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Research article

Physical phenomena of spectral relationships via quadratic third kind mixed integral equation with discontinuous kernel

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Abstract: Spectral relationships explain many physical phenomena, especially in quantum physics and astrophysics. Therefore, in this paper, we first attempt to derive spectral relationships in position and time for an integral operator with a singular kernel. Second, using these relations to solve a mixed integral equation (**MIE**) of the second kind in the space $L_2[-1,1] \times C[0,T],\ T < 1$. The way to do this is to derive a general principal theorem of the spectral relations from the term of the Volterra-Fredholm integral equation (**V-FIE**), with the help of the Chebyshev polynomials (**CPs**), and then use the results in the general **MIE** to discuss its solution. More than that, some special and important cases will be devised that help explain many phenomena in the basic sciences in general. Here, the **FI** term is considered in position, in $L_2[-1,1]$, and its kernel takes a logarithmic form multiplied by a general continuous function. While the **VI** term in time, in $C[0,T],\ T < 1$, and its kernels are smooth functions. Many numerical results are considered, and the estimated error is also established using Maple 2022.

Keywords: mixed integral equation; singular kernel; technique of separation; Chebyshev polynomial; linear algebraic system; the rate of error convergence
Mathematics Subject Classification: 45B05, 65H10, 65R20

1. Introduction

Spectral relationships have played a prominent role in general, in the interpretation of physical phenomena. These phenomena led researchers to formulate them in the forms of mathematical models. Therefore, we find that the mathematical modeling of most phenomena in various branches of science leads to linear and nonlinear integral equations (**IE**s/**NIE**s), integro differential equations (**IDEs**), and fractional integro differential equations (**FIDEs**) of different kinds. Jan [1] used the **CP** method for solving **MIE** in position and time, with a weakly singular kernel in position. Boykov et al. [2] applied a new iterative method for solving hypersingular linear and nonlinear **IE** in automatic control problems. Biazar and Ebrahimi [3] used a modified hat function to solve a class of nonlinear **FVIE** of the second kind. Seifi [4] used the collocation technique method for solving Cauchy singular **IE** of the second kind which has many applications in physics and engineering fields. While Jan [5] used collocation method, based on orthogonal polynomials, to solve a nonlinear **MIE**, Basseem and Alayani [6] used the Toeplitz matrix method and product Nystrom technique to solve a nonlinear quadratic **MIE** of the second kind with a singular kernel. Ghorbanpoor et al. [7] presented a crack problem that is formulated into a singular **IE**. Then, they solved it numerically using the collocation method scheme. Abdou and Basseem [8] used singular integral method to solve, in a complex plane, an **IDE** with a Cauchy kernel. Al-Bugami [9] solved singular Hammerstein-**VIE** by using the Toeplitz matrix method and the product Nystrom method. Abdou et al. [10] used a technique of separating method to discuss the solution of **V-FIE** with a discontinuous kernel. Alhazmi [11] used the separating variables method to discuss the solution of a **MIE** of the first kind with logarithmic and Carleman singular kernels. Doaa [12] used Lerch polynomials to approximate the solution of singular **FIE** with Cauchy kernel. Gao et al. [13] used spectral computation of highly oscillatory systems to solve **IEs** in laser theory. Lienert and Tumulka [14] investigated from relativistic quantum physics and computed its solution numerically. Matoog [15] used orthogonal polynomial method to discuss the solution of the nuclear **IE** in quantum physics problem. Hafez and Youssri [16] used spectral relationships in the form of Legendre-Chebyshev to discuss the numerical solution of nonlinear **VIE** with a continuous kernel. Alalyani et al. [17] computed the numerical solution of **MIDE** with a discontinuous kernel**,** using the orthogonal polynomial method. Also, many authors consider the semi-analytic methods. For example, Noeiaghdam et al. [18] used adomian decomposition for solving **VIE** with discontinuous kernel. A new technique depends on floating point arithmetic using **CADNA** library to find the optimal solution for linear and nonlinear **VIE** of second kind with singular kernel is applied by Noeiaghdam et al. [19], while Noeiaghdam and Micula [20] solved it by using Lagrange collocation technique with the same library. Qiao et al. [21] employed **BDF2** and **ADI** techniques for solving multi-dimensional tempered **FrIDE.** Moreover, Wang et al. [22] used a second order finite difference scheme for nonlinear FrIDE in 3-dimaensions.

In this work, especially in Section 2, the principal **MIE of** this paper and its conditions for having a unique solution are considered. In Section 3, the convergence of the solution is discussed. Moreover, in Section 4, the existence of a unique solution to the **MIE** using the Banach fixed point theorem is investigated. While, in Section 5, the general kernel of position is transformed, using suitable assumptions and methods, is transformed into a logarithmic kernel form. Then, in Section 6, using a separation technique, the **MIE** is reduced to a linear **FIE** of the third kind with a logarithmic kernel. Then, in Section 7, a principal theorem of spectral relationships is proved, after using orthogonal polynomials of Chebyshev's first kind. In Section 8, the system of the third kind of **FIE** is transformed to a linear algebraic system using the principal theorem of spectral relationships. In Section 9, the convergence of the system is studied. Finally, in Section 10, the numerical results are obtained and the error estimate is computed. Finally, a general conclusion is considered to describe the important results of the paper.

2. The principal mixed integral equation and its condition

Consider the following **MIE**

$$\mu_1\Phi(x,t)-\int_0^t\int_{-1}^1 q(x,y)k\left(\left|\frac{y-x}{\sigma}\right|\right)F(t,\tau)\Phi(y,\tau)dyd\tau-\xi(x)\int_0^t G(t,\tau)\Phi(x,\tau)d\tau=f(x,t),$$

$$(|\mathrm{x}|\leq 1,\mu_1-constant),\ \sigma\epsilon(0,\infty), \tag{1}$$

$$k\left(\left|\frac{y-x}{\sigma}\right|\right)=\int_0^\infty\frac{L(u)cosu(\frac{y-x}{\sigma})}{u}du,\ L(u)=\frac{u+m}{1+u},(m\geq 1),m{:}\,finite, \tag{2}$$

under the dynamic condition

$$\int_{-1}^1\Phi(x,t)=P(t),\ t\in[0,T],\ T<1. \tag{3}$$

Here, the time functions $F(t,\tau)$ and $G(t,\tau)$ are continuous in the space $C[0,T],T<1$. The kernel term of position $q(x,y)$ is continuous function, while $k\left(\left|\frac{y-x}{\sigma}\right|\right)$ has a singularity in the space $L_2[-1,1]$. The known function $f(x,t)$ is called free term and defined in the space $L_2[-1,1]\times C[0,T]$ and $\xi(x)$ is continuous in the space $L_2[-1,1]$. The unknown function $\Phi(x,t)$ will be discussed in the space $L_2[-1,1]\times C[0,T]$.

The physical phenomena of the dynamic condition in Eq (3) is that during the period specified for the determination of the unknown function $\Phi(x,t)$ must be equal to the pressure $\mathrm{P}(t)$ on the study area during the time $t,\ where\ t\in[0,T],T<1$.

To discuss the existence of a unique solution of Eq (1), we assume the following conditions:

2.1. The position functions

(a) $k\left(\left|\frac{y-x}{\sigma}\right|\right)$ is the position kernel and in general, satisfies the discontinuity condition $\left[\int_{-1}^1\int_{-1}^1 k\left(\left|\frac{y-x}{\sigma}\right|\right)^2dxdy\right]^{\frac{1}{2}}\leq A,(A-constant)$.

(b) $q(x,y)$ and $\xi(x)$ are bounded and continuous function, where $|q(x,y)|\leq\varsigma,\ |\xi(x)|\leq D,\ (\varsigma,D$ are constants).

