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Research article

Statistical inference of a stochastically restricted linear mixed model

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Abstract: This article compares a predictor with the best linear unbiased predictor (BLUP) for a unified form of all unknown parameters under a stochastically restricted linear mixed model (SRLMM) in terms of the mean squared error matrix (MSEM) criterion. The methodology of block matrix inertias and ranks is employed to compare the MSEMs of these predictors. The comparison results are also demonstrated for a linear mixed model with and without an exact restriction, as well as special cases of the unified form of all unknown parameters in the SRLMM.

Keywords: BLUP; comparison; linear mixed model; linear stochastic restriction; MSEM
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1. Introduction

We will begin by introducing some notation before proceeding. We will write $\mathbf{\Lambda} \in \mathbb{R}_{k,l}$ if $\mathbf{\Lambda}$ is a $k \times l$ real matrix, $\mathbf{\Lambda} \in \mathbb{R}_k^s$ if $\mathbf{\Lambda} \in \mathbb{R}_{k,k}$ and is symmetric, $\mathbf{\Lambda} \in \mathbb{R}_k^{\succcurlyeq}$ if $\mathbf{\Lambda} \in \mathbb{R}_k^s$ and is positive semi-definite. We will use $\boldsymbol{r}(\mathbf{\Lambda})$, $\mathscr{C}(\mathbf{\Lambda})$, $\mathbf{\Lambda}'$ and $\mathbf{\Lambda}^+$ as symbols to present the rank, the column space, the transpose and the Moore-Penrose generalized inverse of $\mathbf{\Lambda} \in \mathbb{R}_{k,l}$, respectively. $\mathbf{\Lambda}^\perp = \mathbf{I}_k - \mathbf{\Lambda}\mathbf{\Lambda}^+$ stands for the orthogonal projector, where $\mathbf{I}_k \in \mathbb{R}_k^s$ is the identity matrix of order k. The symbols $\boldsymbol{i}_+(\mathbf{\Lambda})$ and $\boldsymbol{i}_-(\mathbf{\Lambda})$ represent positive and negative inertias of $\mathbf{\Lambda} \in \mathbb{R}_k^s$, respectively, while $\boldsymbol{i}_\pm(\mathbf{\Lambda})$ and $\boldsymbol{i}_\mp(\mathbf{\Lambda})$ are used to denote both the positive and the negative inertias of $\mathbf{\Lambda} \in \mathbb{R}_k^s$ jointly. The inequality $\mathbf{\Lambda}_1 - \mathbf{\Lambda}_2 \succcurlyeq \mathbf{0}$ or $\mathbf{\Lambda}_1 \succcurlyeq \mathbf{\Lambda}_2$ means $\mathbf{\Lambda}_1 - \mathbf{\Lambda}_2 \in \mathbb{R}_k^{\succcurlyeq}$ in the Löwner partial ordering for $\mathbf{\Lambda}_1, \mathbf{\Lambda}_2 \in \mathbb{R}_k^s$. Similarly, when stating the other inequalities between $\mathbf{\Lambda}_1, \mathbf{\Lambda}_2 \in \mathbb{R}_k^s$, $\mathbf{\Lambda}_1 - \mathbf{\Lambda}_2 \succ \mathbf{0}$ ($\mathbf{\Lambda}_1 \succ \mathbf{\Lambda}_2$), $\mathbf{\Lambda}_1 - \mathbf{\Lambda}_2 \prec \mathbf{0}$ ($\mathbf{\Lambda}_1 \prec \mathbf{\Lambda}_2$), $\mathbf{\Lambda}_1 - \mathbf{\Lambda}_2 \preccurlyeq \mathbf{0}$ ($\mathbf{\Lambda}_1 \preccurlyeq \mathbf{\Lambda}_2$) are used if the difference $\mathbf{\Lambda}_1 - \mathbf{\Lambda}_2$ is positive definite, negative definite and negative semi-definite matrices, respectively. We will write $\mathrm{E}(\boldsymbol{\lambda})$ for the expectation vector and $\mathrm{D}(\boldsymbol{\lambda}) = \mathrm{cov}(\boldsymbol{\lambda}, \boldsymbol{\lambda})$ for the dispersion matrix of a random vector $\boldsymbol{\lambda} \in \mathbb{R}_{k,1}$, where $\mathrm{cov}(\boldsymbol{\lambda}, \boldsymbol{\lambda})$ denotes the covariance matrix of $\boldsymbol{\lambda}$.

A linear mixed model (LMM) is a variant of linear regression model that combines fixed and random effects in the same analysis, allowing for more flexibility in model fitting. In some statistical problems, besides the sample information of LMMs, there may exist additional information, usually produced as prior information for the models. This prior information can be expressed as either a linear stochastic restriction or a linear exact restriction on unknown parameters, and it is usually added to the assumptions of models. Linear exact restrictions occur when there is a specific linear hypothesis being tested or when exact knowledge is available among certain parameters. On the other hand, linear stochastic restrictions arise when prior information is derived from previous investigations or established long-term relationships with relevant studies; see, e.g., [1, 2].

An LMM with a linear stochastic restriction, also known as a stochastically restricted LMM (SRLMM), can be given as follows:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\alpha} + \mathbf{Z}\boldsymbol{\gamma} + \boldsymbol{\varepsilon} \ \text{ with } \ \mathbf{S}\boldsymbol{\alpha} + \mathbf{e} = \mathbf{s}, \tag{1.1}$$

$$\begin{gathered} \mathrm{E}\begin{bmatrix}\boldsymbol{\gamma}\\ \boldsymbol{\varepsilon}\end{bmatrix} = \mathbf{0}, \ \ \mathrm{D}\begin{bmatrix}\boldsymbol{\gamma}\\ \boldsymbol{\varepsilon}\end{bmatrix} = \sigma^2\boldsymbol{\Lambda}_1, \ \ \mathrm{E}(\mathbf{e}) = \mathbf{0}, \ \ \mathrm{D}(\mathbf{e}) = \sigma^2\boldsymbol{\Lambda}_3, \\ \mathrm{cov}\left\{\begin{bmatrix}\boldsymbol{\gamma}\\ \boldsymbol{\varepsilon}\end{bmatrix}, \mathbf{e}\right\} = \sigma^2\boldsymbol{\Lambda}_2, \ \ \text{i.e.,} \ \ \mathrm{D}\begin{bmatrix}\boldsymbol{\gamma}\\ \boldsymbol{\varepsilon}\\ \mathbf{e}\end{bmatrix} = \sigma^2\begin{bmatrix}\boldsymbol{\Lambda}_1 & \boldsymbol{\Lambda}_2\\ \boldsymbol{\Lambda}_2' & \boldsymbol{\Lambda}_3\end{bmatrix} := \sigma^2\boldsymbol{\Lambda}, \end{gathered} \tag{1.2}$$

where $\mathbf{y} \in \mathbb{R}_{n,1}$ is a vector of responses, $\mathbf{X} \in \mathbb{R}_{n,k}$, $\mathbf{Z} \in \mathbb{R}_{n,p}$, $\mathbf{S} \in \mathbb{R}_{m,k}$ are known matrices of arbitrary ranks and $\mathbf{s} \in \mathbb{R}_{m,1}$ is a known vector, $\boldsymbol{\alpha} \in \mathbb{R}_{k,1}$ is a parameter vector of fixed effects, $\boldsymbol{\gamma} \in \mathbb{R}_{p,1}$ is a vector of random effects, $\boldsymbol{\varepsilon} \in \mathbb{R}_{n,1}$ and $\mathbf{e} \in \mathbb{R}_{m,1}$ are vectors of random errors, $\boldsymbol{\Lambda} \in \mathbb{R}^{\succcurlyeq}_{n+p+m}$ of arbitrary ranks and its entries $\boldsymbol{\Lambda}_i$ are known, $i = 1, 2, 3$ and σ^2 is a positive unknown parameter.

In the present study, we consider the SRLMM in (1.1). We derive the best linear unbiased predictors (BLUPs) of a unified form of all unknown parameters in the SRLMM utilizing some quadratic matrix optimization methods, which include block matrix inertias and ranks. Furthermore, we discuss some of the BLUPs' basic characteristics. We specifically address the comparison problem between any predictor and the BLUP for all unknown parameters in a unified form under the SRLMM in the sense of the mean squared error matrix (MSEM) criterion. It is well-known that the MSEM of any predictor or estimator $\widetilde{\boldsymbol{\lambda}}$ of $\boldsymbol{\lambda}$ in a linear regression model is defined as the matrix

$$\mathrm{MSEM}(\widetilde{\boldsymbol{\lambda}}) = \mathrm{E}(\widetilde{\boldsymbol{\lambda}} - \boldsymbol{\lambda})(\widetilde{\boldsymbol{\lambda}} - \boldsymbol{\lambda})'. \tag{1.3}$$

To compare two given predictors or estimators $\widetilde{\boldsymbol{\lambda}}_1$ and $\widetilde{\boldsymbol{\lambda}}_2$, the difference

$$\Delta(\widetilde{\boldsymbol{\lambda}}_1, \widetilde{\boldsymbol{\lambda}}_2) = \mathrm{MSEM}(\widetilde{\boldsymbol{\lambda}}_1) - \mathrm{MSEM}(\widetilde{\boldsymbol{\lambda}}_2) \tag{1.4}$$

is written. In this case, $\widetilde{\boldsymbol{\lambda}}_2$ is said to be superior to $\widetilde{\boldsymbol{\lambda}}_1$ with respect to the MSEM criterion iff $\Delta(\widetilde{\boldsymbol{\lambda}}_1, \widetilde{\boldsymbol{\lambda}}_2) \succcurlyeq \mathbf{0}$ holds. Using the methodology of block matrix inertias and ranks, a number of equalities and inequalities for the comparisons of MSEMs of two predictors, with one being the BLUP, of a unified form of all unknown parameters are established by considering the difference in (1.4). Such comparisons allow us to evaluate whether two predictors, one of which is BLUP, of the same unknown parameter vector are more effective than each other. We also derive comparison results for special cases of a unified form of all unknown parameters under the SRLMM. Furthermore, the results are reduced to both a constrained linear mixed model (CLMM), i.e., an LMM with a linear exact restriction, and an unconstrained LMM (ULMM). We use a numerical example to explain the theoretical results. There have been numerous previous and recent studies in the literature that focus on an exact restriction on unknown parameters under LMMs and linear models from different perspectives; see, e.g., [3–14] and the references therein. There has also been considerable attention on stochastic restriction on unknown parameters in a linear regression model from different approaches; see, e.g., [1,2,15–18], among others. We may also refer to [19–28], among others, for the corresponding studies in which similar approaches were conducted for the comparisons of predictors or estimators by utilizing the methodology of block matrix inertias and ranks. It is worth noting that studies on stochastic or exact restrictions are not limited to the context of linear models and in the field of statistics. The stochastic or exact restrictions on linear or nonlinear equations have been widely investigated from various aspects in a variety of fields, including biostatistics, educational measurement, sociology, finance, biology, chemistry, mechanics and economics, see, e.g., [29–34].

The development of various potential predictors (or, also potential estimators) of unknown parameter vectors in linear regression models is one of the main goals of theoretical and practical studies. In order to define predictors of unknown parameter vectors in the model, various types of optimality criteria have been utilized in mathematics. The unbiasedness of predictors of unknown parameter vectors according to the parameter spaces in the model is one of the crucial characteristics among others. When there are numerous unbiased predictors for the same parameter space, finding the one with the smallest dispersion matrix becomes significant. The BLUP, by its definition, corresponds to these two requirements. Thus, due to the minimizing property of the BLUP's dispersion matrix, the BLUP has a significant role to play in statistical inference theory and is frequently used as the foundation for evaluating the effectiveness of various predictors. As stated in the next section, the dispersion matrix of the BLUP is equal to its MSEM. In this situation, in the context of the SRLMM, comparing any predictor with the BLUP for a unified form of all unknown parameters using the MSEM criterion allows us to determine the optimality of the compared predictor. As it is known, one of the main criteria for determining the superiority of two different predictors of the unknown parameter vectors in the model is the MSEM criterion, which is widely used to measure predictors accurately.

