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Research article

Partial domination of network modelling

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Abstract: Partial domination was proposed in 2017 on the basis of domination theory, which has good practical application background in communication network. Let $G = (V, E)$ be a graph and $\mathcal{F}$ be a family of graphs, a subset $S \subseteq V$ is called an $\mathcal{F}$-isolating set of G if $G[V \backslash N_G[S]]$ does not contain F as a subgraph for all $F \in \mathcal{F}$. The subset S is called an isolating set of G if $\mathcal{F} = \{K_2\}$ and $G[V \backslash N_G[S]]$ does not contain K_2 as a subgraph. The isolation number of G is the minimum cardinality of an isolating set of G, denoted by $\iota(G)$. The hypercube network and n-star network are the basic models for network systems, and many more complex network structures can be built from them. In this study, we obtain the sharp bounds of the isolation numbers of the hypercube network and n-star network.

Keywords: partial domination; isolation number; hypercube network; n-star network
Mathematics Subject Classification: 05C07, 05C69

1. Introduction

In this paper, all graphs considered are non-empty, finite, undirected and simple. Additionally, for standard graph theory terminology not given here, we refer to [1]. Let G be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$, and $|V(G)| = n$, $|E(G)| = m$. For any $v \in V(G)$, the *degree* $d_G(v)$ of v is the number of neighbors of v. The *minimum* and *maximum degree* of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The *open neighborhood* $N_G(v)$ of v is the set $N(v) = \{u \in V | uv \in E\}$ and the *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. For any $S \subseteq V(G)$, the *open neighborhood* of S is the set $N_G(S) = \bigcup_{v\in S} N_G(v)$ and the *closed neighborhood* of S is the set $N_G[S] = N_G(S) \cup S$. Furthermore, we define $N_S(v) = N_G(v) \cap S$ and $N_S[v] = N_S(v) \cup \{v\}$. The subgraph of G induced by S is denoted by $G[S]$. For a graph H, we say that G is H-*free* if G does not contain H as a subgraph. The *cycle* and *clique* on n vertices are denoted as C_n and K_n. Abbreviate $\{1, 2, \cdots, n\}$ to $[n]$ and say "i" is a symbol of $[n]$, where $n \in \mathbb{N}^*$ and $i \in [n]$.

In recent years, with the rapid development of information technology, the domination theory has been widely used in computer technology, cryptography, social network, communication network and many other subjects. In 1958, Claude Berge [2] first proposed the concept of the domination number. A vertex subset $S \subseteq V(G)$ is a *dominating set* of G if $N_G[S] = V(G)$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G, i.e $\gamma(G) = min\{|S| : S\ is\ a\ dominating\ set\ of\ G\}$.

In 2017, Caro and Hansberg [3] extended the domination to the partial domination, and proposed the concept of an $\mathcal{F}$-isolating set of a graph G for the first time. Let $G = (V, E)$ be a graph and $\mathcal{F}$ be a family of graphs.

Definition 1.1. [3] *A subset $S \subseteq V$ is called an $\mathcal{F}$-isolating set of G if $G[V \backslash N_G[S]]$ does not contain F as a subgraph for all $F \in \mathcal{F}$. The $\mathcal{F}$-isolation number of G is the minimum cardinality of an $\mathcal{F}$-isolating set of G, denoted by $\iota(G, \mathcal{F})$.*

In particalur, if $\mathcal{F} = \{K_k\}$, the set S is called a $\{K_k\}$-*isolating set* of G if $G[V \backslash N_G[S]]$ does not contain K_k as a subgraph, and the $\{K_k\}$-*isolation number* of G is the minimum cardinality of a $\{K_k\}$-isolating set of G, denoted by $\iota(G, k)$. For any positive integer k, if $\mathcal{F} = \{K_{1,k+1}\}$, the set S is called a *k-isolating set* of G if $G[V \backslash N_G[S]]$ does not contain $K_{1,k+1}$ as a subgraph, and the *k-isolation number* of G is the minimum cardinality of a k-isolating set of G, denoted by $\iota_k(G)$. Especially, when $k = 0$, the set S is called an *isolating set* of G, and the minimum cardinality of an isolating set of G is the *isolation number* of G, denoted by $\iota(G)$.

