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Research article

Congruences involving generalized Catalan numbers and Bernoulli numbers

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Abstract: In this paper, we establish some congruences mod p^3 involving the sums $\sum_{k=1}^{p-1} k^m B_{p,k}^{2l}$, where $p > 3$ is a prime number and $B_{p,k}$ are generalized Catalan numbers. We also establish some congruences mod p^2 involving the sums $\sum_{k=1}^{p-1} k^m B_{p,k}^{2l_1} B_{p,k-d}^{2l_2}$, where m, l_1, l_2, d are positive integers and $1 \le d \le p-1$.

Keywords: Bernoulli numbers; congruences; generalized Catalan numbers; generalized harmonic numbers
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1. Introduction

In combinatorics,

$$C_k = \frac{1}{k+1}\binom{2k}{k} = \binom{2k}{k} - \binom{2k}{k+1}, \ with\ k \in \mathbb{N},$$

are the well-known Catalan numbers. The meaning of Catalan numbers are the numbers of ways to divide the $(n+2)$-polygon in n triangles. For any positive integer n, the generalized Catalan numbers $B_{n,k}$ are defined (cf. [10, 15]) by

$$B_{n,k} = \frac{k}{n}\binom{2n}{n-k}, \ 0 \le k \le n.$$

In [15], L. W. Shapiro shows that the meaning of the generalized Catalan numbers $B_{n,k}$ are the number of pairs of non-intersecting paths of length n and distance k. For $1 \le k \le n$, we list the first values of generalized Catalan numbers in the following table:

$n \backslash k$	1	2	3	4	5	6
1	1	0	0	0	0	0
2	2	1	0	0	0	0
3	5	4	1	0	0	0
4	14	14	6	1	0	0
5	42	48	27	8	1	0
6	132	165	110	44	10	1

The generalized Catalan numbers satisfy the recurrence relation

$$B_{n,k} = B_{n-1,k-1} + 2B_{n-1,k} + B_{n-1,k+1},\ k \geq 2,\ n \geq 2, \tag{1.1}$$

with the initial conditions $B_{n,0} = B_{n,m} = 0,\ m > n$. When $k = 1$, we have $B_{n,1} = C_n$ for $n \geq 1$.

Now we consider the generating function of the generalized Catalan numbers. Let

$$g(x, y) = \sum_{n=1}^{+\infty} \sum_{k=1}^{n} B_{n,k} \frac{x^k}{k!} y^n = \sum_{n=1}^{+\infty} \sum_{k=1}^{n} \frac{k}{n} \binom{2n}{n-k} \frac{x^k}{k!} y^n.$$

Exchanging the order of summation, we can get

$$g(x, y) = \sum_{k=1}^{+\infty} \frac{x^k}{k!} \sum_{n=k}^{+\infty} \frac{k}{n} \binom{2n}{n-k} y^n = \sum_{k=1}^{+\infty} \frac{x^k y^k}{k!} \sum_{n=0}^{+\infty} \frac{2k}{2(n+k)} \binom{2n+2k}{n} y^n. \tag{1.2}$$

In view of [4, (1.121)], we have

$$\sum_{n=0}^{+\infty} \frac{2k}{2(n+k)} \binom{2n+2k}{n} y^n = z^{2k},\ \textit{where } y = \frac{z-1}{z^2} \textit{ and } |y| < \frac{1}{4}. \tag{1.3}$$

Combining (1.2) and (1.3) yields that

$$g(x, y) = e^{xyz^2} - 1 = e^{x(z-1)} - 1,\ \textit{where } y = \frac{z-1}{z^2} \textit{ and } |y| < \frac{1}{4}.$$

Remark. Taking $k = \frac{1}{2}$ in (1.3), we have

$$z = \sum_{n=0}^{+\infty} \frac{1}{2n+1} \binom{2n+1}{n} y^n = \sum_{n=0}^{+\infty} \frac{1}{n+1} \binom{2n}{n} y^n,$$

which implies that z is the generating function of the Catalan numbers C_n.

There are various identities and congruences involving Catalan numbers (cf. [5, 6, 11]). Differential equations and generating function are often used to manage combinatorial identities involving Catalan numbers (cf. [8, 9]). However, there are few identities involving the numbers $B_{n,k}$. Several applications of $B_{n,k}$ appeared in [1, 6, 15]. Koparal and Ömür [2, 10, 14] studied the congruences involving $B_{p,k}$, where p is prime.

