(\Delta_1 x(i-1,j)\cdot\Delta_1 z(i-1,j))-\sum_{j=1}^{n}\Delta_1 x(i,j)\cdot z(m,j)\right.\\
&\left.+\sum_{i=1}^{m}\sum_{j=1}^{n}(\Delta_2 x(i,j-1)\cdot\Delta_2 z(i-1,j))-\sum_{i=1}^{m}\Delta_2 x(i,j)\cdot z(i,n)\right)\\
&+b\left(\sum_{j=1}^{n}\sum_{i=1}^{m+1}|\Delta_1 x(i-1,j)|^2+\sum_{i=1}^{m}\sum_{j=1}^{n+1}|\Delta_2 x(i,j-1)|^2\right)\cdot\\
&\left(\sum_{j=1}^{n}\sum_{i=1}^{m}(\Delta_1 x(i-1,j)\cdot\Delta_1 z(i-1,j))-\sum_{j=1}^{n}\Delta_1 x(i,j)\cdot z(m,j)\right.\\
&\left.+\sum_{i=1}^{m}\sum_{j=1}^{n}(\Delta_2 x(i,j-1)\cdot\Delta_2 z(i-1,j))-\sum_{i=1}^{m}\Delta_2 x(i,j)\cdot z(i,n)\right)\\
=&-\left[a+b\left(\sum_{j=1}^{n}\sum_{i=1}^{m+1}|\Delta_1 x(i-1,j)|^2+\sum_{i=1}^{m}\sum_{j=1}^{n+1}|\Delta_2 x(i,j-1)|^2\right)\right]\cdot\\
&\left(\sum_{j=1}^{n}\sum_{i=1}^{m}(\Delta_1 x(i-1,j)\cdot\Delta_1 z(i-1,j))+\sum_{i=1}^{m}\sum_{j=1}^{n}(\Delta_2 x(i,j-1)\cdot\Delta_2 z(i-1,j))\right).
\end{aligned}
\tag{2.7}
$$

Recall the definition of $\langle\cdot,\cdot\rangle$ and joint (2.6) with (2.7), it follows that

$$
\begin{aligned}
\langle J'(x), z\rangle =& \langle \Psi'(x)-\Phi'(x), z\rangle\\
=& (a+b\|x\|^2)(\sum_{j=1}^{n}\sum_{i=1}^{m}(\Delta_1 x(i-1,j)\cdot\Delta_1 z(i-1,j))\\
&+\sum_{i=1}^{m}\sum_{j=1}^{n}(\Delta_2 x(i,j-1)\cdot\Delta_2 z(i-1,j)))-\sum_{i=1}^{m}\sum_{j=1}^{n}(f((i,j),x(i,j))\cdot z(i,j)).
\end{aligned}
\tag{2.8}
$$

Accordingly, $\langle J'(x), z\rangle = 0$ is equivalent to $\sum_{j=1}^{n}\sum_{i=1}^{m}((\Delta_1 x(i-1,j)\cdot\Delta_1 z(i-1,j))+(\Delta_2 x(i,j-1)\cdot\Delta_2 z(i-1,j))-(f((i,j),x(i,j))\cdot z(i,j)))=0$. Since z is arbitrary, the critical point of J is just the solution of (1.1) with Dirichlet boundary conditions (1.2). Namely, to seek solutions of (1.1) with Dirichlet boundary conditions (1.2), it is equivalent to look for critical points of the functional J on E. Further, the assumption (H_1) and the strong maximum principle guarantee that nontrivial critical points of J on E are actually positive solutions of (1.1) with Dirichlet boundary conditions (1.2).

Throughout this paper, we denote a universal constant by c unless specified otherwise. To seek critical points of the functional (2.5), we recall the concept of the Cerami condition at level c ($(C)_c$ for short), which is a weak version of the Palais-Smale condition ((PS) for short) and introduced by Cerami [27], as well as a variant version of the mountain pass theorem, which plays an important role in proofs of our main results.

Definition 2.1. *Let $J(x) \in C^1(E, \mathbf{R})$. If any sequence $\{x_\kappa\} \subset E$ satisfying*

$$\{J(x_\kappa)\} \to c \quad and \quad (1 + \|x_\kappa\|)\|J'(x_\kappa)\| \to 0, \qquad as \quad \kappa \to +\infty$$

possesses a convergent subsequence in E, then J satisfies $(C)_c$. For all $c \in \mathbf{R}$, if $J(x)$ satisfies $(C)_c$, then $J(x)$ is called satisfying the (C).

Proposition 2.1. *[25] Let $J \in C^1(E, R)$. Assume that*

$$\max\{J(0), J(x_1)\} \le \alpha < \beta \le \inf_{\|x\|=\rho} J(x)$$

for some $\alpha < \beta$, $\rho > 0$ and $x_1 \in E$ with $\|x_1\| > \beta$. Then there is a sequence $\{x_\kappa\}$ of E satisfying

$$J(x_\kappa) \to c \ge \beta > 0 \quad and \quad (1 + \|x_\kappa\|)\|J'(x_\kappa)\| \to 0, \quad as \quad \kappa \to \infty, \tag{2.9}$$

where

$$c = \inf_{\gamma\in\Gamma} \max_{0\le\tau\le1} J(\gamma(\tau)) \quad and \quad \Gamma = \{\gamma \in C([0, 1], E) : \gamma(0) = 0, \gamma(1) = x_1\}.$$

Further, if $(C)_c$ is satisfied, then J has a critical value c.

Assume that

(H_3) for any $(i, j) \in [1, m] \times [1, n]$,

$$\lim_{x\to0} \frac{f((i, j), x)}{ax} = p(i, j), \qquad \lim_{x\to+\infty} \frac{f((i, j), x)}{x^3} = q(i, j) \not\equiv 0,$$

where $0 \le p(i, j), q(i, j) \le +\infty$ and $\|p\|_{L^\infty} < \lambda_1$.

Remark 2.1. *The assumption (H_3) means that the nonlinearity f possesses asymptotic behavior at zero and infinity. Usually, the asymptotically 4-linear condition*

$$\lim_{x\to0} \frac{f(i, j), x}{ax} = \lambda, \qquad \lim_{x\to+\infty} \frac{f(i, j), x}{bx^3} = \mu, \tag{2.10}$$

or the following classic 4-superlinear condition of Ambrosetti and Rabinowitz (AR)

$$\exists\nu > 4 : \nu F((i, j), x) \le xf((i, j), x), \qquad |x| \quad large, \tag{2.11}$$

is crucial to certify the mountain pass geometry and prove the boundedness of Cerami or Palais-Smale sequences in E. Clearly, our assumption (H_3) is weaker than (2.10) and indicates that (2.11) does not hold any more. Further, $q(i, j) \equiv +\infty$ in (H_3) indicates that f is 4-superlinear at infinity, which is weaker than (2.11).

Set

$$\Lambda = \inf\left\{\|x\|^4 : x \in X, \sum_{i=1}^{m}\sum_{j=1}^{n} q(i,j)x^4(i,j) = 1\right\}. \tag{2.12}$$

Remark 2.2. *By (2.12), we have that Λ is positive.*

First, Λ is attainable. Let a minimizing sequence of Λ be denoted by $\{x_l\} \subset E$, then $\{x_l\}$ is bounded and satisfies $\sum_{i=1}^{m}\sum_{j=1}^{n} q(i,j)x_l^4(i,j) = 1$. Choose a subsequence of $\{x_l\}$, still denoted by $\{x_l\}$. Then there exists $\bar{x}_1 \in E$ such that $x_l \to \bar{x}_1$. Hence,

$$\sum_{i=1}^{m}\sum_{j=1}^{n} q(i,j)x_l^4(i,j) \to \sum_{i=1}^{m}\sum_{j=1}^{n} q(i,j)\bar{x}_1^4(i,j), \qquad as \quad l \to +\infty,$$

and

$$\sum_{i=1}^{m}\sum_{j=1}^{n} q(i,j)\bar{x}_1^4(i,j) = 1.$$

Therefore,

$$\Lambda \le \left(\sum_{j=1}^{n}\sum_{i=1}^{m+1} |\Delta_1\bar{x}_1(i-1,j)|^2 + \sum_{i=1}^{m}\sum_{j=1}^{n+1} |\Delta_2\bar{x}_1(i,j-1)|^2\right)^2 \le \Lambda,$$

which leads to

$$\Lambda = \left(\sum_{j=1}^{n}\sum_{i=1}^{m+1} |\Delta_1\bar{x}_1(i-1,j)|^2 + \sum_{i=1}^{m}\sum_{j=1}^{n+1} |\Delta_2\bar{x}_1(i,j-1)|^2\right)^2 = \|\bar{x}_1\|^4.$$

Namely, Λ is attainable.

