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Research article

Stability of stochastic dynamic systems of a random structure with Markov switching in the presence of concentration points

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Abstract: This article aims to investigate sufficient conditions for the stability of the trivial solution of stochastic differential equations with a random structure, particularly in contexts involving the presence of concentration points. The proof of asymptotic stability leverages the use of Lyapunov functions, supplemented by additional constraints on the magnitudes of jumps and jump times, as well as the Markov property of the system solutions. The findings are elucidated with an example, demonstrating both stable and unstable conditions of the system. The novelty of this work is in the consideration of jump concentration points, which are not considered in classical works. The assumption of the existence of concentration points leads to additional constraints on jumps, jump times and relations between them.

Keywords: system of random structure; Markov switching; concentration point; Lyapunov function; asymptotic stability
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1. Introduction

In the vast majority of works with jump changes of the trajectory, it is assumed that the distance between jumps is not less than some δ, i.e., $|t_k - t_{k-1}| > \delta$. According to this assumption, only a finite number of jumps occur on a finite interval, which is an important condition for proving the stability and exponential boundedness of the solution. In this case, the conditions of existence, unity and stability of the systems of stochastic differential equations with jumps are reduced to the corresponding statements for systems without jumps.

This work considers the case in which jumps can be concentrated at some point, which leads to the following relationship

$$\lim_{k\to\infty} t_k = t_\infty < \infty.$$

In this case, the cumulative effect of jumps can lead to a lack of stability of the system. This effect can be illustrated by a simple example of an ordinary differential equation

$$dx(t) = -x(t)dt,$$

with jumps

$$x(t_k) = x(t_k-)(1 + k^2),$$

at the points

$$t_k = \frac{\alpha}{k}, \alpha > 0.$$

It can be easily concluded that

$$\lim_{t\to\alpha-} |x(t)| = \infty$$

for $x(0) \neq 0$. This simple example indicates that the size of the jumps plays an important role in the presence of concentration points in the system.

The models proposed in this work can be applied in the field of catastrophe theory. Random events with a frequency that exponentially increases over a finite time interval are considered there. Moreover, solving applied problems, one should consider the possibility of the collapse of the system in case of large disturbances and/or small intervals between disturbances g. The conditions of instability in such cases may have the following form

$$|t_k - t_{k-1}| < \delta_{min}$$

for some k.

Among others, resonant systems could be examples of the considered systems. There, the impact of external factors intensifies as the period changes or the influence amplitude increases. More works devoted to real phenomena and processes that can be described using the system (2.1)–(2.3) were referenced in the introduction of the work [1].

One of the main results for ordinary stochastic differential equations is considered in the paper [2]. In this work, sufficient conditions for the existence and uniqueness of the solution, based on the convergence of jumps and the presence of concentration points, are considered. In a more general case of stochastic differential equations, one should take into account not only the average value of the jumps but also the variance of the jumps.

The novelty of this work, in contrast to classical works, is consideration of concentration points of jumps, without setting a limit at jump moments, i.e., $|t_k - t_{k-1}| > \delta_{min}$. The absence of this condition in a real system may lead to an accumulation of jumps and the solution can tend to infinity. Thus, to analyze the stability of a trivial solution, it is necessary to consider additional conditions for the moments and magnitudes of jumps, which are considered in Theorems 2.2 and 4.1.

2. Problem statement

On the probabilistic basis $(\Omega, \mathfrak{F}, F, \mathbf{P})$ [3, 4], we consider a stochastic dynamic system of random structure given by a stochastic differential equation (SDE)

$$dx(t) = a(t, \xi(t), x(t))dt + b(t, \xi(t), x(t))dw(t),\ t \in \mathbb{R}_+ \backslash K, \tag{2.1}$$

with Markov switching

$$\Delta x(t) = g(t_k-, \xi(t_k-), \eta_k, x(t_k-)), \quad t_k \in K = \{t_n \Uparrow\}, \tag{2.2}$$

and initial conditions

$$x(0) = x_0 \in \mathbb{R}^m,\ \xi(0) = y \in \mathbf{Y}, \eta_0 = h \in \mathbf{H}. \tag{2.3}$$

Here $\xi(t), t \geq 0$, is a Markov chain with a finite number of states $\mathbf{Y} = \{1, 2, ..., N_\xi\}$ and generator $Q = \{\tilde{q}_{ij}\}, i, j = \{1, ..., N_\xi\}$; $\{\eta_k, k \geq 0\}$ is a Markov chain with values in space $\mathbf{H}$ and with a transition probability matrix $\mathbb{P}_H$; $x : [0, +\infty) \times \Omega \to \mathbb{R}^m$; $w(t), t \geq 0$, is a m-dimensional standard Wiener process; the processes w, ξ and η are independent random processes [3, 4].

We denote by

$$\mathfrak{F}_{t_k} = \sigma(\xi(s), w(s), \eta_e, s \leq t_k, t_e \leq t_k)$$

the minimal σ-algebra with respect to which $\xi(t), t \in [0, t_k]$ and $\eta_n, n \leq k$, are measured.

As in the works [4, 5], assume that measured by a set of variables functions $a : \mathbb{R}_+ \times \mathbf{Y} \times \mathbb{R}^m \to \mathbb{R}^m$, $b : \mathbb{R}_+ \times \mathbf{Y} \times \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^m$, $g : \mathbb{R}_+ \times \mathbf{Y} \times \mathbf{H} \times \mathbb{R}^m \to \mathbb{R}^m$ satisfy the boundedness condition and the Lipschitz condition

$$|a(t, y, x)|^2 + |b(t, y, x)|^2 + |g(t, y, h, x)|^2 \leq C(1 + |x|^2); \tag{2.4}$$

$$|a(t, y, x_1) - a(t, y, x_2)|^2 + |b(t, y, x_1) - b(t, y, x_2)|^2 \leq L|x_1 - x_2|^2, x_1, x_2 \in \mathbb{R}^m; \tag{2.5}$$

$$|g(t_k, y, h, x_1) - g(t_k, y, h, x_2)|^2 \leq L_k|x_1 - x_2|^2, x_1, x_2 \in \mathbb{R}^m, \sum_{k=1}^{\infty} L_k < \infty. \tag{2.6}$$