With respect to this problem, Borg et al. [4] studied the $\iota(G, k)$ of a connected graph G is at most $\frac{n}{k+1}$ unless $G \cong K_k$, or $k = 2$ and $G \cong C_5$. Caro and Hansberg [3] investigated that $\iota(G) \leq \frac{n}{3}$ for $n \geq 6$, $\iota_k(T) \leq \frac{n}{k+3}$ of a tree T which is different from $K_{1,k+1}$, $\iota_k(G) \leq \frac{n}{4}$ of a maximal outerplanar graph G with $n \geq 4$ and so on. Tokunaga et al. [5] studied that if G is a maximal outerplanar graph of order n with n_2 vertices of degree 2, then $\iota(G) \leq \frac{n+n_2}{5}$ when $n_2 \leq \frac{n}{4}$, and $\iota(G) \leq \frac{n-n_2}{3}$ otherwise. Borg and Kaemawichanurat [6] showed that if G is a maximal outerplanar graph with $n \geq 5$, then $\iota_1(G) \leq \frac{n}{5}$, they also showed that $\iota_1(G) \leq \frac{n+n_2}{6}$ when $n_2 \leq \frac{n}{3}$, and $\iota_1(G) \leq \frac{n-n_2}{3}$ otherwise, where n_2 is the number of vertices of degree 2. Borg and Kaemawichanurat [7] obtained that $\iota_k(G) \leq min\{\frac{n}{k+4}, \frac{n+n_2}{5}, \frac{n-n_2}{3}\}$ for a maximal outerplanar graph G and $k \geq 0$, where n_2 is the number of vertices of degree 2. Vazquez-Araujo [8] analyzed that $\iota(T) = \frac{n}{3}$ implies $\iota(T) = \gamma(T)$ for a tree T, and proposed simple algorithms to build these trees from the connections of stars.

For a $\{K_k\}$-isolating set S, in 2021, Favaron and Kaemawichanurat [9] restriced the induced subgraph $G[S]$ to be an independent set and introduced the concept of the independent $\{K_k\}$-isolation number of G. The vertex subset $S \subseteq V$ is said to be independent $\{K_k\}$-isolating if S is a $\{K_k\}$-isolating set of G and $G[S]$ has no edge. The independent $\{K_k\}$-isolation number of G is the minimum cardinality of an independent $\{K_k\}$-isolating set of G. Based on the concept, we propose the following concepts.

Definition 1.2. *A subset $S \subseteq V$ is called an independent $\mathcal{F}$-isolating set of G if S is an $\mathcal{F}$-isolating set of G and S is an independent set. The independent $\mathcal{F}$-isolation number of G is the minimum cardinality of an independent $\mathcal{F}$-isolating set of G, denoted by $\iota_I(G, \mathcal{F})$.*

Definition 1.3. *A subset $S \subseteq V$ is called an independent isolating set of G if S is an isolating set of G and $G[S]$ has no edge. The independent isolation number of G is the minimum cardinality of an independent isolating set of G, denoted by $\iota_I(G)$.*

In this paper, we investigate respectively the sharp bounds of the isolation number and the independent isolation number of the hypercube network and n-star network.

2. Main results

2.1. Isolation number of the hypercube network

The hypercube network is the basic model for interconnection networks, and it is a popular network because of its attractive properties, including regularity, symmetry, small diameter, strong connectivity, recursive construction, partitionability and relatively low link complexity [10, 11]. In general, a network model can be modeled as a graph. Let n be a positive integer. The *hypercube network* of dimension n, denoted by Q_n, is the simple graph whose vertices are the n-tuples with entries in $\{0, 1\}$ and whose edges are the pairs of n-tuples that differ in exactly one position (see Figure 1). A *m-dimensional subcube* of Q_n is isomorphic to Q_m for any positive integer $1 \leq m \leq n$. A vertex of $V(Q_n)$ is an *even vertex* if the number of 1s is even in its all symbols. A vertex of $V(Q_n)$ is an *odd vertex* if the number of 1s is odd in its all symbols.

The hypercube network have many classic and fascinating topological structural properties, such as those below.

Lemma 2.1. [1] *Hypercube network satisfies the following properties:*
(a) $|V(Q_n)| = 2^n$, $|E(Q_n)| = n \cdot 2^{n-1}$;
(b) Each edge of Q_n *has an even endvertex and an odd endvertex;*
(c) Q_n *is a n-regular bipartite graph;*
(d) Every perfect matching of Q_n *has* 2^{n-1} *edges;*
(e) If $n \geq 3$*, then* Q_n *has* 2^{n-3} *disjoint* 3*-dimensional subcubes of* Q_n.