The numbers $B_{p,k}$ are closely related to generalized harmonic numbers under congruence relation. For $\alpha \in \mathbb{N}$, the generalized harmonic numbers are defined by

$$H_0^{(\alpha)} = 0 \quad \text{and} \quad H_n^{(\alpha)} = \sum_{i=1}^{n} \frac{1}{i^{\alpha}}, \quad \text{for } n \in \mathbb{Z}^+.$$

By the well-known Wolstenholme theorem [20], we have that if $p > 3$ is a prime, then

$$H_{p-1} \equiv 0 \pmod{p^2} \quad \textit{and} \quad H_{p-1}^{(2)} \equiv 0 \pmod{p}. \tag{1.4}$$

For $m \in \{-2, -1, 0, 1, 2, 3\}$ and $n \in \{1, 2, 3\}$, Z.-W. Sun [16] established a kind of congruences mod p involving the sums $\sum_{k=1}^{p-1} k^m H_k^n$. Y. Wang [18, 19] generalized some of these congruences to mod p^2 type. In [16], Z. W. Sun also made two conjectures on supercongruences of Euler-type. These conjectures were conformed in [13, 17], respectively.

In this paper, we focus on the properties of $B_{n,k}$. With the use of the congruences involving harmonic numbers, we establish several congruences mod p^3 involving the sums $\sum_{k=1}^{p-1} k^m B_{p,k}^{2l}$ and mod p^2 involving the sums $\sum_{k=1}^{p-1} k^m B_{p,k}^{2l_1} B_{p,k-d}^{2l_2}$.

Our main results are as follows.

Theorem 1.1. *Let $p > 3$ be a prime and m, l be two positive integers such that $3 \le m < p - 1$. Then*

$$4^{-l} \sum_{k=1}^{p-1} k^m B_{p,k}^{2l} \equiv \begin{cases} (\frac{1}{2} - l) m p^2 B_{m-1} \pmod{p^3}, & if\ 2 \nmid m, \\ (1 - 4l) p B_m + \frac{4l(1-4l)}{m+1} p^2 \sum_{r=0}^{m} \binom{m+1}{r} B_r B_{m-r} \pmod{p^3}, & if\ 2 \mid m, \end{cases}$$

where $B_0, B_1, B_2, \ldots$ are the Bernoulli numbers defined by

$$B_0 = 1 \ \text{ and } \ \sum_{k=0}^{n} \binom{n+1}{k} B_k = 0 \ (n = 1, 2, 3, \ldots).$$

Corollary 1.2. *Let $p > 3$ be a prime and m be an integer such that $3 < m < p - 1$. Then*

$$\sum_{k=1}^{p-1} k^m B_{p,k}^{2l} \equiv \frac{1}{2} \left((-1)^m + 1 \right) (1 - 4l) 4^l p B_m \pmod{p^2}.$$

Corollary 1.3. *Let $p > 3$ be a prime and m be an even integer such that $3 < m < p - 1$. If $p \mid l$ or $p \mid (4l - 1)$ or $m = p - 3$, then*

$$\sum_{k=1}^{p-1} k^m B_{p,k}^{2l} \equiv (1 - 4l) 4^l p B_m \pmod{p^3}.$$

Example 1.4. *Let $p > 5$ be a prime and m be an even integer such that $3 < m < p - 1$. If $4 \mid (\lambda p + 1)$, then*

$$\sum_{k=1}^{p-1} k^m B_{p,k}^{\frac{\lambda p+1}{2}} \equiv (-1)^{\frac{p^2+7}{8}} \lambda 2^{\frac{\lambda+1}{2}} p^2 B_m \pmod{p^3}.$$

In particular, for $\lambda = 3p$, we have

$$\sum_{k=1}^{p-1} k^m B_{p,k}^{\frac{3p^2+1}{2}} \equiv 0 \pmod{p^3}.$$

For a fixed positive integer m, we can use Theorem 1.1 to calculate the corresponding congruence. When m is related to p, in general, we can not give a closed form. With the use of the known congruence, we give the following corollary.