Consider the case of a point of concentration of jumps, i.e.,

$$\lim_{n\to\infty} t_n = t^* \in [0, T].$$

Let's assume that the following relations are true:

$$\sum_{k=1}^{\infty} \gamma_k < \infty, \gamma_k = \sup_{x\in\mathbb{R}^m, y\in\mathbf{Y}, h\in\mathbf{H}} |g(t_k, y, h, x)|, \tag{2.7}$$

and

$$\lim_{\varepsilon\downarrow 0} \left(\ln \varepsilon + N_\varepsilon \sum_{k=1}^{N_\varepsilon} L_k \right) = -\infty, N_\varepsilon := inf \left\{ k \geq 1 : \sum_{m=k}^{\infty} \gamma_m < \varepsilon \right\}. \tag{2.8}$$

The conditions (2.4)–(2.8) guarantee the existence of a strong solution to the Cauchy problem (2.1)–(2.3) [1]. Without loss of generality, we can assume that filtration F is the natural filtration constructed by the random processes $w(t), t \geq 0; \xi(t), t \geq 0; \eta_k, k \geq 0$.

We denote by

$$\mathbf{P}_k((y, h, x), \Gamma \times G \times \mathbf{C}) := \mathbf{P}((\xi(t_{k+1}), \eta_{k+1}, x(t_{k+1})) \in \Gamma \times G \times \mathbf{C} | (\xi(t_k), \eta_k, x(t_k)) = (y, h, x)),$$

the transition probability of the Markov chain $(\xi(t_k), \eta_k, x(t_k))$, that determine the solution to the problem (2.1)–(2.3) on the k-th step.

Definition 2.1. *Discrete Lyapunov operator* $(lv_k)(y, h, x)$ *on a sequence of measurable scalar functions* $v_k(y, h, x)$: $\mathbf{Y} \times \mathbf{H} \times \mathbb{R}^m \to \mathbb{R}^1, k \in \mathbb{N} \cup \{0\}$, *for the SDE* (2.1) *with Markov switching* (2.2) *is defined by the equality*

$$(lv_k)(y, h, x) :=$$
$$\int\limits_{\mathbf{Y} \times \mathbf{H} \times \mathbb{R}^m} \mathbf{P}_k((y, h, x)(du \times dz \times dl)) v_{k+1}(u, z, l) - v_k(y, h, x). \tag{2.9}$$

Here $v_k(y, h, x), k \in \mathbb{N}$, *is a Lyapunov function defined by the following definition.*

Definition 2.2. *The Lyapunov function for the system* (2.1)–(2.3) *is a sequence of non-negative functions* $\{v_k(y, h, x), k \geq 0\}$, *for whom*

(1) for all $k \geq 0, y \in \mathbf{Y}, h \in \mathbf{H}, x \in \mathbb{R}^m$ *the discrete Lyapunov operator* $(lv_k)(y, h, x)$ (2.9) *is defined;*
(2) if $r \to \infty$

$$\bar{v}(r) \equiv \inf_{k \in \mathbb{N}, y \in \mathbf{Y}, h \in \mathbf{H}, |x| \geq r} v_k(y, h, x) \to +\infty;$$

(3) if $r \to 0$

$$\underline{v}(r) \equiv \sup_{k \in \mathbb{N}, y \in \mathbf{Y}, h \in \mathbf{H}, |x| \leq r} v_k(y, h, x) \to 0;$$

where $\bar{v}(r)$ *and* $\underline{v}(r)$ *are continuous and monotonic for* $r > 0$.

Definition 2.3. *A system with a random structure* (2.1)–(2.3) *is called:*

– stable in probability, if for $\forall \varepsilon_1 > 0, \varepsilon_2 > 0$ *it can specify* $\delta > 0$ *such that the inequality* $|x| < \delta$ *implies the inequality*

$$\mathbf{P}\left\{\sup_{t \geq 0} |x(t)| > \varepsilon_1\right\} < \varepsilon_2, \tag{2.10}$$

for all $y \in \mathbf{Y}, h \in \mathbf{H}$*;*

– asymptotically stochastically stable, if it is stable in probability and for any $\varepsilon > 0$ *exists* $\delta_2 > 0$ *such that*

$$\lim_{T \to \infty} \mathbf{P}\left\{\sup_{t \geq T} |x(t)| > \varepsilon\right\} = 0, \tag{2.11}$$

for all $|x| < \delta_2$, $y \in \mathbf{Y}, h \in \mathbf{H}$ *and* $T \geq 0$.

Definition 2.4. *A system with a random structure* (2.1)–(2.3) *is called:*

– mean square stable, if for $\forall \varepsilon > 0$ *it can specify the following* $\delta > 0$, *that the inequality* $|x| < \delta$ *implies the inequality*

$$\mathbf{E}|x(t)|^2 < \varepsilon, \tag{2.12}$$

for all $t \in [0, T], y \in \mathbf{Y}, h \in \mathbf{H}$;

– mean square asymptotically stable, if it is mean square stable for any $T > 0$ and

$$\lim_{t\to\infty} \sup_{y\in\mathbf{Y},h\in\mathbf{H}} \mathbf{E}|x(t)|^2 = 0. \tag{2.13}$$

If (2.10)–(2.13) *hold true for all $x \in \mathbb{R}^m$, then the system is stable in the corresponding probabilistic sense on the whole.*

For solving the problem (2.1)–(2.3) *on the intervals $[t_k, t_{k+1})$, the following estimate is obtained.*

Theorem 2.1. *Let the coefficients a, b of the Eq* (2.1) *satisfy the condition of uniform boundedness* (2.4)*, and the condition* (2.6) *holds for the function g.*

Then for all $k \geq 0$ for a strong solution of the Cauchy problem (2.1)–(2.3) *holds the next inequality*

$$\mathbf{E}\left\{\sup_{t_k\leq t\leq t_{k+1}} |x(t)|^2\right\} \leq 9e^{5C}(1 + 2L_{k+1})\left[\mathbb{E}|x(t_k)|^2 + C(t_{k+1} - t_k)\right]. \tag{2.14}$$