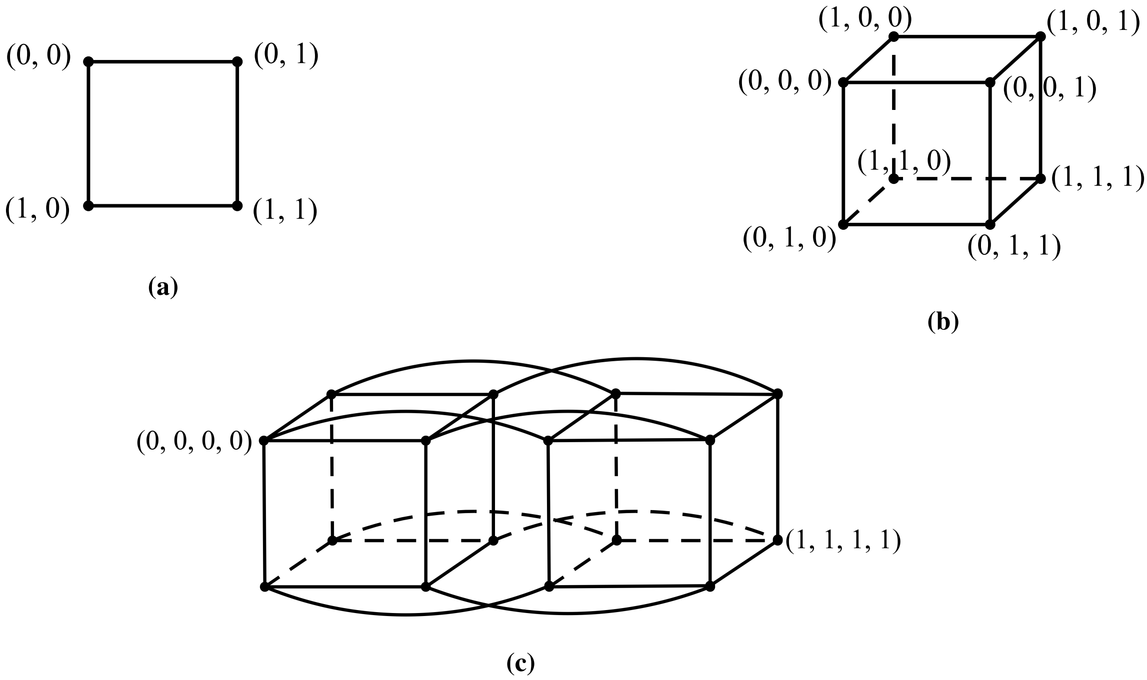

Figure 1. (a)Q_2; (b)Q_3; (c)Q_4.

By Lemma 2.1, we obtain the following results.

Theorem 2.2. $\iota(Q_2) = 1$, $\iota(Q_3) = \iota(Q_4) = 2$.

Proof. According to the structure of Q_n, we know that $Q_2 \cong C_4$, so $\iota(Q_2) = \iota(C_4) = 1$.

It is easy to know $\{(0,0,0), (1,1,1)\}$ is an isolating set of Q_3 (see Figure 1(b)), so $\iota(Q_3) \leq 2$. Let S_1 be a minimum isolating set of Q_3, clearly, $|S_1| \geq 1$. Suppose that $|S_1| = 1$. If the vertex of S_1 is an even vertex, without loss of generality, let $S_1 = \{(1, 1, 0)\}$, then $V(Q_3)\backslash N_{Q_3}[S_1] = \{(0, 0, 0), (0, 0, 1), (1, 0, 1), (0, 1, 1)\}$. Hence, $V(Q_3)\backslash N_{Q_3}[S_1]$ is not an independent set of Q_3, which is a contradiction. If the vertex of S_1 is an odd vertex, without loss of generality, let $S_1 = \{(0, 1, 0)\}$, then $V(Q_3)\backslash N_{Q_3}[S_1] = \{(1, 0, 0), (0, 0, 1), (1, 0, 1), (1, 1, 1)\}$. Hence, $V(Q_3)\backslash N_{Q_3}[S_1]$ is not an independent set of Q_3, which is a contradiction. Thus, $|S_1| \neq 1$, furthermore, $\iota(Q_3) = |S_1| \geq 2$. Hence, $\iota(Q_3) = 2$.