The expressions, which are obtained from the comparison issues of the predictors, include various complicated matrix operations, including the use of Moore-Penrose generalized inverses. Some of the methods to simplify expressions composed by matrices and their Moore-Penrose generalized inverses are based on conventional operations on matrices, as well as some known equalities for inverses or Moore-Penrose generalized inverses of matrices. These methods' efficacy is quite limited, and the related processes are quite tedious. However, the methodology of block matrix inertias and ranks offers an effective approach to simplify such complex matrix expressions. It allows for the representation of results as simple inertia and rank equalities. The theory of matrix inertias and ranks has been developed as an efficient methodology for the simplification of such complex matrix expressions. In linear algebra, the inertia and rank of a matrix are fundamental quantities that are straightforward to understand and calculate. Using the theory of inertias and ranks of the matrix, various inequalities and equalities between the MSEMs of predictors correspond to the comparison of quantities derived from inertias and ranks of matrices.

The paper presents novel theoretical techniques and innovative approaches for comparing any predictor and the BLUP of the same general vector of unknown parameters under the SRLMM. The comparisons are based on the MSEM criterion, and the methodology of block matrix inertias and ranks is employed to establish exact formulas for these comparisons. It is important to note that we do not make any distributional assumptions for the random vectors in the models, except for assuming the existence of first and second moments. Additionally, no restrictions are imposed on the ranks of given matrices. Furthermore, there are no specific requirements concerning the patterns of the submatrices of $\mathbf{\Lambda}$ in (1.2). In statistical practice, if we meet the cases where $\mathbf{\Lambda}$ is unknown, then the estimators of submatrices of $\mathbf{\Lambda}$ can be used. Alternatively, if $\mathbf{\Lambda}$ is given with some parametric forms and some specific patterns, then the estimators obtained from the observed data can be substituted. These are other types of inference work, which we do not consider in this article.

Lastly, we list a few lemmas that are essential for the subsequent sections of the paper, contributing to the overall theoretical framework developed for comparing predictors under the SRLMM.

Lemma 1.1 ([35]). *Let* $\mathbf{\Lambda}_1, \mathbf{\Lambda}_2 \in \mathbb{R}_{k,l}$, *or, let* $\mathbf{\Lambda}_1, \mathbf{\Lambda}_2 \in \mathbb{R}^s_k$. *Then,*

$$\boldsymbol{r}(\mathbf{\Lambda}_1 - \mathbf{\Lambda}_2) = 0 \Leftrightarrow \mathbf{\Lambda}_1 = \mathbf{\Lambda}_2.$$

$$\boldsymbol{i}_-(\mathbf{\Lambda}_1 - \mathbf{\Lambda}_2) = k \Leftrightarrow \mathbf{\Lambda}_1 \prec \mathbf{\Lambda}_2 \ \text{ and } \ \boldsymbol{i}_+(\mathbf{\Lambda}_1 - \mathbf{\Lambda}_2) = k \Leftrightarrow \mathbf{\Lambda}_1 \succ \mathbf{\Lambda}_2.$$

$$\boldsymbol{i}_-(\mathbf{\Lambda}_1 - \mathbf{\Lambda}_2) = 0 \Leftrightarrow \mathbf{\Lambda}_1 \succcurlyeq \mathbf{\Lambda}_2 \ \text{ and } \ \boldsymbol{i}_+(\mathbf{\Lambda}_1 - \mathbf{\Lambda}_2) = 0 \Leftrightarrow \mathbf{\Lambda}_1 \preccurlyeq \mathbf{\Lambda}_2.$$

Lemma 1.2 ([35]). *Let* $\mathbf{\Lambda}_1 \in \mathbb{R}^s_k$, $\mathbf{\Lambda}_2 \in \mathbb{R}_{k,l}$, $\mathbf{\Lambda}_3 \in \mathbb{R}^s_l$ *and* $a \in \mathbb{R}$. *Then,*

$$\boldsymbol{r}(\mathbf{\Lambda}_1) = \boldsymbol{i}_+(\mathbf{\Lambda}_1) + \boldsymbol{i}_-(\mathbf{\Lambda}_1). \tag{1.5}$$

$$\boldsymbol{i}_\pm(a\mathbf{\Lambda}_1) = \boldsymbol{i}_\pm(\mathbf{\Lambda}_1) \ \text{ if } \ a > 0 \ \text{ and } \ \boldsymbol{i}_\pm(a\mathbf{\Lambda}_1) = \boldsymbol{i}_\mp(\mathbf{\Lambda}_1) \ \text{ if } \ a < 0. \tag{1.6}$$

$$\boldsymbol{i}_\pm\begin{bmatrix} \mathbf{\Lambda}_1 & \mathbf{\Lambda}_2 \\ \mathbf{\Lambda}_2' & \mathbf{\Lambda}_3 \end{bmatrix} = \boldsymbol{i}_\pm\begin{bmatrix} \mathbf{\Lambda}_1 & -\mathbf{\Lambda}_2 \\ -\mathbf{\Lambda}_2' & \mathbf{\Lambda}_3 \end{bmatrix} = \boldsymbol{i}_\mp\begin{bmatrix} -\mathbf{\Lambda}_1 & \mathbf{\Lambda}_2 \\ \mathbf{\Lambda}_2' & -\mathbf{\Lambda}_3 \end{bmatrix}. \tag{1.7}$$

$$\boldsymbol{i}_\pm\begin{bmatrix} \mathbf{\Lambda}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_3 \end{bmatrix} = \boldsymbol{i}_\pm(\mathbf{\Lambda}_1) + \boldsymbol{i}_\pm(\mathbf{\Lambda}_3) \ \text{ and } \ \boldsymbol{i}_+\begin{bmatrix} \mathbf{0} & \mathbf{\Lambda}_2 \\ \mathbf{\Lambda}_2' & \mathbf{0} \end{bmatrix} = \boldsymbol{i}_-\begin{bmatrix} \mathbf{0} & \mathbf{\Lambda}_2 \\ \mathbf{\Lambda}_2' & \mathbf{0} \end{bmatrix} = \boldsymbol{r}(\mathbf{\Lambda}_2). \tag{1.8}$$

$$\boldsymbol{i}_\pm\begin{bmatrix} \mathbf{\Lambda}_1 & \mathbf{\Lambda}_2 \\ \mathbf{\Lambda}_2' & \mathbf{0} \end{bmatrix} = \boldsymbol{r}(\mathbf{\Lambda}_2) + \boldsymbol{i}_\pm(\mathbf{\Lambda}_2^\perp \mathbf{\Lambda}_1 \mathbf{\Lambda}_2^\perp). \tag{1.9}$$

$$\boldsymbol{i}_+\begin{bmatrix} \mathbf{\Lambda}_1 & \mathbf{\Lambda}_2 \\ \mathbf{\Lambda}_2' & \mathbf{0} \end{bmatrix} = \boldsymbol{r}\begin{bmatrix} \mathbf{\Lambda}_1, & \mathbf{\Lambda}_2 \end{bmatrix} \ \text{ and } \ \boldsymbol{i}_-\begin{bmatrix} \mathbf{\Lambda}_1 & \mathbf{\Lambda}_2 \\ \mathbf{\Lambda}_2' & \mathbf{0} \end{bmatrix} = \boldsymbol{r}(\mathbf{\Lambda}_2) \ \text{ if } \ \mathbf{\Lambda}_1 \succcurlyeq \mathbf{0}. \tag{1.10}$$

$$\boldsymbol{i}_\pm\begin{bmatrix} \mathbf{\Lambda}_1 & \mathbf{\Lambda}_2 \\ \mathbf{\Lambda}_2' & \mathbf{\Lambda}_3 \end{bmatrix} = \boldsymbol{i}_\pm(\mathbf{\Lambda}_1) + \boldsymbol{i}_\pm(\mathbf{\Lambda}_3 - \mathbf{\Lambda}_2'\mathbf{\Lambda}_1^+\mathbf{\Lambda}_2) \ \text{ if } \ \mathscr{C}(\mathbf{\Lambda}_2) \subseteq \mathscr{C}(\mathbf{\Lambda}_1). \tag{1.11}$$

Lemma 1.3 ([36]). *Let* $\mathbf{\Lambda}_1 \in \mathbb{R}_{k,l}$ *and* $\mathbf{\Lambda}_2 \in \mathbb{R}_{m,l}$ *be given matrices, and let* $\mathbf{H} \in \mathbb{R}^{\succcurlyeq}_k$. *Suppose that there exists* $\mathbf{X} \in \mathbb{R}_{m,k}$ *such that* $\mathbf{X}\mathbf{\Lambda}_1 = \mathbf{\Lambda}_2$. *Then the maximal positive inertia of* $\mathbf{XHX}' - \mathbf{YHY}'$ *s.t. all solutions of* $\mathbf{Y}\mathbf{\Lambda}_1 = \mathbf{\Lambda}_2$ *is*

$$\max_{\mathbf{Y}\mathbf{\Lambda}_1 = \mathbf{\Lambda}_2} \boldsymbol{i}_+(\mathbf{XHX}' - \mathbf{YHY}') = \boldsymbol{r}\begin{bmatrix} \mathbf{XH} \\ \mathbf{\Lambda}_1' \end{bmatrix} - \boldsymbol{r}(\mathbf{\Lambda}_1) = \boldsymbol{r}(\mathbf{XH}\mathbf{\Lambda}_1^\perp).$$

Hence, a solution $\mathbf{X}$ of $\mathbf{X}\mathbf{\Lambda}_1 = \mathbf{\Lambda}_2$ exists such that $\mathbf{XHX}' \preccurlyeq \mathbf{YHY}'$ holds for all solutions of $\mathbf{Y}\mathbf{\Lambda}_1 = \mathbf{\Lambda}_2 \Leftrightarrow$ both the equations $\mathbf{X}\mathbf{\Lambda}_1 = \mathbf{\Lambda}_2$ and $\mathbf{XH}\mathbf{\Lambda}_1^\perp = \mathbf{0}$ are satisfied by $\mathbf{X}$.