Corollary 1.5. *Let $p > 7$ be a prime and l be a positive integer. Then*

$$4^{-l}\sum_{k=1}^{p-1} k^{p-5}B_{p,k}^{2l} \equiv (1-4l)pB_{p-5} + \frac{2}{3}l(1-4l)p^2B_{p-3}^2 \pmod{p^3}.$$

Now, we extend the definition of the generalized Catalan numbers by setting

$$B_{n,k} = \frac{k}{n}\binom{2n}{n-k},\ -n \leq k \leq n.$$

From this, we see that

$$B_{n,-k} = \frac{-k}{n}\binom{2n}{n+k} = \frac{-k}{n}\binom{2n}{n-k} = -B_{n,k}. \tag{1.5}$$

In this case, the generalized Catalan numbers satisfy the recurrence relation

$$B_{n,k} = B_{n-1,k-1} + 2B_{n-1,k} + B_{n-1,k+1},\ |k| \geq 2,\ n \geq 2,$$

with the initial conditions $B_{n,0} = B_{n,m} = 0,\ |m| > n$.

Theorem 1.6. *Let $p > 3$ be a prime and m, l be two positive integers such that $1 \leq m < p-3$. Then*

$$4^{-l}\sum_{k=1}^{p-1}\frac{B_{p,k}^{2l}}{k^m} \equiv \begin{cases} \frac{(2l-1)m^2+(2l-1)m-4l}{2(m+2)}p^2B_{p-2-m} \pmod{p^3}, & if\ 2 \nmid m, \\ (\frac{m}{m+1} - 4l)pB_{p-1-m} \pmod{p^2}, & if\ 2 \mid m. \end{cases}$$

Corollary 1.7. *Let $p > 3$ be a prime and m, l be two positive integers such that $m \in \{1, 3, \ldots, p-4\}$. If $l \equiv \frac{m(m+1)}{2(m-1)(m+2)} \pmod{p}$, then*

$$\sum_{k=1}^{p-1}\frac{B_{p,k}^{2l}}{k^m} \equiv 0 \pmod{p^3}.$$

Theorem 1.8. *Let $p > 5$ be a prime and m, l_1, l_2, d be positive integers such that d is less than $p-1$ and $2 \leq m < p-3$. Then*

$$\begin{aligned} 4^{-l_1-l_2}\sum_{k=1}^{p-1} k^m B_{p,k}^{2l_1}B_{p,k-d}^{2l_2} \equiv &- d^m + (1-4l_1)pB_m + 2l_1d^m(H_d + H_{d-1})p \\ &- 2l_2(md^{m-1} + 2B_m(d))p \pmod{p^2}, \end{aligned}$$

where $B_n(x) = \sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k}$ $(n = 0, 1, 2, \ldots)$ are the Bernoulli polynomials.

Corollary 1.9. *Let $p > 5$ be a prime and m, l_1, l_2, d be positive integers such that d is less than $p-1$ and $2 \leq m < p-3$. Then*

$$4^{-l_1-l_2}\sum_{k=1}^{p-1} k^m B_{p,k}^{2l_1p} B_{p,k-d}^{2l_2} \equiv -d^m + p(B_m - 2l_2md^{m-1} - 4l_2B_m(d)) \pmod{p^2}.$$

Corollary 1.10. *Let $p > 5$ be a prime and $m > 1$ be an odd integer and d be a positive integer less than $p-1$. Then*

$$4^{-1-m}\sum_{k=1}^{p-1} k^m B_{p,k}^{2m} B_{p,k-d}^{2} \equiv -d^m + p(4md^mH_{d-1} - 4B_m(d)) \pmod{p^2}.$$

In particular, for $d = 1$, we have

$$\sum_{k=1}^{p-1} k^m B_{p,k}^{2m} B_{p,k-1}^{2} \equiv -4^{m+1} \pmod{p^2}.$$

Corollary 1.11. *Let $p > 5$ be a prime and m, l_1, l_2, d be positive integers such that d is less than $p-1$ and $2 \leq m < p-3$. Then*

$$\sum_{k=1}^{p-1} k^m B_{p,k}^{2l_1p} B_{p,k-d}^{2l_2p} \equiv 4^{(l_1+l_2)p}(-d^m + pB_m) \pmod{p^2}.$$

In particular, for $l_2 = p-1-l_1$, we have

$$\sum_{\substack{k=1\\k\neq d}}^{p-1} k^m \left(\frac{B_{p,k}}{B_{p,k-d}}\right)^{2l_1p} \equiv -d^m + pB_m \pmod{p^2}.$$

In the next section, we provide some lemmas. In section 3, we show the proof of the main results.

2. Preliminaries

In this section, we first state some basic facts which will be used very often.