Additionally, it is easy to know $\{(0,0,0,0), (1,1,1,1)\}$ is an isolating set of Q_4 (see Figure 1(c)), so $\iota(Q_4) \leq 2$. Let Q_3^1 and Q_3^2 be two disjoint 3-dimensional subcubes of Q_4 by Lemma 2.1(e), and S_2 be a minimum isolating set of Q_4. Obviously, $|S_2| \geq 1$. If $|S_2| = 1$ and $S_2 = \{v\}$, then $v \in Q_3^i$ for any $i \in \{1, 2\}$. Since $Q_3^j \subset G[V(Q_4)\backslash N_{Q_4}[v]]$ for $j \in \{1, 2\}\backslash\{i\}$, $V(Q_4)\backslash N_{Q_4}[S_2]$ is not an independent set of Q_4, which is a contradiction. Thus, $|S_2| \neq 1$, furthermore, $\iota(Q_4) = |S_2| \geq 2$. Hence, $\iota(Q_4) = 2$. □

Theorem 2.3. *Let n be a positive integer and $n \geq 4$, then $\frac{2^{n-1}}{n} \leq \iota(Q_n) \leq 2^{n-3}$. Moreover, the bounds are sharp.*

Proof. Let S be a minimum $\{K_2\}$-isolating set of Q_n. Since every perfect matching of Q_n has 2^{n-1} edges and $V(Q_n)\backslash N_{Q_n}[S]$ is an independent set, every edge of a perfect matching of Q_n has at least one vertex in $N_{Q_n}(S)$, that is, $|N_{Q_n}(S)| \geq \frac{2^n}{2} = 2^{n-1}$. By Lemma 2.1($c$), we have $|N_{Q_n}(S)| \leq \Delta(Q_n)|S| = d(v)|S| = n|S|$ for any vertex $v \in S$. Thus, $\iota(Q_n) = |S| \geq \frac{|N_{Q_n}(S)|}{d(v)} \geq \frac{2^{n-1}}{d(v)} = \frac{2^{n-1}}{n}$.

By Lemma 2.1, we know that Q_n has 2^{n-3} disjoint 3-dimensional subcubes of Q_n for $n \geq 4$ and each edge of Q_n has one even endvertex and one odd endvertex, so each Q_3 has four odd vertices and four even vertices, and every even vertex is adjacent to three odd vertices in Q_3. Without loss of generality, let $x \in V(Q_3^1)$ be an even vertex and $y \in V(Q_3^1)\backslash N_{Q_3^1}[x]$ be an odd vertex of Q_3^1. Since $n \geq 4$, there exists a Q_3^i $(2 \leq i \leq 2^{n-3})$ such that $|N_{Q_n}(x) \cap V(Q_3^i)| = 1$. Let $N_{Q_n}(x) \cap V(Q_3^i) = \{x'\}$ and $y' \in V(Q_3^i)\backslash N_{Q_3^i}[x']$ be an even vertex of Q_3^i. By the structure of Q_n, we know that $N(x) \cap V(Q_3^i) = \{x'\}$, $N(y) \cap V(Q_3^i) = \{y'\}$ and yy' is an edge, then all vertices of $(V(Q_3^1) \cup V(Q_3^i))\backslash N_{Q_n}[\{x, y'\}]$ are even vertices, and $\{x, y'\}$ is an isolating set of $Q_3^1 \cup Q_3^i$. Choose one even vertex in each Q_3^j $(1 \leq j \leq 2^{n-3})$ as the set S such that all vertices of $V(Q_n)\backslash N_{Q_n}[S]$ are even vertices (In Figure 2, a, b, c, d are even vertices of $V(Q_5)$ and $\{a, b, c, d\}$ is an isolating set of Q_5). Since each edge of Q_n has an even endvertex and an odd endvertex, the set $V(Q_n)\backslash N_{Q_n}[S]$ is an independent set of Q_n, thus the set S is an isolating set of Q_n. Since $|S \cap V(Q_3^j)| = 1$ for $1 \leq j \leq 2^{n-3}$, we obtain $\iota(Q_n) \leq |S| = 2^{n-3}$. In conclusion, $\frac{2^{n-1}}{n} \leq \iota(Q_n) \leq 2^{n-3}$ for $n \geq 4$.