Further, $\bar{x}_1(i,j) > 0$ for all $(i,j) \in [1,m] \times [1,n]$. In fact, if $\|q\|_{L^\infty} < +\infty$ and $q(i,j) \ge (\not\equiv)0$, then $\Lambda > 0$ and there exists $\bar{x}_1 \in E$ such that $\|\bar{x}_1\|^4 = 1$ and $\sum_{i=1}^{m}\sum_{j=1}^{n} q(i,j)\bar{x}_1^4(i,j) = 1$. Moreover, $\bar{x}_1(i,j) > 0$ for all $(i,j) \in [1,m] \times [1,n]$. Furthermore, assume $\bar{x}_1(i,j) \ge 0$ on $(i,j) \in [1,m] \times [1,n]$. Otherwise, we can replace $\bar{x}_1$ by $|\bar{x}_1|$. So, the strong maximum principle implies that $\bar{x}_1(i,j) > 0$ for all $(i,j) \in [1,m] \times [1,n]$.

Now we display our main results as following:

Theorem 2.2. *Assume $f((i,j),x)$ satisfies (H_1)–(H_3). Then:*
(i) *If $\Lambda \ge \frac{1}{b}$, there is no any positive solution of (1.1) with Dirichlet boundary conditions (1.2);*
(ii) *If $\Lambda < \frac{1}{b}$, there is at least one positive solution of (1.1) with Dirichlet boundary conditions (1.2).*

Corollary 2.3. *Assume (H_1)–(H_3) hold with $q(i,j) \equiv \ell > 0$. Then:*
(i) *If $\ell \le b\mu_1$, (1.1) with Dirichlet boundary conditions (1.2) admits no any positive solution;*
(ii) *If $b\mu_1 < \ell < +\infty$, (1.1) with Dirichlet boundary conditions (1.2) possesses at least one positive solution.*

Proof. Note that $q(i,j) \equiv \ell > 0$ ensures that $\Lambda = \frac{\mu_1}{\ell}$. Then, Theorem 2.2 guarantees conclusions in Corollary 2.3 are true. □

Remark 2.3. *Owing to* **(ii)** *of Corollary 2.3, (1.1) with Dirichlet boundary conditions (1.2) admits at least one positive solution if* $q(i, j) \equiv b\mu_\iota$, $\iota \geq 2$.

Theorem 2.4. *Assume* $q(i, j) \equiv \infty$ *and* $f((i, j), x)$ *satisfies* (H_1)–(H_3). *If* $4 < k < 6$ *such that*

$$\lim_{x\to+\infty} \frac{f((i, j), x)}{x^{k-1}} = 0, \qquad (i, j) \in [1, m] \times [1, n].$$

Then, there exists at least one positive solution for (1.1) with Dirichlet boundary conditions (1.2).

Remark 2.4. *It is necessary to point out that it is not difficult to find many functions satisfying our conditions in Theorems 2.2 and 2.4, but (2.11) is not satisfied.*

Example 2.1. Given $M > 0$, for any $(i, j) \in [1, m] \times [1, n]$, set

$$f((i, j), x) = \begin{cases} 0, & -\infty < x \leq 0; \\ x^3, & 0 < x \leq M; \\ Mx^3, & M < x < +\infty. \end{cases}$$

By simple calculation, we have that $f((i, j), x)$ satisfies (H_1)–(H_3) with $p(i, j) \equiv 0$, $q(i, j) \equiv M$ and

$$F((i, j), x) = \frac{1}{4}Mx^4 - \frac{1}{20}M^5, \qquad x > M.$$

If (2.11) is met, then there exists $\nu > 4$ such that

$$Mx^4\left(\frac{\nu}{4} - \frac{M^4}{20x^4}\right) \leq Mx^4, \qquad \text{for large } x,$$

which means that $\nu \leq 4$. And it is a contradiction.

Example 2.2. For any $(i, j) \in [1, m] \times [1, n]$, set

$$f((i, j), x) = \begin{cases} 0, & -\infty < x \leq 0; \\ x^3 \ln x, & 0 < x < +\infty; \end{cases} \tag{2.13}$$

(2.13) means that $p(i, j) \equiv 0$, $q(i, j) \equiv +\infty$ and (H_1)–(H_3) are satisfied. Meanwhile,

$$F((i, j), x) = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + \frac{1}{16}e^4, \qquad x > e,$$

which indicates that

$$x^4 \ln x\left(\frac{\nu}{4} - \frac{\nu}{16 \ln x}\right) + \frac{\nu e^4}{16} \leq x^4 \ln x, \qquad \text{for large } x$$

holds for $\nu > 4$ impossible. Subsequently, (2.11) is not satisfied.

Theorem 2.5. *Suppose* (H_1) *holds and* $q(i, j) = +\infty$. *Moreover:*
(H_4) *There exist some positive constant C and* $4 < k < 6$ *satisfying*

$$|f((i, j), x)| \leq C(|x|^{k-1} + 1), \qquad \forall (i, j) \in [1, m] \times [1, n], \quad x \in \mathbf{R};$$

(H_5) *Let* $G((i, j), x) = xf((i, j), x) - 4F((i, j), x)$, *there exists* $\theta \geq 1$ *such that*

$$\theta G((i, j), x) \geq G((i, j), \omega x), \qquad \forall (i, j) \in [1, m] \times [1, n], \quad x \in \mathbf{R} \quad \textit{and} \quad 0 \leq \omega \leq 1;$$

(H_6) *There exists* $\delta > 0$ *such that* $F((i, j), x) \leq \frac{a}{2}\lambda_1 x^2$ *holds for* $|x| < \delta$.

Then, problem (1.1) with Dirichlet boundary conditions (1.2) admits at least one positive solution.

Remark 2.5. *In some sense, Theorem 2.5 extends Theorem 2.2 in two ways. On the one hand,* (H_5) *is equivalent to* (H_2) *when* $\theta = 1$ *and gives some general when* $\theta > 1$*. For example, set*

$$f((i,j),x) = 4x^3\ln(1+x^4) + 2\sin x, \qquad \forall (i,j) \in [1,m]\times[1,n],$$

direct computation yields that f *satisfies* (H_5) *but does not satisfy* (H_2)*. On the other hand,* (H_6) *is weaker than* (H_3)*. For example, set*

$$f((i,j),x) = \begin{cases} 0, & -\infty < x \le 0; \\ 4x^3\ln(1+x^4) + 2x\ln x, & x > 0. \end{cases}$$

Then, f *satisfies both* (H_5) *and* (H_6)*, but neither* (H_2) *nor* (H_3)*.*

3. Proofs of main results

Proof of Theorem 2.2. We give the proof of **(i)** by contradiction. As to the proof of **(ii)**, we complete it by Proposition 2.1 in 2 steps: First, we are to verify that there exists a sequence $\{x_\kappa\}_{\kappa\in\mathbf{N}} \subset E$ such that (2.9) in Proposition 2.1 is true. Second, we show the functional J satisfies the $(C)_c$ in E. Since E is finite dimensional, we only need to prove $\{x_\kappa\}$ is bounded.

(i) Suppose that $x \in E$ is positive and solves (1.1) with Dirichlet boundary conditions (1.2), by (2.8), it follows that

$$(a + b\|x\|^2)\|x\|^2 = \sum_{i=1}^{m}\sum_{j=1}^{n}(f((i,j),x(i,j))\cdot x(i,j)).$$

Together with (H_1)–(H_3), we obtain

$$b\|x\|^4 < \sum_{i=1}^{m}\sum_{j=1}^{n}(f((i,j),x(i,j))\cdot x(i,j)) \le \sum_{i=1}^{m}\sum_{j=1}^{n}(q(i,j)\cdot x^4(i,j)),$$

which implies that $\Lambda < \frac{1}{b}$. This is a contradiction and Theorem 2.2**(i)** is verified.