Proof. We use the same methodology as in [6, 7]. A strong solution of the Cauchy problem (2.1), (2.3) for all $t \in [t_k, t_{k+1}), k \geq 0$, can be written in the integral form

$$x(t) = x(t_k) + \int_{t_k}^{t} a(\tau, \xi(\tau), x(\tau))d\tau + \int_{t_k}^{t} b(\tau, \xi(\tau), x(\tau))dw(\tau). \tag{2.15}$$

After squaring the left and right sides of (2.15), calculating sup, and applying the Cauchy–Schwarz inequality, we obtain:

$$\sup_{t_k\leq t<t_{k+1}} |x(t)|^2 \leq 3 \cdot \sup_{t_k\leq t<t_{k+1}} \left\{|x(t_k)|^2 + |\int_{t_k}^{t} a(\tau, \xi(\tau), x(\tau))d\tau|^2 + |\int_{t_k}^{t} b(\tau, \xi(\tau), x(\tau))dw(\tau)|^2\right\}$$
$$\leq 3\left[\sup_{t_k\leq t<t_{k+1}} |x(t_k)|^2 + \sup_{t_k\leq t<t_{k+1}} \int_{t_k}^{t} |a(\tau, \xi(\tau), x(\tau))|^2 d\tau + \sup_{t_k\leq t<t_{k+1}} |\int_{t_k}^{t} b(\tau, \xi(\tau), x(\tau))dw(\tau)|^2\right].$$

To the last inequality, we apply the conditional mathematical expectation operation with respect to the σ-algebra $\mathfrak{F}_{t_k}$ and, taking into account the properties of the Ito integral and Markov property, we obtain

$$\mathbb{E}\left\{\sup_{t_k\leq t<t_{k+1}} |x(t)|^2/\mathfrak{F}_{t_k}\right\} \leq 3\left[\mathbb{E}|x(t_k)|^2 + C(t_{k+1} - t_k) + C\int_{t_k}^{t_{k+1}} \mathbb{E}|x(\tau)|^2 d\tau + 4C\int_{t_k}^{t_{k+1}} \mathbb{E}|x(\tau)|^2 d\tau\right]$$
$$= 3\left[\mathbb{E}|x(t_k)|^2 + C(t_{k+1} - t_k) + 5C\int_{t_k}^{t_{k+1}} \mathbb{E}|x(\tau)|^2 d\tau\right].$$

Using the Gronwall inequality, we obtain an estimate of

$$\mathbb{E}\left\{\sup_{t_k\leq t<t_{k+1}} |x(t)|^2/\mathfrak{F}_{t_k}\right\} \leq 3\left[\mathbb{E}|x(t_k)|^2 + C(t_{k+1} - t_k)\right]e^{5C}.$$

For $t = t_{k+1}$ the strong solution of the system (2.1)–(2.3), obviously, must satisfy the inequality

$$\begin{aligned}\mathbb{E}\left\{|x(t_{k+1})|^2/\mathfrak{F}_{t_k}\right\} \le 3\Big[&\mathbb{E}\left\{|x(t_{k+1}-)|^2/\mathfrak{F}_{t_k}\right\}\\ &+ 2\mathbb{E}\left\{|g(t_{k+1}-,\xi(t_{k+1}-),\eta_{k+1},x(t_{k+1}-)) - g(t_{k+1}-,\xi(t_{k+1}-),\eta_{k+1},0)|^2/\mathfrak{F}_{t_k}\right\}\\ &+ 2\mathbb{E}\left\{|g(t_{k+1}-,\xi(t_{k+1}-),\eta_{k+1},0)|^2/\mathfrak{F}_{t_k}\right\}\Big]\\ \le 3&\left[(1+2L_{k+1})\mathbb{E}\left\{\sup_{t_k\le t\le t_{k+1}}|x(t)|^2/\mathfrak{F}_{t_k}\right\} + C\right].\end{aligned}$$

Combining the last two inequalities, we get the desired estimate (2.14). □

Remark 2.1. *We will consider the stability of the trivial solution* $x \equiv 0$*, i.e. the satisfying of* (2.4)*, if* $C = 0$ *[5], [8], [9].*

Remark 2.2. *Note that the Lipschitz condition (2.5) was not used in the proof of the Theorem 2.1, i.e., any (not necessarily unique) solution to the problem* (2.1)–(2.3) *satisfies the condition of the Theorem 2.1.*

Theorem 2.2. *Let:*

1) the conditions (2.4)–(2.8) *are hold;*

2) the Lyapunov functions $v_k(y,h,x)$ *and* $a_k(y,h,x), k \ge 0$*, exist, such that, based on the system, the following inequality*

$$(lv_k)(y,h,x(t)) \le -a_k(y,h,x(t)), k \ge 0, \tag{2.16}$$

is correct.

Then the system of random structure (2.1)–(2.3) *is asymptotically stochastically stable on the whole.*

Proof. Define by $\mathfrak{F}_{t_k} = \sigma(\xi(s),\eta_e, s \le t_k, t_e \le t_k)$ a minimal σ-algebra, relative to which are measured $\xi(t)$ for all $t \in [0,t_k]$ and η_n for $n \le k$. The conditional mathematical expectation is calculated by the formula

$$\mathbf{E}\left\{v_{k+1}(\xi(t_{k+1}),\eta_{k+1},x(t_{k+1}))/\mathfrak{F}_{t_k}\right\} = \int_{\mathbf{Y}\times\mathbf{H}\times\mathbb{R}^m}\mathbf{P}_k((\xi(t_k),\eta_k,x)(du\times dz\times dl)v_{k+1}(u,z,l)). \tag{2.17}$$

Then, by the definition of the discrete Lyapunov operator $(lv_k)(y,h,x)$ (see (2.9)) from equality (2.17), considering (2.16), we get the inequality

$$\mathbf{E}\left\{v_{k+1}(\xi(t_{k+1}),\eta_{k+1},x(t_{k+1}))/\mathfrak{F}_{t_k}\right\} = v_k(\xi(t_k),\eta_k,x(t_k)) + (lv_k)(\xi(t_k),\eta_k,x(t_k)) \le \bar{v}(|x(t_k)|). \tag{2.18}$$

From Theorem 2.1 (because the existence of the second moment implies the existence of the first moment) and from properties of the function $\bar{v}$ follows the existence of a conditional mathematical expectation of the left-hand side of the inequality (2.18).