Lemma 2.1. *Let $p > 3$ be a prime. If $m \in \{1, 2, \dots, p-2\}$, then*

$$H_{p-1}^{(m)} \equiv \frac{m}{m+1}pB_{p-1-m} \pmod{p^2}. \tag{2.1}$$

In particular, for $m \in \{1, 3, 5, \dots, p-4\}$, we have

$$H_{p-1}^{(m)} \equiv \frac{m(m+1)}{2}\frac{B_{p-2-m}}{p-2-m}p^2 \pmod{p^3}. \tag{2.2}$$

Proof. These two congruences are due to J. W. L. [3]. □

Lemma 2.2. *Let $p > 3$ be a prime and k be an integer such that $1 \leq k \leq p-1$. Then*

$$B_{p,k}^2 \equiv 4 + 8p\left(\frac{1}{k} - 2H_k\right) + 4p^2\left(\frac{3}{k^2} - \frac{8}{k}H_k + 8H_k^2\right) \pmod{p^3}. \tag{2.3}$$

Proof. According to the definition of generalized Catalan numbers, it follows that

$$B_{p,k} = \frac{k}{p}\binom{2p}{p-k} = \frac{2k}{p-k}\binom{2p-1}{p-k-1}.$$

Observe that

$$\binom{2p-1}{p-k-1} = \frac{(2p-1)_{p-k-1}}{(p-k-1)!} = (-1)^{p-k-1}\prod_{i=1}^{p-k-1}\left(1 - \frac{2p}{i}\right).$$

From this it is not difficult to deduce that

$$\binom{2p-1}{p-k-1} \equiv (-1)^k\left(1 - 2pH_{p-k-1} + 2p^2(H_{p-k-1}^2 - H_{p-k-1}^{(2)})\right) \pmod{p^3}. \tag{2.4}$$

By the definition of generalized harmonic numbers, we have

$$H_{p-k}^{(n)} = H_{p-1}^{(n)} - \sum_{i=1}^{k-1}\frac{1}{(p-i)^n} \equiv H_{p-1}^{(n)} - (-1)^n\left(H_{k-1}^{(n)} + npH_{k-1}^{(n+1)}\right) \pmod{p^2}.$$

It follows from (2.1) that

$$H_{p-k}^{(n)} \equiv \frac{n}{n+1}pB_{p-1-n} - (-1)^n\left(H_{k-1}^{(n)} + npH_{k-1}^{(n+1)}\right) \pmod{p^2}. \tag{2.5}$$

Combining (2.4) and (2.5) gives that

$$\binom{2p-1}{p-k-1} \equiv (-1)^k\left(1 - 2pH_k + 2p^2H_k^2\right) \pmod{p^3}. \tag{2.6}$$

Hence

$$B_{p,k}^2 \equiv \left(1 - 2pH_k + 2p^2H_k^2\right)^2\frac{4k^2}{(p-k)^2} \pmod{p^3}. \tag{2.7}$$

A simple calculation gives (2.3). □

Lemma 2.3. *Let $p > 3$ be a prime and m be a non-negative integer less than $p-1$. Then*

$$\sum_{k=0}^{p-1}k^mH_k \equiv B_m - \frac{p}{m+1}\sum_{r=0}^{m}\binom{m+1}{r}B_rB_{m-r} \pmod{p^2}. \tag{2.8}$$

For $m \geq 3$, we have

$$\sum_{k=0}^{p-1}k^mH_k^2 \equiv \begin{cases} B_{m-1} \pmod{p}, & if\ 2 \nmid m, \\ \frac{-2}{m+1}\sum_{r=0}^{m}\binom{m+1}{r}B_rB_{m-r} \pmod{p}, & if\ 2 \mid m. \end{cases} \tag{2.9}$$

Proof. Sun [16, (2.3)] showed that $\sum_{k=1}^{p-1} H_k \equiv 1-p \pmod{p^3}$ and Wang [18, Theorem3.1] proved the rest cases of (2.8). The congruence (2.9) is the special case of the result of Wang [18, Theorem3.2]. □

Lemma 2.4. *Let $p>5$ be a prime. Then*

$$\sum_{i=0}^{p-3}\binom{p-2}{r}B_iB_{p-3-r}\equiv 0 \pmod{p}. \tag{2.10}$$

For $p>7$, we have

$$\sum_{r=0}^{p-5}\binom{p-4}{r}B_rB_{p-5-r}\equiv -\frac{2}{3}B_{p-3}^2 \pmod{p}. \tag{2.11}$$