(ii) We are to complete the proof by applying Proposition 2.1. Thus we begin the proof with showing that there exist $\rho, \beta > 0$ such that $J(x) \ge \beta$ for $x \in E$ with $\|x\| = \rho$, and $J(\tau x_1) \to -\infty$, as $\tau \to +\infty$. Indeed, thanks to (H_1) and (H_3), for any $\epsilon > 0$, there exists $\hat{M} = \hat{M}(\epsilon) > 0$ such that

$$F((i,j),x) \le \frac{1}{2}a(\|p\|_{L^\infty} + \epsilon)x^2 + \hat{M}x^4, \qquad \forall (i,j) \in [1,m]\times[1,n],\ x \in \mathbf{R}. \tag{3.1}$$

Choosing a suitable $\epsilon > 0$ such that $(\|p\|_{L^\infty} + \epsilon) < \lambda_1$. Combining (3.1) with (2.2), (2.3) and (2.5), it follows that

$$\begin{aligned} J(x) =& \frac{a}{2}\|x\|^2 + \frac{b}{4}\|x\|^4 - \sum_{i=1}^{m}\sum_{j=1}^{n}F((i,j),x(i,j)) \\ \ge& \frac{a}{2}\|x\|^2 + \frac{b}{4}\|x\|^4 - \frac{1}{2}a(\|p\|_{L^\infty} + \epsilon)\sum_{i=1}^{m}\sum_{j=1}^{n}x^2(i,j) - \hat{M}\sum_{i=1}^{m}\sum_{j=1}^{n}x^4(i,j) \\ \ge& \frac{a}{2}\left(1 - \frac{(\|p\|_{L^\infty} + \epsilon)}{\lambda_1}\right)\|x\|^2 + \frac{b}{4}\|x\|^4 - \hat{M}\eta_4^4\|x\|^4. \end{aligned}$$

Therefore, we can select small $\rho > 0$ such that

$$J(x) \geq \frac{a}{4}\left(1 - \frac{(\|p\|_{L^\infty} + \epsilon)}{\lambda_1}\right)\rho^2 \triangleq \beta > 0, \qquad x \in E \quad \text{with} \quad \|x\| = \rho.$$

Since $\Lambda < \frac{1}{b}$, we have

$$\begin{aligned}
\lim_{\tau\to+\infty} \frac{J(\tau\bar{x}_1(i,j))}{\tau^4} = \lim_{\tau\to+\infty} &\left[\frac{a\|\bar{x}_1\|^2}{2\tau^2} + \frac{b}{4}\|\bar{x}_1\|^4 - \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{F((i,j),\tau\bar{x}_1(i,j))}{\tau^4}\right] \\
=&\frac{b}{4}\|\bar{x}_1\|^4 - \sum_{i=1}^{m}\sum_{j=1}^{n} \lim_{\tau\to+\infty} \frac{F((i,j),\tau\bar{x}_1(i,j))}{\tau^4\bar{x}_1^4(i,j)} \cdot \bar{x}_1^4(i,j) \\
=&\frac{b}{4}\|\bar{x}_1\|^4 - \frac{1}{4}\sum_{i=1}^{m}\sum_{j=1}^{n} q(i,j)\bar{x}_1^4(i,j) \\
=&\frac{b}{4}\Lambda - \frac{1}{4} < 0,
\end{aligned}$$

which implies that $J(\tau\bar{x}_1) \to -\infty$ as $\tau \to +\infty$ for all $(i, j) \in [1, m] \times [1, n]$. Therefore, there exists $\tau_0 > 0$ large enough such that

$$J(\tau_0\bar{x}_1) < 0 \qquad \text{and} \qquad \max\{J(0), J(\tau_0\bar{x}_1)\} \leq 0 < \beta \leq \inf_{\|x\|=\rho} J(x).$$

Define

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = \tau_0\bar{x}_1\},$$

and

$$c = \inf_{\gamma\in\Gamma} \max_{0\leq\varpi\leq1} J(\gamma(\varpi)).$$

According to Proposition 2.1, it yields that $c \geq \beta > 0$ and there exists a sequence $\{x_\kappa\}_{\kappa\in\mathbf{N}} \subset E$ such that

$$J(x_\kappa) \to c, \qquad (1 + \|x_\kappa\|)\|J'(x_\kappa)\| \to 0, \qquad \text{as} \quad \kappa \to \infty. \tag{3.2}$$

Let $\{x_\kappa\} \subset E$. Our next task is to prove that $\{x_\kappa\}$ has a convergent subsequence, also written by $\{x_\kappa\}$. Since E is an mn-dimensional Hilbert space, it suffices to show the boundedness of $\{x_\kappa\}$. Arguing indirectly, suppose that $\|x_\kappa\| \to +\infty$ as $\kappa \to +\infty$. Write $y_\kappa = \frac{x_\kappa}{\|x_\kappa\|}$, which follows that $\|y_\kappa\| = 1$. Therefore, $\{y_\kappa\}$ possesses a subsequence, still denoted by $\{y_\kappa\}$, satisfying $y_\kappa(i, j) \to y(i, j)$ as $\kappa \to \infty$ for all $(i, j) \in [1, m] \times [1, n]$.

We assume $y = 0$ and denote $\Omega_1 \triangleq \{(i, j) : (i, j) \in [1, m] \times [1, n]$ such that $x(i, j) > 0\}$ and $y^+ = \max\{y, 0\}$. In view of (H_1) and (H_3), there exists $\tilde{M} > 0$ such that

$$\frac{f((i,j),x)}{x^3} \leq \tilde{M}, \quad (i,j) \in \Omega_1. \tag{3.3}$$

Owing to (H_1), (2.8), (3.2) and (3.3), we have

$$\begin{aligned}
b =& \lim_{\kappa\to+\infty} \frac{\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} f((i,j), x_\kappa((i,j)) \cdot x_\kappa((i,j)}{\|x_\kappa\|^4} \\
=& \lim_{\kappa\to+\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{f((i,j), x_\kappa((i,j))}{x_\kappa^3(i,j)/y_\kappa^3(i,j)} \cdot y_\kappa(i,j) \\
=& \lim_{\kappa\to+\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{f((i,j), x_\kappa((i,j))}{x_\kappa^3(i,j)} \cdot y_\kappa^4(i,j) \\
=& \lim_{\kappa\to+\infty} \sum\sum\nolimits_{(i,j)\in\Omega_1} \frac{f((i,j), x_\kappa((i,j))}{x_\kappa^3(i,j)} \cdot (y_\kappa^+(i,j))^4 \\
\le& \tilde{M} \lim_{\kappa\to+\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} (y_\kappa^+(i,j))^4 \\
=& 0.
\end{aligned}$$

Obviously, it is impossible.

We assume $y \neq 0$ and set

$$p_\kappa(i,j) = \begin{cases} 0, & \text{if} \quad x_\kappa(i,j) \le 0; \\ \dfrac{f((i,j), x_\kappa(i,j))}{x_\kappa^3(i,j)}, & \text{if} \quad x_\kappa(i,j) > 0. \end{cases}$$

We obtain that $0 \le p_\kappa(i,j) \le \tilde{M}$ for all $(i,j) \in [1,m] \times [1,n]$. Consequently, we can assume that there exists a function $h(i,j)$ such that

$$p_\kappa(i,j) \to h(i,j), \quad \text{as} \quad \kappa \to +\infty.$$

Now, for any $z \in E$, we have

$$\begin{aligned}
\sum_{i=1}^{m}\sum_{j=1}^{n} p_\kappa(i,j) y_\kappa^3(i,j) z(i,j) &= \sum_{i=1}^{m}\sum_{j=1}^{n} p_\kappa(i,j)(y_\kappa^+(i,j))^3 z(i,j) \\
&\to \sum_{i=1}^{m}\sum_{j=1}^{n} h(i,j)(y^+(i,j))^3 z(i,j).
\end{aligned} \tag{3.4}$$

Furthermore, recall $y_\kappa = \frac{x_\kappa}{\|x_\kappa\|}$, for all $z \in E$, (2.8), (3.2) and (H_1) induce that

$$\begin{aligned}
\frac{(J'(x_\kappa), z)}{\|x_\kappa\|^3} &= \frac{a + b\|x_\kappa\|^2}{\|x_\kappa\|^3}\langle x_\kappa, z\rangle - \sum_{i=1}^{m}\sum_{j=1}^{n} \left(\frac{f((i,j), x_\kappa(i,j))}{\|x_\kappa\|^3}, z(i,j) \right) \\
&= \frac{a + b\|x_\kappa\|^2}{\|x_\kappa\|^2}\langle y_\kappa, z\rangle - \sum_{i=1}^{m}\sum_{j=1}^{n} \left(\frac{f((i,j), x_\kappa^+(i,j))}{\|x_\kappa\|^3}, z(i,j) \right) \\
&\to 0, \qquad \text{as} \quad \kappa \to \infty.
\end{aligned} \tag{3.5}$$

