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Research article

Ground states to a Kirchhoff equation with fractional Laplacian

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Abstract: The aim of this paper is to deal with the Kirchhoff type equation involving fractional Laplacian operator

$$\left(\alpha+\beta\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\psi|^2\,\mathrm{d}x\right)(-\Delta)^s\psi+\kappa\psi=|\psi|^{p-2}\psi\quad\text{in }\mathbb{R}^3,$$

where $\alpha,\beta,\kappa>0$ are constants. By constructing a Palais-Smale-Pohozaev sequence at the minimax value c_{mp}, the existence of ground state solutions to this equation for all $p\in(2,2_s^*)$ is established by variational arguments. Furthermore, the decay property of the ground state solution is also investigated.

Keywords: fractional Kirchhoff equation; ground state solution; (PSP)-sequence; variational methods
Mathematics Subject Classification: 35J60, 35B40, 35R11

1. Introduction

We are concerned with the existence and decay property of ground state solutions for the following Kirchhoff equation involving fractional Laplacian operator:

$$\begin{cases}\left(\alpha+\beta\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\psi|^2\,\mathrm{d}x\right)(-\Delta)^s\psi+\kappa\psi=|\psi|^{p-2}\psi\quad\text{in }\mathbb{R}^3,\\ \psi(x)\in H^s(\mathbb{R}^3),\end{cases}\tag{1.1}$$

where $\alpha,\beta,\kappa>0$ are positive constants, $s\in(0,1)$, $2<p<2_s^*=\frac{6}{3-2s}$ and the fractional Laplacian $(-\Delta)^s$ is given by

$$(-\Delta)^s\psi(x)=C_s\mathrm{P.V.}\int_{\mathbb{R}^3}\frac{\psi(x)-\psi(y)}{|x-y|^{3+2s}}\mathrm{d}y,$$

where

$$C_s = s(1-s)4^s \frac{\Gamma(\frac{3}{2}+s)}{\pi^{\frac{3}{2}}\Gamma(2-s)}.$$

The fractional Laplacian operator $(-\Delta)^s$ has a wide range of applications arising in some physical phenomena such as fractional quantum mechanics, flames propagation, etc. (see [10, 13]). In recent years, problems involving fractional Laplacian operators and Kirchhoff-type nonlocal terms have been discussed by lots of researchers for their broad applications. Some remarkable results have been yielded, see [1, 2, 6–8, 12, 14, 15] and the references therein. In particular, when $\alpha = 1, \beta = 0$ and $\mathbb{R}^3$ is replaced by $\mathbb{R}^N (N \geq 2)$, equation (1.1) turns into the classical fractional Laplacian problem

$$\begin{cases} (-\Delta)^s\psi + \kappa\psi = |\psi|^{p-2}\psi \quad \text{in } \mathbb{R}^N, \\ \psi(x) \in H^s(\mathbb{R}^N). \end{cases} \tag{1.2}$$

In [4], employing the constrained variational methods, Dipierro et al. studied the existence and symmetry of nontrivial solutions for (1.2) as $s \in (0,1)$ and $p \in (2, \frac{2N}{N-2s})$.

In this paper, we intend to consider the fractional Kirchhoff equation (1.1) with $p \in (2, 2_s^*)$ using variational arguments, and we encounter several difficulties to overcome. First, note that solutions of Eq (1.1) correspond to critical points of the following functional:

$$\mathcal{E}(\psi) = \frac{\alpha}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}\psi|^2\,\mathrm{d}x + \frac{\beta}{4}\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}\psi|^2\,\mathrm{d}x\right)^2 + \frac{\kappa}{2}\int_{\mathbb{R}^3}|\psi|^2\,\mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^3}|\psi|^p\,\mathrm{d}x.$$

Since the nonlocal term $(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}\psi|^2\,\mathrm{d}x)^2$ included in the energy functional $\mathcal{E}(\psi)$ is homogeneous of degree 4, and the nonlinearity $|\psi|^{p-2}\psi$ does not satisfy the global Ambrosetti-Rabinowitz type condition for $p \in (2, 2_s^*)$, it would bring about more difficulties to establish the boundedness of (PS)-sequence for $\mathcal{E}(\psi)$ when $p \leq 4$. Second, in general, from $\psi_n \rightharpoonup \psi$ in $H^s(\mathbb{R}^3)$, we do not know whether there holds

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}\psi_n|^2\,\mathrm{d}x \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}}\psi_n(-\Delta)^{\frac{s}{2}}\xi\,\mathrm{d}x \to \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}\psi|^2\,\mathrm{d}x \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}}\psi(-\Delta)^{\frac{s}{2}}\xi\,\mathrm{d}x,\ \forall\, \xi \in H^s(\mathbb{R}^3),$$

which is vital when we consider the convergence of the (PS)-sequence.

We now give the main result.

Theorem 1.1. *Let $s \in (\frac{3}{4}, 1)$ and $p \in (2, 2_s^*)$. Then, Eq (1.1) has a ground state solution $\psi_0(x) \in H^s(\mathbb{R}^3)$, namely*

$$\mathcal{E}(\psi_0) = \mathfrak{m} := \inf\left\{\mathcal{E}(\psi) :\ \mathcal{E}'(\psi) = 0, \psi \in H^s(\mathbb{R}^3) \setminus \{0\}\right\}.$$

Moreover, $\psi_0(x) \leq \frac{C}{1+|x|^{3+2s}}$ for some constant $C > 0$.