Now, using (2.17), (2.18), we write the discrete Lyapunov's operator $(lv_k)(y,h,x)$, which given on the solutions (2.1)–(2.3):

$$\begin{aligned}lv_k(\xi(t_k),\eta_k,x(t_k)) &= \mathbf{E}\left\{v_{k+1}(\xi(t_{k+1}),\eta_{k+1},x(t_{k+1}))/\mathfrak{F}_{t_k}\right\} - v_k(\xi(t_k),\eta_k,x(t_k))\\ &\le -a_k(\xi(t_k),\eta_k,x(t_k)) \le 0.\end{aligned} \tag{2.19}$$

Then, at $k \geq 0$ the next inequality holds

$$\mathbf{E}\left\{v_{k+1}(\xi(t_{k+1}), \eta_{k+1}, x(t_{k+1}))/\mathfrak{F}_{t_k}\right\} \leq v_k(\xi(t_k), \eta_k, x(t_k)).$$

This means that a sequence of random variables

$$v_k(\xi(t_k), \eta_k, x(t_k)),$$

forms a supermartingale in relation to $\mathfrak{F}_{t_k}$ [10].

Taking the mathematical expectation of both parts of inequality (2.19), we summarize the obtained expressions for k from $n \geq 0$ to N, and obviously, we have the next inequality:

$$\begin{aligned} &\mathbf{E}\left\{v_{N+1}(\xi(t_{N+1}), \eta_{N+1}, x(t_{N+1}))\right\} - \mathbf{E}\left\{v_n(\xi(t_n), \eta_n, x(t_n))\right\} \\ &= \sum_{k=n}^{N} \mathbf{E}\left\{l v_k(\xi(t_k), \eta_k, x(t_k))\right\} \\ &\leq -\sum_{k=n}^{N} \mathbf{E}\left\{a_k(\xi(t_k), \eta_k, x(t_k))\right\} \leq 0. \end{aligned} \tag{2.20}$$

Since a random variable $\sup\limits_{t_k \leq t \leq t_{k+1}} |x(t)|^2$ does not depend on events of σ-algebra $\mathfrak{F}_{t_k}$ [11], then

$$\mathbf{E}\left\{\sup_{t_k \leq t \leq t_{k+1}} |x(t)|^2 \Big/ \mathfrak{F}_{t_k}\right\} = E\left\{\sup_{t_k \leq t \leq t_{k+1}} |x(t)|^2\right\}, \tag{2.21}$$

that is, the inequality (2.14) also holds for the simple mathematical expectation

$$\mathbf{E}\left\{\sup_{t_k \leq t \leq t_{k+1}} |x(t)|^2\right\} \leq 3\mathbb{E}|x|^2.$$

Next, we have

$$\begin{aligned} &\mathbf{P}\left\{\sup_{t \geq 0} |x(t)| > \varepsilon_1\right\} \\ &= \mathbf{P}\left\{\sup_{n \in \mathbb{N}} \sup_{t_{n-1} \leq t \leq t_n} |x(t)| > \varepsilon_1\right\} \\ &\leq \mathbf{P}\left\{\sup_{n \in \mathbb{N}} 3|x(t_{n-1})| > \varepsilon_1\right\} \\ &\leq \mathbf{P}\left\{\sup_{n \in \mathbb{N}} |x(t_{n-1})| > \frac{\varepsilon_1}{3}\right\} \\ &\leq \mathbf{P}\left\{\sup_{n \in \mathbb{N}} v_{n-1}(\xi(t_{n-1}), \eta_{n-1}, x(t_{n-1})) \geq \bar{v}(\frac{\varepsilon_1}{3})\right\}. \end{aligned} \tag{2.22}$$

If $\sup|x(t_k)| \geq r$, then, based on the definition of the Lyapunov function, the next inequality holds:

$$\sup_{k \geq 0} v_k(\xi(t_k), \eta_k, x(t_k)) \geq \inf_{k \geq 0, y \in \mathbf{Y}, h \in \mathbf{H}, |x| \geq r} v_k(y, h, x) = \bar{v}(r). \tag{2.23}$$

Now let's use the well-known inequality for nonnegative supermartingales [3, 10] to evaluate the right-hand side of (2.22):

$$\mathbf{P}\left\{\sup_{n\in\mathbb{N}} v_{n-1}(\xi(t_{n-1}),\eta_{n-1},x(t_{n-1}))\geq \bar{v}(\frac{\varepsilon_1}{3})\right\}\leq \frac{1}{\bar{v}(\frac{\varepsilon_1}{3})}v_k(y,h,x)\leq \frac{\bar{v}(|x|)}{\bar{v}(\frac{\varepsilon_1}{3})}. \tag{2.24}$$

Given inequality (2.22), inequality (2.24) make it possible to guarantee the fulfillment of inequality (2.10) of stability in probability on the whole of the system (2.1)–(2.3).

From the inequality (2.20) follows the estimate

$$\mathbf{E}\{v_{N+1}(\xi(t_{N+1}),\eta_{N+1},x(t_{N+1}))\}\leq v_0(y,h,x)-\sum_{k=0}^{N}\mathbf{E}\{a_k(\xi(t_k),\eta_k,x(t_k))\}\leq v_0(y,h,x), \tag{2.25}$$

for all $N\geq 0, y\in\mathbf{Y}, h\in\mathbf{H}, x\in\mathbb{R}^m$.

Since the sequence $\{a_k\}, k\geq 0$, forms Lyapunov functions, there must exist continuous strictly monotone functions $\underline{a}(r)$ and $\bar{a}(r)$, which are zero if $r=0$ [12] and such that

$$\bar{a}(|x|)\leq a_k(y,h,x)\leq \underline{a}(|x|), \tag{2.26}$$

for $\forall k\in\mathbb{N}, y\in\mathbf{Y}, h\in\mathbf{H}$ and $x\in\mathbb{R}^m$.

Thus, from the convergence of the series on the left side of the inequality (2.25) (which will be convergent in the case of convergence of the series $\sum\limits_{k=1}^{\infty} L_k$) follows the convergence of the series $\sum\limits_{k=0}^{\infty}\mathbf{E}\{\bar{a}(|x(t)|\}$ for $\forall t\geq t_k, y\in\mathbf{Y}, h\in\mathbf{H}, x\in\mathbb{R}^m$.