2. BLUPs under SRLMMs

Let us consider the SRLMM in (1.1). By unifying two given equation parts in (1.1), we obtain

$$\begin{aligned} \mathcal{S}:\ \mathbf{y}_s &= \mathbf{X}_s\boldsymbol{\alpha} + \mathbf{Z}_s\boldsymbol{\gamma} + \boldsymbol{\varepsilon}_s = \mathbf{X}_s\boldsymbol{\alpha} + \begin{bmatrix}\mathbf{Z}_s, & \mathbf{I}_{n+m}\end{bmatrix}\begin{bmatrix}\boldsymbol{\gamma} \\ \boldsymbol{\varepsilon}_s\end{bmatrix}, \\ \mathbf{y}_s &= \begin{bmatrix}\mathbf{y} \\ \mathbf{s}\end{bmatrix},\ \ \mathbf{X}_s = \begin{bmatrix}\mathbf{X} \\ \mathbf{S}\end{bmatrix},\ \ \mathbf{Z}_s = \begin{bmatrix}\mathbf{Z} \\ \mathbf{0}\end{bmatrix},\ \ \boldsymbol{\varepsilon}_s = \begin{bmatrix}\boldsymbol{\varepsilon} \\ \mathbf{e}\end{bmatrix}. \end{aligned} \tag{2.1}$$

Through the unifying operation that we use in (2.1), which is a common procedure for approaching two or more equations, the SRLMM in (1.1) is converted to the implicitly linear stochastically restricted LMM in (2.1). We can take into consideration the following general vector to produce conclusions on predictors of all unknown vectors under $\mathcal{S}$:

$$\boldsymbol{\lambda} = \mathbf{K}\boldsymbol{\alpha} + \mathbf{L}\boldsymbol{\gamma} + \mathbf{M}\boldsymbol{\varepsilon}_s = \mathbf{K}\boldsymbol{\alpha} + \begin{bmatrix}\mathbf{L}, & \mathbf{M}\end{bmatrix}\begin{bmatrix}\boldsymbol{\gamma} \\ \boldsymbol{\varepsilon}_s\end{bmatrix} = \mathbf{K}\boldsymbol{\alpha} + \begin{bmatrix}\mathbf{L}, & \mathbf{M}_1, & \mathbf{M}_2\end{bmatrix}\begin{bmatrix}\boldsymbol{\gamma} \\ \boldsymbol{\varepsilon} \\ \mathbf{e}\end{bmatrix}, \tag{2.2}$$

where $\mathbf{K} \in \mathbb{R}_{t,k}$, $\mathbf{L} \in \mathbb{R}_{t,p}$, $\mathbf{M}_1 \in \mathbb{R}_{t,n}$ and $\mathbf{M}_2 \in \mathbb{R}_{t,m}$ with $\mathbf{M} = \begin{bmatrix}\mathbf{M}_1, & \mathbf{M}_2\end{bmatrix}$ are given matrices of arbitrary ranks. Then, from (1.2), (2.1) and (2.2),

$$\begin{aligned} \mathrm{E}(\mathbf{y}_s) = \mathbf{X}_s\boldsymbol{\alpha},\ \ \mathrm{D}(\mathbf{y}_s) &= \sigma^2\begin{bmatrix}\mathbf{Z}_s, & \mathbf{I}_{n+m}\end{bmatrix}\boldsymbol{\Lambda}\begin{bmatrix}\mathbf{Z}_s, & \mathbf{I}_{n+m}\end{bmatrix}' := \sigma^2\boldsymbol{\Phi}\boldsymbol{\Lambda}\boldsymbol{\Phi}', \\ \mathrm{E}(\boldsymbol{\lambda}) = \mathbf{K}\boldsymbol{\alpha},\ \ \mathrm{D}(\boldsymbol{\lambda}) &= \sigma^2\begin{bmatrix}\mathbf{L}, & \mathbf{M}\end{bmatrix}\boldsymbol{\Lambda}\begin{bmatrix}\mathbf{L}, & \mathbf{M}\end{bmatrix}' := \sigma^2\boldsymbol{\Upsilon}\boldsymbol{\Lambda}\boldsymbol{\Upsilon}', \\ \mathrm{cov}(\boldsymbol{\lambda}, \mathbf{y}_s) &= \sigma^2\begin{bmatrix}\mathbf{L}, & \mathbf{M}\end{bmatrix}\boldsymbol{\Lambda}\begin{bmatrix}\mathbf{Z}_s, & \mathbf{I}_{n+m}\end{bmatrix}' := \sigma^2\boldsymbol{\Upsilon}\boldsymbol{\Lambda}\boldsymbol{\Phi}', \end{aligned} \tag{2.3}$$

where $\boldsymbol{\Phi} = \begin{bmatrix}\mathbf{Z}_s, & \mathbf{I}_{n+m}\end{bmatrix}$ and $\boldsymbol{\Upsilon} = \begin{bmatrix}\mathbf{L}, & \mathbf{M}\end{bmatrix}$. Further, in this study, the model $\mathcal{S}$ is assumed to be consistent, i.e., $\mathbf{y}_s \in \mathscr{C}\begin{bmatrix}\mathbf{X}_s, & \boldsymbol{\Phi}\boldsymbol{\Lambda}\boldsymbol{\Phi}'\end{bmatrix}$ holds with probability 1; see, e.g., [37].

We introduce the following definition of the predictability of $\boldsymbol{\lambda}$ and its subcases; see, e.g., [38, 39]. Later, the definition of BLUP of $\boldsymbol{\lambda}$ under $\mathcal{S}$ is given; see, e.g., [40, 41].

Definition 2.1. *Let consider* $\mathcal{S}$ *in* (2.1) *and* $\boldsymbol{\lambda}$ *in* (2.2)*. Then,*

1) If a matrix $\mathbf{F} \in \mathbb{R}_{t,(n+m)}$ exists with $\mathrm{E}(\mathbf{F}\mathbf{y}_s - \boldsymbol{\lambda}) = \mathbf{0}$, then $\boldsymbol{\lambda}$ is defined to be predictable, that is, $\boldsymbol{\lambda}$ is predictable $\Leftrightarrow \mathscr{C}(\mathbf{K}') \subseteq \mathscr{C}(\mathbf{X}_s') \Leftrightarrow \mathbf{K}\boldsymbol{\alpha}$ is estimable.
2) $\mathbf{X}_s\boldsymbol{\alpha} + \mathbf{Z}_s\boldsymbol{\gamma} + \boldsymbol{\varepsilon}_s$ is always predictable and $\mathbf{X}_s\boldsymbol{\alpha}$ is always estimable.
3) $\boldsymbol{\alpha}$ is estimable $\Leftrightarrow r(\mathbf{X}_s) = k$.
4) $\boldsymbol{\gamma}$, $\boldsymbol{\varepsilon}$ and $\mathbf{e}$ are always predictable.

Definition 2.2. *Suppose that* $\boldsymbol{\lambda}$ *in* (2.2) *is predictable under* $\mathcal{S}$ *in* (2.1)*.* $\mathbf{F}\mathbf{y}_s$ *is said to be the* BLUP *of* $\boldsymbol{\lambda}$*, represented by* $\mathbf{F}\mathbf{y}_s = \widetilde{\boldsymbol{\lambda}}_{\mathrm{BLUP}}$*, if there exists* $\mathbf{F}\mathbf{y}_s$ *such that*

$$\mathrm{D}(\mathbf{F}\mathbf{y}_s - \boldsymbol{\lambda}) = \ \min \text{ s.t. } \ \mathrm{E}(\mathbf{F}\mathbf{y}_s - \boldsymbol{\lambda}) = \mathbf{0}$$

holds in the Löwner partial ordering. If $\mathbf{L} = \mathbf{0}$ and $\mathbf{M} = \mathbf{0}$ in $\boldsymbol{\lambda}$, $\mathbf{F}\mathbf{y}_s$ is the well-known the best linear unbiased estimator (BLUE) of $\mathbf{K}\boldsymbol{\alpha}$, represented by $\widetilde{\mathbf{K}\boldsymbol{\alpha}}_{\mathrm{BLUE}}$.

According to the definition of the MSEM given in (1.3), we can give the following lemma, see; [42, 43].

Lemma 2.1. *Let $\widetilde{\lambda}$ be a predictor for a general vector of all unknown vectors λ in an* LMM. *Then,*

$$\mathrm{MSEM}(\widetilde{\lambda}) = \mathrm{D}(\widetilde{\lambda}) + \mathrm{D}(\lambda) - \mathrm{cov}(\widetilde{\lambda}, \lambda) - \mathrm{cov}(\lambda, \widetilde{\lambda}) + [\mathrm{bias}(\widetilde{\lambda})][\mathrm{bias}(\widetilde{\lambda})]', \tag{2.4}$$

where $\mathrm{bias}(\widetilde{\lambda}) = \mathrm{E}(\widetilde{\lambda} - \lambda)$. If $\widehat{\lambda}$ is an estimator for a general vector of all unknown parameters λ in an LMM, then,

$$\mathrm{MSEM}(\widehat{\lambda}) = \mathrm{D}(\widehat{\lambda}) + [\mathrm{bias}(\widehat{\lambda})][\mathrm{bias}(\widehat{\lambda})]',$$

where $\mathrm{bias}(\widehat{\lambda}) = \mathrm{E}(\widehat{\lambda}) - \lambda$.

The fundamental BLUP equation and associated properties are summarized in the following lemma; see [25], for different approaches; see also, [37,41,44].

Lemma 2.2. *Suppose that λ in* (2.2) *is predictable under $\mathcal{S}$ in* (2.1). *Consider any two unbiased linear predictors $\mathbf{F}\mathbf{y}_s$ and $\mathbf{G}\mathbf{y}_s$ for λ. Then,*

$$\max_{\mathrm{E}(\mathbf{G}\mathbf{y}_s - \lambda)=\mathbf{0}} \boldsymbol{i}_{+}\left(\mathrm{D}(\mathbf{F}\mathbf{y}_s - \lambda) - \mathrm{D}(\mathbf{G}\mathbf{y}_s - \lambda)\right) = \boldsymbol{r}\left(\begin{bmatrix}\mathbf{F}, & -\mathbf{I}_t\end{bmatrix}\begin{bmatrix}\mathbf{\Phi\Lambda\Phi}' & \mathbf{\Phi\Lambda\Upsilon}' \\ \mathbf{\Upsilon\Lambda\Phi}' & \mathbf{\Upsilon\Lambda\Upsilon}'\end{bmatrix}\begin{bmatrix}\mathbf{X}_s \\ \mathbf{K}\end{bmatrix}^{\perp}\right) \tag{2.5}$$

is the maximal positive inertia of $\mathrm{D}(\mathbf{F}\mathbf{y}_s - \lambda) - \mathrm{D}(\mathbf{G}\mathbf{y}_s - \lambda)$ s.t. $\mathbf{G}\mathbf{X}_s = \mathbf{K}$. Hence, $\mathrm{D}(\mathbf{F}\mathbf{y}_s - \lambda) = \min$ s.t. $\mathrm{E}(\mathbf{F}\mathbf{y}_s - \lambda) = \mathbf{0} \Leftrightarrow \mathbf{F}\mathbf{y}_s = \widetilde{\lambda}_{\mathrm{BLUP}}$

$$\Leftrightarrow \mathbf{F}\begin{bmatrix}\mathbf{X}_s, & \mathbf{\Phi\Lambda\Phi}'\mathbf{X}_s^{\perp}\end{bmatrix} = \begin{bmatrix}\mathbf{K}, & \mathbf{\Upsilon\Lambda\Phi}'\mathbf{X}_s^{\perp}\end{bmatrix}. \tag{2.6}$$