Remark 1.1. *In Theorem 1.1, we give the existence result for all $p \in (2, 2_s^*)$, our result could be viewed as an extension of one of the main results in [11] (see in particular Theorem 1.4 there), which only dealt with the case $p \in (3, 2_s^*)$ with $s = 1$, the case $p \in (2, 3]$ having been left open.*

2. Proof of Theorem 1.1

In this paper, we use the notation $\|\psi\|_{L^q(\mathbb{R}^3)} = (\int_{\mathbb{R}^3} |\psi|^q\,\mathrm{d}x)^{\frac{1}{q}}$ to denote the norm of $L^q(\mathbb{R}^3), q \in [1, +\infty)$. For $s \in (0, 1)$, the fractional Sobolev space $H^s(\mathbb{R}^3)$ is defined as

$$H^s(\mathbb{R}^3) = \left\{\psi \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{3+2s}}\,\mathrm{d}x\mathrm{d}y < +\infty\right\},$$

endowed with the norm

$$\|\psi\|_{H^s} = \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{3+2s}}\,\mathrm{d}x\mathrm{d}y + \|\psi\|^2_{L^2(\mathbb{R}^3)}\right)^{\frac{1}{2}}.$$

Note that by Propositions 3.4 and 3.6 in [13], one has

$$\|(-\Delta)^{\frac{s}{2}}\psi\|^2_{L^2(\mathbb{R}^3)} = \frac{C_s}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{3+2s}}\,\mathrm{d}x\mathrm{d}y,\ \forall\, \psi \in H^s(\mathbb{R}^3),$$

and so $\|\psi\|_{H^s(\mathbb{R}^3)} = (\int_{\mathbb{R}^3}(|(-\Delta)^{\frac{s}{2}}\psi|^2 + |\psi|^2)\,\mathrm{d}x)^{\frac{1}{2}}$ is an equivalent norm to $\|\psi\|_{H^s}$ for $\psi \in H^s(\mathbb{R}^3)$.

For fixed $\alpha, \kappa > 0$, we also employ the norm $\|\psi\| = (\int_{\mathbb{R}^3}(\alpha|(-\Delta)^{\frac{s}{2}}\psi|^2 + \kappa|\psi|^2)\,\mathrm{d}x)^{\frac{1}{2}}$, which is an equivalent norm to $\|\psi\|_{H^s(\mathbb{R}^3)}$. $\mathcal{D}^{s,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|\psi\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)} = (\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\psi|^2\,\mathrm{d}x)^{\frac{1}{2}}$. Now, we recall the following fractional Sobolev embedding results.

Lemma 2.1. *(See [13]) Let $s \in (0,1)$. Then, the embeddings $\mathcal{D}^{s,2}(\mathbb{R}^3) \hookrightarrow L^{2_s^*}(\mathbb{R}^3)$ and $H^s(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)(q \in [2, 2_s^*])$ are continuous, and the embedding $H^s(\mathbb{R}^3) \hookrightarrow L^q_{loc}(\mathbb{R}^3)(q \in [1, 2_s^*))$ is compact.*

We define the minimax value

$$c_{mp} := \inf_{\zeta\in\Lambda}\max_{t\in[0,1]} \mathcal{E}(\zeta(t)), \tag{2.1}$$

where

$$\Lambda = \{\zeta \in C([0,1], H^s(\mathbb{R}^3)) : \zeta(0) = 0, \mathcal{E}(\zeta(1)) < 0\}.$$

First, we show that $\Lambda \neq \emptyset$, it is sufficient to prove the following lemma.

Lemma 2.2. *Let $s \in (\frac{3}{4}, 1)$ and $p \in (2, 2_s^*)$. Then, there exists $\psi_* \in H^s(\mathbb{R}^3)$ such that $\mathcal{E}(\psi_*) < 0$.*

Proof. We consider the following perturbation functional $\mathcal{E}_\eta$ defined by

$$\mathcal{E}_\eta(\psi) = \frac{\alpha}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\psi|^2\,\mathrm{d}x + \frac{\beta}{4}\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\psi|^2\,\mathrm{d}x\right)^2 - \int_{\mathbb{R}^3} G_\eta(\psi)\,\mathrm{d}x, \tag{2.2}$$

where $G_\eta(\psi) := \frac{\eta}{p}|\psi|^p - \frac{\kappa}{2}|\psi|^2$ and $\eta \in [\eta_0, 1]$ is a parameter, $\eta_0 \in (0,1)$ is a positive constant. Now, we take $t_0 > 0$ such that

$$G_{\eta_0}(t_0) = \frac{\eta_0}{p}t_0^p - \frac{\kappa}{2}t_0^2 > 0,$$

and for $\ell > 0$ define

$$\varphi_\ell(x) = \begin{cases} t_0, \text{ if } |x| \le \ell, \\ (\ell + 1 - |x|)t_0, \text{ if } \ell < |x| \le \ell + 1, \\ 0, \text{ if } |x| > \ell + 1. \end{cases}$$