Then, taking into account the continuity of $\underline{a}(r)$ and the equality $\underline{a}(0)=0$, we have:

$$\lim_{k\to\infty}|x(t)|=0, t\geq t_k. \tag{2.27}$$

And from (2.27) it follows tends to zero in probability of the sequence $\bar{v}\,(|x(t)|)$ for $k\to\infty$ for all $t\geq t_k, y\in\mathbf{Y}, h\in\mathbf{H}, x\in\mathbb{R}^m$.

So, from the properties of the Lyapunov function, we conclude that the non-negative supermartingale $v_k(\xi(t_k),\eta_k,x(t_k))$ for $k\to+\infty$ tends to zero in probability for all realizations of the process ξ and sequence η_k.

Further, the nonnegative bounded supermartingale has a bound with probability 1 [3]. Based on Theorem 2.1 (inequality (2.14) for the usual mathematical expectation), we obtain the asymptotic stochastically stability on the whole of the system (2.1)–(2.3) by the Definition 2.3 (see (2.11)). Theorem 2.2 is proven. □

Theorem 2.3. *Suppose that the conditions of Theorem 2.2 are satisfied, and the Lyapunov functions* $\{v_k\},\{a_k\}, k\geq 0$, *satisfy the inequalities*

$$c_1|x|^2\leq v_k(y,h,x)\leq c_2|x|^2, \tag{2.28}$$

$$c_3|x|^2\leq a_k(y,h,x)\leq c_4|x|^2, \tag{2.29}$$

for some $c_i>0, i=\overline{1,4}$, *for all* $k\in\mathbb{N}, y\in\mathbf{Y}, h\in\mathbf{H}, x\in\mathbb{R}^m$.

Then, the system of random structure (2.1)–(2.3) *is asymptotically stable in the mean square.*

The proof is similar to the proof of Theorem 3 in [7].

Theorem 2.4 (Corollary). *If the conditions of Theorem 2.3 are fulfilled and the inequality* (2.28) *holds, then the system of random structure* (2.1)–(2.3) *is stable in the mean square on the whole.*

3. Computation of the weak infinitesimal operator

Based on the method [13], we will obtain an expression for calculating the explicit form of the weak infinitesimal operator (WIO) based on the system (2.1)–(2.3), which plays the role of the Lyapunov operator.

Let $U(t, y, h, x)$ be such a scalar integral function, that the sequence

$$\{v_k(y, h, x) \equiv U(t_k, y, h, x), k \geq 0\}$$

is a Lyapunov function.

It is possible to prove [3] that the pair $(\xi(t), x(t), t \geq 0,)$ is a Markov process and it is possible to introduce WIO

$$(\mathcal{L}U)(t, y, h, x) := \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t}[\mathbf{E}^{(t)}_{y,h,x}\{U(t + \Delta t, \xi(t + \Delta t), \eta(t + \Delta t), x(t + \Delta t)) - U(t, y, h, x)\}], \tag{3.1}$$

where $\mathbf{E}^{(t)}_{y,h,x}U = \mathbf{E}\{U|\xi(t) = y, \eta(t) = h, x(t) = x\}$, $\eta(t) := \eta_k$ and $t_k \leq t < t_{k+1}, k \geq 0$. It is natural to assume that the function U, defined above, belongs to the domain of definition of the operator $\mathcal{L}$, if the limit (3.1) exists in the sense of uniform convergence in some neighborhood of the point (y, x) uniformly by $h \in \mathbf{H}$.

Let's introduce the operator $\mathcal{L}_0$ which is related to Markov switching (2.2) at the moment $t_k, k \geq 0$:

$$(\mathcal{L}_0U) := \mathbb{I}_{t\in K}\left[\int_{\mathbf{H}} U(t, y, h, x)\mathbf{P}_k(h, dz) - U(t, y, h, x)\right], \tag{3.2}$$

where $\mathbf{P}_k(h, dz)$ is the transition probability of the Markov chain at the k-th step, $\mathbb{I}$ is the indicator of the set K.

At the moment τ of changing of the structure of the parameter ξ of the system $y_i \to y_j$ there is a jump-like change in the phase vector x with transition probability,

$$p_{ij}(\tau, x, A) := \mathbf{P}\{x(\tau) \in A|x(\tau-) = x, \xi(\tau-) = y_i, \xi(\tau) = y_j\}, A \subset \mathbb{R}^m. \tag{3.3}$$

Theorem 3.1. *Let the conditions* (2.4)–(2.8) *are hold. Then weak infinitesimal operator* $\mathcal{L}$ *on the solutions of the system* (2.1)–(2.3) *of the function U is calculated by the formula*

$$(\mathcal{L}U)(t, y, h, x) = (\mathcal{L}_tU)(t, y, h, x) + (\mathcal{L}_xU)(t, y, h, x) + (\mathcal{L}_yU)(t, y, h, x) + (\mathcal{L}_0U)(t, y, h, x), \tag{3.4}$$

where

$$(\mathcal{L}_tU)(t, y, h, x) = \frac{\partial U(t, y, h, x)}{\partial t}, \tag{3.5}$$

$$(\mathcal{L}_xU)(t, y, h, x) = (\nabla_x U, a(t, y, x)) + \frac{1}{2}Sp(\nabla^2_{xx}Ub(t, y, x), b^T(t, y, x)), \tag{3.6}$$

$$(\mathcal{L}_yU)(t, y, h, x) = \sum_{i\neq j}\left[\int_{\mathbb{R}^m} U(t, y_j, h, \zeta)p_{ij}(t, x, d\zeta) - U(t, y_i, h, x)\right]q_{ij}. \tag{3.7}$$