Thus,

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left(\frac{f((i,j),x_\kappa^+(i,j))}{\|x_\kappa\|^3},z(i,j)\right)\to b\langle y,z\rangle. \tag{3.6}$$

Combining (3.4) with (3.6), we obtain that

$$\begin{aligned}&\lim_{\kappa\to+\infty}\left[\sum_{i=1}^{m}\sum_{j=1}^{n}\left(\frac{f((i,j),x_\kappa^+(i,j))}{\|x_\kappa\|^3},z(i,j)\right)-\sum_{i=1}^{m}\sum_{j=1}^{n}p_\kappa(i,j)(y_\kappa^+(i,j))^3z(i,j)\right]\\ =&b\langle y,z\rangle-\sum_{i=1}^{m}\sum_{j=1}^{n}h(i,j)(y^+(i,j))^3z(i,j)\\ =&0, \qquad\qquad \forall z\in E.\end{aligned}$$

By (H_3), it yields that

$$b\langle y,z\rangle-\sum_{i=1}^{m}\sum_{j=1}^{n}q(i,j)(y^+(i,j))^3z(i,j)=0. \tag{3.7}$$

Set $z=y^-$, then $\|y^-\|^2=0$ and $y\equiv y^+\geq 0$. Consider (3.7) with the boundary conditions (1.2), we have

$$\begin{cases}-b(\Delta_1^2y(i-1,j)+\Delta_2^2y(i,j-1))=q(i,j)(y^+(i,j))^3, & (i,j)\in[1,m]\times[1,n],\\ y(i,0)=y(i,n+1)=0, \quad y(0,j)=y(m+1,j)=0, & i\in[0,m+1], \quad j\in[0,n+1],\end{cases}$$

and the maximum principle implies $y=y^+>0$. Hence,

$$b\langle y,z\rangle-\sum_{i=1}^{m}\sum_{j=1}^{n}q(i,j)y^3(i,j)z(i,j)=0, \quad \forall z\in E.$$

Let $z=y_\kappa-y$. Note that $\|y\|=1$, (3.5) gives

$$\|y\|^2\langle y,z\rangle=\frac{1}{b}\sum_{i=1}^{m}\sum_{j=1}^{n}q(i,j)y^3(i,j)z(i,j)=0, \quad \forall z\in E,$$

which contradicts $\Lambda<\frac{1}{b}$. Therefore, $\{x_\kappa\}$ is bounded in E and J satisfies $(C)_c$.

Consequently, Proposition 2.1 ensures that x is a nontrivial critical point of J, which means that there exists at least one positive solution for (1.1) with Dirichlet boundary conditions (1.2). Thus, we have verified Theorem 2.2.

Proof of Theorem 2.4. Applying Proposition 2.1, we finish the proof of Theorem 2.4 by 3 steps.

Step 1. We show that there exist constants $\rho,\beta>0$ such that $J(x)\geq\beta$ for $x\in E$ with $\|x\|=\rho$.

Set $0<\epsilon<\lambda_1-\|p\|_{L^\infty}$. Since $\lim\limits_{x\to+\infty}\frac{f((i,j),x)}{x^{k-1}}=0$ holds for all $(i,j)\in[1,m]\times[1,n]$, by (H_1) and (H_3), there exists constant $\alpha>0$ such that

$$|f((i,j),x)|\leq a(\|p\|_{L^\infty}+\epsilon)|x|+a|x|^{k-1}, \qquad \forall(i,j)\in[1,m]\times[1,n],$$

which induces that

$$F((i,j),x)\leq\frac{a(\|p\|_{L^\infty}+\epsilon)}{2}x^2+\frac{a}{k}|x|^k, \qquad \forall(i,j)\in[1,m]\times[1,n]. \tag{3.8}$$

Then, (2.2), (2.3) and (3.8) yield that

$$\begin{aligned}J(x) =&\frac{a}{2}\|x\|^2 + \frac{b}{4}\|x\|^4 - \sum_{i=1}^{m}\sum_{j=1}^{n} F((i,j), x(i,j))\\ \geq&\frac{a}{2}\|x\|^2 + \frac{b}{4}\|x\|^4 - \frac{1}{2}a(\|p\|_{L^\infty} + \epsilon)\sum_{i=1}^{m}\sum_{j=1}^{n} x^2(i,j) - \frac{a}{k}\sum_{i=1}^{m}\sum_{j=1}^{n} x^k(i,j)\\ \geq&\frac{a}{2}\left(1 - \frac{(\|p\|_{L^\infty} + \epsilon)}{\lambda_1}\right)\|x\|^2 + \frac{b}{4}\|x\|^4 - \frac{a}{k}\eta_k^k\|x\|^k.\end{aligned}$$

Note that $4 < k < 6$, there exist constants $\rho, \beta > 0$ such that

$$J(x) \geq \frac{a}{4}\left(1 - \frac{(\|p\|_{L^\infty} + \epsilon)}{\lambda_1}\right)\rho^2 \triangleq \beta > 0$$

for $x \in E$ with $\|x\| = \rho$.

Step 2. We claim that $J(t\psi_1) \to -\infty$ as $t \to +\infty$, where ψ_1 is the eigenfunction corresponding to μ_1.

Write $\Omega \triangleq \{(i,j) : (i,j) \in [1,m] \times [1,n]\}$. Then, there exists some $\alpha > 0$ such that

$$\min_{(i,j)\in\overline{\Omega_0}} \psi_1(x) \geq \alpha > 0,$$

where $\Omega_0 \triangleq \{(i,j) : (i,j) \in \Omega \text{ such that } \psi_1(x(i,j)) > 0\}$. Obviously, $\Omega_0 \neq \emptyset$ and $\Omega_0 \subset \overline{\Omega_0} \subset\subset \Omega$. Hence, $t\psi_1(x) \to +\infty$ as $t \to +\infty$ in $\overline{\Omega_0}$. In view of (H_1) and (H_2), it follows that

$$0 \leq 4F((i,j), x) \leq f((i,j), x)x, \quad (i,j) \in \Omega, \quad \forall x \geq 0,$$

and $\frac{F((i,j),x)}{x^4}$ is nondecreasing in $x > 0$. Since $q(i,j) \equiv +\infty$, we get

$$\frac{F((i,j), t\psi_1)}{t^4\psi_1^4} \geq \frac{F((i,j), t\alpha)}{t^4\alpha^4} \to +\infty \quad \text{as} \quad t \to +\infty, \quad (i,j) \in \Omega_0.$$

Hence, for any $K > 0$, there exists $T > 0$ such that

$$\frac{F((i,j), t\alpha)}{t^4\alpha^4} \geq K > 0, \quad \forall t \geq T, \quad (i,j) \in \Omega_0.$$

Therefore, for $t \geq T$, choose $K > 0$ large enough such that

$$\begin{aligned}\frac{J(t\psi_1)}{t^4} =&\frac{a}{2}\frac{\|\psi_1\|^2}{t^2} + \frac{b}{4}\|\psi_1\|^4 - \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{F((i,j), t\psi_1(i,j))}{t^4}\\ \leq&\frac{a}{2}\frac{\|\psi_1\|^2}{t^2} + \frac{b}{4}\|\psi_1\|^4 - \sum\sum\nolimits_{(i,j)\in\Omega_0}\frac{F((i,j), t\psi_1(i,j))}{t^4\psi_1^4(i,j)}\psi_1^4(i,j)\\ \leq&\frac{a}{2T^2}\|\psi_1\|^2 + \frac{b}{4}\|\psi_1\|^4 - K\sum\sum\nolimits_{(i,j)\in\Omega_0}\psi_1^4(i,j)\\ \leq&\frac{a}{2T^2}\|\psi_1\|^2 + \frac{b}{4}\|\psi_1\|^4 - K\alpha^4\Omega_0^\sharp\\ <&0,\end{aligned}$$

where $\Omega_0^\sharp$ denotes the number of (i,j) and $(i,j)\in\Omega_0$. Therefore, $J(t\psi_1)\to+\infty$ as $t\to+\infty$.