This equation in (2.6) is consistent and the BLUP of λ under $\mathcal{S}$ can be written as follows by considering the general solution of this equation:

$$\widetilde{\lambda}_{\mathrm{BLUP}} = \mathbf{F}\mathbf{y}_s = \left(\begin{bmatrix}\mathbf{K}, & \mathbf{\Upsilon\Lambda\Phi}'\mathbf{X}_s^{\perp}\end{bmatrix}\mathbf{J}^{+} + \mathbf{U}\mathbf{J}^{\perp}\right)\mathbf{y}_s, \tag{2.7}$$

where $\mathbf{J} = \begin{bmatrix}\mathbf{X}_s, & \mathbf{\Phi\Lambda\Phi}'\mathbf{X}_s^{\perp}\end{bmatrix}$ and $\mathbf{U} \in \mathbb{R}_{t,(n+m)}$ is an arbitrary matrix. Further,

$$\mathrm{D}(\widetilde{\lambda}_{\mathrm{BLUP}}) = \sigma^2\begin{bmatrix}\mathbf{K}, & \mathbf{\Upsilon\Lambda\Phi}'\mathbf{X}_s^{\perp}\end{bmatrix}\mathbf{J}^{+}\mathbf{\Phi\Lambda\Phi}'\left(\begin{bmatrix}\mathbf{K}, & \mathbf{\Upsilon\Lambda\Phi}'\mathbf{X}_s^{\perp}\end{bmatrix}\mathbf{J}^{+}\right)', \tag{2.8}$$

$$\mathrm{cov}(\widetilde{\lambda}_{\mathrm{BLUP}}, \lambda) = \sigma^2\begin{bmatrix}\mathbf{K}, & \mathbf{\Upsilon\Lambda\Phi}'\mathbf{X}_s^{\perp}\end{bmatrix}\mathbf{J}^{+}\mathbf{\Phi\Lambda\Upsilon}', \tag{2.9}$$

$$\mathrm{D}(\lambda - \widetilde{\lambda}_{\mathrm{BLUP}}) = \sigma^2\left(\begin{bmatrix}\mathbf{K}, & \mathbf{\Upsilon\Lambda\Phi}'\mathbf{X}_s^{\perp}\end{bmatrix}\mathbf{J}^{+}\mathbf{\Phi} - \mathbf{\Upsilon}\right)\mathbf{\Lambda}\left(\begin{bmatrix}\mathbf{K}, & \mathbf{\Upsilon\Lambda\Phi}'\mathbf{X}_s^{\perp}\end{bmatrix}\mathbf{J}^{+}\mathbf{\Phi} - \mathbf{\Upsilon}\right)', \tag{2.10}$$

and the MSEM of $\widetilde{\lambda}_{\mathrm{BLUP}}$ is

$$\mathrm{MSEM}(\widetilde{\lambda}_{\mathrm{BLUP}}) = \mathrm{D}(\lambda - \widetilde{\lambda}_{\mathrm{BLUP}}). \tag{2.11}$$

In particular,

1) $\mathscr{C}(\mathbf{J}) = \mathbb{R}_{(n+m),1} \Leftrightarrow \mathbf{F}$ is unique and $\mathcal{S}$ is consistent $\Leftrightarrow \widetilde{\lambda}_{\mathrm{BLUP}}$ is unique.
2) The following equalities hold:

$$\boldsymbol{r}(\mathbf{J}) = \boldsymbol{r}\begin{bmatrix}\mathbf{X}_s, & \mathbf{\Phi\Lambda\Phi}'\end{bmatrix} \quad \text{and} \quad \mathscr{C}(\mathbf{J}) = \mathscr{C}\begin{bmatrix}\mathbf{X}_s, & \mathbf{\Phi\Lambda\Phi}'\end{bmatrix}. \tag{2.12}$$

Proof of Lemma 2.2. For an unbiased linear predictor $\mathbf{F}\mathbf{y}_s$ of λ in $\mathcal{S}$,

$$\begin{aligned} &\mathrm{E}(\mathbf{F}\mathbf{y}_s - \lambda) = \mathbf{0} \Leftrightarrow \mathbf{F}\mathbf{X}_s = \mathbf{K}, \ \text{ i.e., } \begin{bmatrix}\mathbf{F}, & -\mathbf{I}_t\end{bmatrix}\begin{bmatrix}\mathbf{X}_s \\ \mathbf{K}\end{bmatrix} = \mathbf{0}, \\ &\mathrm{D}(\mathbf{F}\mathbf{y}_s - \lambda) = \sigma^2 \begin{bmatrix}\mathbf{F}, & -\mathbf{I}_t\end{bmatrix}\begin{bmatrix}\mathbf{\Phi} \\ \mathbf{\Upsilon}\end{bmatrix}\mathbf{\Lambda}\begin{bmatrix}\mathbf{\Phi} \\ \mathbf{\Upsilon}\end{bmatrix}' \begin{bmatrix}\mathbf{F}, & -\mathbf{I}_t\end{bmatrix}' := f(\mathbf{F}) \end{aligned} \tag{2.13}$$

are written. For the other unbiased predictor $\mathbf{G}\mathbf{y}_s$ of λ, the similar expressions can be written as in (2.13) by putting $\mathbf{G}$ instead of $\mathbf{F}$. Finding solution $\mathbf{F}$ of the consistent equation $\mathbf{F}\mathbf{X}_s = \mathbf{K}$ such that

$$f(\mathbf{F}) \preccurlyeq f(\mathbf{G}) \ \text{ s.t. } \ \mathbf{G}\mathbf{X}_s = \mathbf{K} \tag{2.14}$$

corresponds to find the BLUP of λ under $\mathcal{S}$. According to Lemma 1.3, (2.14) is a typical constrained quadratic matrix-valued function optimization problem in the Löwner partial ordering. Applying Lemma 1.3 to (2.14), (2.5) is obtained. Using Lemma 1.3, we obtain the fundamental BLUP equation of λ in (2.6). The consistency of the equation in (2.6) is seen from the following column space inclusions:

$$\mathscr{C}(\mathbf{K}') \subseteq \mathscr{C}(\mathbf{X}_s') \text{ and } \mathscr{C}(\mathbf{\Phi\Lambda\Upsilon}') \subseteq \mathscr{C}(\mathbf{\Phi\Lambda}) = \mathscr{C}(\mathbf{\Phi\Lambda\Phi}'). \tag{2.15}$$

The well-known general solution of (2.6) s.t. $\mathbf{F}$ can be written in the parametric form as in (2.7). (2.8) and (2.9) are directly seen from (2.3) and (2.7). $\mathrm{D}(\lambda - \widetilde{\lambda}_{\mathrm{BLUP}})$ is written as

$$\mathrm{D}(\lambda - \widetilde{\lambda}_{\mathrm{BLUP}}) = \mathrm{D}(\widetilde{\lambda}_{\mathrm{BLUP}}) + \mathrm{D}(\lambda) - \mathrm{cov}(\widetilde{\lambda}_{\mathrm{BLUP}}, \lambda) - \mathrm{cov}(\lambda, \widetilde{\lambda}_{\mathrm{BLUP}}). \tag{2.16}$$

By setting the equalities in (2.3), (2.8) and (2.9) into (2.16), (2.10) is obtained. (2.4) and (2.16) give (2.11) since $\mathrm{bias}(\widetilde{\lambda}_{\mathrm{BLUP}}) = \mathbf{0}$. Item 1 follows from (2.7). The expressions in (2.12) are well-known results; see [45, Lemma 2.1 (a)]. □

We note that the fundamental equations of BLUPs and BLUEs for unknown vectors in λ and their related results can be derived from 2.2, by special choices of the matrices $\mathbf{K}$, $\mathbf{L}$ and $\mathbf{M}$.

3. Comparisons under SRLMM

The results of the relationships between the MSEM of any predictor for λ and the MSEM of the BLUP of λ under $\mathcal{S}$ are collected in the theorem given below.

Theorem 3.1. *Suppose that λ in* (2.2) *is predictable under $\mathcal{S}$ in* (2.1)*. Let $\widetilde{\lambda}$ be any predictor (unbiased or biased) for λ under $\mathcal{S}$. Denote*

$$\Delta(\widetilde{\lambda}, \widetilde{\lambda}_{\mathrm{BLUP}}) = \mathrm{MSEM}(\widetilde{\lambda}) - \mathrm{MSEM}(\widetilde{\lambda}_{\mathrm{BLUP}})$$

and

$$\mathbf{E} = \begin{bmatrix} \mathbf{\Phi\Lambda\Phi}' & \mathbf{\Phi\Lambda\Upsilon}' & \mathbf{X}_s \\ \mathbf{\Upsilon\Lambda\Phi}' & \mathbf{\Upsilon\Lambda\Upsilon}' - \sigma^{-2}\,\mathrm{MSEM}(\widetilde{\lambda}) & \mathbf{K} \\ \mathbf{X}_s' & \mathbf{K}' & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \sigma^{-2}\,\mathrm{D}(\mathbf{y}_s) & \sigma^{-2}\,\mathrm{cov}(\mathbf{y}_s, \lambda) & \mathbf{X}_s \\ \sigma^{-2}\,\mathrm{cov}(\lambda, \mathbf{y}_s) & \sigma^{-2}\left(\mathrm{D}(\lambda) - \mathrm{MSEM}(\widetilde{\lambda})\right) & \mathbf{K} \\ \mathbf{X}_s' & \mathbf{K}' & \mathbf{0} \end{bmatrix}.$$

Then,

$$i_{+}(\mathrm{MSEM}(\widetilde{\lambda}) - \mathrm{MSEM}(\widetilde{\lambda}_{\mathrm{BLUP}})) = i_{-}(\mathbf{E}) - r(\mathbf{X}_s), \tag{3.1}$$

$$i_{-}(\mathrm{MSEM}(\widetilde{\lambda}) - \mathrm{MSEM}(\widetilde{\lambda}_{\mathrm{BLUP}})) = i_{+}(\mathbf{E}) - r\begin{bmatrix}\mathbf{X}_s, & \mathbf{\Phi\Lambda\Phi}'\end{bmatrix}, \tag{3.2}$$

$$r(\mathrm{MSEM}(\widetilde{\lambda}) - \mathrm{MSEM}(\widetilde{\lambda}_{\mathrm{BLUP}})) = r(\mathbf{E}) - r(\mathbf{X}_s) - r\begin{bmatrix}\mathbf{X}_s, & \mathbf{\Phi\Lambda\Phi}'\end{bmatrix}. \tag{3.3}$$

In consequence,

1) $\Delta(\widetilde{\lambda}, \widetilde{\lambda}_{\mathrm{BLUP}}) \succcurlyeq \mathbf{0}$, i.e., $\mathrm{MSEM}(\widetilde{\lambda}) \succcurlyeq \mathrm{MSEM}(\widetilde{\lambda}_{\mathrm{BLUP}}) \Leftrightarrow i_{+}(\mathbf{E}) = r\begin{bmatrix}\mathbf{X}_s, & \mathbf{\Phi\Lambda\Phi}'\end{bmatrix}$.
2) $\Delta(\widetilde{\lambda}, \widetilde{\lambda}_{\mathrm{BLUP}}) \succ \mathbf{0}$, i.e., $\mathrm{MSEM}(\widetilde{\lambda}) \succ \mathrm{MSEM}(\widetilde{\lambda}_{\mathrm{BLUP}}) \Leftrightarrow i_{-}(\mathbf{E}) = r(\mathbf{X}_s) + t$.
3) $\Delta(\widetilde{\lambda}, \widetilde{\lambda}_{\mathrm{BLUP}}) \preccurlyeq \mathbf{0}$, i.e., $\mathrm{MSEM}(\widetilde{\lambda}) \preccurlyeq \mathrm{MSEM}(\widetilde{\lambda}_{\mathrm{BLUP}}) \Leftrightarrow i_{-}(\mathbf{E}) = r(\mathbf{X}_s)$.
4) $\Delta(\widetilde{\lambda}, \widetilde{\lambda}_{\mathrm{BLUP}}) \prec \mathbf{0}$, i.e., $\mathrm{MSEM}(\widetilde{\lambda}) \prec \mathrm{MSEM}(\widetilde{\lambda}_{\mathrm{BLUP}}) \Leftrightarrow i_{+}(\mathbf{E}) = r\begin{bmatrix}\mathbf{X}_s, & \mathbf{\Phi\Lambda\Phi}'\end{bmatrix} + t$.
5) $\Delta(\widetilde{\lambda}, \widetilde{\lambda}_{\mathrm{BLUP}}) = \mathbf{0}$, i.e., $\mathrm{MSEM}(\widetilde{\lambda}) = \mathrm{MSEM}(\widetilde{\lambda}_{\mathrm{BLUP}}) \Leftrightarrow r(\mathbf{E}) = r\begin{bmatrix}\mathbf{X}_s, & \mathbf{\Phi\Lambda\Phi}'\end{bmatrix} + r(\mathbf{X}_s)$.