2.2. The time functions

For $t,\tau\in[0,T]$, $T<1$ the two functions of times $F(t,\tau)$ and $G(t,\tau)$ belong to the class $C[0,T]$ and satisfy $|F(t,\tau)|\le\beta_1$, $|G(t,\tau)|\le\beta_2$, β_1,β_2 are constants.

2.3. The mixed free function

The given function $f(x,t)$ with its partial derivatives is continuous in the space $L_2[-1,1]\times C[0,T]$ and its norm is defined as $\|f\|=\max_{0\le t\le T}\left(\int_{-1}^{1} f^2(x,t)dx\right)^{\frac{1}{2}}$.

3. Convergence of the solution

To discuss the solution behavior of Eq (1), we construct a sequence of $\{\Phi_n(x,t)\}_0^\infty$ in which $\{\Phi_1(x,t),\Phi_2(x,t),\dots,\Phi_{n-1}(x,t),\Phi_n(x,t),\dots\}\in\{\Phi_n(x,t)\}$. Hence, we can pick up the two equations $\Phi_{n-1}(x,t),\Phi_n(x,t)$, to have

$$\mu_1(\Phi_n(x,t)-\Phi_{n-1}(x,t))$$
$$=\int_0^t\int_{-1}^1 q(x,y)k\left(\left|\frac{y-x}{\sigma}\right|\right)F(t,\tau)\{\Phi_{n-1}(x,t)-\Phi_{n-2}(y,\tau)\}dyd\tau$$
$$+\xi(x)\int_0^t G(t,\tau)\{\Phi_{n-1}(x,t)-\Phi_{n-2}(x,\tau)\}d\tau,$$

$$\left(\Phi_0(x,t)=\frac{f(x,t)}{\mu_1}\right). \tag{4}$$

For this, we assume

$$\Psi_n=\Phi_n-\Phi_{n-1},\ \Psi_0(x,t)=\frac{f(x,t)}{\mu_1}. \tag{5}$$

It is easily to establish

$$\Phi_n(\mathrm{x},\mathrm{t})=\sum_{\boldsymbol{i=0}}^{\boldsymbol{n}}\Psi_i.$$

Theorem 1. (The solution convergence)

A solution sequence $\{\Phi_\mathrm{n}\}$ of Eq (4) is uniformly convergent under the condition

$$\mu_1>\mathrm{T}[\zeta\mathrm{A}\beta_1+\mathrm{D}\beta_2]. \tag{6}$$

Proof: By applying Cauchy Schwarz inequality, the formula (4) yields

$$|\mu_1|\|\Phi_n-\Phi_{n-1}\|\le$$

$$\left(\left|\int_0^t\int_{-1}^1 q(x,y)k\left(\left|\frac{y-x}{\sigma}\right|\right)F(t,\tau)dyd\tau\right|+\left|\xi(x)\int_0^t G(t,\tau)d\tau\right|\right)\|\Phi_{n-1}-\Phi_{n-2}\|. \tag{7}$$

Using the conditions (1) and (2), we have

$$|\mu_1|\|\Phi_n-\Phi_{n-1}\| \le T[\zeta A\beta_1+D\beta_2]\|\Phi_{n-1}-\Phi_{n-2}\|.$$

Therefore,

$$|\mu_1|\|\Psi_n\| \le T[\zeta A\beta_1+D\beta_2]\|\Psi_{n-1}\|. \tag{8}$$

By induction, with the aid of (5),

$$\|\Psi_n\| \le \rho^n\|f\|,\ \rho=\frac{T}{|\mu_1|}[\zeta A\beta_1+D\beta_2]. \tag{9}$$

So, $\Phi_\mathrm{n}(x,t)=\sum_{i=0}^{n}\Psi_i$ is uniformly convergent, provided that $\rho<1$.
As $n\ \to\infty$, $\Phi_\mathrm{n}(x,t)\to\Phi(x,t)$, hence the solution $\Phi(x,t)$ is uniformly convergent under the same assumption.

4. Existence and uniqueness of the MIE (1)

To discuss the existence of a unique solution of Eq (1), in view of Banach fixed point theorem, we write it in the integral operator form as

$$\boldsymbol{\mu_1}\Phi(\mathrm{x},\mathrm{t})=f(x,y)+\boldsymbol{\chi}\Phi(\mathrm{x},\mathrm{t}),\ \boldsymbol{\chi}\Phi(\mathrm{x},\mathrm{t})=\boldsymbol{K_1}\Phi+\boldsymbol{K_2}\Phi, \tag{10}$$

where,

$$\boldsymbol{K_1}\Phi=\int_0^t\int_{-1}^1 q(x,y)k\left(\left|\frac{y-x}{\sigma}\right|\right)F(t,\tau)\Phi(y,\tau)dyd\tau, \tag{11}$$

$$\boldsymbol{K_2}\Phi=\xi(x)\int_0^t G(t,\tau)\Phi(x,\tau)d\tau. \tag{12}$$

Theorem 2. (Existence and uniqueness)

The MIE (1) has the existence and unique solution under the condition

$$\mu_1>\mathrm{T}[\zeta\mathrm{A}\beta_1+\mathrm{D}\beta_2].$$

Proof: To prove this theorem, we go to prove that the integral operator (10) is bounded, continuous and a contraction mapping in the following two lemmas.
Lemma 1. The integral operator χ is bounded.
Proof: Taking the norm of Eq (10), we get

$$\|\chi\Phi(\mathrm{x},\mathrm{t})\| \le [\|\boldsymbol{K_1}\Phi(\mathrm{x},\mathrm{t})\|+\|\boldsymbol{K_2}\Phi(\mathrm{x},\mathrm{t})\|], \tag{13}$$

$$\|\boldsymbol{K_1}\Phi(\mathrm{x},\mathrm{t})\|=\left\|\int_0^t\int_{-1}^1 q(x,y)k\left(\left|\frac{y-x}{\sigma}\right|\right)F(t,\tau)\Phi(y,\tau)dyd\tau\right\|.$$

Using Cauchy Schwarz inequality and the previous conditions (1) and (2), one has

$$\|\boldsymbol{K_1}\Phi(\mathrm{x},\mathrm{t})\| \leq |q(x,y)|\left[\int_{-1}^{1}\int_{-1}^{1} k^2\left(\left|\frac{y-x}{\sigma}\right|\right)dx\,dy\right]^{\frac{1}{2}}|F(t,\tau)|T\|\Phi\| \leq \zeta A\beta_1\|\Phi\|T, \tag{14}$$

and

$$\|\boldsymbol{K_2}\Phi(\mathrm{x},\mathrm{t})\| = \left\|\xi(x)\int_0^t G(t,\tau)\Phi(x,\tau)d\tau\right\| \leq D\beta_2\|\Phi\|. \tag{15}$$

Hence, Eq (13) becomes

$$\|\chi\Phi\| \leq [\zeta A\beta_1 + D\beta_2]\|\Phi\|T. \tag{16}$$

Therefore, the operator χ maps the ball $S_r \subset L_2$ into itself where

$$r = \frac{\delta}{1-\rho},\ \delta = \frac{\|f\|}{\mu_1} \tag{17}$$

$$\rho = \frac{T}{|\mu_1|}[\zeta A\beta_1 + D\beta_2] < 1.$$

The inequality (16) involves the boundedness of the operator $\boldsymbol{\chi}$.

Lemma 2. The integral operator χ is a contraction mapping in its space.