Step 3. Let

$$\Gamma=\{\gamma\in C([0,1],E):\gamma(0)=0,\gamma(1)=t_0\psi_1\}\quad\text{and}\quad c=\inf_{\gamma\in\Gamma}\max_{0\le\varpi\le1}J\left(\gamma(\varpi)\right).$$

Proposition 2.1 means that $c\ge\beta>0$ and there exists a sequence $\{x_\kappa\}_{\kappa\in\mathbf{N}}\subset E$ such that

$$J(x_\kappa)\to c,\qquad (1+\|x_\kappa\|)\|J'(x_\kappa)\|\to0,\qquad\text{as}\quad\kappa\to+\infty.$$

In the following, we are to show that the sequence $\{x_\kappa\}_{\kappa\in\mathbf{N}}\subset E$ possesses a convergent subsequence, still denoted by $\{x_\kappa\}$. In fact, owing to the finite dimension of space E, it is necessary to verify that $\{x_\kappa\}$ is bounded. Or else, we suppose $\|x_\kappa\|\to+\infty$. Let $\varrho=\sqrt{2}\left[(\frac{c}{a})^{\frac12}+(\frac{c}{b})^{\frac14}\right]$. Set

$$t_\kappa=\frac{\varrho}{\|x_\kappa\|},\quad \widetilde{y_\kappa}=t_\kappa x_\kappa=\frac{\varrho x_\kappa}{\|x_\kappa\|}.\tag{3.9}$$

Then, $\|\widetilde{y_\kappa}\|=\varrho$ and $\{\widetilde{y_\kappa}\}$, $\{\widetilde{y_\kappa}^+\}$ are bounded in E, which imply that there exists $\tilde{y}\in E$ such that

$$\widetilde{y_\kappa}(i,j)\to\tilde{y}(i,j),\quad \{\widetilde{y_\kappa}^+\}(i,j)\to\tilde{y}^+(i,j),\quad\text{for}\quad(i,j)\in\Omega.$$

Denote $\Omega_1=\{(i,j):(i,j)\in\Omega_0\quad\text{and}\quad y^+(i,j)>0\}$. Thus (3.9) implies that $x_\kappa^+(i,j)\to+\infty$ in Ω_1. Therefore, by $q(i,j)\equiv+\infty$, for any $\check{M}>0$ and κ large enough, we have

$$\frac{f((i,j),x_\kappa^+)}{(x_\kappa^+)^3}\ge\check{M},\quad\forall(i,j)\in\Omega_1.$$

Notice that $J(x_\kappa)\to c$, (2.8), (3.9) and (H_1) lead to

$$\begin{aligned}\frac{(J'(x_\kappa),x_\kappa)}{\|x_\kappa\|^4}&=\frac{a\|x_\kappa\|^2+b\|x_\kappa\|^4}{\|x_\kappa\|^4}-\sum_{i=1}^m\sum_{j=1}^n\left(\frac{f((i,j),x_\kappa^+(i,j))x_\kappa(i,j)}{\|x_\kappa\|^4}\right)\\&=\frac{a+b\|x_\kappa\|^2}{\|x_\kappa\|^2}-\sum_{i=1}^m\sum_{j=1}^n\left(\frac{f((i,j),x_\kappa^+(i,j))}{\varrho^4x_\kappa(i,j)^3}(\widetilde{y_\kappa}(i,j))^4\right)\\&\to0,\qquad\text{as}\quad\kappa\to\infty.\end{aligned}$$

Hence, there holds

$$\begin{aligned}b&=\lim_{\kappa\to+\infty}\sum_{i=1}^m\sum_{j=1}^n\left(\frac{f((i,j),x_\kappa^+(i,j))}{\varrho^4x_\kappa(i,j)^3}(\widetilde{y_\kappa}(i,j))^4\right)\\&=\frac{1}{\varrho^4}\lim_{\kappa\to+\infty}\sum_{i=1}^m\sum_{j=1}^n\left(\frac{f((i,j),x_\kappa^+(i,j))}{x_\kappa^+(i,j)^3}(\widetilde{y_\kappa^+}(i,j))^4\right)\\&\ge\frac{1}{\varrho^4}\lim_{\kappa\to+\infty}\sum\sum\nolimits_{(i,j)\in\Omega_1}\left(\frac{f((i,j),x_\kappa^+(i,j))}{x_\kappa^+(i,j)^3}(\widetilde{y_\kappa^+}(i,j))^4\right)\\&\ge\frac{\check{M}}{\varrho^4}\lim_{\kappa\to+\infty}\sum\sum\nolimits_{(i,j)\in\Omega_1}(\widetilde{y_\kappa^+}(i,j))^4\\&\ge\frac{\check{M}}{\varrho^4}\sum\sum\nolimits_{(i,j)\in\Omega_1}(\widetilde{y^+}(i,j))^4.\end{aligned}\tag{3.10}$$

Evidently, (3.10) means $\Omega_1 \equiv \emptyset$. Otherwise, for large $\check{M}$, (3.10) is impossible. Thus, $\widetilde{y^+}(i,j) = 0$ for all $(i,j) \in [1,m] \times [1,n]$. Moreover, by (3.8), we get

$$\lim_{\kappa\to+\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} F((i,j), y_\kappa^+(i,j)) = 0.$$

Recall $\varrho = \sqrt{2}\left[(\frac{c}{a})^{\frac{1}{2}} + (\frac{c}{b})^{\frac{1}{4}}\right]$. Therefore,

$$\begin{aligned}
\lim_{\kappa\to+\infty} J(\widetilde{y_\kappa}) &= \lim_{\kappa\to+\infty}\left(\frac{a}{2}\|\widetilde{y_\kappa}\|^2 + \frac{b}{4}\|\widetilde{y_\kappa}\|^4 - \sum_{i=1}^{m}\sum_{j=1}^{n} F((i,j), \widetilde{y_\kappa}(i,j))\right)\\
&= \frac{a\varrho^2}{2} + \frac{b\varrho^4}{4}\\
&= c + \frac{a\varrho^2}{2} + b\left((\frac{c}{a})^2 + 4(\frac{c}{a})^{\frac{3}{2}}(\frac{c}{b})^{\frac{1}{4}} + 6\frac{c}{a}(\frac{c}{b})^{\frac{1}{2}} + 4(\frac{c}{a})^{\frac{1}{2}}(\frac{c}{b})^{\frac{3}{4}}\right)\\
&= c + \frac{a\varrho^2}{2} + L,
\end{aligned} \tag{3.11}$$

where $L = b\left((\frac{c}{a})^2 + 4(\frac{c}{a})^{\frac{3}{2}}(\frac{c}{b})^{\frac{1}{4}} + 6\frac{c}{a}(\frac{c}{b})^{\frac{1}{2}} + 4(\frac{c}{a})^{\frac{1}{2}}(\frac{c}{b})^{\frac{3}{4}}\right) > 0$.

To get a contradiction, first of all, we prove that if (H_1) and (H_3) hold and a sequence $\{x_\kappa\} \subset E$ satisfying

$$J'(x_\kappa)x_\kappa \to 0, \quad \text{as} \quad \kappa \to +\infty,$$

then $\{x_\kappa\}$ possesses a subsequence, denoted by $\{x_\kappa\}$ once more, such that

$$J(tx_\kappa) \le \frac{at^2}{2}\|x_\kappa\|^2 + \frac{1+t^4}{4\kappa} + J(x_\kappa), \quad \forall t > 0, \quad \kappa \ge 1. \tag{3.12}$$

Since $J'(x_\kappa)x_\kappa \to 0$ as $\kappa \to +\infty$, for a subsequence $\{x_\kappa\}$, we may assume that

$$-\frac{1}{\kappa} < J'(x_\kappa)x_\kappa = (a + b\|x_\kappa\|^2)\|x_\kappa\|^2 - \sum_{i=1}^{m}\sum_{j=1}^{n} f((i,j), x_\kappa(i,j))x_\kappa(i,j) < \frac{1}{\kappa}, \quad \forall \kappa \ge 1. \tag{3.13}$$

Hence, for any $t > 0$ and positive integer κ, by (3.13), it follows that

$$\begin{aligned}
J(tx_\kappa) &= \frac{at^2}{2}\|x_\kappa\|^2 + \frac{bt^4}{4}\|x_\kappa\|^4 - \sum_{i=1}^{m}\sum_{j=1}^{n} F((i,j), tx_\kappa(i,j))\\
&\le \frac{at^2}{2}\|x_\kappa\|^2 + \frac{t^4}{4}[(a+b)\|x_\kappa\|^2]\|x_\kappa\|^2 - \sum_{i=1}^{m}\sum_{j=1}^{n} F((i,j), tx_\kappa(i,j))\\
&\le \frac{at^2}{2}\|x_\kappa\|^2 + \frac{t^4}{4}\left(\frac{1}{\kappa} + \sum_{i=1}^{m}\sum_{j=1}^{n} f((i,j), x_\kappa(i,j))x_\kappa(i,j)\right) - \sum_{i=1}^{m}\sum_{j=1}^{n} F((i,j), tx_\kappa(i,j))\\
&= \frac{at^2}{2}\|x_\kappa\|^2 + \frac{t^4}{4\kappa} + \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\frac{t^4}{4} f((i,j), x_\kappa(i,j))x_\kappa(i,j) - F((i,j), tx_\kappa(i,j))\right).
\end{aligned} \tag{3.14}$$