Proof. Let say $\mathbf{C} = \mathrm{MSEM}(\widetilde{\lambda})$ and apply (1.11) to the difference between $\mathrm{MSEM}(\widetilde{\lambda})$ and $\mathrm{MSEM}(\widetilde{\lambda}_{\mathrm{BLUP}})$ in (2.11). Then, we obtain

$$\begin{aligned}
i_{\pm}\left(\Delta(\widetilde{\lambda}, \widetilde{\lambda}_{\mathrm{BLUP}})\right) &= i_{\pm}\left(\mathrm{MSEM}(\widetilde{\lambda}) - \mathrm{MSEM}(\widetilde{\lambda}_{\mathrm{BLUP}})\right) \\
&= i_{\pm}\left(\mathbf{C} - \mathrm{D}(\lambda - \widetilde{\lambda}_{\mathrm{BLUP}})\right) \\
&= i_{\pm}\left(\mathbf{C} - \sigma^2\left(\mathbf{DJ}^{+}\mathbf{\Phi} - \mathbf{\Upsilon}\right)\mathbf{\Lambda}\left(\mathbf{DJ}^{+}\mathbf{\Phi} - \mathbf{\Upsilon}\right)'\right) \\
&= i_{\pm}\begin{bmatrix}\mathbf{\Lambda} & \mathbf{\Lambda\Phi}'(\mathbf{J}^{+})'\mathbf{D}' - \mathbf{\Lambda\Upsilon}' \\ \mathbf{DJ}^{+}\mathbf{\Phi\Lambda} - \mathbf{\Upsilon\Lambda} & \sigma^{-2}\mathbf{C}\end{bmatrix} - i_{\pm}(\mathbf{\Lambda}) \\
&= i_{\pm}\left(\begin{bmatrix}\mathbf{\Lambda} & -\mathbf{\Lambda\Upsilon}' \\ -\mathbf{\Upsilon\Lambda} & \sigma^{-2}\mathbf{C}\end{bmatrix} + \begin{bmatrix}\mathbf{\Lambda\Phi}' & \mathbf{0} \\ \mathbf{0} & \mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{0} & \mathbf{J} \\ \mathbf{J}' & \mathbf{0}\end{bmatrix}^{+}\begin{bmatrix}\mathbf{\Phi\Lambda} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}'\end{bmatrix}\right) - i_{\pm}(\mathbf{\Lambda}),
\end{aligned} \tag{3.4}$$

where $\mathbf{D} = \begin{bmatrix}\mathbf{K}, & \mathbf{\Upsilon\Lambda\Phi}'\mathbf{X}_s^{\perp}\end{bmatrix}$ and $\mathbf{J} = \begin{bmatrix}\mathbf{X}_s, & \mathbf{\Phi\Lambda\Phi}'\mathbf{X}_s^{\perp}\end{bmatrix}$. By using column space equalities and inclusions in (2.12) and (2.15), reapplying (1.11) to (3.4) and also using (1.6) and (1.8) with the elementary block matrix operations, the expression in (3.4) is equivalently written as

$$i_{\pm}\begin{bmatrix}\mathbf{0} & -\mathbf{J} & \mathbf{\Phi\Lambda} & \mathbf{0} \\ -\mathbf{J}' & \mathbf{0} & \mathbf{0} & \mathbf{D}' \\ \mathbf{\Lambda\Phi}' & \mathbf{0} & \mathbf{\Lambda} & -\mathbf{\Lambda\Upsilon}' \\ \mathbf{0} & \mathbf{D} & -\mathbf{\Upsilon\Lambda} & \sigma^{-2}\mathbf{C}\end{bmatrix} - i_{\mp}\begin{bmatrix}\mathbf{0} & \mathbf{J} \\ \mathbf{J}' & \mathbf{0}\end{bmatrix} - i_{\pm}(\mathbf{\Lambda}) = i_{\pm}\begin{bmatrix}-\mathbf{\Phi\Lambda\Phi}' & -\mathbf{J} & \mathbf{\Phi\Lambda\Upsilon}' \\ -\mathbf{J}' & \mathbf{0} & \mathbf{D}' \\ \mathbf{\Upsilon\Lambda\Phi}' & \mathbf{D} & \sigma^{-2}\mathbf{C} - \mathbf{\Upsilon\Lambda\Upsilon}'\end{bmatrix} - r(\mathbf{J}). \tag{3.5}$$

By setting $\mathbf{D}$ and $\mathbf{J}$ into (3.5) and also using (1.7)–(1.9) and (2.12) with the elementary block matrix operations,

$$\begin{aligned}
& i_{\pm}\begin{bmatrix}-\mathbf{\Phi\Lambda\Phi}' & -\mathbf{X}_s & -\mathbf{\Phi\Lambda\Phi}'\mathbf{X}_s^{\perp} & \mathbf{\Phi\Lambda\Upsilon}' \\ -\mathbf{X}_s' & \mathbf{0} & \mathbf{0} & \mathbf{K}' \\ -\mathbf{X}_s^{\perp}\mathbf{\Phi\Lambda\Phi}' & \mathbf{0} & \mathbf{0} & \mathbf{X}_s^{\perp}\mathbf{\Phi\Lambda\Upsilon}' \\ \mathbf{\Upsilon\Lambda\Phi}' & \mathbf{K} & \mathbf{\Upsilon\Lambda\Phi}'\mathbf{X}_s^{\perp} & \sigma^{-2}\mathbf{C} - \mathbf{\Upsilon\Lambda\Upsilon}'\end{bmatrix} - r\begin{bmatrix}\mathbf{X}_s, & \mathbf{\Phi\Lambda\Phi}'\end{bmatrix} \\
&= i_{\pm}\begin{bmatrix}-\mathbf{\Phi\Lambda\Phi}' & -\mathbf{X}_s & \mathbf{\Phi\Lambda\Upsilon}' \\ -\mathbf{X}_s' & \mathbf{0} & \mathbf{K}' \\ \mathbf{\Upsilon\Lambda\Phi}' & \mathbf{K} & \sigma^{-2}\mathbf{C} - \mathbf{\Upsilon\Lambda\Upsilon}'\end{bmatrix} + i_{\pm}\left(\mathbf{X}_s^{\perp}\mathbf{\Phi\Lambda\Phi}'\mathbf{X}_s^{\perp}\right) - r\begin{bmatrix}\mathbf{X}_s, & \mathbf{\Phi\Lambda\Phi}'\end{bmatrix}
\end{aligned}$$

$$= i_{\mp}\begin{bmatrix} \mathbf{\Phi\Lambda\Phi}' & \mathbf{\Phi\Lambda\Upsilon}' & \mathbf{X}_s \\ \mathbf{\Upsilon\Lambda\Phi}' & \mathbf{\Upsilon\Lambda\Upsilon}' - \sigma^{-2}\mathbf{C} & \mathbf{K} \\ \mathbf{X}_s' & \mathbf{K}' & \mathbf{0} \end{bmatrix} + i_{\pm}\begin{bmatrix} \mathbf{\Phi\Lambda\Phi}' & \mathbf{X}_s \\ \mathbf{X}_s' & \mathbf{0} \end{bmatrix} - r(\mathbf{X}_s) - r\begin{bmatrix} \mathbf{X}_s, & \mathbf{\Phi\Lambda\Phi}' \end{bmatrix} \tag{3.6}$$

is obtained. From (1.10),

$$i_{+}\begin{bmatrix} \mathbf{\Phi\Lambda\Phi}' & \mathbf{X}_s \\ \mathbf{X}_s' & \mathbf{0} \end{bmatrix} = r\begin{bmatrix} \mathbf{X}_s, & \mathbf{\Phi\Lambda\Phi}' \end{bmatrix} \text{ and } i_{-}\begin{bmatrix} \mathbf{\Phi\Lambda\Phi}' & \mathbf{X}_s \\ \mathbf{X}_s' & \mathbf{0} \end{bmatrix} = r(\mathbf{X}_s) \tag{3.7}$$

are written and then, after setting $\mathbf{C} = \mathrm{MSEM}(\widetilde{\boldsymbol{\lambda}})$ in the first matrix in (3.6), (3.1) and (3.2) are obtained from (3.6) and (3.7). According to (1.5), adding the equalities in (3.1) and (3.2) yields (3.3). Applying Lemma 1.1 to (3.1)–(3.3) yields 1–5. □

Corollary 3.1. *Let $\mathcal{S}$ and λ be as given in* (2.1) *and* (2.2)*, respectively. Suppose that $\widetilde{\boldsymbol{\gamma}}$, $\widetilde{\boldsymbol{\varepsilon}}$ and $\widetilde{\mathbf{e}}$ are predictors for $\boldsymbol{\gamma}$, $\boldsymbol{\varepsilon}$ and $\mathbf{e}$, respectively. Let say $\boldsymbol{\mu} = \mathbf{X}_s\boldsymbol{\alpha}$ and $\widehat{\boldsymbol{\mu}}$ be any estimator for $\boldsymbol{\mu}$ under $\mathcal{S}$. Denote*

$$\mathbf{E}_1 = \begin{bmatrix} \mathbf{\Phi\Lambda\Phi}' & \mathbf{0} & \mathbf{X}_s \\ \mathbf{0} & -\sigma^{-2}\,\mathrm{MSEM}(\widehat{\boldsymbol{\mu}}) & \mathbf{X}_s \\ \mathbf{X}_s' & \mathbf{X}_s' & \mathbf{0} \end{bmatrix},$$

$$\mathbf{E}_2 = \begin{bmatrix} \mathbf{\Phi\Lambda\Phi}' & \mathbf{\Phi\Lambda}\breve{\mathbf{I}}_p' & \mathbf{X}_s \\ \breve{\mathbf{I}}_p\mathbf{\Lambda\Phi}' & \breve{\mathbf{I}}_p\mathbf{\Lambda}\breve{\mathbf{I}}_p' - \sigma^{-2}\,\mathrm{MSEM}(\widetilde{\boldsymbol{\gamma}}) & \mathbf{0} \\ \mathbf{X}_s' & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\mathbf{E}_3 = \begin{bmatrix} \mathbf{\Phi\Lambda\Phi}' & \mathbf{\Phi\Lambda}\breve{\mathbf{I}}_n' & \mathbf{X}_s \\ \breve{\mathbf{I}}_n\mathbf{\Lambda\Phi}' & \breve{\mathbf{I}}_n\mathbf{\Lambda}\breve{\mathbf{I}}_n' - \sigma^{-2}\,\mathrm{MSEM}(\widetilde{\boldsymbol{\varepsilon}}) & \mathbf{0} \\ \mathbf{X}_s' & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\mathbf{E}_4 = \begin{bmatrix} \mathbf{\Phi\Lambda\Phi}' & \mathbf{\Phi\Lambda}\breve{\mathbf{I}}_m' & \mathbf{X}_s \\ \breve{\mathbf{I}}_m\mathbf{\Lambda\Phi}' & \breve{\mathbf{I}}_m\mathbf{\Lambda}\breve{\mathbf{I}}_m' - \sigma^{-2}\,\mathrm{MSEM}(\widetilde{\mathbf{e}}) & \mathbf{0} \\ \mathbf{X}_s' & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

where $\breve{\mathbf{I}}_p = \begin{bmatrix} \mathbf{I}_p, & \mathbf{0}, & \mathbf{0} \end{bmatrix}$, $\breve{\mathbf{I}}_n = \begin{bmatrix} \mathbf{0}, & \mathbf{I}_n, & \mathbf{0} \end{bmatrix}$ and $\breve{\mathbf{I}}_m = \begin{bmatrix} \mathbf{0}, & \mathbf{0}, & \mathbf{I}_m \end{bmatrix}$. Then,