Proof: Let the functions Φ_1 and Φ_2 be two solutions of Eq (1), then Eq (10) with the aid of Eqs (11) and (12) leads to

$$\|\chi\Phi_1 - \chi\Phi_2\| \leq \frac{T}{|\mu_1|}[(\zeta A\beta_1 + D\beta_2)\|\Phi_1 - \Phi_2\|]. \tag{18}$$

So, χ is continuous and it is a contraction mapping under the condition $\rho < 1$.

According to Banach fixed point theorem, since χ is bounded and a contraction mapping, then the Eq (1) has a unique and existing solution, concluding the proof of Theorem 2.

5. The behavior of position kernel

The kernel of the **IE** in position or in time plays an important role in the interpretation of physical phenomena. Therefore, the authors must take into account how to solve the **IEs** according to its kernels. In this section, we study the phenomenon of a general singular kernel and how to convert this kernel into a singular logarithmic kernel as well. Thus, it is possible to search how solving the **IE** using the spectral relationships method.

The function $L(u)$ of Eq (2) is continuous and positive, for $u \in (0,\infty)$. This function is studied physically at two values of the variable u when it is very small or when it is very large. This is expressed by the following two relationships

$$L(u) = m - (m-1)u + O(u^3),\ u \to 0,$$

$$L(u) = 1 - \frac{m-1}{u} + O(u^{-2}),\ u \to \infty,\ 1 \leq m \leq M. \tag{19}$$

When $m = 1$, in Eq (19), and $\sigma \to \infty$ in Eq (1), such that the term $\left(\left|\frac{y-x}{\sigma}\right|\right)$ is very small. In this case, the kernel of position takes a logarithmic function form (Popov [23]).

$$\int_0^\infty \frac{cosuz}{u} du = -[\ln|x-y| - d],\ (d = \ln\frac{4\sigma}{\pi}). \tag{20}$$

In view of (19) and (20), the **MIE** yields

$$\mu_1\Phi(x,t) + \int_0^t \int_{-1}^1 q(x,y)[\ln|x-y| - d]F(t,\tau)\Phi(y,\tau)dyd\tau - \xi(x)\int_0^t G(t,\tau)\Phi(x,\tau)d\tau = f(x,t). \tag{21}$$

In many physical applications, the researcher may be required to study the case $\mathrm{u} \to 0, \mathrm{m} > 1$. Hence, the following equations must be used: (Gradstein and Ryzhik [24]).

(i) $\frac{1}{\pi}\int_0^\infty cos\,\mathrm{\nu x d\nu} = \delta(\mathrm{x})$, δ(x) is the Dirac delta function

$$\text{(ii)}\ \int_a^b \phi(y)\delta(y-x)\,\mathrm{dy} = \begin{cases} 0, & x < a, \\ \frac{1}{2}[h(x-a) + h(x+a)], & a < x < b, \\ 0, & x > b. \end{cases} \tag{22}$$

The importance of logarithmic kernel with some applications can be found in Abdou et al. [25], and the relation between σ, m and the logarithmic kernel was obtained in the form

$$\left\{\int_{-1}^1 \int_{-1}^1 \ln^2|x-y|dxdy\right\}^{\frac{1}{2}} < \frac{1}{\sigma},\ \sigma = 1 + \frac{1}{m} + \frac{1}{m^2} + \cdots + \frac{1}{m^N}. \tag{23}$$

Also, some different methods for solving the integral equations with logarithmic kernel have been discussed in (Frankel [26], Anastasias and Aral [27]).

6. The technique of separation

Many authors have solved integral (differential) equations of all kinds by neglecting the effect of time. Also, when some authors studied the effect of time their works, Laplace transform or Fourier transform have been used to obtain equations in position. These methods have problems when trying to obtain the transformation inverse. Also, the authors often find it difficult to explain the phenomenon of time in the problem to be solved.

Also, one of the famous methods has been used is dividing the time into periods to obtain a complete system of **IEs** specific to the position only. In this direction we find that the **MIE** is transformed into an algebraic system for **FIEs** (Abdou [25]). While in this section we use the separation method, in the form of a new technique, to obtain the **FIE** with time coefficients and these functions are described as an integral operator in time. Thus, this technique enables the authors to study the behavior of the solution with the time dimension more broadly and deeper than the previous one. Assume the following

$$\Phi(x,t) = M(t)B(x),\ f(x,t) = g(x)M(t). \tag{24}$$

Hence, after using (24), the formula (21) yields,

$$\mu(x,t)B(\mathrm{x})+\frac{1}{M(t)}\int_0^t\int_{-1}^{1}q(x,y)[\ln|x-y|-d]B(y)F(t,\tau)M(\tau)dyd\tau=g(x), \tag{25}$$

where

$$\mu(x,t)=\mu_1-\xi(x)\gamma(t),\ \gamma(t)=\frac{1}{M(t)}\int_0^t G(t,\tau)\mathrm{M}(\tau)d\tau. \tag{26}$$

The coefficient of the unknown function $\mu(x,t)$ determines the type of the **IE**. If this coefficient is equal to zero, then it becomes an **IE** of the first kind, while if it is equal to a constant, then the equation is of the second kind. The equation is of the third kind, if this coefficient is a function of *x*. In view of the above, and although, μ_1 is a constant but through the separation method, we obtain directly a third kind **FIE** that is created from the coefficient of the **VI** operator. Also, we have a time focused view where we get a **FIE** with time-related coefficients that can be computed explicitly at any point in time. To obtain the solution of Eq (25), we use the **CPs** of the first kind $T_n(x)$ with its famous relations. For this, we write the unknown functions $B(x)$ and the known functions $q(x,y)$ and $g(x)$ as a linear combination between eigenvalues and eigenfunctions $T_n(x)$ in the forms

$$(a)\ B(x)=\frac{1}{\sqrt{1-x^2}}\sum_{n=0}^{\infty}b_nT_n(x),\ (b)\ q(x,y)=\sum_{m=0}^{M}T_m(x)\ T_m(y),$$

$$(c)\ g(x)=\sum_{n=0}^{\infty}g_n\ T_n(x), g_0=\frac{1}{\pi}\int_{-1}^{1}\frac{g(x)dx}{\sqrt{1-x^2}},\ g_n=\frac{2}{\pi}\int_{-1}^{1}\frac{g(x)dx}{\sqrt{1-x^2}}, n>0. \tag{27}$$

In (27-a) b_n, $n\geq 0$, are unknown constants with respect to position only and its values will be discovered it depends upon the time. The function $\sqrt{1-x^2}$ is called the weight function of Chebyshev polynomials of the first kind.

It is difficult to obtain the solution of Eq (25) numerically in the form of (27-a). For this, the formula (27-a) can be truncated to

$$B_n(x)=\frac{1}{\sqrt{1-x^2}}\sum_{n=0}^{N}b_n\ T_n(x), \lim_{N\to\infty}B_N(x)=B(x). \tag{28}$$

First, we derive the spectral relationships for the 1st kind **V-FIE** which takes the form

$$\frac{1}{M(t)}\int_0^t\int_{-1}^{1}q(x,y)[\ln|x-y|-d]F(t,\tau)M(\tau)B_N(y)dyd\tau=g(x). \tag{29}$$

Then, we use it to obtain the solution of Eq (25).