Here, $(\cdot,\cdot)$ *is a scalar product;* $(\Delta U) = (\frac{\partial U}{\partial x_1}, ..., \frac{\partial U}{\partial x_m})^T$, $\frac{\partial U}{\partial x_i}, i = \overline{1,m}$ *is the derivative of the i-th coordinate of the vector* $x \in \mathbb{R}^m$; $\nabla^2_{xx}U = \left[\frac{\partial^2 U}{\partial x_i \partial x_j}\right]_{i,j=1}^m$ *is a matrix of second derivatives;* Sp *is a trace of the matrix;* $q_{ij} = -\frac{\tilde{q}_{ij}}{\tilde{q}_i}$; $(\mathcal{L}_0 U)(t,y,h,x)$ *calculated by formula* (3.2)*;* U *is a function differentiable with respect to t, which has derivatives of the 1st and 2nd order by the last argument.*

Proof. By Definition (3.1)

$$(\mathcal{L}U)(t,y,h,x) := \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t}[\mathbf{E}^{(t)}_{y,h,x}\{U(t+\Delta t, \xi(t+\Delta t), \eta(t+\Delta t), x(t+\Delta t)) - U(t,y,h,x)\}].$$

Next,

$$\begin{aligned}(\mathcal{L}U)(t,y,h,x) := \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t}[\mathbf{E}^{(t)}_{y,h,x}\{&U(t+\Delta t, \xi(t+\Delta t), \eta(t+\Delta t), x(t+\Delta t)) \\ &- U(t,y,h,x) \pm U(t, \xi(t+\Delta t), \eta(t+\Delta t), x(t+\Delta t)) \\ &\pm U(t,y,\eta(t+\Delta t), x(t+\Delta t)) \pm U(t,y,h,x(t+\Delta t))\}].\end{aligned}$$

Therefore, $\mathcal{L}$ can be represented as

$$\begin{aligned}(\mathcal{L}U)(t,y,h,x) := &\lim_{\Delta t \downarrow 0} \frac{1}{\Delta t}[\mathbf{E}^{(t)}_{y,h,x}\{U(t+\Delta t, \xi(t+\Delta t), \eta(t+\Delta t), x(t+\Delta t)) \\ &- U(t, \xi(t+\Delta t), \eta(t+\Delta t), x(t+\Delta t))\}] \\ &+ \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t}[\mathbf{E}^{(t)}_{y,h,x}\{U(t, \xi(t+\Delta t), \eta(t+\Delta t), x(t+\Delta t)) \\ &- U(t,y,\eta(t+\Delta t), x(t+\Delta t))\}] \\ &+ \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t}[\mathbf{E}^{(t)}_{y,h,x}\{U(t,y,\eta(t+\Delta t), x(t+\Delta t)) \\ &- U(t,y,h,x(t+\Delta t))\}] + \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t}[\mathbf{E}^{(t)}_{y,h,x}\{U(t,y,h,x(t+\Delta t)) - U(t,y,h,x\}].\end{aligned}$$

Let's consider each term separately.

The form of the first term $\mathcal{L}_t U$ is obvious.

Let's establish the explicit form of the term $\mathcal{L}_x U$. Consider a complete group of disjoint events constructed as follows: denote by H_i the event which means that the structure (2.1) does not change in the interval $(t, t+\Delta t]$, i.e., $\xi(\tau) = y_i$ at $\tau \in (t, t+\Delta t]$. Then, with an accuracy of $o(\Delta t)$, we obtain [14]

$$\mathbf{P}(H_i) = -q_i \Delta t.$$

Next, denote by H_{ij} event, which means that in the interval $(t, t+\Delta t]$ a change $y_i \to y_j \neq y_i$ occurs. Then, with accuracy up to $o(\Delta t)$, we have

$$\mathbf{P}(H_{ij}) = -q_{ij}\Delta t.$$

Denote by $\Delta_i U := U(t+\Delta t, \xi(t+\Delta t), h, x(t+\Delta t) - U(t, y_i, h, x)$ and by $\Delta_{ij} U$ the increment ΔU upon occurrence of the event H_{ij}. Let's calculate the increments $\Delta_i U$ and $\Delta_{ij} U$ of the function U when events $H_i, H_{ij}, i \neq j$, occur, neglecting terms of order $o(\Delta t)$:

$$\Delta_i H = \left[\frac{\partial U}{\partial t} + (\nabla_x U, a(t, y_i, x)) + \frac{1}{2} Sp(\nabla^2_{xx} U b(t, y_i, x), b^T(t, y_i, x))\right]\Delta t + o(\Delta t). \tag{3.8}$$

Here, the partial derivatives are calculated at a point (t, y_i, x), where x is the solution of Eq (2.1) with initial condition $\xi(t) = y_i, x(t) = x, s > t \geq 0$. Next, for $\mathcal{L}_y U$ in the case of a change in the structure $y_i \to y_j$ in the interval $(t.t + \Delta t]$, we will get an increase

$$\Delta_{ij} U = U(t + \Delta t, y_j, h, x(t + \Delta t)) - U(t, y_i, h, x) \tag{3.9}$$

with the probability $q_{ij}\Delta t$.

The terms that illustrate the possibility of changing the structure of ξ are not included in the last equality and there are no Markov switching. This is because, after averaging, they have the order of $o(\Delta t)$ and we can ignore them.

To calculate $\mathbf{E}\{\Delta U | \xi(t) = y_i, \eta_0 = h, x(t) = x\}$, we use the full probability formula

$$\mathbf{E}\{\Delta U | \xi(t-) = y_i, \eta_0 = h, x(t) = x\} = \mathbf{E}\{\mathbf{E}\{\Delta U | \xi(t) = y_j, \xi(t-) = y_i, \eta_0 = h, x(t) = x\}\},$$

where the external mathematical expectation on the right-hand side is calculated by the variable ξ at the moment t.

Ignoring terms of order $o(\Delta t)$, from (3.8) and (3.9) we obtain

$$\begin{aligned} &\mathbf{E}\{\Delta U | \xi(t) = y_i, \eta_0 = h, x(t) = x\} \\ &= \left[\frac{\partial U}{\partial t} + (\nabla_x U, a(t, y_i, x)) + \frac{1}{2} Sp(\nabla^2_{xx} U b(t, y_i, x), b^T(t, y_i, x))\right](1 - q_i \Delta t)\Delta t \\ &\quad + \sum_{i \neq j} \left[\int_{\mathbb{R}^m} U(t, y_j, h, \zeta) p_{ij}(t, x, d\zeta) - U(t, y_i, h, x)\right] q_{ij} \Delta t + o(\Delta t). \end{aligned}$$

When calculating the third term, we used the property $x^T B x = Sp(Bxx^T)$ and the property of the Wiener process with respect to the covariance of the increment [3, 10].