By the definition of $\varphi_\ell(x)$, clearly $\varphi_\ell(x) \in H^s(\mathbb{R}^3)$ and $\|\varphi_\ell\|_{H^s(\mathbb{R}^3)} \to +\infty$ as $\ell \to +\infty$. Moreover, by direct calculations (see Lemma 2.6 in [4]), we conclude that

$$\|(-\Delta)^{\frac{s}{2}}\varphi_\ell\|^2_{L^2(\mathbb{R}^3)} \le C(s,\ell)t_0^2,$$

and

$$\int_{\mathbb{R}^3} G_{\eta_0}(\varphi_\ell(x))\,\mathrm{d}x = \int_{B_\ell} G_{\eta_0}(\varphi_\ell(x))\,\mathrm{d}x + \int_{B_{\ell+1}\setminus B_\ell} G_{\eta_0}(\varphi_\ell(x))\,\mathrm{d}x$$
$$\geq |B_\ell| G_{\eta_0}(t_0) - |B_{\ell+1}\setminus B_\ell| \max_{t\in[0,t_0]} |G_{\eta_0}(t)|$$
$$\geq \frac{4\pi\ell^3}{3} G_{\eta_0}(t_0) - C_0((\ell+1)^3 - \ell^3)$$
$$\geq C_1\ell^3 - C_2\ell^2,$$

where C_0, C_1, C_2 are constants depending on t_0. Thus, for sufficiently large $\ell_0 > 0$ we can get

$$\int_{\mathbb{R}^3} G_{\eta_0}(\varphi_{\ell_0}(x))\,\mathrm{d}x \geq 1.$$

Let $\varphi_{\ell_0,\sigma}(x) = \varphi_{\ell_0}(\frac{x}{\sigma})$ for $\sigma > 0$, then the following hold true:

$$\|(-\Delta)^{\frac{s}{2}}\varphi_{\ell_0,\sigma}\|^2_{L^2(\mathbb{R}^3)} = \sigma^{3-2s}\|(-\Delta)^{\frac{s}{2}}\varphi_{\ell_0}\|^2_{L^2(\mathbb{R}^3)},$$

$$\int_{\mathbb{R}^3} G_{\eta_0}(\varphi_{\ell_0,\sigma})\,\mathrm{d}x = \sigma^3 \int_{\mathbb{R}^3} G_{\eta_0}(\varphi_{\ell_0})\,\mathrm{d}x \geq \sigma^3.$$

Hence, by (2.2) and note that $6 - 4s < 3$ we have

$$\mathcal{E}_{\eta_0}(\varphi_{\ell_0,\sigma}) = \frac{\alpha\sigma^{3-2s}}{2}\|(-\Delta)^{\frac{s}{2}}\varphi_{\ell_0}\|^2_{L^2(\mathbb{R}^3)} + \frac{\beta\sigma^{6-4s}}{4}\|(-\Delta)^{\frac{s}{2}}\varphi_{\ell_0}\|^4_{L^2(\mathbb{R}^3)} - \sigma^3\int_{\mathbb{R}^3} G_{\eta_0}(\varphi_{\ell_0})\,\mathrm{d}x$$
$$\leq \frac{\alpha\sigma^{3-2s}}{2}\|(-\Delta)^{\frac{s}{2}}\varphi_{\ell_0}\|^2_{L^2(\mathbb{R}^3)} + \frac{\beta\sigma^{6-4s}}{4}\|(-\Delta)^{\frac{s}{2}}\varphi_{\ell_0}\|^4_{L^2(\mathbb{R}^3)} - \sigma^3$$
$$\to -\infty \text{ as } \sigma \to +\infty.$$

In addition, we notice that $\mathcal{E}(\psi) \leq \mathcal{E}_{\eta_0}(\psi)$ for any $\psi \in H^s(\mathbb{R}^3)$, so we obtain $\mathcal{E}(\varphi_{\ell_0,\sigma}) \to -\infty$ as $\sigma \to +\infty$. Thus, we can take $\psi_* = \varphi_{\ell_0,\sigma_0}$ such that $\mathcal{E}(\psi_*) < 0$ for $\sigma_0 > 0$ large enough.

By Lemma 2.1, for all $\psi \in H^s(\mathbb{R}^3)$, we know that $\|\psi\|^p_{L^p(\mathbb{R}^3)} \leq c\|\psi\|^p$ for some positive constant $c > 0$, and noting that $p > 2$, we deduce that

$$\mathcal{E}(\psi) = \frac{1}{2}\|\psi\|^2 + \frac{\beta}{4}\|(-\Delta)^{\frac{s}{2}}\psi\|^4_{L^2(\mathbb{R}^3)} - \frac{1}{p}\|\psi\|^p_{L^p(\mathbb{R}^3)}$$
$$\geq \frac{1}{2}\|\psi\|^2 - \frac{c}{p}\|\psi\|^p \geq \varrho_0 > 0,$$

if $\|\psi\| = \varepsilon_0 > 0$ is sufficiently small. Thus, combining with Lemma 2.2, we know that $c_{mp} \in (0, +\infty)$. Note that if $\psi \in H^s(\mathbb{R}^3)$ is a critical point of $\mathcal{E}$, then ψ satisfies the Pohozaev identity (see Lemma 2.2 in [15]):

$$\mathcal{P}(\psi) := \frac{\alpha(3-2s)}{2}\|(-\Delta)^{\frac{s}{2}}\psi\|^2_{L^2(\mathbb{R}^3)} + \frac{3\kappa}{2}\|\psi\|^2_{L^2(\mathbb{R}^3)} + \frac{\beta(3-2s)}{2}\|(-\Delta)^{\frac{s}{2}}\psi\|^4_{L^2(\mathbb{R}^3)} - \frac{3}{p}\|\psi\|^p_{L^p(\mathbb{R}^3)} = 0. \quad (2.3)$$

Next, we expound that there is a Palais-Smale-Pohozaev sequence ((PSP)-sequence, for short) at the minimax level c_{mp} defined by (2.1).