Proof. For any integer m and r, we have the congruence

$$\binom{p-m}{r}\equiv(-1)^r\binom{r+m-1}{m-1} \pmod{p}.$$

Suppose m is an even integer. Then we have $(-1)^rB_rB_{p-1-m-r}=B_rB_{p-1-m-r}$ for $p>m+3$. Therefore,

$$\sum_{r=0}^{p-1-m}\binom{p-m}{r}B_rB_{p-1-m-r}\equiv\sum_{r=0}^{p-1-m}\binom{r+m-1}{m-1}B_rB_{p-1-m-r} \pmod{p}. \tag{2.12}$$

Taking $m=2$ in (2.12), we have

$$\sum_{r=0}^{p-3}\binom{p-2}{r}B_rB_{p-3-r}\equiv\sum_{r=0}^{p-3}(r+1)B_rB_{p-3-r} \pmod{p}. \tag{2.13}$$

Observe that

$$\sum_{r=0}^{p-3}rB_rB_{p-3-r}=\sum_{r=0}^{p-3}(p-3-r)B_rB_{p-3-r}\equiv-\sum_{r=0}^{p-3}(r+3)B_rB_{p-3-r} \pmod{p},$$

which implies that

$$\sum_{r=0}^{p-3}rB_rB_{p-3-r}\equiv-\frac{3}{2}\sum_{r=0}^{p-3}B_rB_{p-3-r} \pmod{p}. \tag{2.14}$$

Zhao [21, (3.19)] showed that

$$\sum_{i=0}^{p-3}B_iB_{p-3-i}\equiv 0 \pmod{p}. \tag{2.15}$$

Substituting (2.14) and (2.15) into (2.13) gives (2.10).

Taking $m=4$ in (2.12), we have

$$\sum_{r=0}^{p-5}\binom{p-4}{r}B_rB_{p-5-r}\equiv\sum_{r=0}^{p-5}\binom{r+3}{3}B_rB_{p-5-r} \pmod{p}. \tag{2.16}$$

Matiyasevich [12] proved that for an even integer $n \geq 4$, we have

$$(n+2)\sum_{i=2}^{n-2} B_i B_{n-i} - 2\sum_{i=2}^{n-2}\binom{n+2}{i} B_i B_{n-i} = n(n+1)B_n.$$

Taking $n = p-5$ in the above identity, we can obtain

$$-\sum_{i=2}^{p-7} i^2 B_i B_{p-5-i} - \sum_{i=2}^{p-7} i B_i B_{p-5-i} \equiv 20B_{p-5} \pmod{p}. \tag{2.17}$$

Zhao [21, Proposition 3.13] proved that

$$\sum_{r=0}^{p-5} B_r B_{p-5-r} \equiv -\frac{2}{3}B_{p-3}^2 \pmod{p}. \tag{2.18}$$

For any positive integer i, we have

$$\sum_{r=0}^{p-5} r^i B_r B_{p-5-r} \equiv (-1)^i \sum_{r=0}^{p-5} (r+5)^i B_r B_{p-5-r} \pmod{p}. \tag{2.19}$$

Combining (2.16) through (2.19) gives (2.11). □

Lemma 2.5. *Let p be an odd prime. Suppose s and t are two positive integers of same parity such that $p > s+t+1$. Then*

$$\sum_{1\leq i\leq j\leq p-1} \frac{1}{i^s j^t} \equiv p\left[(-1)^s t\binom{s+t+1}{s} - (-1)^s s\binom{s+t+1}{t} + s + t\right] \times \frac{B_{p-s-t-1}}{2(s+t+1)} \pmod{p^2}. \tag{2.20}$$

Proof. The congruence (2.20) is proved by Zhao [21, Theorem 3.2]. □

3. Proof of the main results

Proof of Theorem 1.1. It follows from (2.3) that

$$B_{p,k}^{2l} \equiv 4^l\left(1 + 2p\left(\frac{1}{k} - 2H_k\right) + p^2\left(\frac{3}{k^2} - \frac{8}{k}H_k + 8H_k^2\right)\right)^l \pmod{p^3},$$

which implies that

$$B_{p,k}^{2l} \equiv 4^l\left(1 + 2pl\left(\frac{1}{k} - 2H_k\right) + 2p^2l^2\left(\frac{1}{k^2} - \frac{4}{k}H_k + 4H_k^2\right) + \frac{p^2 l}{k^2}\right) \pmod{p^3}. \tag{3.1}$$