7. Spectral relationships of integral Eq (25)

The objective of this section is to establish a theory of spectral relationships between the eigenvalues and its corresponding eigenvectors of the following integral operator

$$L = \frac{1}{M(t)} \int_0^t \int_{-1}^1 q(x,y)[\ln|x-y|-d] \frac{b_n T_n(y)}{\sqrt{1-y^2}} F(t,\tau) M(\tau) dy d\tau.$$

Theorem 3. (Main Theorem of spectral relationships)

$$\frac{1}{M(t)} \int_0^t \int_{-1}^1 q(x,y)[\ln|x-y|-d] \frac{T_n(y)}{\sqrt{1-y^2}} F(t,\tau) M(\tau) dy d\tau =$$

$$\begin{cases} \pi\lambda(t)(\ln 2 - d), & n = m = 0 \\ \pi\lambda(t) \sum_{m=1}^{M} \frac{1}{m}\left(\frac{1-2m^2}{1-4m^2}\right), & n = 0, m \geq 1 \\ \lambda(t) \frac{\pi T_n(x)}{n}, & m = 0, n \geq 1 \\ \frac{\pi\lambda(t)}{2}\left[\frac{T_{3n}}{4n} + \left(\frac{1}{4n} + \ln 2 - d\right) T_n\right], & m = n \geq 1 \\ \frac{\pi\lambda(t)}{4} \sum_{m=1}^{M}\left[\frac{T_{2m+n}}{n+m} + \left(\frac{1}{n+m} + \frac{1}{|n-m|}\right) T_n(x) + \frac{T_{|n-2m|}}{|n-m|}\right], & m \neq n \neq 0, \end{cases} \tag{30}$$

where

$$\lambda(t) = \frac{1}{M(t)} \int_0^t F(t,\tau) \mathrm{M}(\tau) d\tau. \tag{31}$$

Proof: Consider the **V-FIE** of the 1st kind in the form

$$\frac{1}{M(t)} \int_0^t \int_{-1}^1 q(x,y)[\ln|x-y|-d] F(t,\tau) M(\tau) B_N(y) dy d\tau = g(x). \tag{32}$$

After using Eqs (27) and (28), we have

$$\lambda(t) \sum_{m=0}^{M} b_n T_m(x) \int_{-1}^1 \frac{[\ln|x-y|-d] T_m(y) T_n(y)}{\sqrt{1-y^2}} dy = g_n T_n(x). \tag{33}$$

Consider the following famous relations (Gradstein and Ryzhik [24])

(a) Algebraic relation: $T_m(x)T_n(x) = \frac{1}{2}\left[T_{n+m}(x) + T_{|n-m|}(x)\right]$.

(b) Integral relation: $\int_{-1}^1 T_n(x) dx = \frac{2}{1-n^2},\ n = 0,2,4, \ldots.$

(c) Orthogonal relation: $\int_{-1}^1 \frac{[\ln|x-y|] T_n(y)}{\sqrt{1-y^2}} dy = \begin{cases} \pi ln2, & n = 0, \\ \frac{\pi}{n} T_n(x), & n \geq 1. \end{cases}$ (34)

In view of (34-c), we can establish the following:

Case (1): When $n = m = 0$, we get

$$\pi\lambda(t)(\ln 2 - d) b_0 = g_0. \tag{35}$$

Case (2): If $n = 0, m \neq 0$, then Eq (32), after using (34-c) becomes

$$\frac{\pi}{m}\lambda(t)T_m^2 b_0 = g_0. \tag{36}$$

Using (34-a) then integrating from –1 to 1, by aid of (34-b), we get

$$\frac{\pi}{m}\lambda(t)\left(\frac{2}{1-4m^2}+2\right)b_0 = 2g_0. \tag{37}$$

Therefore,

$$b_0 = \frac{m(1-4m^2)}{\pi\lambda(t)(1-2m^2)}g_0. \tag{38}$$

Case (3): If $n \neq 0, m = 0$, then with the aid of (34), the formula (32) yields,

$$b_n = \frac{n}{\pi\lambda(t)}g_n. \tag{39}$$

Case (4): For $n \neq 0, m \neq 0$,

$$\frac{\lambda(t)}{2}b_n\sum_{m=1}^{M}T_m(x)\int_{-1}^{1}\frac{[\ln|x-y|-d]\left(T_{n+m}(y)+T_{|n-m|}(y)\right)}{\sqrt{1-y^2}}dy = g_nT_n(x). \tag{40}$$

(i) For $n = m \neq 0$

$$\frac{\pi\lambda(t)}{2}b_n\left[\frac{T_{3n}(x)}{4n}+\left(\frac{1}{4n}+\ln 2-d\right)T_n(x)\right] = g_nT_n(x). \tag{41}$$

(ii) If $n \neq m \neq 0$ the formula (32), with the aid of (34) yields

$$\frac{\pi\lambda(t)}{4}b_n\sum_{m=1}^{M}\left[\frac{T_{n+2m}(x)}{n+m}+\left(\frac{1}{n+m}+\frac{1}{|n-m|}\right)T_n(x)+\frac{T_{|n-2m|}(x)}{|n-m|}\right] = g_nT_n(x). \tag{42}$$

Introducing the formulae (35), (38), (39), (41) and (42) in Eq (32), the theorem is completely proved.

7.1. On a description of case 4 (ii)

The importance of our description is clarification the different levels of eigenvalues and its corresponding eigenvectors in the presence of time. So, the spectral relationships between its eigenvalues corresponding to its eigenvectors with time parameter $M(t) = 0.45e^{-t}, F(t,\tau) = \tau^3$, can be seen from the following Figures 1–8:

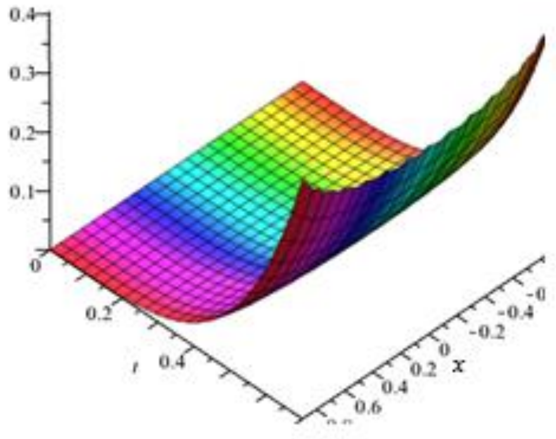


Figure 1. [$n = 0, p(x,y) = \sum_{m=1}^{20} T_m(x)T_m(y)$].

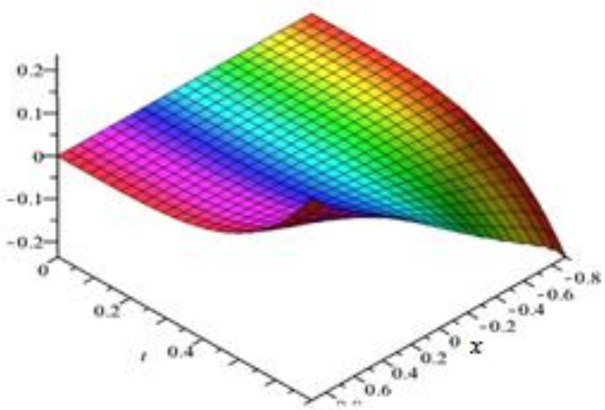


Figure 2. [$n = 1, p(x,y) = \sum_{m=2}^{20} T_m(x)T_m(y)$].

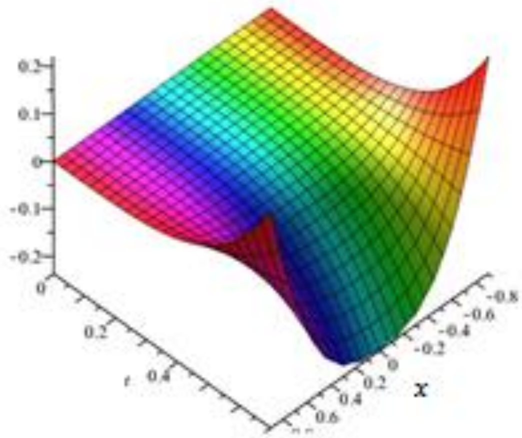


Figure 3. [$n = 2, p(x,y) = \sum_{\substack{m=1 \\ m\neq 3}}^{20} T_m(x)T_m(y)$].