Lemma 2.3. *Let $s \in (\frac{3}{4}, 1)$, $p \in (2, 2_s^*)$ and $\kappa > 0$. Then, there is a sequence $\{\psi_n\} \subset H^s(\mathbb{R}^3)$ such that*

(i) $\mathcal{E}(\psi_n) \to c_{mp}$ *as* $n \to \infty$;
(ii) $\mathcal{E}'(\psi_n) \to 0$ *in* $(H^s(\mathbb{R}^3))^*$ *as* $n \to \infty$;
(iii) $\mathcal{P}(\psi_n) \to 0$ *as* $n \to \infty$.

Proof. For $\tau \in \mathbb{R}$ and $\varphi \in H^s(\mathbb{R}^3)$, we set $(\mathbf{S}_\tau \varphi)(x) = \varphi(e^{-\tau} x)$, and we denote $\widetilde{\mathcal{E}} : \mathbb{R} \times H^s(\mathbb{R}^3) \to \mathbb{R}$ the functional defined by

$$\begin{aligned}\widetilde{\mathcal{E}}(\tau, \varphi) =& \mathcal{E}(\mathbf{S}_\tau \varphi)\\ =& \frac{\alpha e^{(3-2s)\tau}}{2} \|(-\Delta)^{\frac{s}{2}} \varphi\|^2_{L^2(\mathbb{R}^3)} + \frac{\kappa e^{3\tau}}{2} \|\varphi\|^2_{L^2(\mathbb{R}^3)} + \frac{\beta e^{2(3-2s)\tau}}{4} \|(-\Delta)^{\frac{s}{2}} \varphi\|^4_{L^2(\mathbb{R}^3)} - \frac{e^{3\tau}}{p} \|\varphi\|^p_{L^p(\mathbb{R}^3)}.\end{aligned}$$

Here, $\mathbb{R} \times H^s(\mathbb{R}^3)$ is equipped with the norm $\|(\tau, \varphi)\|_{\mathbb{R} \times H^s(\mathbb{R}^3)} = (|\tau|^2 + \|\varphi\|^2)^{\frac{1}{2}}$. Clearly, $\widetilde{\mathcal{E}}(\tau, \varphi) \in C^1(\mathbb{R} \times H^s(\mathbb{R}^3))$ and satisfies the following properties:

$$\widetilde{\mathcal{E}}(0, \varphi) = \mathcal{E}(\varphi), \quad \widetilde{\mathcal{E}}(\tau, \varphi) = \mathcal{E}(\varphi(e^{-\tau} x)). \tag{2.4}$$

We define a minimax value for $\widetilde{\mathcal{E}}(\tau, \varphi)$ by

$$d_{mp} = \inf_{\widetilde{\zeta} \in \widetilde{\Lambda}} \sup_{t \in [0,1]} \widetilde{\mathcal{E}}(\widetilde{\zeta}(t)),$$

where

$$\widetilde{\Lambda} := \left\{ \widetilde{\zeta} \in C([0, 1], \mathbb{R} \times H^s(\mathbb{R}^3)) : \widetilde{\zeta}(0) = (0, 0), \widetilde{\mathcal{E}}(\widetilde{\zeta}(1)) < 0 \right\}.$$

It is not hard to see that $\widetilde{\Lambda} \neq \emptyset$, and thus the minimax value d_{mp} is well defined. Now, we claim that $d_{mp} = c_{mp}$. Indeed, for any $\zeta(t) \in \Lambda$ we can check that $(0, \zeta(t)) \in \widetilde{\Lambda}$, so $\{0\} \times \Lambda \subset \widetilde{\Lambda}$, thus for $\widetilde{\mathcal{E}}(0, \varphi) = \mathcal{E}(\varphi)$ we get $d_{mp} \leq c_{mp}$. On the other hand, for every given $\widetilde{\zeta}(t) = (\tau(t), \eta(t)) \in \widetilde{\Lambda}$, letting $\zeta(t)(x) = \eta(t)(e^{-\tau(t)}(x))$, we can verify that $\zeta(t) \in \Lambda$, and from (2.4), we obtain $\mathcal{E}(\zeta(t)) = \widetilde{\mathcal{E}}(\widetilde{\zeta}(t))$. This yields that $d_{mp} \geq c_{mp}$. Hence, the claim follows.