1) $\mathrm{MSEM}(\widehat{\boldsymbol{\mu}}) \succcurlyeq \mathrm{MSEM}(\widehat{\boldsymbol{\mu}}_{\mathrm{BLUE}}) \Leftrightarrow i_{+}(\mathbf{E}_1) = r\begin{bmatrix} \mathbf{X}_s, & \mathbf{\Phi\Lambda\Phi}' \end{bmatrix}$.
2) $\mathrm{MSEM}(\widehat{\boldsymbol{\mu}}) \succ \mathrm{MSEM}(\widehat{\boldsymbol{\mu}}_{\mathrm{BLUE}}) \Leftrightarrow i_{-}(\mathbf{E}_1) = r(\mathbf{X}_s) + n + m$.
3) $\mathrm{MSEM}(\widehat{\boldsymbol{\mu}}) \preccurlyeq \mathrm{MSEM}(\widehat{\boldsymbol{\mu}}_{\mathrm{BLUE}}) \Leftrightarrow i_{-}(\mathbf{E}_1) = r(\mathbf{X}_s)$.
4) $\mathrm{MSEM}(\widehat{\boldsymbol{\mu}}) \prec \mathrm{MSEM}(\widehat{\boldsymbol{\mu}}_{\mathrm{BLUE}}) \Leftrightarrow i_{+}(\mathbf{E}_1) = r\begin{bmatrix} \mathbf{X}_s, & \mathbf{\Phi\Lambda\Phi}' \end{bmatrix} + n + m$.
5) $\mathrm{MSEM}(\widehat{\boldsymbol{\mu}}) = \mathrm{MSEM}(\widehat{\boldsymbol{\mu}}_{\mathrm{BLUE}}) \Leftrightarrow r(\mathbf{E}_1) = r\begin{bmatrix} \mathbf{X}_s, & \mathbf{\Phi\Lambda\Phi}' \end{bmatrix} + r(\mathbf{X}_s)$.
6) $\mathrm{MSEM}(\widetilde{\boldsymbol{\gamma}}) \succcurlyeq \mathrm{MSEM}(\widetilde{\boldsymbol{\gamma}}_{\mathrm{BLUP}}) \Leftrightarrow i_{+}(\mathbf{E}_2) = r\begin{bmatrix} \mathbf{X}_s, & \mathbf{\Phi\Lambda\Phi}' \end{bmatrix}$.
7) $\mathrm{MSEM}(\widetilde{\boldsymbol{\gamma}}) \succ \mathrm{MSEM}(\widetilde{\boldsymbol{\gamma}}_{\mathrm{BLUP}}) \Leftrightarrow i_{-}(\mathbf{E}_2) = r(\mathbf{X}_s) + p$.
8) $\mathrm{MSEM}(\widetilde{\boldsymbol{\gamma}}) \preccurlyeq \mathrm{MSEM}(\widetilde{\boldsymbol{\gamma}}_{\mathrm{BLUP}}) \Leftrightarrow i_{-}(\mathbf{E}_2) = r(\mathbf{X}_s)$.
9) $\mathrm{MSEM}(\widetilde{\boldsymbol{\gamma}}) \prec \mathrm{MSEM}(\widetilde{\boldsymbol{\gamma}}_{\mathrm{BLUP}}) \Leftrightarrow i_{+}(\mathbf{E}_2) = r\begin{bmatrix} \mathbf{X}_s, & \mathbf{\Phi\Lambda\Phi}' \end{bmatrix} + p$.
10) $\mathrm{MSEM}(\widetilde{\boldsymbol{\gamma}}) = \mathrm{MSEM}(\widetilde{\boldsymbol{\gamma}}_{\mathrm{BLUP}}) \Leftrightarrow r(\mathbf{E}_2) = r\begin{bmatrix} \mathbf{X}_s, & \mathbf{\Phi\Lambda\Phi}' \end{bmatrix} + r(\mathbf{X}_s)$.
11) $\mathrm{MSEM}(\widetilde{\boldsymbol{\varepsilon}}) \succcurlyeq \mathrm{MSEM}(\widetilde{\boldsymbol{\varepsilon}}_{\mathrm{BLUP}}) \Leftrightarrow i_{+}(\mathbf{E}_3) = r\begin{bmatrix} \mathbf{X}_s, & \mathbf{\Phi\Lambda\Phi}' \end{bmatrix}$.

12) $\mathrm{MSEM}(\widetilde{\boldsymbol{\varepsilon}}) \succ \mathrm{MSEM}(\widetilde{\boldsymbol{\varepsilon}}_{\mathrm{BLUP}}) \Leftrightarrow \boldsymbol{i}_{-}(\mathbf{E}_3) = \boldsymbol{r}(\mathbf{X}_s) + n$.
13) $\mathrm{MSEM}(\widetilde{\boldsymbol{\varepsilon}}) \preccurlyeq \mathrm{MSEM}(\widetilde{\boldsymbol{\varepsilon}}_{\mathrm{BLUP}}) \Leftrightarrow \boldsymbol{i}_{-}(\mathbf{E}_3) = \boldsymbol{r}(\mathbf{X}_s)$.
14) $\mathrm{MSEM}(\widetilde{\boldsymbol{\varepsilon}}) \prec \mathrm{MSEM}(\widetilde{\boldsymbol{\varepsilon}}_{\mathrm{BLUP}}) \Leftrightarrow \boldsymbol{i}_{+}(\mathbf{E}_3) = \boldsymbol{r}\begin{bmatrix}\mathbf{X}_s, & \mathbf{\Phi\Lambda\Phi}'\end{bmatrix} + n$.
15) $\mathrm{MSEM}(\widetilde{\boldsymbol{\varepsilon}}) = \mathrm{MSEM}(\widetilde{\boldsymbol{\varepsilon}}_{\mathrm{BLUP}}) \Leftrightarrow \boldsymbol{r}(\mathbf{E}_3) = \boldsymbol{r}\begin{bmatrix}\mathbf{X}_s, & \mathbf{\Phi\Lambda\Phi}'\end{bmatrix} + \boldsymbol{r}(\mathbf{X}_s)$.
16) $\mathrm{MSEM}(\widetilde{\mathbf{e}}) \succcurlyeq \mathrm{MSEM}(\widetilde{\mathbf{e}}_{\mathrm{BLUP}}) \Leftrightarrow \boldsymbol{i}_{+}(\mathbf{E}_4) = \boldsymbol{r}\begin{bmatrix}\mathbf{X}_s, & \mathbf{\Phi\Lambda\Phi}'\end{bmatrix}$.
17) $\mathrm{MSEM}(\widetilde{\mathbf{e}}) \succ \mathrm{MSEM}(\widetilde{\mathbf{e}}_{\mathrm{BLUP}}) \Leftrightarrow \boldsymbol{i}_{-}(\mathbf{E}_4) = \boldsymbol{r}(\mathbf{X}_s) + m$.
18) $\mathrm{MSEM}(\widetilde{\mathbf{e}}) \preccurlyeq \mathrm{MSEM}(\widetilde{\mathbf{e}}_{\mathrm{BLUP}}) \Leftrightarrow \boldsymbol{i}_{-}(\mathbf{E}_4) = \boldsymbol{r}(\mathbf{X}_s)$.
19) $\mathrm{MSEM}(\widetilde{\mathbf{e}}) \prec \mathrm{MSEM}(\widetilde{\mathbf{e}}_{\mathrm{BLUP}}) \Leftrightarrow \boldsymbol{i}_{+}(\mathbf{E}_4) = \boldsymbol{r}\begin{bmatrix}\mathbf{X}_s, & \mathbf{\Phi\Lambda\Phi}'\end{bmatrix} + m$.
20) $\mathrm{MSEM}(\widetilde{\mathbf{e}}) = \mathrm{MSEM}(\widetilde{\mathbf{e}}_{\mathrm{BLUP}}) \Leftrightarrow \boldsymbol{r}(\mathbf{E}_4) = \boldsymbol{r}\begin{bmatrix}\mathbf{X}_s, & \mathbf{\Phi\Lambda\Phi}'\end{bmatrix} + \boldsymbol{r}(\mathbf{X}_s)$.

4. Comparisons under CLMMs and ULMMs

Now, let us consider the LMM in (1.1) with an exact restriction $\mathbf{S}\boldsymbol{\alpha} = \mathbf{s}$. Then, the model $\mathcal{S}$ in (2.1) corresponds the following CLMM:

$$\mathcal{R}: \ \mathbf{y}_s = \mathbf{X}_s\boldsymbol{\alpha} + \mathbf{Z}_s\boldsymbol{\gamma} + \boldsymbol{\varepsilon}_r = \mathbf{X}_s\boldsymbol{\alpha} + \begin{bmatrix}\mathbf{Z}_s, & \mathbf{I}_{n+m}\end{bmatrix}\begin{bmatrix}\boldsymbol{\gamma} \\ \boldsymbol{\varepsilon}_r\end{bmatrix}, \tag{4.1}$$

where $\mathbf{y}_s$, $\mathbf{X}_s$ and $\mathbf{Z}_s$ are given as in (2.1) and $\boldsymbol{\varepsilon}_r = \begin{bmatrix}\boldsymbol{\varepsilon} \\ \mathbf{0}\end{bmatrix}$. λ in (2.2) corresponds the vector

$$\mathbf{r} = \mathbf{K}\boldsymbol{\alpha} + \mathbf{L}\boldsymbol{\gamma} + \mathbf{M}\boldsymbol{\varepsilon}_r = \mathbf{K}\boldsymbol{\alpha} + \begin{bmatrix}\mathbf{L}, & \mathbf{M}\end{bmatrix}\begin{bmatrix}\boldsymbol{\gamma} \\ \boldsymbol{\varepsilon}_r\end{bmatrix} = \mathbf{K}\boldsymbol{\alpha} + \begin{bmatrix}\mathbf{L}, & \mathbf{M}_1, & \mathbf{M}_2\end{bmatrix}\begin{bmatrix}\boldsymbol{\gamma} \\ \boldsymbol{\varepsilon} \\ \mathbf{0}\end{bmatrix}. \tag{4.2}$$

In this case, the assumptions under $\mathcal{R}$ are written as

$$\mathrm{D}(\mathbf{y}_s) = \sigma^2\mathbf{\Phi\Lambda}_r\mathbf{\Phi}', \ \ \mathrm{D}(\mathbf{r}) = \sigma^2\mathbf{\Upsilon\Lambda}_r\mathbf{\Upsilon}', \ \ \mathrm{cov}(\mathbf{r}, \mathbf{y}_s) = \sigma^2\mathbf{\Upsilon\Lambda}_r\mathbf{\Phi}',$$

where $\mathbf{\Lambda}_r = \begin{bmatrix}\mathbf{\Lambda}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}$.

In the theorem given below, the results on the relationships between the MSEM of any predictor for $\mathbf{r}$ and the MSEM of the BLUP of $\mathbf{r}$ under $\mathcal{R}$ are collected. These results are obtained from Theorem 3.1.