Especially, if $\frac{2^{n-1}}{n} = 2^{n-3}$, then $n = 4$. So the upper bound is equal to the lower bound if and only if $n = 4$. If $n = 4$, then $\iota(Q_n) = 2 = 2^{n-3} = \frac{2^{n-1}}{n}$ by Theorem 2.2. Thus, the upper and lower bounds are sharp. □

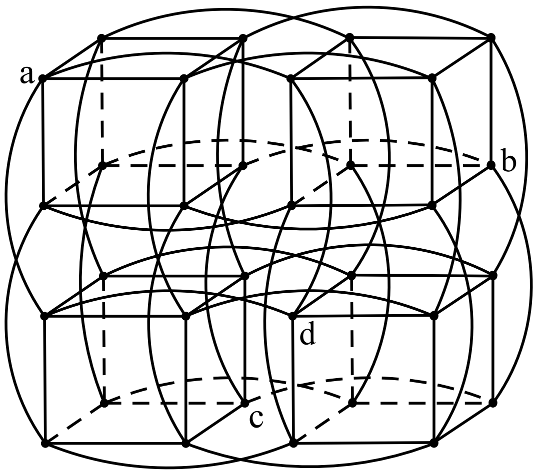

Figure 2. $a = \{0, 0, 0, 0, 0\}$, $b = \{0, 1, 1, 1, 1\}$, $c = \{1, 0, 1, 1, 1\}$, $d = \{1, 1, 0, 0, 0\}$.

Corollary 2.4. *Let n be a positive integer and $n \geq 4$, then $\frac{2^{n-1}}{n} \leq \iota_I(Q_n) \leq 2^{n-3}$. Moreover, the bounds are sharp.*

Proof. Let S_I be a minimum independent isolaing set of Q_n. Obviously, S_I is an isolating set of Q_n. Thus, by Theorem 2.2, we have $\iota_I(Q_n) \geq \frac{2^{n-1}}{n}$. Let $\{Q_3^1, Q_3^2, \cdots, Q_3^{2^{n-3}}\}$ be the set of 2^{n-3} disjoint 3-dimensional subcubes of Q_n. Choose one even vertex in each Q_3^i $(1 \leq i \leq 2^{n-3})$ as the set S such that all vertices of $V(Q_n) \backslash N_{Q_n}[S]$ are even vertices. According to the proof of Theorem 2.2, the set S is an isolating set of Q_n and $\iota(Q_n) \leq 2^{n-3}$. Since all vertices of S are even vertices, the set S is an independent isolating set of Q_n. Thus $\iota_I(Q_n) \leq 2^{n-3}$. In conclusion, $\frac{2^{n-1}}{n} \leq \iota_I(Q_n) \leq 2^{n-3}$ for $n \geq 4$.

Especially, if $n = 4$, then $\iota(Q_n) = 2 = 2^{n-3} = \frac{2^{n-1}}{n}$ by Theorem 2.2. Thus, the upper and lower bounds are sharp. □

2.2. *Isolation number of the n-star network*

The n-star network was proposed by Akers, Harel and Krishnamurthy [12] as an attractive alternative to the hypercube network for interconnecting processors on a parallel computer. For a positive integer $n (n \geq 2)$, the *n-star network* on n symbols, denoted by S_n, is the graph with $n!$ vertices, whose the vertex set $V(S_n)$ is all permutations of symbols $1, 2, \cdots, n$, and each vertex $v \in V(S_n)$ is connected to $n-1$ vertices which can be obtained by interchanging the first symbol of v with the ith symbol of v for $2 \leq i \leq n$ (S_4 is shown as an example in Figure 3).

Lemma 2.5. [13, 14] *Let G be an n-vertex graph with minimum degree $\delta(G)$. If $\delta(G) \geq 1$, then $\gamma(G) \leq \frac{n}{2}$.*

Theorem 2.6. *Let n be a positive integer and $n \geq 2$, then $\frac{n \cdot (n-2)!}{2} \leq \iota(S_n) \leq (n-1)!$. Moreover, the bounds are sharp.*

Proof. Let S be a minimum isolating set of S_n. According to the structure of S_n, we know that S_n is a $(n-1)$-regular bipartite graph, and every perfect matching of S_n has $\frac{n!}{2}$ edges. Note that S is a minimum isolating set of S_n and $V(S_n) \backslash N_{S_n}[S]$ is an independent set, then every edge of a perfect matching of S_n has at least one endvertex in $N_{S_n}(S)$, that is, $|N_{S_n}(S)| \geq \frac{n!}{2}$. For any vertex $v \in S$, we have $|N_{S_n}(S)| \leq$