By (2.1), we may choose $\{\zeta_n\} \subset \Lambda$ such that

$$\sup_{0 \leq t \leq 1} \mathcal{E}(\zeta_n(t)) \leq c_{mp} + \frac{1}{n}.$$

Let $\tilde{\zeta}_n(t) := (0, \zeta_n(t))$, then $\tilde{\zeta}_n \in \widetilde{\Lambda}$ and so, we obtain

$$\sup_{0 \leq t \leq 1} \widetilde{\mathcal{E}}(\tilde{\zeta}_n(t)) \leq c_{mp} + \frac{1}{n}.$$

Then, by Lemma 2.3 in [9], we can get that a sequence $\{(\tau_n, \varphi_n)\} \subset \mathbb{R} \times H^s(\mathbb{R}^3)$ satisfies

$$\widetilde{\mathcal{E}}(\tau_n, \varphi_n) \to c_{mp}, \quad \widetilde{\mathcal{E}}'(\tau_n, \varphi_n) \to 0, \tag{2.5}$$

and

$$\min_{0 \leq t \leq 1} \|(\tau_n, \varphi_n) - \tilde{\zeta}_n(t)\|_{\mathbb{R} \times H^s(\mathbb{R}^3)} \to 0 \text{ as } n \to \infty. \tag{2.6}$$

Remark that for any $(h, U) \in \mathbb{R} \times H^s(\mathbb{R}^3)$,

$$o_n(1) = \langle \widetilde{\mathcal{E}}'(\tau_n, \varphi_n), (h, U) \rangle = \langle \mathcal{E}'(\mathbf{S}_{\tau_n}\varphi_n), \mathbf{S}_{\tau_n} U \rangle + \mathcal{P}(\mathbf{S}_{\tau_n}\varphi_n) h. \tag{2.7}$$

Then, the conclusion of Lemma 2.3 follows by taking $\psi_n = \mathbf{S}_{\tau_n}\varphi_n$. Indeed, by (2.6) we can get

$$|\tau_n| = |\tau_n - 0| \leq \min_{0 \leq t \leq 1} \|(\tau_n, \varphi_n) - (0, \zeta_n(t))\|_{\mathbb{R} \times H^s(\mathbb{R}^3)} \to 0.$$

By the fact that $\tau_n \to 0$ as $n \to \infty$, via (2.5) and (2.4) one can obtain that $\mathcal{E}(\psi_n) \to c_{mp}$ as $n \to \infty$.

For any $\varphi \in H^s(\mathbb{R}^3)$, we choose $h = 0$, $U(x) = \varphi(e^{\tau_n} x)$ in (2.7), and note that $\tau_n \to 0$, then we obtain

$$\langle \mathcal{E}'(\psi_n), \varphi \rangle = \langle \widetilde{\mathcal{E}}'(\tau_n, \varphi_n), (0, \varphi(e^{\tau_n} x)) \rangle = o_n(1) \|\varphi(e^{\tau_n} x)\| = o_n(1) \|\varphi\|.$$

Hence, $\mathcal{E}'(\psi_n) \to 0$ as $n \to \infty$. Moreover, taking $(h, U) = (1, 0)$ in (2.7), we get $\mathcal{P}(\psi_n) \to 0$ as $n \to \infty$.

To sum up, we have obtained a sequence $\{\psi_n\} \subset H^s(\mathbb{R}^3)$ that satisfies

$$\mathcal{E}(\psi_n) \to c_{mp}, \ \ \mathcal{E}'(\psi_n) \to 0, \ \ \mathcal{P}(\psi_n) \to 0 \ \text{ as } n \to \infty. \tag{2.8}$$

The proof is completed.

Lemma 2.4. *The (PSP)-sequence $\{\psi_n\}$ in (2.8) is bounded in $H^s(\mathbb{R}^3)$.*

Proof. From (2.8), we get

$$c_{mp} + o_n(1) = \mathcal{E}(\psi_n) - \frac{1}{3}\mathcal{P}(\psi_n) = \frac{\alpha s}{3} \|(-\Delta)^{\frac{s}{2}} \psi_n\|^2_{L^2(\mathbb{R}^3)} + \frac{\beta(4s-3)}{12} \|(-\Delta)^{\frac{s}{2}} \psi_n\|^4_{L^2(\mathbb{R}^3)}. \tag{2.9}$$

Note that by (2.9), $\|(-\Delta)^{\frac{s}{2}} \psi_n\|_{L^2(\mathbb{R}^3)}$ is bounded. By Lemma 2.1, the fractional Sobolev embedding $D^{s,2}(\mathbb{R}^3) \hookrightarrow L^{2_s^*}(\mathbb{R}^3)$ is continuous, so we have

$$\|\psi_n\|_{L^{2_s^*}(\mathbb{R}^3)} \leq C \|(-\Delta)^{\frac{s}{2}} \psi_n\|_{L^2(\mathbb{R}^3)},$$

and thus $\|\psi_n\|_{L^{2_s^*}(\mathbb{R}^3)}$ is bounded. Next, we prove $\{\psi_n\}$ is bounded in $L^2(\mathbb{R}^3)$. By the fact that $\mathcal{E}'(\psi_n) \to 0$ and $\|(-\Delta)^{\frac{s}{2}} \psi_n\|_{L^2(\mathbb{R}^3)}$ is bounded, we can deduce that

$$\kappa \|\psi_n\|^2_{L^2(\mathbb{R}^3)} \leq \|\psi_n\|^p_{L^p(\mathbb{R}^3)} + C$$

for some constant $C > 0$. Since $2 < p < 2_s^*$, then for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$\|\psi_n\|^p_{L^p(\mathbb{R}^3)} \leq \varepsilon \|\psi_n\|^2_{L^2(\mathbb{R}^3)} + C_\varepsilon \|\psi_n\|^{2_s^*}_{L^{2_s^*}(\mathbb{R}^3)}. \tag{2.10}$$