For any fixed $(i, j) \in [1, m] \times [1, n]$ and $\kappa \geq 1$, set

$$h(t) = \frac{t^4}{4} f((i, j), x_\kappa)x_\kappa - F((i, j), tx_\kappa).$$

By (H_1) and (H_2), direct computation yields that

$$\frac{dh(t)}{dt} = t^3 f((i, j), x_\kappa)x_\kappa - f((i, j), tx_\kappa)x_\kappa = t^3 x_\kappa \left(f((i, j), x_\kappa) - \frac{f((i, j), tx_\kappa)}{t^3} \right),$$

and

$$\frac{dh(t)}{dt} \geq 0, \quad 0 < t \leq 1; \qquad \frac{dh(t)}{dt} \leq 0, \quad t \geq 1.$$

Then,

$$h(t) \leq h(1) = \frac{1}{4} f((i, j), x_\kappa)x_\kappa - F((i, j), x_\kappa), \quad \forall t > 0.$$

Therefore, (3.14) implies that

$$J(tx_\kappa) \leq \frac{at^2}{2}\|x_\kappa\|^2 + \frac{t^4}{4\kappa} + \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\frac{1}{4} f((i, j), x_\kappa(i, j))x_\kappa(i, j) - F((i, j), x_\kappa(i, j))\right). \tag{3.15}$$

Moreover, by (3.13), we have

$$\begin{aligned} J(x_\kappa) &= \frac{a}{2}\|x_\kappa\|^2 + \frac{b}{4}\|x_\kappa\|^4 - \sum_{i=1}^{m}\sum_{j=1}^{n} F((i, j), x_\kappa(i, j)) \\ &= \frac{1}{4}(2a + b\|x_\kappa\|^2)\|x_\kappa\|^2 - \sum_{i=1}^{m}\sum_{j=1}^{n} F((i, j), x_\kappa(i, j)) \\ &\geq \frac{1}{4}(a + b\|x_\kappa\|^2)\|x_\kappa\|^2 - \sum_{i=1}^{m}\sum_{j=1}^{n} F((i, j), x_\kappa(i, j)) \\ &\geq \frac{1}{4}\sum_{i=1}^{m}\left(f((i, j), x_\kappa(i, j))x_\kappa(i, j) - \frac{1}{\kappa}\right) - \sum_{i=1}^{m}\sum_{j=1}^{n} F((i, j), x_\kappa(i, j)), \end{aligned}$$

which ensures that

$$\sum_{i=1}^{m}\left(\frac{1}{4} f((i, j), x_\kappa(i, j))x_\kappa(i, j) - F((i, j), x_\kappa(i, j))\right) \leq \frac{1}{4\kappa} + J(x_\kappa). \tag{3.16}$$

As a result, (3.15) and (3.16) guarantee that (3.12) holds.

In view of (3.12), we have

$$\begin{aligned} \lim_{\kappa\to+\infty} J(\widetilde{y_\kappa}) = \lim_{\kappa\to+\infty} J(t_\kappa x_\kappa) &\leq \lim_{\kappa\to+\infty}\left(\frac{at_\kappa^2}{2}\|x_\kappa\|^2 + \frac{1 + t_\kappa^4}{4\kappa} + J(x_\kappa)\right) \\ &\leq \frac{a\varrho^2}{2} + \lim_{\kappa\to+\infty}\left(\frac{1 + t_\kappa^4}{4\kappa} + J(x_\kappa)\right) \\ &= c + \frac{a\varrho^2}{2}. \end{aligned} \tag{3.17}$$

Evidently, (3.11) contradicts (3.17). Subsequently, $\{x_\kappa\}$ is bounded in E. Thus, all conditions of Proposition 2.1 are verified and the proof of Theorem 2.4 is completed.

Proof of Theorem 2.5. In order to complete the proof by Proposition 2.1, what is first to do is to prove that if (H_1), (H_4) and (H_5) are satisfied and $q(i,j) \equiv +\infty$, then $J(x)$ satisfies $(C)_c$. To this end, we assume that $\{x_\kappa\}_{\kappa\in\mathbf{N}} \subset E$ is the $(C)_c$ sequence, that is, for $c \in \mathbf{R}$,

$$J(x_\kappa) \to c, \quad (1 + \|x_\kappa\|)\|J'(x_\kappa)\| \to 0, \quad \text{as} \quad \kappa \to +\infty. \tag{3.18}$$

Then, for κ large enough, (2.5) and (2.8) yield that

$$1 + c \geq J(x_\kappa) - \frac{1}{4}(J'(x_\kappa), x_\kappa) = \frac{a}{4}\|x_\kappa\|^2 + \sum_{i=1}^{m}\left(\frac{1}{4}f((i,j), x_\kappa(i,j))x_\kappa(i,j) - F((i,j), x_\kappa(i,j))\right). \tag{3.19}$$

Since E is an mn-dimensional Hilbert space, it is sufficient to show that $\{x_\kappa\}$ possesses a bounded subsequence, still denoted by $\{x_\kappa\}$. Or else, we may assume that $\|x_\kappa\| \to +\infty$ as $\kappa \to +\infty$. Set $\widehat{y_\kappa} = \frac{x_\kappa}{\|x_\kappa\|}$, then $\|\widehat{y_\kappa}\| = 1$ and $\{y_\kappa\}$ is bounded. Thus, there exists $\hat{y} \in E$ such that $\widehat{y_\kappa}(i,j) \to \hat{y}(i,j)$ holds for all $(i,j) \in [1,m] \times [1,n]$.

We suppose that $\hat{y} \neq 0$. Because $\|x_\kappa\| \to +\infty$, we have $|x_\kappa| \to +\infty$ as $\kappa \to +\infty$. For $q(i,j) \equiv +\infty$, it follows that

$$\lim_{\kappa\to+\infty} \frac{f((i,j), x_\kappa^+(i,j))}{(x_\kappa^+(i,j))^3} = +\infty, \quad \forall (i,j) \in [1,m] \times [1,n]. \tag{3.20}$$

Meanwhile, (2.8) and (H_1) induce that

$$\begin{aligned}\langle J'(x_\kappa), x_\kappa\rangle &= a\|x_\kappa\|^2 + b\|x_\kappa\|^4 - \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{4}(f((i,j), x_\kappa(i,j)) \cdot x_\kappa(i,j)) \\ &= \|x_\kappa\|^4 \cdot \left(\frac{a}{\|x_\kappa\|^2} + b - \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\frac{f((i,j), x_\kappa^+(i,j))}{(x_\kappa^+(i,j))^3} \cdot \widehat{y_\kappa}^4(i,j)\right)\right).\end{aligned}$$

Together with (3.18), it follows that

$$\frac{\langle J'(x_\kappa), x_\kappa\rangle}{\|x_\kappa\|^4} = \frac{a}{\|x_\kappa\|^2} + b - \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{f((i,j), x_\kappa^+(i,j))}{(x_\kappa^+(i,j))^3} \cdot \widehat{y_\kappa}^4(i,j) \to 0, \quad \text{as} \quad \kappa \to +\infty.$$

Hence,

$$b \geq \liminf_{\kappa\to+\infty} \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{f((i,j), x_\kappa(i,j))}{x_\kappa(i,j)^3} \cdot \widehat{y_\kappa}^4(i,j) = +\infty,$$

which is a contradiction.