Hence

$$4^{-l}\sum_{k=1}^{p-1} k^m B_{p,k}^{2l} \equiv \sum_{k=1}^{p-1} k^m + 2pl\sum_{k=1}^{p-1} k^{m-1} - 4pl\sum_{k=1}^{p-1} k^m H_k + 2p^2l^2\sum_{k=1}^{p-1} k^{m-2}$$

$$-8p^2l^2\sum_{k=1}^{p-1}k^{m-1}H_k+8p^2l^2\sum_{k=1}^{p-1}k^mH_k^2+p^2l\sum_{k=1}^{p-1}k^{m-2}\pmod{p^3}. \tag{3.2}$$

It is well-known that for $m, p \in \mathbb{Z}^+$,

$$\sum_{k=1}^{p-1}k^m=\frac{1}{m+1}\sum_{r=0}^{m}\binom{m+1}{r}B_rp^{m+1-r}. \tag{3.3}$$

(cf. [7, pp.230–238].) Hence for $3 \le m < p-1$, the congruence (3.2) reduces to

$$\begin{aligned}4^{-l}\sum_{k=1}^{p-1}k^mB_{p,k}^{2l}\equiv&pB_m+\left(\frac{m}{2}+2l\right)p^2B_{m-1}-4pl\sum_{k=1}^{p-1}k^mH_k\\&-8p^2l^2\sum_{k=1}^{p-1}k^{m-1}H_k+8p^2l^2\sum_{k=1}^{p-1}k^mH_k^2\pmod{p^3}.\end{aligned} \tag{3.4}$$

Suppose m is an odd integer. It follows from (2.8) and (2.9) that

$$\sum_{k=1}^{p-1}k^{m-1}H_k\equiv\sum_{k=1}^{p-1}k^mH_k^2\equiv B_{m-1}\pmod{p}. \tag{3.5}$$

For the odd integer m, the integers r and $m-r$ are of different parity. Recall that $B_1=-1/2$ and $B_i=0$ if $i \ge 3$ and $2 \nmid i$. Therefore,

$$\frac{1}{m+1}\sum_{r=0}^{m}\binom{m+1}{r}B_rB_{m-r}=-\frac{m+2}{4}B_{m-1}. \tag{3.6}$$

Together with (2.8) and (3.6), we obtain

$$\sum_{k=1}^{p-1}k^mH_k\equiv\frac{m+2}{4}pB_{m-1}\pmod{p^2}. \tag{3.7}$$

Substituting (3.5) and (3.7) into (3.4) gives the first congruence of Theorem 1.1.

If m is an even integer such that $4 \le m < p-1$, then $\sum_{k=1}^{p-1}k^{m-1}H_k\equiv 0 \pmod{p}$. Substituting (2.8) and the second congruence of (2.9) into (3.4) gives the second congruence of Theorem 1.1. The proof of Theorem 1.1 is completed. □

Proof of Corollary 1.2. It follows from Theorem 1.1. □

Proof of Corollary 1.3. Case $p \mid l$ or $p \mid 4l-1$ is obvious by the second congruence of Theorem 1.1. Case $m = p-3$ follows from (2.10). □

Proof of Corollary 1.5. Combining (2.11) and the second congruence of Theorem 1.1 gives the desired congruence. □

Proof of Theorem 1.6. In view of (3.1), we have

$$4^{-l}\sum_{k=1}^{p-1}\frac{B_{p,k}^{2l}}{k^m} \equiv H_{p-1}^{(m)} + 2plH_{p-1}^{(m+1)} - 4pl\sum_{k=1}^{p-1}\frac{H_k}{k^m} + 2p^2l^2H_{p-1}^{(m+2)}$$
$$- 8p^2l^2\sum_{k=1}^{p-1}\frac{H_k}{k^{m+1}} + 8p^2l^2\sum_{k=1}^{p-1}\frac{H_k^2}{k^m} + p^2lH_{p-1}^{(m+2)} \pmod{p^3}. \tag{3.8}$$

Suppose m is an odd integer. It follows from (2.1) that $H_{p-1}^{(m+2)} \equiv 0 \pmod p$ and

$$H_{p-1}^{(m+1)} \equiv \frac{m+1}{m+2}pB_{p-2-m} \pmod{p^2}. \tag{3.9}$$

By Fermat's little theorem, we have $\sum_{k=1}^{p-1} H_k/k^{m+1} \equiv \sum_{k=1}^{p-1} k^{p-2-m}H_k \pmod p$. In view of (2.8), we have $\sum_{k=1}^{p-1} H_k/k^{m+1} \equiv B_{p-2-m} \pmod p$. Similarly, we can obtain $\sum_{k=1}^{p-1} H_k/k^m \equiv B_{p-1-m} \equiv 0 \pmod p$, since m is odd and $1 \le m \le p-4$. Also, we can get $\sum_{k=1}^{p-1} H_k^2/k^m \equiv B_{p-2-m} \pmod p$ by (2.9). Combining the above and (2.2), we obtain

$$4^{-l}\sum_{k=1}^{p-1}\frac{B_{p,k}^{2l}}{k^m} \equiv \frac{(m+1)(4l-m)}{2(m+2)}p^2B_{p-2-m} - 4pl\sum_{k=1}^{p-1}\frac{H_k}{k^m} \pmod{p^3}. \tag{3.10}$$