Using division by Δt and passing to the boundary at $\Delta \downarrow 0$, we obtain the first, second, and third terms in (3.4). The idea of calculating the fourth term $\mathcal{L}_0 U$ can be found in [8], pp. 163–164. Theorem 3.1 is proved.

□

4. Stability in probability on the whole of a linear stochastic system of random structure

One-dimensional linear stochastic system of random structure given by SDE

$$dx(t) = a(\xi(t))x(t)dt + b(\xi(t)), x(t)dw(t), \ t \in \mathbb{R}_+ \backslash K, \tag{4.1}$$

with Markov switching

$$\begin{gathered} \Delta x(t) = g(t_k-, \xi(t_k-), \eta_k, x(t_k-)), \\ t_k \in K = \{t_n \Uparrow\}, \lim_{n \to \infty} t_n = t^* \in [0, T < \infty], \end{gathered} \tag{4.2}$$

and initial conditions

$$x(0) = x_0 \in \mathbb{R}^1, \ \xi(0) = y \in \mathbf{Y}, \eta_0 = h \in \mathbf{H}, \tag{4.3}$$

where $x \in \mathbb{R}^1$ is a strong solution of the SDE (4.1); t^* is a concentration point; ξ is a Markov chain with a finite number of states $\mathbf{Y} = \{1, 2, ..., N_\xi\}$ and generator $Q = \{\tilde{q}_{ij}\}, i, j = \{1, ..., N_\xi\}$; $\{\eta_k, k \geq 0\}$ is a Markov chain with values in space $\mathbf{H}$ and the transition probability at the k-th step $\mathbb{P}_k(h, dz)$; $w(t), t \geq 0$ is a one-dimensional standard Wiener process; the processes w, ξ and η are independent [3, 4].

We obtain sufficient conditions for the stability of the system (4.1)–(4.3) in probability on the whole.

Let's choose a Lyapunov function in the form [14]

$$v(\xi(t), h, x) = \gamma\xi(t)|x|^\beta, \gamma > 0. \tag{4.4}$$

Let the functions $a(i) = a_i$, $b(i) = b_i$ be such that for all $i = \{1, ..., N_\xi\}$

$$a_i - \frac{b_i^2}{2} < -\varepsilon. \tag{4.5}$$

Then in (4.4)

$$\beta = \varepsilon b^{-2}, b = \max_{i=\{1,...,N\}} \{b_i\}. \tag{4.6}$$

We can show which restrictions must satisfy the transitional probabilities q_{ij} of the Markov chain ξ and $\mathbb{P}_k(h, dz)$ of the Markov chain η, so that the system (4.1)–(4.3) is stable in probability on the whole.

Calculating $\mathcal{L}v$ on the solutions of the system (4.1)–(4.3), we obtain

$$(\mathcal{L}v)(t_k, y, h, x) = \gamma|x|^\beta \left\{ b_i \left(a_i + \frac{\beta - 1}{2} b_i \right) + \sum_{j \neq i}^{k} (j - i) q_{ij} \right\} + \int_{\mathbf{H}} \gamma i |x + g(t_k, y, h, x)|^\beta \mathbb{P}_k(h, dz) - \gamma i |x|^\beta.$$

Considering (4.4)–(4.6), at the point $(\xi(t) = i, x)$ we have

$$(\mathcal{L}v)(t_k, y, h, x) = \gamma|x|^\beta \left[-\frac{\beta i \varepsilon}{2} + a_i \right] + i\gamma \left[\int_{\mathbf{H}} |x + g(t_k, y, h, x)|^\beta \mathbb{P}_k(h, dz) - |x|^\beta \right], \tag{4.7}$$

where $a_i = \sum_{j>i}^{k} (j - i) q_{ij}, a_k = 0$.

Assuming that for $\forall h \in \mathbf{H}$ of the Markov chain η the transition probability at the k-th step $\mathbb{P}_k(h, dz)$ such that

$$\int_{\mathbf{H}} |x + g(t_k, y, h, x)|^\beta \mathbb{P}_k(h, dz) \leq 2|x|^\beta, \tag{4.8}$$

then the right-hand side of (4.7) will take the form

$$(\mathcal{L}v)(t_k, y, h, x) = \gamma|x|^\beta \left[-\frac{\beta i \varepsilon}{2} + a_i + i \right] = \gamma|x|^\beta \left[-\frac{i(\beta\varepsilon + 2)}{2} + a_i \right].$$

The function (4.4) satisfies the condition $\mathcal{L}v < 0$ if the expression in square brackets is negative. Thus, we can formulate the following statement.

Theorem 4.1. *If the conditions* (4.5), (4.6) *are met and*

$$a_i < \frac{i(\beta\varepsilon + 2)}{2}, i = \{1, ..., N_\xi\}, \tag{4.9}$$

then the solution of the system (4.1)–(4.3) *is stable in probability on the whole for all fixed* $y \in \mathbf{Y}$ *and* $h \in \mathbf{H}$.

5. Model example

Consider the linear stochastic differential equation

$$dx(t) = a(\xi(t))x(t)dt + b(\xi(t))x(t)dw(t), t \geq 0, r > 0, \tag{5.1}$$

with impulse action

$$\Delta x\left(2-\frac{1}{k}\right) = x\left(2-\frac{1}{k}-\right) + e^{-\alpha k \eta_k}\left(x\left(2-\frac{1}{k}\right)\wedge 1\right), k \to \infty, \tag{5.2}$$

and initial condition

$$x(0) = 10, \xi(0) = y_0 \in \mathbf{Y}, \eta_0 = 1. \tag{5.3}$$

Here a and b are constants that depend on Markov process ξ with values in dimensional space $(\mathbf{Y}, \mathcal{Y})$ with generator Q, and $\eta_k, k \geq 0$, is Markov chain with two non-absorbing states $h_1 = 0$ and $h_2 = 1$.

According to [1] the solution of the system (5.1)–(5.3) exists, for example, when $\alpha = 1.673$.