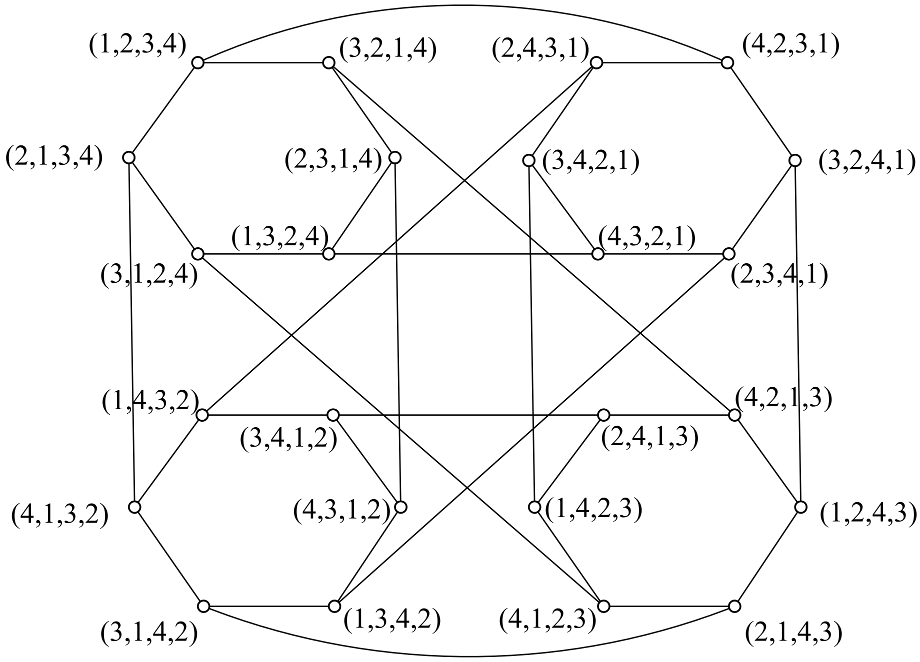

Figure 3. S_4.

$\Delta(S_n)|S| = d(v)|S| = (n-1)|S|$. Thus, $\iota(S_n) = |S| \geq \frac{|N_{S_n}(S)|}{d(v)} \geq \frac{\frac{n!}{2}}{n-1} = \frac{n\cdot(n-2)!}{2}$.

Inspired by Alon and Spencer [15] and Caroa and Hansbergb [3], we show that $\iota(S_n) \leq (n-1)!$ by the probabilistic method. Since $n \geq 2$, $d(v) = n-1 \geq 1$ for any vertex $v \in V(S_n)$. Let $p \in [0, 1]$, and we independently select a vertex subset $A \subset V(S_n)$ at random such that $P(v \in A) = p$. Let I be the set of the isolated vertices of $V(S_n)\backslash A$. Meanwhile, let $B = V(S_n)\backslash(N_{S_n}[A] \cup I)$ and D be a minimum dominating set of $G[B]$. Since there is no isolated vertice in B, $\delta(G[B]) \geq 1$, furthermore, $\gamma(G[B]) = |D| \leq \frac{|B|}{2}$ by Lemma 2.5. Thus, $A \cup D$ is an isolating set of S_n. Note that the expectated value $E[|D|] \leq E[\frac{|B|}{2}] = \frac{1}{2}E[|B|]$. Hence, we have

$$P(v \in B) = P(v \in B) = P(v \in V(S_n)\backslash(N_{S_n}[A] \cup I)) = P(v \in V(S_n)\backslash N_{S_n}[A])$$

$$= (1-p)^{d(v)+1} = (1-p)^{n-1+1} = (1-p)^n.$$

Thus, we obtain that

$$E[|A \cup D|] \leq E[|A|] + \frac{1}{2}E[|B|] = p|V(S_n)| + \frac{1}{2}(1-p)^n|V(S_n)| = (p + \frac{1}{2}(1-p)^n) \cdot (n!).$$