Theorem 4.1. *Suppose that* $\mathbf{r}$ *in* (4.2) *is predictable under* $\mathcal{R}$ *in* (4.1). *Let* $\widetilde{\mathbf{r}}$ *be any predictor (unbiased or biased) for* $\mathbf{r}$ *under* $\mathcal{R}$. *Denote*

$$\mathbf{E}_r = \begin{bmatrix}\mathbf{\Phi\Lambda}_r\mathbf{\Phi}' & \mathbf{\Phi\Lambda}_r\mathbf{\Upsilon}' & \mathbf{X}_s \\ \mathbf{\Upsilon\Lambda}_r\mathbf{\Phi}' & \mathbf{\Upsilon\Lambda}_r\mathbf{\Upsilon}' - \sigma^{-2}\,\mathrm{MSEM}(\widetilde{\mathbf{r}}) & \mathbf{K} \\ \mathbf{X}_s' & \mathbf{K}' & \mathbf{0}\end{bmatrix} = \begin{bmatrix}\sigma^{-2}\,\mathrm{D}(\mathbf{y}_s) & \sigma^{-2}\,\mathrm{cov}(\mathbf{y}_s, \mathbf{r}) & \mathbf{X}_s \\ \sigma^{-2}\,\mathrm{cov}(\mathbf{r}, \mathbf{y}_s) & \sigma^{-2}\,(\mathrm{D}(\mathbf{r}) - \mathrm{MSEM}(\widetilde{\mathbf{r}})) & \mathbf{K} \\ \mathbf{X}_s' & \mathbf{K}' & \mathbf{0}\end{bmatrix}.$$

Then, the following results hold.

1) $\mathrm{MSEM}(\widetilde{\mathbf{r}}) \succcurlyeq \mathrm{MSEM}(\widetilde{\mathbf{r}}_{\mathrm{BLUP}}) \Leftrightarrow \boldsymbol{i}_{+}(\mathbf{E}_r) = \boldsymbol{r}\begin{bmatrix}\mathbf{X}_s, & \mathbf{\Phi\Lambda}_r\mathbf{\Phi}'\end{bmatrix}$.
2) $\mathrm{MSEM}(\widetilde{\mathbf{r}}) \succ \mathrm{MSEM}(\widetilde{\mathbf{r}}_{\mathrm{BLUP}}) \Leftrightarrow \boldsymbol{i}_{-}(\mathbf{E}_r) = \boldsymbol{r}(\mathbf{X}_s) + t$.

3) $\mathrm{MSEM}(\widetilde{\mathbf{r}}) \preccurlyeq \mathrm{MSEM}(\widetilde{\mathbf{r}}_{\mathrm{BLUP}}) \Leftrightarrow \boldsymbol{i}_{-}(\mathbf{E}_r) = \boldsymbol{r}(\mathbf{X}_s)$.
4) $\mathrm{MSEM}(\widetilde{\mathbf{r}}) \prec \mathrm{MSEM}(\widetilde{\mathbf{r}}_{\mathrm{BLUP}}) \Leftrightarrow \boldsymbol{i}_{+}(\mathbf{E}_r) = \boldsymbol{r}\left[\mathbf{X}_s, \quad \mathbf{\Phi}\mathbf{\Lambda}_r\mathbf{\Phi}'\right] + t$.
5) $\mathrm{MSEM}(\widetilde{\mathbf{r}}) = \mathrm{MSEM}(\widetilde{\mathbf{r}}_{\mathrm{BLUP}}) \Leftrightarrow \boldsymbol{r}(\mathbf{E}_r) = \boldsymbol{r}\left[\mathbf{X}_s, \quad \mathbf{\Phi}\mathbf{\Lambda}_r\mathbf{\Phi}'\right] + \boldsymbol{r}(\mathbf{X}_s)$.

Now, let us consider the LMM in (1.1) without any restrictions on parameters and show this model $\mathcal{U}$, i.e.,

$$\mathcal{U}: \ \mathbf{y} = \mathbf{X}\boldsymbol{\alpha} + \mathbf{Z}\boldsymbol{\gamma} + \boldsymbol{\varepsilon} = \mathbf{X}\boldsymbol{\alpha} + \begin{bmatrix}\mathbf{Z}, & \mathbf{I}_n\end{bmatrix}\begin{bmatrix}\boldsymbol{\gamma} \\ \boldsymbol{\varepsilon}\end{bmatrix}, \tag{4.3}$$

and $\boldsymbol{\lambda}$ in (2.2) corresponds the vector

$$\mathbf{u} = \mathbf{K}\boldsymbol{\alpha} + \mathbf{L}\boldsymbol{\gamma} + \mathbf{M}_1\boldsymbol{\varepsilon} = \mathbf{K}\boldsymbol{\alpha} + \begin{bmatrix}\mathbf{L}, & \mathbf{M}_1\end{bmatrix}\begin{bmatrix}\boldsymbol{\gamma} \\ \boldsymbol{\varepsilon}\end{bmatrix}. \tag{4.4}$$

In this case,

$$\begin{aligned} \mathrm{D}(\mathbf{y}) &= \sigma^2 \begin{bmatrix}\mathbf{Z}, & \mathbf{I}_n\end{bmatrix} \mathbf{\Lambda}_1 \begin{bmatrix}\mathbf{Z}, & \mathbf{I}_n\end{bmatrix}', \\ \mathrm{D}(\mathbf{u}) &= \sigma^2 \begin{bmatrix}\mathbf{L}, & \mathbf{M}_1\end{bmatrix} \mathbf{\Lambda}_1 \begin{bmatrix}\mathbf{L}, & \mathbf{M}_1\end{bmatrix}', \\ \mathrm{cov}(\mathbf{u}, \mathbf{y}) &= \sigma^2 \begin{bmatrix}\mathbf{L}, & \mathbf{M}_1\end{bmatrix} \mathbf{\Lambda}_1 \begin{bmatrix}\mathbf{Z}, & \mathbf{I}_n\end{bmatrix}'. \end{aligned}$$

Theorem 4.2. *Suppose that* $\mathbf{u}$ *in* (4.4) *is predictable under* $\mathcal{U}$ *in* (4.3). *Let* $\widetilde{\mathbf{u}}$ *be any predictor (unbiased or biased) for* $\mathbf{u}$ *under* $\mathcal{U}$. *Denote*

$$\mathbf{E}_u = \begin{bmatrix} \mathbf{\Phi}_1\mathbf{\Lambda}_1\mathbf{\Phi}_1' & \mathbf{\Phi}_1\mathbf{\Lambda}_1\mathbf{\Upsilon}_1' & \mathbf{X} \\ \mathbf{\Upsilon}_1\mathbf{\Lambda}_1\mathbf{\Phi}_1' & \mathbf{\Upsilon}_1\mathbf{\Lambda}_1\mathbf{\Upsilon}_1' - \sigma^{-2}\,\mathrm{MSEM}(\widetilde{\mathbf{u}}) & \mathbf{K} \\ \mathbf{X}' & \mathbf{K}' & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \sigma^{-2}\,\mathrm{D}(\mathbf{y}) & \sigma^{-2}\,\mathrm{cov}(\mathbf{y}, \mathbf{u}) & \mathbf{X} \\ \sigma^{-2}\,\mathrm{cov}(\mathbf{u}, \mathbf{y}) & \sigma^{-2}\,(\mathrm{D}(\mathbf{u}) - \mathrm{MSEM}(\widetilde{\mathbf{u}})) & \mathbf{K} \\ \mathbf{X}' & \mathbf{K}' & \mathbf{0} \end{bmatrix},$$

where $\mathbf{\Phi}_1 = \begin{bmatrix}\mathbf{Z}, & \mathbf{I}_n\end{bmatrix}$ and $\mathbf{\Upsilon}_1 = \begin{bmatrix}\mathbf{L}, & \mathbf{M}_1\end{bmatrix}$. Then, the following results hold.

1) $\mathrm{MSEM}(\widetilde{\mathbf{u}}) \succcurlyeq \mathrm{MSEM}(\widetilde{\mathbf{u}}_{\mathrm{BLUP}}) \Leftrightarrow \boldsymbol{i}_{+}(\mathbf{E}_u) = \boldsymbol{r}\left[\mathbf{X}, \quad \mathbf{\Phi}_1\mathbf{\Lambda}_1\mathbf{\Phi}_1'\right]$.
2) $\mathrm{MSEM}(\widetilde{\mathbf{u}}) \succ \mathrm{MSEM}(\widetilde{\mathbf{u}}_{\mathrm{BLUP}}) \Leftrightarrow \boldsymbol{i}_{-}(\mathbf{E}_u) = \boldsymbol{r}(\mathbf{X}) + t$.
3) $\mathrm{MSEM}(\widetilde{\mathbf{u}}) \preccurlyeq \mathrm{MSEM}(\widetilde{\mathbf{u}}_{\mathrm{BLUP}}) \Leftrightarrow \boldsymbol{i}_{-}(\mathbf{E}_u) = \boldsymbol{r}(\mathbf{X})$.
4) $\mathrm{MSEM}(\widetilde{\mathbf{u}}) \prec \mathrm{MSEM}(\widetilde{\mathbf{u}}_{\mathrm{BLUP}}) \Leftrightarrow \boldsymbol{i}_{+}(\mathbf{E}_u) = \boldsymbol{r}\left[\mathbf{X}, \quad \mathbf{\Phi}_1\mathbf{\Lambda}_1\mathbf{\Phi}_1'\right] + t$.
5) $\mathrm{MSEM}(\widetilde{\mathbf{u}}) = \mathrm{MSEM}(\widetilde{\mathbf{u}}_{\mathrm{BLUP}}) \Leftrightarrow \boldsymbol{r}(\mathbf{E}_u) = \boldsymbol{r}\left[\mathbf{X}, \quad \mathbf{\Phi}_1\mathbf{\Lambda}_1\mathbf{\Phi}_1'\right] + \boldsymbol{r}(\mathbf{X})$.

As is well-known, two types of commonly used estimators under a general linear model are the BLUEs and the ordinary least squares estimators (OLSEs). Both of these estimators have a variety of simple and remarkable properties. Therefore, they have attracted statisticians' attention throughout the historical development of regression theory, and numerous results regarding the BLUEs and the OLSEs have been established. Since these estimators are defined by different optimality criteria and thereby their expressions and properties are not necessarily the same, it is natural to seek possible connections between them. As a subject area within the regression analysis, the relationships between the BLUEs and OLSEs have been widely considered in the statistical literature and various identifying conditions for their equivalence or comparisons have been obtained. Based on these explanations, to further clarify the results obtained above, since both the BLUE and the OLSE of $\mathbf{X}_s\boldsymbol{\alpha}$, denoted as $\widehat{\mathbf{X}_s\boldsymbol{\alpha}}_{\mathrm{OLSE}}$, are well-known estimators, we present the following result on the relationship between them.