Thus, by (2.10) we obtain

$$\kappa \|\psi_n\|^2_{L^2(\mathbb{R}^3)} \leq \varepsilon \|\psi_n\|^2_{L^2(\mathbb{R}^3)} + C_\varepsilon \|\psi_n\|^{2_s^*}_{L^{2_s^*}(\mathbb{R}^3)} + C.$$

Choosing $\varepsilon = \frac{\kappa}{2}$, and meanwhile $\|\psi_n\|_{L^{2_s^*}(\mathbb{R}^3)}$ is bounded, we obtain the boundedness of $\|\psi_n\|_{L^2(\mathbb{R}^3)}$ and therefore $\{\psi_n\}$ is bounded in $H^s(\mathbb{R}^3)$.

Proof of Theorem 1.1. The result of Lemma 2.3 reveals that there is a (PSP)-sequence $\{\psi_n\} \subset H^s(\mathbb{R}^3)$ satisfying

$$\mathcal{E}(\psi_n) \to c_{mp}, \ \ \mathcal{E}'(\psi_n) \to 0, \ \ \mathcal{P}(\psi_n) \to 0 \ \text{ as } n \to \infty.$$

Moreover, by Lemma 2.4 the (PSP)-sequence $\{\psi_n\}$ must be bounded in $H^s(\mathbb{R}^3)$. Then, passing to a subsequence if necessary, we may suppose that

$$\psi_n \rightharpoonup \psi_0 \text{ weekly in } H^s(\mathbb{R}^3), \text{ and } \psi_n(x) \to \psi_0(x) \text{ a.e. in } x \in \mathbb{R}^3. \tag{2.11}$$

Next, we divide our arguments into several steps.

Step 1: We claim that ψ_0 solves Eq (1.1). In fact, by (2.11) for any $\xi \in C_0^\infty(\mathbb{R}^3)$, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} (\alpha(-\Delta)^{\frac{s}{2}}\psi_n(-\Delta)^{\frac{s}{2}}\xi + \kappa\psi_n\xi)\,\mathrm{d}x = \int_{\mathbb{R}^3} (\alpha(-\Delta)^{\frac{s}{2}}\psi_0(-\Delta)^{\frac{s}{2}}\xi + \kappa\psi_0\xi)\,\mathrm{d}x \tag{2.12}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} |\psi_n|^{p-2}\psi_n\xi\,\mathrm{d}x = \int_{\mathbb{R}^3} |\psi_0|^{p-2}\psi_0\xi\,\mathrm{d}x \ \text{ for } p \in (2, 2_s^*). \tag{2.13}$$

Moreover, suppose that $\|(-\Delta)^{\frac{s}{2}}\psi_n\|^2_{L^2(\mathbb{R}^3)} \to \mathscr{B}$ for some $\mathscr{B} \geq 0$, then from (2.12) and (2.13), for $\xi \in C_0^\infty(\mathbb{R}^3)$, we deduce that

$$\begin{aligned}
0 =& \lim_{n\to\infty} \langle \mathcal{E}'(\psi_n), \xi\rangle \\
=& \int_{\mathbb{R}^3} (\alpha(-\Delta)^{\frac{s}{2}}\psi_0(-\Delta)^{\frac{s}{2}}\xi + \kappa\psi_0\xi)\,\mathrm{d}x + \beta\mathscr{B}\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}}\psi_0(-\Delta)^{\frac{s}{2}}\xi\,\mathrm{d}x - \int_{\mathbb{R}^3} |\psi_0|^{p-2}\psi_0\xi\,\mathrm{d}x \\
=& \langle G'(\psi_0), \xi\rangle,
\end{aligned} \tag{2.14}$$

where

$$G(\psi) = \frac{\alpha + \beta\mathscr{B}}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}\psi|^2\,\mathrm{d}x + \frac{\kappa}{2}\int_{\mathbb{R}^3} |\psi|^2\,\mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^3} |\psi|^p\,\mathrm{d}x.$$

Using Fatou's lemma, we have

$$\begin{cases}
\liminf_{n\to\infty} \|\psi_n\|^2_{L^2(\mathbb{R}^3)} \geq \|\psi_0\|^2_{L^2(\mathbb{R}^3)}, \\
\|(-\Delta)^{\frac{s}{2}}\psi_0\|^2_{L^2(\mathbb{R}^3)} \leq \liminf_{n\to\infty} \|(-\Delta)^{\frac{s}{2}}\psi_n\|^2_{L^2(\mathbb{R}^3)} = \mathscr{B}.
\end{cases} \tag{2.15}$$

Noting that $\mathcal{E}'(\psi_n) \to 0$ and $\|(-\Delta)^{\frac{s}{2}}\psi_n\|^2_{L^2(\mathbb{R}^3)} \to \mathscr{B}$, one can obtain that

$$\lim_{n\to\infty} \langle G'(\psi_n), \psi_n\rangle = 0.$$

Now, by combining (2.13)–(2.15) we conclude that

$$\begin{aligned}
\limsup_{n\to\infty} (\alpha + \beta\mathscr{B})\|(-\Delta)^{\frac{s}{2}}\psi_n\|^2_{L^2(\mathbb{R}^3)} =& \limsup_{n\to\infty} \left(\|\psi_n\|^p_{L^p(\mathbb{R}^3)} - \kappa\|\psi_n\|^2_{L^2(\mathbb{R}^3)}\right) \\
\leq& \|\psi_0\|^p_{L^p(\mathbb{R}^3)} - \kappa\|\psi_0\|^2_{L^2(\mathbb{R}^3)} \\
=& (\alpha + \beta\mathscr{B})\|(-\Delta)^{\frac{s}{2}}\psi_0\|^2_{L^2(\mathbb{R}^3)}.
\end{aligned} \tag{2.16}$$