Observe that

$$\sum_{1\le i\le j\le p-1}\frac{1}{ij^m} = \sum_{k=1}^{p-1}\frac{H_k}{k^m}.$$

Hence taking $s = 1, t = m$ in (2.20), we have

$$\sum_{k=1}^{p-1}\frac{H_k}{k^m} \equiv \frac{-m^2+m+4}{4(m+2)}pB_{p-2-m} \pmod{p^2}. \tag{3.11}$$

Substituting (3.11) into (3.10), we get the first congruence.

Now we assume that m is an even integer. With the use of (2.8), we have $\sum_{k=1}^{p-1} H_k/k^m \equiv B_{p-1-m} \pmod p$. Combining (2.1) and (3.8), we obtain the second congruence. □

Proof of Corollary 1.7. It follows from Theorem 1.6. □

Proof of Theorem 1.8. According to the definition of the generalized Catalan numbers, it is easy to see that $B_{p,0} = 0$ and

$$\sum_{k=1}^{p-1}k^mB_{p,k}^{2l_1}B_{p,k-d}^{2l_2} = \sum_{k=1}^{d-1}k^mB_{p,k}^{2l_1}B_{p,k-d}^{2l_2} + \sum_{k=d+1}^{p-1}k^mB_{p,k}^{2l_1}B_{p,k-d}^{2l_2}. \tag{3.12}$$

With the use of (1.5) and (2.3), we can obtain the following congruences. If $k > d$, then

$$B_{p,k}^{2l_1}B_{p,k-d}^{2l_2} \equiv 4^{l_1+l_2}\left(1 + \frac{2l_1p}{k} - 4pl_1H_k + \frac{2l_2p}{k-d} - 4pl_2H_{k-d}\right) \pmod{p^2}. \tag{3.13}$$

If $k < d$, then

$$B_{p,k}^{2l_1} B_{p,k-d}^{2l_2} \equiv 4^{l_1+l_2} \left(1 + \frac{2l_1 p}{k} - 4pl_1 H_k + \frac{2l_2 p}{d-k} - 4pl_2 H_{d-k}\right) \pmod{p^2}. \tag{3.14}$$

Combining (3.12) through (3.14), it follows that

$$\begin{aligned}\sum_{k=1}^{p-1} k^m B_{p,k}^{2l_1} B_{p,k-d}^{2l_2} \equiv & 4^{l_1+l_2} \left(\sum_{k=1}^{d-1} k^m \left(1 + \frac{2l_1 p}{k} - 4pl_1 H_k + \frac{2l_2 p}{d-k} - 4pl_2 H_{d-k}\right)\right. \\ & \left. + \sum_{k=d+1}^{p-1} k^m \left(1 + \frac{2l_1 p}{k} - 4pl_1 H_k + \frac{2l_2 p}{k-d} - 4pl_2 H_{k-d}\right)\right) \pmod{p^2}.\end{aligned}$$

The above congruence can be written as

$$\begin{aligned}4^{-l_1-l_2} \sum_{k=1}^{p-1} k^m B_{p,k}^{2l_1} B_{p,k-d}^{2l_2} \equiv & \sum_{k=1}^{p-1} k^m \left(1 + \frac{2l_1 p}{k} - 4pl_1 H_k\right) + 2l_2 p(\Sigma_1 + \Sigma_2) \\ & - d^m \left(1 + \frac{2l_1 p}{d} - 4pl_1 H_d\right) \pmod{p^2},\end{aligned} \tag{3.15}$$

where $\Sigma_1 = \sum_{k=d+1}^{p-1} k^m \left(\frac{1}{k-d} - 2H_{k-d}\right)$ and $\Sigma_2 = \sum_{k=1}^{d-1} k^m \left(\frac{1}{d-k} - 2H_{d-k}\right)$. Observe that

$$\Sigma_1 = \sum_{k=1}^{p-1} (k+d)^m \left(\frac{1}{k} - 2H_k\right) - \sum_{k=0}^{d-1} (p+k)^m \left(\frac{1}{p-d+k} - 2H_{p-d+k}\right).$$