Let us consider the LMM in (1.1) by setting $\mathbf{Z} = \mathbf{0}$ with an exact restriction $\mathbf{S}\boldsymbol{\alpha} = \mathbf{s}$. Then, the model $\mathcal{S}$ in (2.1) corresponds the following linear model:

$$\mathcal{L}: \ \mathbf{y}_s = \mathbf{X}_s\boldsymbol{\alpha} + \boldsymbol{\varepsilon}_r, \tag{4.5}$$

where $\mathbf{y}_s$ and $\mathbf{X}_s$ are given as in (2.1) and $\boldsymbol{\varepsilon}_r$ is given as in (4.1). By taking $\boldsymbol{\lambda} = \mathbf{X}_s\boldsymbol{\alpha}$ in (2.2), the matrix $\mathbf{E}$ in Theorem 3.1 corresponds

$$\mathbf{E}_l = \begin{bmatrix} \boldsymbol{\Lambda}_r & \mathbf{0} & \mathbf{X}_s \\ \mathbf{0} & -\sigma^{-2}\,\mathrm{MSEM}(\widehat{\mathbf{X}_s\boldsymbol{\alpha}}_{\mathrm{OLSE}}) & \mathbf{X}_s \\ \mathbf{X}_s' & \mathbf{X}_s' & \mathbf{0} \end{bmatrix}, \tag{4.6}$$

where $\boldsymbol{\Lambda}_r = \begin{bmatrix} \boldsymbol{\Lambda}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ and $\mathrm{D}(\boldsymbol{\varepsilon}) = \boldsymbol{\Lambda}_{11}$. Then,

$$\boldsymbol{i}_+(\mathbf{E}_l) = \boldsymbol{i}_+\begin{bmatrix} \boldsymbol{\Lambda}_r & \mathbf{0} & \mathbf{X}_s \\ \mathbf{0} & -\mathbf{X}_s\mathbf{X}_s^+\boldsymbol{\Lambda}_r\mathbf{X}_s\mathbf{X}_s^+ & \mathbf{X}_s \\ \mathbf{X}_s' & \mathbf{X}_s' & \mathbf{0} \end{bmatrix} = \boldsymbol{r}\begin{bmatrix} \boldsymbol{\Lambda}_r & \mathbf{0} & \mathbf{X}_s \\ \mathbf{0} & -\mathbf{X}_s\mathbf{X}_s^+\boldsymbol{\Lambda}_r\mathbf{X}_s\mathbf{X}_s^+ & \mathbf{X}_s \end{bmatrix} = \boldsymbol{r}\begin{bmatrix} \boldsymbol{\Lambda}_r, & \mathbf{X}_s \end{bmatrix}$$

always holds, i.e., $\mathrm{MSEM}(\widehat{\mathbf{X}_s\boldsymbol{\alpha}}_{\mathrm{OLSE}}) \succcurlyeq \mathrm{MSEM}(\widehat{\mathbf{X}_s\boldsymbol{\alpha}}_{\mathrm{BLUE}})$. This inequality is already a well-known result in statistical theory.

We now use Example 4.1.8 from Section 4.1 of [46] to illustrate our theoretical findings in response to the referees' advice. Consider the linear model $\mathbf{y} = \mathbf{X}\boldsymbol{\alpha} + \boldsymbol{\varepsilon}$ with

$$\mathbf{X} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix} \text{ and } \boldsymbol{\alpha} = \begin{bmatrix} \mu \\ \beta_1 \\ \beta_2 \\ \tau_1 \\ \tau_2 \end{bmatrix},$$

where $\mathbf{1} \in \mathbb{R}_{10,1}$ and $\mathbf{0} \in \mathbb{R}_{10,1}$. Let consider

$$\mathrm{D}(\boldsymbol{\varepsilon}) = \boldsymbol{\Lambda}_{11} = \begin{bmatrix} 3\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & \mathbf{1}\mathbf{1}' \\ 2\mathbf{1}\mathbf{1}' & 3\mathbf{1}\mathbf{1}' & \mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' \\ 2\mathbf{1}\mathbf{1}' & \mathbf{1}\mathbf{1}' & 3\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' \\ \mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & 2\mathbf{1}\mathbf{1}' & 3\mathbf{1}\mathbf{1}' \end{bmatrix}$$

as in [2]. This designed set-up is typical in agricultural experiments where several treatments are applied to various blocks of land. The experiment is often conducted to assess the differential impact of the treatments. Here, the parameter μ represents a general effect that is present in all the observations, and the parameters β_1 and β_2 represent the respective effects of two blocks, and the parameters τ_1 and τ_2 represent the respective effects of two treatments. Suppose that

$$\tau_1 - \tau_2 = 0, \ \text{i.e.,} \ \mathbf{S}\boldsymbol{\alpha} = 0,$$

where $\mathbf{S} = \begin{bmatrix} 0, & 0, & 0, & 1, & -1 \end{bmatrix}$. This restriction may be a known fact from the theory or experiment view.

Direct calculations will show that

$$\mathrm{MSEM}(\widehat{\mathbf{X}_s\boldsymbol{\alpha}}_{\mathrm{OLSE}}) = \begin{bmatrix} 2.91\mathbf{11}' & 1.91\mathbf{11}' & 2.08\mathbf{11}' & 1.08\mathbf{11}' & 0.82\mathbf{1} \\ 1.91\mathbf{11}' & 2.91\mathbf{11}' & 1.08\mathbf{11}' & 2.08\mathbf{11}' & 0.82\mathbf{1} \\ 2.08\mathbf{11}' & 1.08\mathbf{11}' & 2.91\mathbf{11}' & 1.91\mathbf{11}' & -0.82\mathbf{1} \\ 1.08\mathbf{11}' & 2.08\mathbf{11}' & 1.91\mathbf{11}' & 2.91\mathbf{11}' & -0.82\mathbf{1} \\ 0.82\mathbf{1}' & 0.82\mathbf{1}' & -0.82\mathbf{1}' & -0.82\mathbf{1}' & 1.65 \end{bmatrix}$$

and

$$\mathrm{MSEM}(\widehat{\mathbf{X}_s\boldsymbol{\alpha}}_{\mathrm{BLUE}}) = \begin{bmatrix} 2.5\mathbf{11}' & 1.5\mathbf{11}' & 2.5\mathbf{11}' & 1.5\mathbf{11}' & 5.97\times10^{-15}\mathbf{1} \\ 1.5\mathbf{11}' & 2.5\mathbf{11}' & 1.5\mathbf{11}' & 2.5\mathbf{11}' & 4.02\times10^{-15}\mathbf{1} \\ 2.5\mathbf{11}' & 1.5\mathbf{11}' & 2.5\mathbf{11}' & 1.5\mathbf{11}' & 5.97\times10^{-15}\mathbf{1} \\ 1.5\mathbf{11}' & 2.5\mathbf{11}' & 1.5\mathbf{11}' & 2.5\mathbf{11}' & 4.02\times10^{-15}\mathbf{1} \\ 5.97\times10^{-15}\mathbf{1}' & 4.02\times10^{-15}\mathbf{1}' & 5.97\times10^{-15}\mathbf{1}' & 4.02\times10^{-15}\mathbf{1}' & 1.19\times10^{-28} \end{bmatrix},$$

when we consider above matrices for the model $\mathcal{L}$ in (4.5). Now it is easy to see that

$$\begin{bmatrix} 2.91\mathbf{11}' & 1.91\mathbf{11}' & 2.08\mathbf{11}' & 1.08\mathbf{11}' & 0.82\mathbf{1} \\ 1.91\mathbf{11}' & 2.91\mathbf{11}' & 1.08\mathbf{11}' & 2.08\mathbf{11}' & 0.82\mathbf{1} \\ 2.08\mathbf{11}' & 1.08\mathbf{11}' & 2.91\mathbf{11}' & 1.91\mathbf{11}' & -0.82\mathbf{1} \\ 1.08\mathbf{11}' & 2.08\mathbf{11}' & 1.91\mathbf{11}' & 2.91\mathbf{11}' & -0.82\mathbf{1} \\ 0.82\mathbf{1}' & 0.82\mathbf{1}' & -0.82\mathbf{1}' & -0.82\mathbf{1}' & 1.65 \end{bmatrix} \succcurlyeq$$

$$\begin{bmatrix} 2.5\mathbf{11}' & 1.5\mathbf{11}' & 2.5\mathbf{11}' & 1.5\mathbf{11}' & 5.97\times10^{-15}\mathbf{1} \\ 1.5\mathbf{11}' & 2.5\mathbf{11}' & 1.5\mathbf{11}' & 2.5\mathbf{11}' & 4.02\times10^{-15}\mathbf{1} \\ 2.5\mathbf{11}' & 1.5\mathbf{11}' & 2.5\mathbf{11}' & 1.5\mathbf{11}' & 5.97\times10^{-15}\mathbf{1} \\ 1.5\mathbf{11}' & 2.5\mathbf{11}' & 1.5\mathbf{11}' & 2.5\mathbf{11}' & 4.02\times10^{-15}\mathbf{1} \\ 5.97\times10^{-15}\mathbf{1}' & 4.02\times10^{-15}\mathbf{1}' & 5.97\times10^{-15}\mathbf{1}' & 4.02\times10^{-15}\mathbf{1}' & 1.19\times10^{-28} \end{bmatrix},$$

i.e.,

$$\mathrm{MSEM}(\widehat{\mathbf{X}_s\boldsymbol{\alpha}}_{\mathrm{OLSE}}) \succcurlyeq \mathrm{MSEM}(\widehat{\mathbf{X}_s\boldsymbol{\alpha}}_{\mathrm{BLUE}}).$$

Using the numpy library in Python and setting the above findings in the matrix $\mathbf{E}_l$ in (4.6), we find the eigenvalues of matrix $\mathbf{E}_l$ as 1.44, 20.98, 20.98, 80.99, -80.99, -20.98, -19.36 and -1.25. Thus, we can see that the number of positive eigenvalues of $\mathbf{E}_l$ is 4 and the number of negative eigenvalues of $\mathbf{E}_l$ is 4, i.e., $\boldsymbol{i}_+(\mathbf{E}_l) = 4$ and $\boldsymbol{i}_-(\mathbf{E}_l) = 4$. It is easily seen from (1.5) that $\boldsymbol{r}(\mathbf{E}_l) = 8$. Also, we obtain $\boldsymbol{r}\begin{bmatrix} \mathbf{X}_s, & \boldsymbol{\Lambda}_r \end{bmatrix} = 4$ and $\boldsymbol{r}(\mathbf{X}_s) = 3$. Therefore, $\boldsymbol{i}_+(\mathbf{E}_l) = \boldsymbol{r}\begin{bmatrix} \mathbf{X}_s, & \boldsymbol{\Lambda}_r \end{bmatrix} = 4 \Leftrightarrow \mathrm{MSEM}(\widehat{\mathbf{X}_s\boldsymbol{\alpha}}_{\mathrm{BLUE}}) \preccurlyeq \mathrm{MSEM}(\widehat{\mathbf{X}_s\boldsymbol{\alpha}}_{\mathrm{OLSE}})$ holds. Since $\boldsymbol{r}(\mathbf{X}_s) = 3$, i.e., $\boldsymbol{i}_-(\mathbf{E}_l) \neq \boldsymbol{r}(\mathbf{X}_s)$, $\mathrm{MSEM}(\widehat{\mathbf{X}_s\boldsymbol{\alpha}}_{\mathrm{OLSE}}) \preccurlyeq \mathrm{MSEM}(\widehat{\mathbf{X}_s\boldsymbol{\alpha}}_{\mathrm{BLUE}})$ is not provided.

5. Conclusions

The findings of this study, according to the MSEM criterion, provide a broad perspective on the SRLMMs in the context of predictor comparability problems using the methodology of block matrix inertias and ranks. We establish the comparison results after converting an explicitly SRLMM to an implicitly linear stochastically restricted LMM. We also reduce our results to models CLMMs and ULMMs.

We derive the formulas of comparisons of any predictor and the BLUP for the same general vector of all unknown vectors under SRLMM by considering the MSEM sense. Thus, the performance of a new biased or unbiased predictor according to the dispersion matrices of BLUP can be examined with various inequalities and equalities derived among MSEMs based on inertias and ranks. These kinds of inequalities or equalities for comparisons have a useful and strong statistical explanation in the theory of linear statistical models and applications. The comparisons of MSEMs to determine the effectiveness of any predictor according to BLUP are explained by the comparisons of some quantities that come from inertias and ranks.

As a result, we can say that our contribution to issues involving inference and prediction under an LMM with a linear stochastic restriction on unknown parameters in the literature is to serve as a theoretically comprehensive and significant search for the comparison results between any predictors and BLUPs.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors state that they have no competing interest.

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