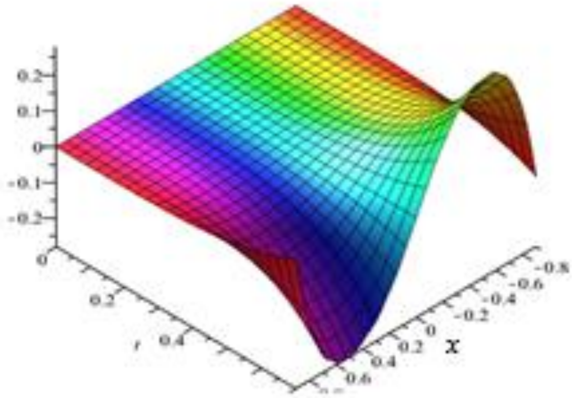
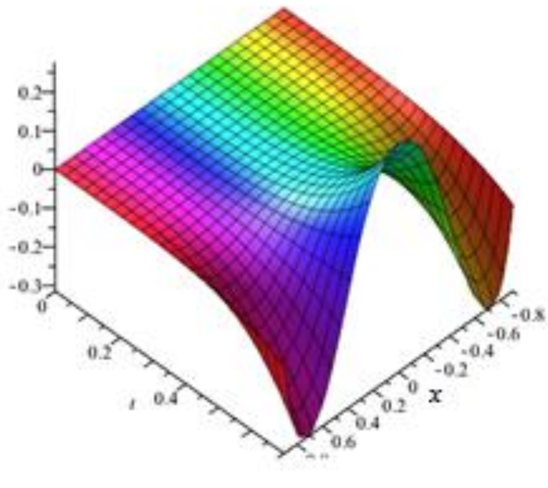


Figure 4. [$n = 3, p(x,y) = \sum_{\substack{m=1 \\ m\neq 4}}^{20} T_m(x)T_m(y)$].



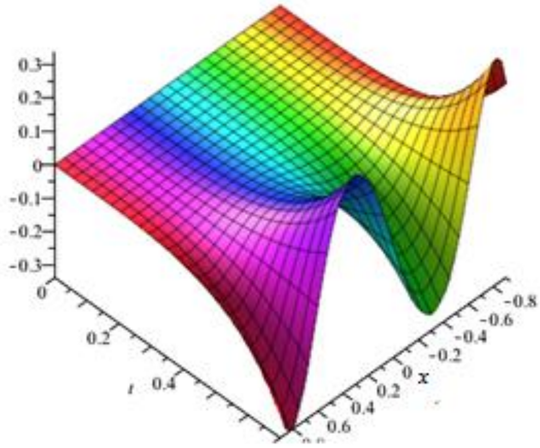
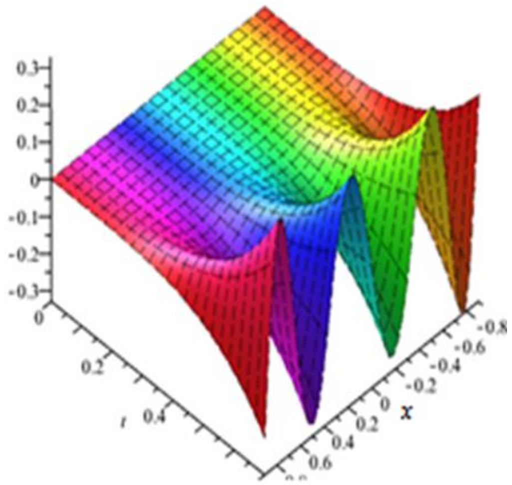


Figure 5. [$n = 4,\ p(x,y) = \sum_{\substack{m=1\\ m\neq 4}}^{20} T_m(x)T_m(y)$]. **Figure 6.** [$n = 5\ \ p(x,y) = \sum_{\substack{m=1\\ m\neq 5}}^{20} T_m(x)T_m(y)$].



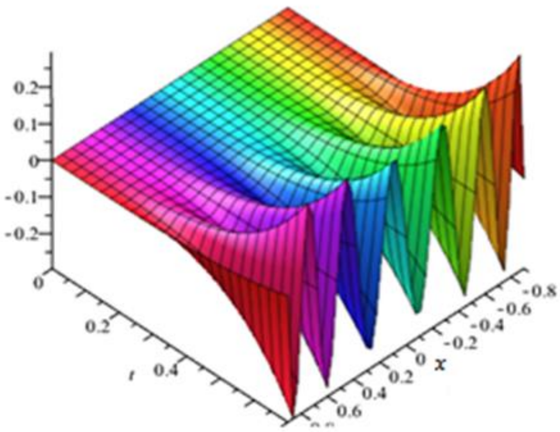


Figure 7. [$n = 9,\ p(x,y) = \sum_{\substack{m=1\\ m\neq 9}}^{20} T_m(x)T_m(y)$]. **Figure 8.** [$n = 17,\ p(x,y) = \sum_{\substack{m=1\\ m\neq 17}}^{20} T_m(x)T_m(y)$].

8. Mixed integral equation of the 3rd kind

Here, we use the main theorem of spectral relationship (30) to deduce the solution of third kind **MIE**. For this, return to Eqs (25) and (26) by considering the known function $\xi(x)$ as a linear combination between eigenvalues ξ_k and eigenfunctions $T_k(x)$as:

$$\xi(x) = \sum_{k=0}^{\infty} \xi_k T_k(x)\,,\ \ \xi_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{\xi(x)}{\sqrt{1-x^2}}dx\,,\ \ \xi_k = \frac{2}{\pi}\int_{-1}^{1} \frac{\xi(x)T_k(x)}{\sqrt{1-x^2}}dx\,,\ \ k \neq 0. \tag{43}$$

In view of Eq (30), we can establish the following cases:

Case (1): When $n = m = 0$ and $k = 0$, we get

$$b_0 = \frac{2g_0}{\pi[\mu_1 - \xi_0\gamma(t) + 2\lambda(t)(\ln 2 - d)]}. \tag{44}$$

The solution in this case, takes the form

$$\Phi(x,t) = \frac{1}{\pi\sqrt{1-x^2}}\left(\frac{2f(x,t)}{\pi[\mu_1 - \xi_0\gamma(t) + 2\lambda(t)(\ln 2 - d)]}\right). \tag{45}$$

If $n = m = 0$ and $k \neq 0$, we obtain

$$b_0 = \frac{4g_0}{\pi\left[2\mu_1 - \gamma(t)\sum_{k=1}^{\infty}\xi_k + 2\lambda(t)(\ln 2 - d)\right]}. \tag{46}$$

Case (2): If $n = 0,\ m \neq 0$ and $k = 0$. Substitute in (25), after using (30) becomes

$$b_0 = \frac{2m(1-4m^2)g_0}{\pi\left[\left(\mu_1 - \xi_0\gamma(t)\right)m(1-4m^2) + 2\lambda(t)(1-2m^2)\right]}. \tag{47}$$

If $n = 0,\ m \neq 0$ and $k \neq 0$, we have

$$b_0 = \frac{4m(1-4m^2)g_0}{\pi\left[\left(2\mu_1 - \gamma(t)\sum_{k=1}^{\infty}\xi_k\right)m(1-4m^2) + 4\lambda(t)(1-2m^2)\right]}. \tag{48}$$

Case (3): If $n \neq 0,\ m = 0$ and $k = 0$, then with the aid of (30), the formula (25) yields

$$b_n = \frac{4n(1-2n^2)g_n}{\pi[(\mu_1 - \gamma(t)\xi_0)n(1-4n^2) + 4\lambda(t)(1-2n^2)]}. \tag{49}$$