Putting together (2.16) and (2.15), we get

$$\lim_{n\to\infty} \|(-\Delta)^{\frac{s}{2}}\psi_n\|^2_{L^2(\mathbb{R}^3)} = \|(-\Delta)^{\frac{s}{2}}\psi_0\|^2_{L^2(\mathbb{R}^3)} = \mathscr{B}. \tag{2.17}$$

Accordingly, using (2.12), (2.13) and (2.17), we can derive that $\psi_n \to \psi_0$ strongly in $H^s(\mathbb{R}^3)$ and so

$$0 = \lim_{n\to\infty}\langle \mathcal{E}'(\psi_n), \xi\rangle = \langle \mathcal{E}'(\psi_0), \xi\rangle$$

for all $\xi \in C_0^\infty(\mathbb{R}^3)$, that is, $\mathcal{E}'(\psi_0) = 0$.

Moreover $\psi_0 \not\equiv 0$. Otherwise, if $\psi_0 \equiv 0$, that is $\psi_n \to 0$ in $H^s(\mathbb{R}^3)$, which leads to $\mathcal{E}(\psi_n) \to 0$, this is a contradiction since $\mathcal{E}(\psi_n) \to c_{mp} > 0$.

Step 2: Next, we claim that $\mathcal{E}(\psi_0) = c_{mp} = \mathfrak{m}$, that is, ψ_0 is a ground state solution of (1.1). Indeed, note that $\mathcal{P}(\psi_n) \to 0$ as $n \to \infty$, one has $\mathcal{P}(\psi_0) = 0$. Therefore, from (2.3),

$$\begin{aligned}
\mathcal{E}(\psi_0) =& \mathcal{E}(\psi_0) - \frac{1}{3}\mathcal{P}(\psi_0) = \frac{\alpha s}{3}\|(-\Delta)^{\frac{s}{2}}\psi_0\|^2_{L^2(\mathbb{R}^3)} + \frac{\beta(4s-3)}{12}\|(-\Delta)^{\frac{s}{2}}\psi_0\|^4_{L^2(\mathbb{R}^3)}\\
\le& \liminf_{n\to\infty}\left(\frac{\alpha s}{3}\|(-\Delta)^{\frac{s}{2}}\psi_n\|^2_{L^2(\mathbb{R}^3)} + \frac{\beta(4s-3)}{12}\|(-\Delta)^{\frac{s}{2}}\psi_n\|^4_{L^2(\mathbb{R}^3)}\right)\\
=& \liminf_{n\to\infty}\,(\mathcal{E}(\psi_n) - \frac{1}{3}\mathcal{P}(\psi_n))\\
=& \liminf_{n\to\infty}\mathcal{E}(\psi_n) = c_{mp}.
\end{aligned}$$

Clearly, by the definition of $\mathfrak{m}$, there holds $\mathfrak{m} \le \mathcal{E}(\psi_0)$, and hence $\mathfrak{m} \le c_{mp}$.

On the other hand, we prove that $c_{mp} \le \mathfrak{m}$. Let $w(x) \in H^s(\mathbb{R}^3)\setminus\{0\}$ be another solution of (1.1) and satisfy $\mathcal{E}(w) \le \mathcal{E}(\psi_0)$. We set $\zeta^*(\tau)(x) = w(\frac{x}{\tau})$ for $\tau > 0$ and $\zeta^*(0) = 0$. It is clear that $\zeta^*(\tau) \in C([0,+\infty), H^s(\mathbb{R}^3))$. From (2.3), for $\tau > 0$ we obtain that

$$\begin{aligned}
\mathcal{E}(\zeta^*(\tau)) =& \frac{\alpha\tau^{3-2s}}{2}\|(-\Delta)^{\frac{s}{2}}w\|^2_{L^2(\mathbb{R}^3)} + \frac{\kappa\tau^3}{2}\|w\|^2_{L^2(\mathbb{R}^3)} + \frac{\beta\tau^{2(3-2s)}}{4}\|(-\Delta)^{\frac{s}{2}}w\|^4_{L^2(\mathbb{R}^3)} - \frac{\tau^3}{p}\|w\|^p_{L^p(\mathbb{R}^3)}\\
=& \frac{3\tau^{3-2s} - (3-2s)\tau^3}{6}\alpha\|(-\Delta)^{\frac{s}{2}}w\|^2_{L^2(\mathbb{R}^3)} + \frac{3\tau^{2(3-2s)} - 2(3-2s)\tau^3}{12}\beta\|(-\Delta)^{\frac{s}{2}}w\|^4_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