Recall that $H_{p-d+k} \equiv H_{d-k-1} \pmod{p}$ for $k < d$. Then

$$\Sigma_1 \equiv \sum_{k=1}^{p-1} (k+d)^m \left(\frac{1}{k} - 2H_k\right) - \sum_{k=1}^{d-1} k^m \left(\frac{1}{k-d} - 2H_{d-k-1}\right) \pmod{p},$$

which implies that

$$\Sigma_1 + \Sigma_2 \equiv \sum_{k=1}^{p-1} (k+d)^m \left(\frac{1}{k} - 2H_k\right) \pmod{p}.$$

With the use of the binomial theorem, we have

$$\Sigma_1 + \Sigma_2 \equiv \sum_{k=1}^{p-1} \sum_{r=0}^{m} \binom{m}{r} k^r d^{m-r} \left(\frac{1}{k} - 2H_k\right) \pmod{p}.$$

Exchanging the summation order gives that

$$\Sigma_1 + \Sigma_2 = \sum_{r=0}^{m} \binom{m}{r} d^{m-r} \sum_{k=1}^{p-1} k^{r-1} - 2 \sum_{r=0}^{m} \binom{m}{r} d^{m-r} \sum_{k=1}^{p-1} k^r H_k \pmod{p}. \tag{3.16}$$

For $m \in \mathbb{Z}$, there holds

$$\sum_{k=1}^{p-1} k^m \equiv \begin{cases} -1 \pmod{p}, & if\ p-1 \mid m, \\ 0 \pmod{p}, & if\ p-1 \nmid m. \end{cases}$$

(cf. [7, pp.235].) Hence

$$\sum_{r=0}^{m} \binom{m}{r} d^{m-r} \sum_{k=1}^{p-1} k^{r-1} \equiv -md^{m-1} \pmod{p}. \tag{3.17}$$

In view of (2.8) and the definition of Bernoulli polynomials, we have

$$\sum_{r=0}^{m} \binom{m}{r} d^{m-r} \sum_{k=1}^{p-1} k^r H_k \equiv \sum_{r=0}^{m} \binom{m}{r} d^{m-r} B_r = B_m(d) \pmod{p}. \tag{3.18}$$

Writing (3.17) and (3.18) into (3.16), we obtain

$$\Sigma_1 + \Sigma_2 \equiv -md^{m-1} - 2B_m(d) \pmod{p}. \tag{3.19}$$

In view of (3.3), we have $\sum_{k=1}^{p-1} k^m \equiv pB_m \pmod{p^2}$ and $\sum_{k=1}^{p-1} k^{m-1} \equiv 0 \pmod{p}$ for $m > 2$. Hence with the help of (2.8), we have

$$\sum_{k=1}^{p-1} k^m \left(1 + \frac{2l_1 p}{k} - 4l_1 pH_k\right) \equiv (1 - 4l_1)pB_m \pmod{p^2}. \tag{3.20}$$

Substituting (3.19) and (3.20) into (3.15), we complete the proof of the theorem. □

Proof of Corollary 1.9 and 1.10. These congruences can be obtained directly from Theorem 1.8. □

Proof of Corollary 1.11. Replacing l_1, l_2 by $l_1 p, l_2 p$ in Theorem 1.8 respectively, we get the first congruence. Then taking $l_2 = p - 1 - l_1$ and with the use of Euler theorem, we obtain the second one. □

4. Conclusions

Koparal and Ömür [10] established a kind of mod p^2 congruences involving the sum $\sum_{k=1}^{p-1} k^2 B_{p,k} B_{p,k-d}$ and $\sum_{k=1}^{p-1} \frac{B_{p,k} B_{p,k-d}}{k}$, where $B_{p,k}$ is generalized Catalan numbers and the power of $B_{p,k}$ is odd. This article studies the congruences involving generalized Catalan numbers with even powers and establishes a kind of mod p^3 congruences involving the sum $\sum_{k=1}^{p-1} k^m B_{p,k}^{2l}$ and mod p^2 congruences involving the sum $\sum_{k=1}^{p-1} k^m B_{p,k}^{2l_1} B_{p,k-d}^{2l_2}$, where m is an integer and l, l_1, l_2 are positive integers.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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