Case 1. Let's consider the same coefficients as in [1]:

- if $\xi = 1$: $a = 1, b = 0.3$;
- if $\xi = 2$: $a = -0.5, b = 2.1$;
- $\eta_k \in \{1, 2\}$.

In this case condition (4.5) is not hold for $i = 1$ because

$$1 - \frac{0.3^2}{2} = 0.955 > 0.$$

Therefore, the solution can be unstable. Indeed, if we consider an example of the realization of the solution of the system (5.1)–(5.3) with indicated parameters, then we observe a rapid growth (see Figure 1a).

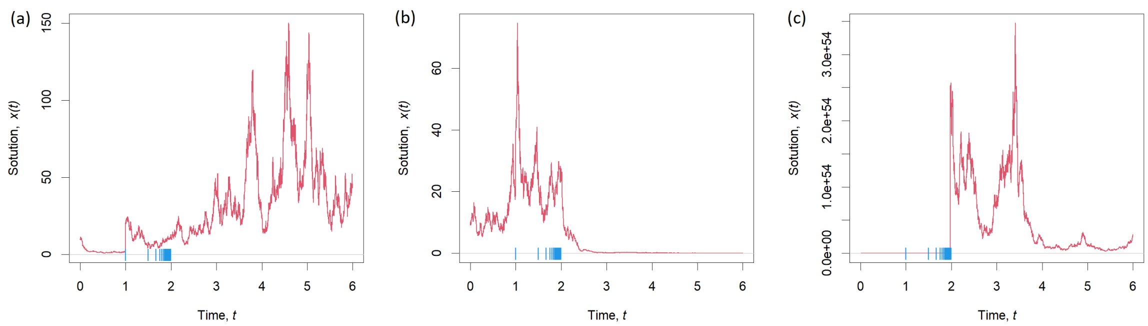

Figure 1. Estimated solution trajectories (by Euler-Maruyama method): (a) case 1 (unstable), (b) case 2 (stable), (c) case 3 (unstable with an extreme growth at $t = 2$). The red line corresponds to the system's solution $x(t)$ evolution, blue marks – moments of impulse actions.

Case 2. Next, we consider the values of the coefficients:

- if $\xi = 1$: $a = -1, b = 0.3$;

- if $\xi = 2$: $a = 0.5, b = 2$;
- $\eta_k \in \{1, 2\}$.

Condition (4.5) for $i = 1$ has the next form

$$-1 - \frac{0.3^2}{2} = -1.045 < -\varepsilon,$$

and for $i = 2$ has the form

$$0.5 - \frac{2^2}{2} = -1.5 < -\varepsilon,$$

and holds for $\varepsilon = 0.1$.

According to (4.6)

$$\beta = \frac{0.1}{2^2} = 0.025.$$

And (4.9) hold:
- if $i = 1$: $-1 < \frac{1\cdot(0.025\cdot 0.1+2)}{2} = 1.00125$;
- if $\xi = 2$: $0.5 < \frac{2\cdot(0.025\cdot 0.1+2)}{2} = 2.0025$.

So, all conditions of Theorem 4.1 are held and the solution of the system (5.1)–(5.3) with indicated parameters is stable in probability on the whole. Indeed, in the realization (see Figure 1b) we observe a direction to zero after the point $t = 3$.

Case 3. Here, the values of the coefficients are the same as in Case 2, but the impulse action has the next form

$$\Delta x\left(2 - \frac{1}{k}\right) = x\left(2 - \frac{1}{k}-\right) + e^{\alpha k \eta_k}\left(x\left(2 - \frac{1}{k}\right) \wedge 1\right), k \to \infty.$$

In this case, condition (4.8) does not hold and we cannot guarantee stability in the probability of solution of the system (5.1)–(5.3): we observe a very rapid growth (see Figure 1c).

6. Discussion

In this work, we consider dynamic stochastic systems with Markov parameters and switching that is condensed in one or several time points. For such a system, we obtain sufficient conditions for the asymptotic stochastic stability and asymptotic stability in mean square. We find an explicit form of a weak infinitesimal operator on the solutions of the system, which plays the role of the Lyapunov operator. For a linear case of the stochastic system, we find a condition that defines the stability area.

As was previously reported in [15], the condition (4.5) means that stability in probability can be ensured due to larger values of the coefficients and the fulfillment of the condition (4.8), even when the system is unstable

$$dx(t) = a_i x(t)dt.$$

For example, if we consider the second case of the model example with the coefficients $a = 0.5, b = 2$, we will see that the solution of the system corresponding to the deterministic part is not Lyapunov stable, but the solution of the stochastic system, as was demonstrated, is stable in probability.

The limitation of this work is the assumption that real systems should be described by Ito's differential equations. In this way, an assumption is made about the influence of a large number of independent factors. These differential equations are widely used in financial mathematics and information transmission systems. Another limitation concerns the absence of an aftereffect, as a result of which we can use the Markov properties of systems, but we lose a wide field of applications of the theory.

7. Conclusions

This paper explores sufficient conditions for the asymptotic stability of stochastic differential equations with a random structure, particularly in the context of jump concentration points. Our main result is presented in Theorem 2.2, which leverages the second Lyapunov method and involves the construction of corresponding Lyapunov functions. An important consideration in analyzing systems of random structure is the relationship between the magnitudes of jumps, denoted as L_k, and the jump times, denoted as τ_k. The implications of Theorem 2.2 are demonstrated through an example system whose stability can be modulated by varying parameters. We also highlight a remarkable observation that the system can maintain asymptotic stability even if, for some fixed value of the random process $\xi(t)$, the system described by Eq (2.1) becomes unstable when jumps (2.2) are absent.

In future studies, we plan to investigate the stability of stochastic differential equations with a random structure, particularly when the jump moments, denoted as τ_k, are random variables satisfying the condition

$$P\left(\lim_{k\to\infty} t_k = t_\infty < \infty\right) > 0.$$

This implies a non-zero occurrence of concentration points. Furthermore, the weak independence between the jumps and their corresponding moments will also be considered as part of this analysis.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

The authors declare the usage of AI tools only for spelling correction (Grammarly, https://app.grammarly.com/) and word/phrase translation between Ukrainian/Russian/English languages (DeepL, https://www.deepl.com/).

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Conflict of interest

The authors declare no conflict of interest.

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