With a simple calculation, we conclude that

$$\max_{\tau\ge 0}\mathcal{E}(\zeta^*(\tau)) = \mathcal{E}(\zeta^*(1)) = \mathcal{E}(w),$$

and it follows that $\mathcal{E}(\zeta^*(\tau)) \le \mathcal{E}(w)$. Observe that $\mathcal{E}(\zeta^*(\tau)) \to -\infty$ as $\tau \to +\infty$. Then, with appropriate scaling change we can get a path $\zeta(t) \in C([0,1], H^s(\mathbb{R}^3))$ such that $\zeta(0) = 0$ and $\mathcal{E}(\zeta(1)) < 0$; $\zeta(t_0) = w$ for some $t_0 \in (0,1)$; $\max_{0\le t\le 1}\mathcal{E}(\zeta(t)) = \mathcal{E}(\zeta(t_0)) = \mathcal{E}(w)$. Then, by the definition of c_{mp} in (2.1), we know that $c_{mp} \le \mathcal{E}(w)$, which shows that $c_{mp} \le \mathfrak{m}$. Thus as desired $\mathcal{E}(w) = \mathcal{E}(\psi_0) = c_{mp} = \mathfrak{m}$ has been proved.

Step 3: We estimate the decay properties of $\psi_0(x)$. Following [3], by the standard regularity arguments we can deduce that $\psi_0(x) \in H^{2s}(\mathbb{R}^3) \cap C^r(\mathbb{R}^3)$ for all $r \in (0, 2s)$ and $\lim_{|x|\to\infty}\psi_0(x) = 0$. Note that $p > 2$. Then, we can pick $\rho > 0$ such that for all $|x| \ge \rho$,

$$\frac{|\psi_0|^{p-2}}{\alpha + \beta\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\psi_0|^2\,dx} \le \frac{\kappa}{2(\alpha+\beta L)},$$

where $L > 0$ such that $\|\psi_0\|^2_{H^s(\mathbb{R}^3)} \leq L$, and we conclude that

$$\begin{aligned}(-\Delta)^s\psi_0(x) + \frac{\kappa}{\alpha+\beta L}\psi_0(x) &\leq (-\Delta)^s\psi_0(x) + \frac{\kappa\psi_0(x)}{\alpha+\beta\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\psi_0|^2\,\mathrm{d}x} \\ &= \frac{|\psi_0(x)|^{p-2}\psi_0(x)}{\alpha+\beta\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\psi_0|^2\,\mathrm{d}x} \leq \frac{\kappa}{2(\alpha+\beta L)}\psi_0(x).\end{aligned}$$

Therefore,

$$(-\Delta)^s\psi_0(x) + \frac{\kappa}{2(\alpha+\beta L)}\psi_0(x) \leq 0, \ \forall\, x \in \mathbb{R}^3 \setminus B_\rho(0). \tag{2.18}$$

According to Lemma 4.3 of [5], we can find a continuous function $\Phi(x)$ satisfying $0 < \Phi(x) \leq \frac{C}{1+|x|^{3+2s}}$ and

$$(-\Delta)^s\Phi(x) + \frac{\kappa}{2(\alpha+\beta L)}\Phi(x) \geq 0, \ \forall\, x \in \mathbb{R}^3 \setminus B_{R_1}(0) \tag{2.19}$$

for some suitable $R_1 > 0$. Let $R = \max\{\rho, R_1\}$, and set

$$a = \min_{|x|\leq R}\Phi(x), \quad b = \max_{|x|\leq R}\psi_0(x).$$

Define $U(x) = \frac{b}{a}\Phi(x) - \psi_0(x)$. From (2.18) and (2.19), consequently, we can obtain

$$\begin{cases} (-\Delta)^s U(x) + \dfrac{\kappa}{2(\alpha+\beta L)}U(x) \geq 0 & \text{for } |x| \geq R, \\ U(x) \geq 0 & \text{for } |x| = R, \\ \lim\limits_{|x|\to\infty} U(x) = 0. \end{cases}$$

Then, by the maximum principle we infer that $U(x) \geq 0$ for all $|x| \geq R$. In addition, by the definition of $U(x)$, obviously, $U(x) \geq 0$ for $|x| \leq R$. Thus, we get $U(x) \geq 0$ for all $x \in \mathbb{R}^3$, furthermore, we have

$$\psi_0(x) \leq \frac{b}{a}\Phi(x) \leq \frac{C}{1+|x|^{3+2s}}, \ \forall\, x \in \mathbb{R}^3.$$

The proof of Theorem 1.1 is finished.

3. Conclusions

In this paper, we are interested in the existence and decay property of ground state solutions for a Kirchhoff equation involving fractional Laplacian operator. Since the nonlocal term $\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\psi|^2\,\mathrm{d}x\right)^2$ included in the energy functional $\mathcal{E}(\psi)$ is homogeneous of degree 4, when $p \leq 4$, it brings about two obstacles to the standard mountain-pass arguments both in checking the geometrical assumptions in the corresponding energy functional and in proving the boundedness of the Palais-Smale sequence for $\mathcal{E}(\psi)$. By constructing a Palais-Smale-Pohozaev sequence at the minimax value c_{mp}, the existence of ground state solutions to this equation for all $p \in (2, 2^*_s)$ is established by variational arguments. Furthermore, the decay property of the ground state solution is also investigated. Our result extends and improves the recent results in the literature. We believe that the proposed approach in the present paper can also be applied to studying other related variational problems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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