Considering the function $f(p) = (p+\frac{1}{2}(1-p)^n)\cdot(n!)$ and its derivative $f'(p) = (1-\frac{1}{2}n(1-p)^{n-1}\cdot(n!)$, we can see that $f'(p) = 0$ when $p = 1-(\frac{2}{n})^{\frac{1}{n-1}}$. It follows that

$$\iota(S_n) \leq E[|A \cup D|] \leq (p + \frac{1}{2}(1-p)^n) \cdot (n!) \leq (1-(\frac{2}{n})^{\frac{1}{n-1}} + \frac{1}{2}((\frac{2}{n})^{\frac{1}{n-1}})^n) \cdot (n!)$$

$$= (1-(\frac{2}{n})^{\frac{1}{n-1}} + \frac{1}{2}(\frac{2}{n})(\frac{2}{n})^{\frac{1}{n-1}}) \cdot (n!).$$

Since $n \geq 2$, we have $(\frac{2}{n})^{\frac{1}{n-1}} \leq 1$. Then $\iota(S_n) \leq (1 - 1 + \frac{1}{2}(\frac{2}{n}) \cdot (n!) = \frac{1}{n} \cdot (n!) = (n-1)!$.

In conclusion, $\frac{n \cdot (n-2)!}{2} \leq \iota(S_n) \leq (n-1)!$ for any positive integer $n \geq 2$.

Especially, if $n = 2$, then $S_2 \cong K_2$, and $\iota(K_2) = 1 = \frac{2 \cdot 0!}{2} = \frac{n \cdot (n-2)!}{2}$. If $n = 3$, then $S_3 \cong C_6$, and $\iota(C_6) = 2 = (3-1)! = (n-1)!$. Hence, the bounds are sharp. □

Corollary 2.7. *Let n be a positive integer and $n \geq 2$, then $\frac{n \cdot (n-2)!}{2} \leq \iota_I(S_n) \leq (n-1)!$. Moreover, the bounds are sharp.*

Proof. Let S_I be a minimum independent isolaing set of S_n. Obviously, S_I is also an isolating set of S_n. By Theorem 2.6, we have $\iota_I(S_n) \geq \frac{n \cdot (n-2)!}{2}$. Let V_1 be the set of all vertices with the first symbol is 1. Clearly, the set V_1 is an independent set of S_n. According the structure of S_n, we know that any vertex of V_1 has different $n-1$ neighbors, and the first symbol of these $n-1$ neighbors is $2, 3, \cdots, n-1, n$ respectively. Let $x, y \in V_1$ and $x \neq y$. If $N_{S_n}(x) \cap N_{S_n}(y) \neq \emptyset$, then there exists a vertex to be adjacent to x and y, which contradicts the definition of S_n. Thus, $N_{S_n}(x) \cap N_{S_n}(y) = \emptyset$, that is, the neighborhoods of any two vertices of V_1 are disjoint. Hence, $|N_{S_n}[V_1]| = (n-1) \cdot |V_1| + |V_1| = n \cdot |V_1| = n \cdot (n-1)! = n! = |V(S_n)|$, this means that $G[V(S_n) \backslash N_{S_n}[V_1]]$ has no edge. Since V_1 is independent, the set V_1 is an independent isolating set of S_n. Then $\iota_I(S_n) \leq |V_1| = (n-1)!$. In conclusion, $\frac{n \cdot (n-2)!}{2} \leq \iota_I(S_n) \leq (n-1)!$ for any positive integer $n \geq 2$.

Especially, if $n = 2$, then $\{(1, 2)\}$ is an independent isolating set of S_2. Thus, $\iota_I(S_2) = 1 = \frac{2 \cdot 0!}{2} = \frac{n \cdot (n-2)!}{2}$. If $n = 3$, then $\{(1, 2, 3), (1, 3, 2)\}$ is an independent isolating set of S_3. Thus, $\iota_I(S_3) = 2 = (3-1)! = (n-1)!$. Hence, the bounds are sharp. □

3. Conclusions and problems

The hypercube network Q_n and n-star network S_n are both recursively constructed networks, and they have many known and attractive topological properties. This paper demonstrates the sharp bounds of the isolation number and the independent isolation number of the hypercube network and n-star network. In view of this research direction, there are still many academic issues worth studying:

Problem 1. Let $m \geq 1$ be a positive integar and $\mathcal{F} = \{F_1, F_2, \cdots, F_m\}$. For any $F_i \in \mathcal{F}$, if $F_i \ncong K_2$, what F_i can be used to explore the $\mathcal{F}$-isolation number of the hypercube network or n-star network?

Problem 2. Consider the $\mathcal{F}$-isolation number of some other network models.

For future work, it would be interesting and meaningful to probe and research the $\mathcal{F}$-isolation number of some other network models.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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