If $n \neq 0,\ m = 0$ and $k \neq 0$, Eq (25) becomes

$$\mu_1 \frac{1}{\sqrt{1-x^2}}\sum_{n=1}^{\infty} b_n T_n(x) - \frac{1}{2\sqrt{1-x^2}}\gamma(t)\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\xi_k\left(T_{n+k}(x) + T_{|n-k|}(x)\right)$$

$$+\pi\lambda(t)\sum_{n=1}^{\infty}\frac{T_n(x)}{n} = \sum_{n=1}^{\infty} g_n T_n(x). \tag{50}$$

Multiplying both sides by $T_l(x)$ and integrating from -1 to 1, we have

$$\pi\left[2\mu_1\delta_{n,l} - \gamma(t)\sum_{k=1}^{\infty}\xi_k\left(\delta_{n+k,l} + \delta_{|n-k|,l}\right) + \lambda(t)\frac{8\left(1-2n^2\right)}{n(1-4n^2)}\right] = \frac{8\left(1-2n^2\right)}{n(1-4n^2)}g_n. \tag{51}$$

If $n + k = l$, then

$$b_n = \frac{8n(1-2n^2)g_n}{\pi\left[\left(2\mu_1 - \gamma(t)\sum_{k=1}^{\infty}\xi_k\right)n(1-4n^2) + 8\lambda(t)(1-2n^2)\right]}. \tag{52}$$

Case 4 (i): If $n = m \neq 0$, Eq (25) becomes

$$\left(\mu_1-\gamma(t)\sum_{k=1}^{\infty}\xi_k T_k(x)\right)\frac{1}{\sqrt{1-x^2}}\sum_{n=1}^{\infty}b_n T_n(x)$$

$$+\frac{\pi\lambda(t)}{2}\sum_{n=1}^{\infty}\left(\frac{T_{3n}(x)+T_{3n}(x)}{n}+\ln 2-d\right)T_n(x)=\sum_{n=1}^{\infty}g_n T_n(x). \tag{53}$$

Multiplying both sides by $T_l(x)$ and integrating from -1 to 1, we get the following:
If $k=0$, we obtain

$$b_n=\frac{4n(1-2n^2)(1-16n^2)\pi^{-1}g_n}{(\mu_1-\gamma(t)\xi_0)n(1-4n^2)(1-16n^2)+\lambda(t)[(1-14n^2+16n^4)+2n(1-2n^2)(1-16n^2)(\ln 2-d)]}. \tag{54}$$

If $k\neq 0, n+k=l$,

$$b_n=\frac{4n(1-2n^2)(1-16n^2)\pi^{-1}g_n}{\left(\mu_1-\frac{\gamma(t)}{2}\sum_{k=1}^{\infty}\xi_k\right)n(1-4n^2)(1-16n^2)+\lambda(t)[(1-14n^2+16n^4)+2n(1-2n^2)(1-16n^2)(\ln 2-d)]}. \tag{55}$$

Case 4 (ii): If $n\neq m\neq 0$, Eq (25) becomes

$$\left(\mu_1-\gamma(t)\sum_{k=1}^{\infty}\xi_k T_k(x)\right)\frac{1}{\sqrt{1-x^2}}\sum_{n=1}^{\infty}b_n T_n(x)$$

$$+\frac{\pi\lambda(t)}{4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{T_{2m+n}(x)+T_n(x)}{m+n}+\frac{T_{|2m-n|}(x)+T_n(x)}{|m-n|}\right)=\sum_{n=1}^{\infty}g_n T_n(x). \tag{56}$$

Multiplying both sides by $T_l(x)$ and integrating from -1 to 1, we get the following:
If $k=0$, we obtain

$$\mathrm{b_n}=\frac{4\pi^{-1}\eta_{n,m}^{(2)}g_n}{2(\mu_1-\gamma(t)\xi_0)+\lambda(t)\sum_{m=1}^{\infty}\left[\frac{\eta_{n,m}^{(1)}}{n+m}+\left\{\frac{1}{n+m}+\frac{1}{|n-m|}\right\}\eta_{n,m}^{(2)}+\frac{\eta_{n,m}^{(3)}}{|n-m|}\right]}. \tag{57}$$

If $k\neq 0, n+k=l$, we get

$$b_n=\frac{4\pi^{-1}\eta_{n,m}^{(2)}g_n}{2\left(\mu_1-\frac{\gamma(t)}{2}\sum_{k=1}^{\infty}\xi_k\right)+\lambda(t)\sum_{m=1}^{\infty}\left[\frac{\eta_{n,m}^{(1)}}{n+m}+\left\{\frac{1}{n+m}+\frac{1}{|n-m|}\right\}\eta_{n,m}^{(2)}+\frac{\eta_{n,m}^{(3)}}{|n-m|}\right]}, \tag{58}$$

where

$$\eta_{n,m}^{(1)}=\frac{1}{1-4(n+m)^2}+\frac{1}{1-4m^2},\ \eta_{n,m}^{(2)}=\frac{1}{1-4n^2}+1,\ \eta_{n,m}^{(3)}=\frac{1}{1-4(n-m)^2}+\frac{1}{1-4m^2}. \tag{59}$$

9. Convergence of the linear algebraic system

To discuss the convergence of algebraic systems (57) and (58), consider

$$\beta_n = \frac{4\eta_{n,m}^{(2)} g_n}{\pi\lambda(t)\sum_{m=1}^{M}\chi_{n,m}}, \tag{60}$$

where

$$\chi_{n,m} = \frac{\eta_{n,m}^{(1)}}{n+m} + \left\{\frac{1}{n+m} + \frac{1}{|n-m|}\right\}\eta_{n,m}^{(2)} + \frac{\eta_{n,m}^{(3)}}{|n-m|}. \tag{61}$$

Lemma 3. The series $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\chi_{n,m}$ is convergent.
Proof: Applying Cauchy-Minkowski, we get

$$\|\chi_{n,m}\| = \left|\sum_{n=1}^{\infty}\sum_{m=1}^{M}\chi_{n,m}^2\right|^{\frac{1}{2}} \leq \left|\sum_{n=1}^{\infty}\sum_{m=1}^{M}\left(\frac{1}{n+m}\right)^2\right|^{\frac{1}{2}}\left|\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\eta_{n,m}^{(1)}\right)^2\right|^{\frac{1}{2}}$$

$$+\left|\sum_{n=1}^{\infty}\sum_{m=1}^{M}\left(\frac{1}{n+m}\right)^2\right|^{\frac{1}{2}}\left|\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\eta_{n,m}^{(2)}\right)^2\right|^{\frac{1}{2}}$$

$$+\left|\sum_{n=1}^{\infty}\sum_{m=1}^{M}\left(\frac{1}{n-m}\right)^2\right|^{\frac{1}{2}}\left|\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\eta_{n,m}^{(2)}\right)^2\right|^{\frac{1}{2}}$$

$$+\left|\sum_{n=1}^{\infty}\sum_{m=1}^{M}\left(\frac{1}{n-m}\right)^2\right|^{\frac{1}{2}}\left|\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\eta_{n,m}^{(3)}\right)^2\right|^{\frac{1}{2}}. \tag{62}$$

Since the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent, and the two series $\sum_{n,\ell=1}^{\infty}\sum_{m=1}^{M}(\frac{1}{(n+m)})^2$ and $\sum_{n,\ell=1}^{\infty}\sum_{m=1}^{M}(\frac{1}{(n-m)})^2$ behave like $\sum_{n=1}^{\infty}\frac{1}{n^2}$. Then, the two series are convergent and tends to zero as $n\to\infty$. Moreover,

$$\eta_{n,m}^{(1)} = \left[\frac{1}{1-4(n+m)^2} + \frac{1}{1-4m^2}\right] \leq \left[\frac{1}{4(n+m)^2} + \frac{1}{4m^2}\right] \leq \frac{1}{2n^2}. \tag{63}$$

Similarly,

$$\eta_{n,m}^{(2)} \leq \frac{1}{2n^2},\ \eta_{n,m}^{(3)} \leq \frac{1}{2n^2}. \tag{64}$$

Therefore, $\chi_{n,m} \to 0$ as $n\to\infty$. Finally, from (58) and (60), we have

$$|b_n| \leq |\beta_n| = \frac{4\left|\eta_{n,m}^{(2)}\right||g_n|}{\pi|\lambda(t)|\left|\chi_{n,m}\right|} \leq \text{constant}. \tag{65}$$

10. Numerical results

In this section, some numerical applications are considered to show the accuracy and applicable of the proposed methods.