We suppose that $\hat{y} = 0$. Let $\{\ell_\kappa\}$ be a sequence of real numbers such that $J(\ell_\kappa x_\kappa) = \max_{\ell\in[0,1]} J(\ell x_\kappa)$. For any integer $s > 0$, set $\widehat{y_\kappa}^s = (\frac{8s}{b})^{1/4}\widehat{y_\kappa}$. By (H_4), we have

$$|F((i,j), x)| \leq c|x|^k + \tilde{c}|x|.$$

Note that $\widehat{y_\kappa}^s \to 0$ as $\kappa \to +\infty$ and $F(\cdot, x)$ is continuous in x, we achieve

$$\lim_{\kappa\to+\infty} F((i,j), \widehat{y_\kappa}^s) = 0, \qquad \forall (i,j) \in [1,m] \times [1,n]. \tag{3.21}$$

Since $\|x_\kappa\| \to +\infty$ as $\kappa \to +\infty$, we obtain $0 \le \frac{(\frac{8s}{b})^{1/4}}{\|x_\kappa\|} \le 1$ is true as κ large enough. Together with the definitions of $J(\ell_\kappa x_\kappa)$ and ℓ_κ, it yields that

$$\begin{aligned} J(\ell_\kappa x_\kappa) &\ge J\left(\frac{(\frac{8s}{b})^{1/4}}{\|x_\kappa\|}x_\kappa\right) = J(\widehat{y_\kappa}^s) = \frac{a}{2}\|\widehat{y_\kappa}^s\|^2 + \frac{b}{4}\|\widehat{y_\kappa}^s\|^4 - \sum_{i=1}^{m}\sum_{j=1}^{n} F((i,j),\widehat{y_\kappa}^s(i,j)) \\ &\ge 2s - \sum_{i=1}^{m}\sum_{j=1}^{n} F((i,j),\widehat{y_\kappa}^s(i,j)). \end{aligned} \tag{3.22}$$

Consider $s > 0$ is arbitrary, (3.21) and (3.22) imply that

$$J(\ell_\kappa x_\kappa) \to +\infty, \qquad \kappa \to +\infty. \tag{3.23}$$

For $0 \le \ell_\kappa \le 1$, (H_5) means that there exists $\theta \ge 1$ such that $\theta G((i,j),x_\kappa) \ge G((i,j),\ell_\kappa x_\kappa)$. Notice that $J(0) = 0$ and $J(x_\kappa) \to c$, then $0 < \ell_\kappa < 1$ for κ large enough. Therefore,

$$\begin{aligned} \langle J'(\ell_\kappa x_\kappa), \ell_\kappa x_\kappa\rangle &= a\|\ell_\kappa x_\kappa\|^2 + b\|\ell_\kappa x_\kappa\|^4 - \sum_{i=1}^{m}\sum_{j=1}^{n} \ell_\kappa x_\kappa \cdot f((i,j),\ell_\kappa x_\kappa(i,j)) \\ &= \ell_\kappa \frac{dJ(\ell x_\kappa)}{d\ell}\Big|_{\ell=\ell_\kappa} = 0, \end{aligned}$$

that is,

$$a\|\ell_\kappa x_\kappa\|^2 + b\|\ell_\kappa x_\kappa\|^4 = \sum_{i=1}^{m}\sum_{j=1}^{n} \ell_\kappa x_\kappa \cdot f((i,j),\ell_\kappa x_\kappa(i,j)). \tag{3.24}$$

Combining (3.23) with (3.24), it follows that

$$\begin{aligned} &\frac{a}{4}\|x_\kappa\|^2 + \frac{1}{4}\sum_{i=1}^{m}\sum_{j=1}^{n} G((i,j),x_\kappa(i,j)) \\ \ge&\frac{a}{4\theta}\|\ell_\kappa x_\kappa\|^2 + \frac{1}{4\theta}\sum_{i=1}^{m}\sum_{j=1}^{n} G((i,j),\ell_\kappa x_\kappa(i,j)) \\ =&\frac{1}{\theta}\left[\frac{a}{4}\|\ell_\kappa x_\kappa\|^2 + \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\frac{1}{4}\ell_\kappa x_\kappa \cdot f((i,j),\ell_\kappa x_\kappa(i,j)) - F((i,j),\ell_\kappa x_\kappa(i,j))\right)\right] \\ =&\frac{1}{\theta}\left[\frac{a}{2}\|\ell_\kappa x_\kappa\|^2 + \frac{b}{4}\|\ell_\kappa x_\kappa\|^4 - \sum_{i=1}^{m}\sum_{j=1}^{n} F((i,j),\ell_\kappa x_\kappa(i,j))\right] \\ =&\frac{1}{\theta}J(\ell_\kappa x_\kappa) \to +\infty, \qquad \text{as} \quad \kappa \to +\infty. \end{aligned}$$

Namely, as $\kappa \to +\infty$, there has

$$\frac{a}{4}\|\ell_\kappa x_\kappa\|^2 + \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\frac{1}{4}\ell_\kappa x_\kappa \cdot f((i,j),\ell_\kappa x_\kappa(i,j)) - F((i,j),\ell_\kappa x_\kappa(i,j))\right) \to +\infty,$$

which contradicts (3.19). Thus, $\{x_\kappa\}$ is bounded. Therefore, $J(x)$ satisfies $(C)_c$.

Next, we claim that there exist some $\rho, \beta > 0$ such that $J(x) \geq \beta$ for all $x \in E$ with $\|x\| = \rho$.

In fact, (H_5) and (H_6) imply that there exists $C_1 > 0$ such that

$$F((i,j),x) \leq \frac{a}{2}\lambda_1 x^2 + C_1|x|^k, \quad [i,j] \in [1,m] \times [1,n], \quad x \in \mathbf{R}. \tag{3.25}$$

Thus, by (2.2), (2.3), (2.5) and (3.25), it follows that

$$\begin{aligned} J(x) =& \frac{a}{2}\|x\|^2 + \frac{b}{4}\|x\|^4 - \sum_{i=1}^{m}\sum_{j=1}^{n} F((i,j),x(i,j)) \\ \geq& \frac{a}{2}\|x\|^2 + \frac{b}{4}\|x\|^4 - \frac{a\lambda_1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} x^2(i,j) - C_1\sum_{i=1}^{m}\sum_{j=1}^{n}|x(i,j)|^k \\ \geq& \frac{b}{4}\|x\|^4 - C_1\eta_k^k\|x\|^k, \end{aligned} \tag{3.26}$$

where $k > 4$. Given a small $\rho > 0$, (3.26) means that

$$J(x) \geq \beta \triangleq \frac{b}{4}\rho^4 - C_1\eta_k^k\rho^k > 0, \qquad x \in E \quad \text{and} \quad \|x\| = \rho.$$

Last, we show there exists $\breve{x} \in E$ with $\|\breve{x}\| > \rho$ such that $J(\breve{x}) < 0$. Since $q(i,j) \equiv +\infty$, that is, $\lim\limits_{x\to+\infty} \frac{f((i,j),x)}{x^3} \equiv +\infty$, for any $\varepsilon > 0$, there exists $\breve{M} > 0$ such that $\frac{f((i,j),x)}{x^3} \geq \frac{1}{\varepsilon}$ holds for all $x > \breve{M}$ and $(i,j) \in [0,1+m] \times [0,n+1]$. Set $c(\varepsilon) = \frac{\breve{M}^3}{\varepsilon}$, then,

$$f((i,j),x) \geq \frac{1}{\varepsilon}x^3 - c(\varepsilon), \qquad x \geq 0, \quad (i,j) \in [0,1+m] \times [0,n+1].$$

Therefore, for all $x \geq 0$, $(i,j) \in [0,1+m] \times [0,n+1]$ and $0 \leq \omega \leq 1$, we have

$$f((i,j),\omega x) \geq \frac{1}{\varepsilon}\omega^3 x^3 - c(\varepsilon).$$

Then,

$$f((i,j),\omega x)x \geq \frac{1}{\varepsilon}\omega^3 x^4 - c(\varepsilon)x, \quad x \geq 0. \tag{3.27}$$

Integrating both sides of (3.27) on [0, 1] with respect to ω, we have

$$F((i,j),x) \geq \frac{1}{4\varepsilon}x^4 - c(\varepsilon)x, \quad x \geq 0,$$

which ensures that

$$F((i,j),t\varphi_1) \geq \frac{1}{4\varepsilon}t^4\varphi_1^4 - c(\varepsilon)t\varphi_1, \tag{3.28}$$

where φ_1 is the normal eigenfunction corresponding to λ_1. Dividing by t^4, (3.28) means that

$$\frac{F((i,j),t\varphi_1)}{t^4} \geq \frac{1}{4\varepsilon}\varphi_1^4 - \frac{c(\varepsilon)\varphi_1}{t^3}.$$

Thus,

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{F((i,j),t\varphi_1)}{t^4} \geq \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\frac{1}{4\varepsilon}\varphi_1^4 - \frac{c(\varepsilon)\varphi_1}{t^3}\right). \tag{3.29}$$

Let $t \to +\infty$, (3.29) leads to

$$\liminf_{t\to+\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{F((i,j),t\varphi_1)}{t^4} \geq \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{4\varepsilon}\varphi_1^4. \tag{3.30}$$

Notice that $\varepsilon > 0$ is arbitrary and let $\varepsilon \to 0$, then (3.30) indicates that

$$\liminf_{t\to+\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{F((i,j),t\varphi_1)}{t^4} = +\infty.$$

Therefore,

$$\frac{J(t\varphi_1)}{t^4} = \frac{a\|\varphi_1\|^2}{2t^2} + \frac{b\|\varphi_1\|^2}{4} - \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{F((i,j),t\varphi_1)}{t^4} \to -\infty, \quad \text{as} \quad t \to +\infty.$$

Therefore, there exists t_0 large enough such that $J(\check{x}) < 0$, where $\check{x} = t_0\varphi_1$.