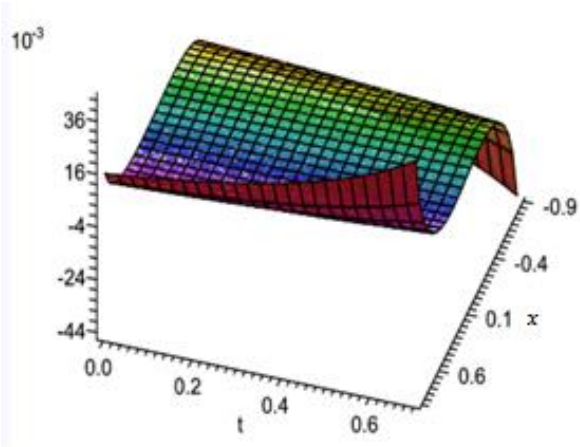
Example 1. Consider the **MIE** of the second kind

$$1.7\Phi(x,t)-\int_0^t\int_{-1}^1 q(x,y)[\ln|x-y|-0.01]\tau^3\Phi(y,\tau)dyd\tau-(1+x)^3\int_0^t\tau^2\Phi(x,\tau)d\tau=$$
$$(0.04+0.05t+0.03t^2)g(x), \text{ and } q(x,y)=\sum_{m=0}^{M}T_m(x)T_m(y). \tag{66}$$

Table 1 represents the solution $\Phi(x,t)$ and its corresponding errors for different time and Figures 9–12 show the behavior of the approximate solution $\Phi(x,t)$ and its corresponding errors for a certain g(x).

Table 1. The solution $\Phi(x,t)$ and its corresponding errors for different time.

		x	N=20, M=0		N=20, M=13	
			$\Phi(x,t)$	Error	$\Phi(x,t)$	Error
$g(x)=x^2\sin x$	T=0.8	–0.8	–0.027702	9.7e-15	–0.02836	9.2e-15
		–0.4	0.00194	1.8e-14	0.00036	1.7e-14
		0.0	8.1e-12	9.4e-15	7.0e-12	8.9e-15
		0.4	–0.00194	1.7e-15	–0.00036	1.6e-15
		0.8	0.02770	1.0e-14	0.02836	9.8e-15
	T=0.3	–0.8	–0.010611	3.3e-15	–0.01061	3.3e-15
		–0.4	0.00046	5.9e-15	0.00045	6.0e-15
		0.0	2.9e-12	3.1e-14	2.8e-12	3.2e-15
		0.4	–0.00046	1.1e-14	–0.00045	5.7e-16
		0.8	0.01061	3.6e-16	0.01061	3.5e-15
	T=0.01	–0.8	–0.00727	2.3e-15	–0.00727	2.2e-15
		–0.4	0.00031	4.1e-15	0.00031	4.1e-15
		0.0	2.0e-12	2.2e-15	1.9e-10	2.5e-15
		0.4	–0.00031	3.9e-16	–0.00031	3.9e-16
		0.8	–1.7e-13	2.4e-15	0.00727	2.4e-15
$g(x)=\sqrt{1+x}$	T=0.8	–0.8	0.15689	6.3e-5	2.47942	5.9 e-5
		–0.4	0.11810	1.1e-4	1.63829	1.0e-4
		0.0	0.11665	6.0e-5	1.51011	5.7e-5
		0.4	0.13391	1.0e-5	1.65458	1.0e-5
		0.8	0.21272	6.7e-5	2.53604	6.3e-5
	T=0.3	–0.8	0.05567	2.1e-5	0.85399	2.1e-5
		–0.4	0.04225	3.8e-5	0.56487	3.8e-5
		0.0	0.04194	2.1e-5	0.52093	2.6e-5
		0.4	0.04832	3.5e-6	0.57094	3.5e-6
		0.8	0.07696	2.3e-5	0.87529	2.2e-5
	T=0.01	–0.8	0.54236	1.4e-5	0.58433	1.5e-5
		–0.4	0.36502	2.6e-5	0.38651	2.6e-5
		0.0	0.33710	1.4e-5	0.35645	1.4e-5
		0.4	0.36502	2.4e-6	0.39067	2.4e-6
		0.8	0.54236	1.5e-5	0.59892	1.5e-5



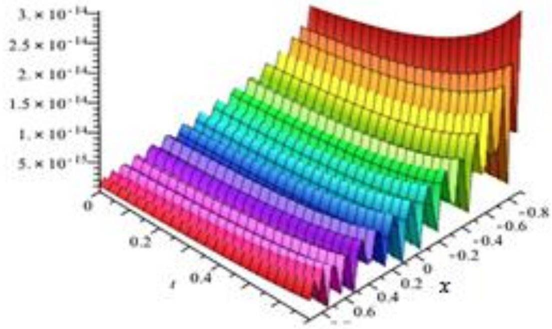
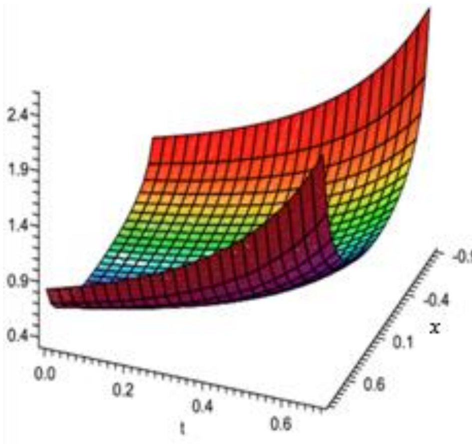


Figure 9. The solution $\Phi(x,t)$ where $g(x) = x^2 \sin x$. **Figure 10.** The error where $g(x) = x^2 \sin x$.



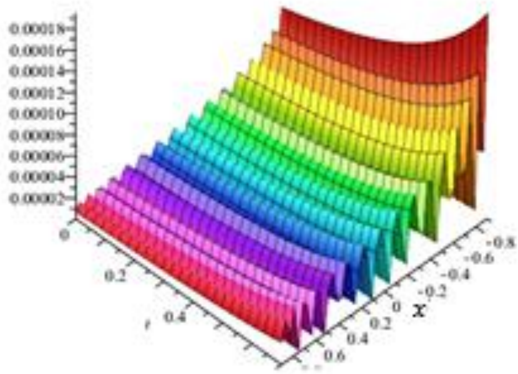


Figure 11. The solution $\Phi(x,t)$ where $g(x) = \sqrt{1+x}$. **Figure 12.** The error where $g(x) = \sqrt{1+x}$.

Example 2. Consider the **MIE** of the second kind

$$1.7\Phi(x,t) - \sum_{m=0}^{13} T_m(x) \int_0^t \int_{-1}^1 T_m(y)[\ln|x-y| - 0.01]\tau^3 \Phi(y,\tau) dy d\tau$$

$$-\xi(x) \int_0^t \tau^2 \Phi(x,\tau) d\tau = M(t) g(x). \tag{67}$$

In this example, we solve the **MIE** by using Chebyshev polynomial method after separation method technique where the position interval was dividing in 3, 9, 27 and 81 parts, the mean errors and the convergence rate of errors are computing, see Tables 2–4.

Table 2. $\xi(x) = x\cos(x),\ M(t) = 0.7\sin(t)$.

N	$k = 0$		$k = 13$	
	Mean Error	Rate of convergence	Mean Error	Rate of convergence
3	2.17×10^{-3}	1.497	2.19×10^{-3}	1.497
9	4.19×10^{-4}	2.009	4.23×10^{-4}	2.038
27	4.61×10^{-5}	1.861	4.51×10^{-5}	1.859
81	5.97×10^{-6}	---	5.85×10^{-6}	---

Table 3. $\xi(x) = \frac{1}{1+x^2}, M(t) = 0.45\sqrt{1+t}$.

N	$k = 0$		$k = 13$	
	Mean Error	Rate of convergence	Mean Error	Rate of convergence
3	5.32×10^{-3}	1.544	5.12×10^{-3}	1.451
9	9.76×10^{-4}	2.004	1.04×10^{-3}	1.997
27	1.08×10^{-4}	1.821	1.16×10^{-4}	1.874
81	1.46×10^{-5}	---	1.48×10^{-5}	---

Table 4. $\xi(x) = e^{x^2},\ M(t) = \ln(1.3 + t)$.

N	$k = 0$		$k = 13$	
	Mean Error	Rate of convergence	Mean Error	Rate of convergence
3	4.95×10^{-3}	1.492	4.98×10^{-3}	1.490
9	9.61×10^{-4}	1.998	9.69×10^{-4}	2.022
27	1.07×10^{-4}	1.891	1.05×10^{-4}	1.854
81	1.34×10^{-5}	---	1.37×10^{-5}	---

Figures 13–15 represent the behavior of $\Phi(x,t)$ for different $\xi(x)$ and $M(t)$.

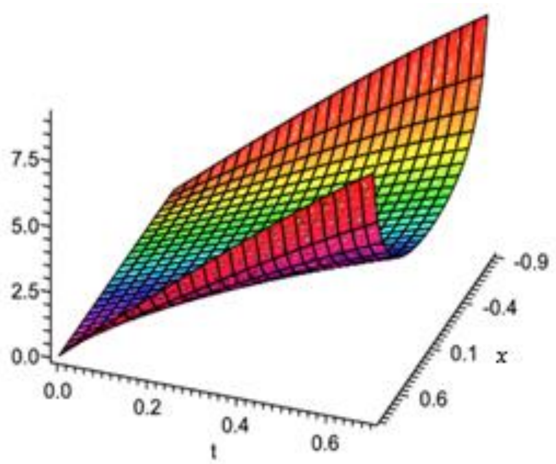


Figure 13. The solution $\Phi(x,t)$ where $\xi(x) = x\cos(x),\ M(t) = 0.7\sin(t)$.

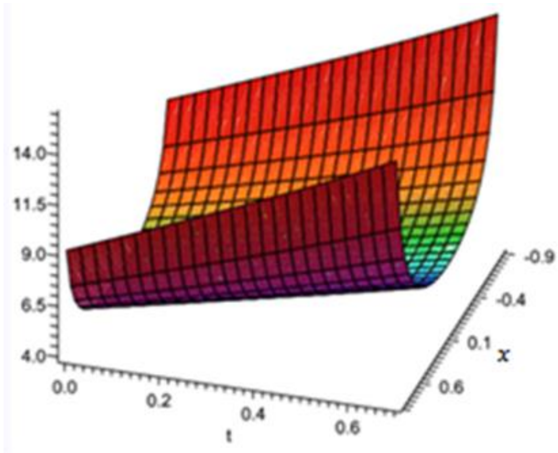


Figure 14. The solution $\Phi(x,t)$ wher $\xi(x)=\frac{1}{1+x^2}, M(t)=0.45\sqrt{1+t}$e.

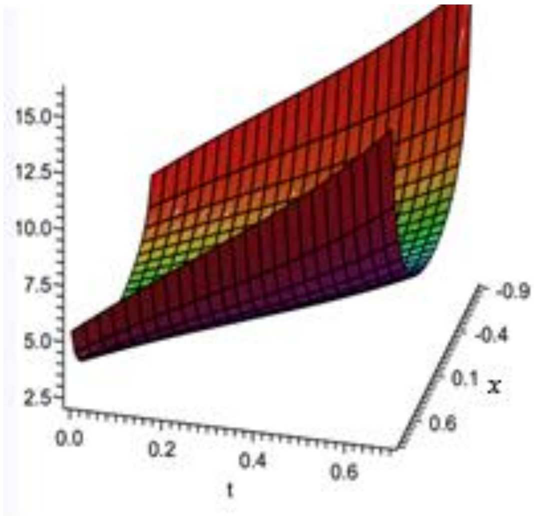


Figure 15. The solution $\Phi(x,t)$ where $\xi(x)=e^{x^2},\ M(t)=\ln(1.3+t)$.

11. Conclusions

From the previous work and discussion we can establish the following:

1) Many natural phenomena and contact problems in mathematical physics turn the problem into an **IE** with different kernel.

2) A mixed **IE** has its own kernel in time and is always a positive and continuous time function. While the position kernel may be continuous or discontinuous (which is the most important in the study).

3) The technique of separating variables by the direct method, helped to treat the scientific shortcomings in the previous methods. In addition, it enables authors to choose the necessary and appropriate time explicitly to solve the problem.

4) The method of orthogonal polynomials, with the help of some special functions, enables the authors to directly represent the solution in the form of a linear relationship of eigenvalues and eigenfunctions. The eigenvalues play a very important role in describing the solution and its behavior.
5) The importance of Eq (25) comes from the special cases that can be deduced from it. For example, when we put $q(x,y)=1$, $\xi(x)=1$, $\lambda(t)$ given by Eq (31) and differentiate the equation with respect to the variable x, we get the equation

$$\frac{dB(x)}{dx}+\frac{\lambda(t)}{\mu(t)}\int_{-1}^{1}\frac{B(y)dy}{x-y}=\frac{1}{\mu(t)}\frac{dg(x)}{dx}. \tag{68}$$

Taking the transformations $y=2u-1, x=2v-1$, the above **IDE**, on noting the difference notations, becomes

$$\frac{d\Theta(\mathrm{v})}{dv}+\check{\lambda}\int_{0}^{1}\frac{\Theta(u)du}{v-u}=z(v),\left(\check{\lambda}=\frac{\lambda(t)}{\mu(t)}\right). \tag{69}$$

This equation has appeared in both combined infrared gaseous radiations and molecular conduction, where $\check{\lambda}$ in (69), represents a relationship between the radiative conduction of the maximum path length and time, and represents the only parameter of the dimensionless system and its relationship to time. If we expand $t=\sum_{i=0}^{T}t_i$ and consider $z(v)=\frac{1}{2}-v$, then, the **IDE** (69), in the zero approximate of time and under the conditions $\Theta(0)=\Theta(1)$, was considered by Frankel [26], where $\Theta(t)$ represents the unknown temperature.
6) In this paper, the method of separating the variables helped in converting the **MIE** in position and time into **FIE** in position with time parameters. In addition, spectral relationships have been deduced, which help in solving many problems of mathematical physics.
7) From the above results, we can deduce that, the error is extremely stable with time, see Table 1. While, the error decreases by increasing the number of iterations (N), see Tables 2–4.
Future Work: Future work will attempt to solve Eq (1) with delay time.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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