Define

$$\Gamma = \{\gamma \in C([0,1],E) : \gamma(0) = 0, \gamma(1) = \check{x}\}, \qquad c = \inf_{\gamma\in\Gamma}\max_{0\leq\varpi\leq1} J(\gamma(\varpi)),$$

then $c \geq \beta > 0$. Hence, Proposition 2.1 guarantees that J has at least a nontrivial critical point. Therefore, the proof of Theorem 2.5 is completed.

4. Conclusions

Due to their wide applications, partial difference equations have been studied extensively. We all know that the discrete Kirchhoff term

$$b(\sum_{j=1}^{n}\sum_{i=1}^{m+1}|\Delta_1 x(i-1,j)|^2 + \sum_{i=1}^{m}\sum_{j=1}^{n+1}|\Delta_2 x(i,j-1)|^2)(\Delta_1^2 x(i-1,j) + \Delta_2^2 x(i,j-1))$$

makes it not only more difficult but also more interesting to study. In this paper, we investigate the existence and nonexistence of positive solutions to a class of partial difference equations which involve the discrete Kirchhoff term. First, we established the corresponding variational functional on a suitable variational function space. Then, we obtained a series of results on the existence and nonexistence of positive solutions via a variant version of the mountain pass theorem. The conditions in our obtained results release the classical (AR) condition.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no competing interest in this paper.

References

1. C. O. Alves, F. J. S. A. Correa, T. F. Ma, Positive solutions for a quasilinear elliptic equation of Kirchhoff type, *Comput. Math. Appl.*, **49** (2005), 85–93. https://doi.org/10.1016/j.camwa.2005.01.008
2. K. Perera, Z. T. Zhang, Nontrivial solutions of Kirchhoff-type problems via the Yang index, *J. Differ. Equ.*, **221** (2006), 246–255. https://doi.org/10.1016/j.jde.2005.03.006
3. Z. T. Zhang, K. Perera, Sign changing solutions of Kirchhoff type problems via invariant sets of descent flow, *J. Math. Anal. Appl.*, **317** (2006), 456–463. https://doi.org/10.1016/j.jmaa.2005.06.102
4. A. M. Mao, Z. T. Zhang, Sign-changing and multiple solutions of Kirchhoff type problems without P.S. condition, *Nonlinear Anal.*, **70** (2009), 1275–1287. https://doi.org/10.1016/j.na.2008.02.011
5. S. Gupta, G. Dwivedi, Ground state solution to N-Kirchhoff equation with critical exponential growth and without Ambrosetti-Rabinowitz condition, *Rend. Circ. Mat. Palermo Ser. 2*, 2023. https://doi.org/10.1007/s12215-023-00902-7
6. X. M. He, W. M. Zou, Existence and concentration behavior of positive solutions for a Kirchhoff equation in $\mathbb{R}^3$, *J. Differ. Equ.*, **252** (2012), 1813–1834. https://doi.org/10.1016/j.jde.2011.08.035
7. K. Wu, F. Zhou, G. Z. Gu, Some remarks on uniqueness of positive solutions to Kirchhoff type equations, *Appl. Math. Lett.*, **124** (2022), 107642. https://doi.org/10.1016/j.aml.2021.107642
8. Y. H. Long, L. Wang, Global dynamics of a delayed two-patch discrete SIR disease model, *Commun. Nonlinear Sci. Numer. Simul.*, **83** (2020), 105117. https://doi.org/10.1016/j.cnsns.2019.105117
9. J. S. Yu, J. Li, Discrete-time models for interactive wild and sterile mosquitoes with general time steps, *Math. Biosci.*, **346** (2022), 108797. https://doi.org/10.1016/j.mbs.2022.108797
10. J. S. Yu, Z. M. Guo, X. F. Zou, Periodic solutions of second order self-adjoint difference equations, *J. Lond. Math. Soc.*, **71** (2005), 146–160. https://doi.org/10.1112/S0024610704005939
11. Z. Zhou, J. X. Ling, Infinitely many positive solutions for a discrete two point nonlinear boundary value problem with ϕ_c-Laplacian, *Appl. Math. Lett.*, **91** (2019), 28–34. https://doi.org/10.1016/j.aml.2018.11.016
12. J. H. Kuang, Z. M. Guo, Heteroclinic solutions for a class of p-Laplacian difference equations with a parameter, *Appl. Math. Lett.*, **100** (2020), 106034. https://doi.org/10.1016/j.aml.2019.106034

13. A. Nastasi, S. Tersian, C. Vetro, Homoclinic solutions of nonlinear Laplacian difference equations without Ambrosetti-Rabinowitz condition, *Acta Math. Sci.*, **41** (2021), 712–718. https://doi.org/10.1007/s10473-021-0305-z

14. S. S. Cheng, *Partial difference equations*, London: Taylor and Francis, 2003.

15. H. Zhang, Y. H. Long, Multiple existence results of nontrivial solutions for a class of second-order partial difference equations, *Symmetry*, **15** (2023), 1–14. https://doi.org/10.3390/sym15010006

16. Y. H. Long, H. Zhang, Three nontrivial solutions for second-order partial difference equation via Morse theory, *J. Funct. Spaces*, **2022** (2022), 1–9. https://doi.org/10.1155/2022/1564961

17. Y. H. Long, D. Li, Multiple nontrivial periodic solutions to a second-order partial difference equation, *Electron. Res. Arch.*, **31** (2023), 1596–1612. https://doi.org/10.3934/era.2023082

18. Y. H. Long, D. Li, Multiple periodic solutions of a second-order partial difference equation involving p-Laplacian, *J. Appl. Math. Comput.*, **69** (2023), 3489–3508. https://doi.org/10.1007/s12190-023-01891-7

19. S. H. Wang, Z. Zhou, Periodic solutions for a second-order partial difference equation, *J. Appl. Math. Comput.*, **69** (2023), 731–752. https://doi.org/10.1007/s12190-022-01769-0

20. M. Bohner, G. Caristi, A. Ghobadi, S. Heidarkhani, Three solutions for discrete anisotropic Kirchhoff-type problems, *Demonstratio Math.*, **56** (2023), 1–13. https://doi.org/10.1515/dema-2022-0209

21. Y. H. Long, X. Q. Deng, Existence and multiplicity solutions for discrete Kirchhoff type problems, *Appl. Math. Lett.*, **126** (2022), 107817. https://doi.org/10.1016/j.aml.2021.107817

22. Y. H. Long, Multiple results on nontrivial solutions of discrete Kirchhoff type problems, *J. Appl. Math. Comput.*, **69** (2023), 1–17. https://doi.org/10.1007/s12190-022-01731-0

23. Y. H. Long, Q. Q. Zhang, Infinitely many large energy solutions to a class of nonlocal discrete elliptic boundary value problems, *Commun. Pure Appl. Math.*, **22** (2023), 1545–1564. https://doi.org/10.3934/cpaa.2023037

24. Y. H. Long, Nontrivial solutions of discrete Kirchhoff-type problems via Morse theory, *Adv. Nonlinear Anal.*, **11** (2022), 1352–1364. https://doi.org/10.1515/anona-2022-0251

25. I. Ekeland, *Convexity methods in Hamiltonian mechanics*, Berlin, Heidelberg: Springer, 1990. https://doi.org/10.1007/978-3-642-74331-3

26. J. Ji, B. Yang, Eigenvalue comparisons for boundary value problems of the discrete elliptic equation, *Commun. Appl. Anal.*, **12** (2008), 189–198.

27. G. Cerami, Un criterio di esistenza per i punti critici su varietà illimitate, *Rend. Instituto Lombardo Sci. Lett.*, **112** (1978), 332–336.