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Research article

Generalized primal topological spaces

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Abstract: In the present article, a new category of mathematical structure is described based on the topological structure "primal" and the notion of "generalized". Such a structure is discussed in detail in terms of topological properties and some basic theories. Also, we introduced some operators using the concepts "primal" and "generalized primal neighbourhood", which have a lot of nice properties.

Keywords: generalized primal topology; generalized primal neighbourhood; $(\mathfrak{g}, \mathcal{P})$-open sets; $cl^{\diamond}$-operator; Φ-operator
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1. Introduction

It is noted from the literature that many topologies with important applications in mathematics have been defined using new mathematical structures, for example, filter [20], ideal [19] and partially ordered [25]. The grill is one of the classic structures presented by Choquet [11] around 1947, which is a good tool for studying topological notions and has a wide range of important applications in topological theory (see [9, 10, 30]). Some structures based on both "filter" and "grill" are defined and investigated, as in [22, 23], while others are based on "partially ordered set" as [7]. A grill's associated topology has been defined and investigated by Roy and Mukherjee [26]. They presented the basic topological concepts in terms of this new structure and studied the fundamental theorems.

Moreover, some operators have been defined and their properties studied with clarity to determine whether they satisfy Kuratowskis closure axioms (see [27–29]). The notion of continuity occupies a great place of study in the wide range of literature [18, 24, 32].

Császár [12] established the theory of generalizations in 1997 by presenting the notion of generalized open sets. Furthermore, he introduced the generalized topological spaces (breifly GTS) around 2002 [13], which differ from topological spaces in the condition of intersection. In more specific terms, they are closed under arbitrary unions.

In [13,14,31,33], a lot of its characteristics are studied in detail, such as the generalized interior of a set A, the generalized closure of a set A, the generalized neighbourhood, separation axioms, and so on. The notion of continuity gets a lot of attention under the new appellations "generalized continuity", "θ-continuity", "γ-continuity", etc. (see [13, 16]). Various kinds of generalized topology appeared and were studied, such as "extremely disconnected generalized topology", "δ- and θ-modifications generalized topology", etc. (see [15,16,33]).

The dual structure of a grill was presented around 2022. It was called "primal". This new structure has a lot of interesting properties, studied in detail by Acharjee et al. [1].

Furthermore, a new topology has been defined by using this new structure, called "primal topology" and some of its topological properties have been investigated. Some articles have appeared that study primal topological spaces in different fields, such as vector spaces induced by metric or soft environments (see [2,8,21]).

In 1990, [17] different kinds of topological operators were defined based on the structure "ideal". Then, other studies followed the same approach. Al-Omari and Noiri studied some types of operators based on the grill notion (see [4–6]). Moreover, [1,3] present types of operators have been defined by using the notion of primal with deep studies of their various properties.

In this paper, our aim is to define a new structure based on both the notions of "primal" and "generalized". This new space will be called "generalized primal topological spaces".

Also, the authors will introduce some operators based on the notion of generalized primal neighbourhood such as $cl^{\diamond}(A)$ and $\Phi(A)$, with a study of their fundamental characteristics and give an answer to the important question, "do they satisfy Kuratowskis closure axioms?" A lot of nice properties with examples will be present.

Thereafter, we will define a new form of generalized primal topological spaces induced by the previous operators, with a comparison between the structures.

2. Preliminaries

We will go over the fundamental definitions and conclusions in this section. We represent the power set of $X \neq \phi$ by 2^X throughout our work.

Definition 2.1. *[11] Suppose that $X \neq \phi$. A collection $\mathcal{G} \subseteq X$ of 2^X is named a grill if the next hold:*
(i) $\phi \notin \mathcal{G}$.
(ii) For two subsets A and B of X, with $A \subseteq B$, we have $B \in \mathcal{G}$, if $A \in \mathcal{G}$.
(iii) For two subsets A and B of X, $A \in \mathcal{G}$ or $B \in \mathcal{G}$, if $A \cup B \in \mathcal{G}$.

Definition 2.2. *[12] Let $X \neq \phi$. A family $\mathfrak{g}$ of 2^X is named a generalized topology (GT) on X if the next hold:*
(i) $\phi \in \mathfrak{g}$.
(ii) The countable union of $G_i \in \mathfrak{g}$, for $i \in I \neq \phi$ is belongs to $\mathfrak{g}$.

A generalized topological space is the pair $(X, \mathfrak{g})$.

Remark 2.1. *According to [12], we note that:*

(i) This space's elements are identified as $\mathfrak{g}$*-open and their own complements are identified as* $\mathfrak{g}$*-closed.*

(ii) $C_{\mathfrak{g}}(X)$ *represents the set of all* $\mathfrak{g}$*-closed sets of* X*.*

(iii) $c_{\mathfrak{g}}A$ *represents the closure of* $A \subseteq X$*, which is described as the intersection of all* $\mathfrak{g}$*-closed sets that contain* A*.*

(iv) $i_{\mathfrak{g}}A$ *represents the interior of* $A \subseteq X$*, which is described as the union of all* $\mathfrak{g}$*-open sets that contained in* A*.*

(v) $c_{\mathfrak{g}}(c_{\mathfrak{g}}A) = c_{\mathfrak{g}}A$*,* $i_{\mathfrak{g}}(i_{\mathfrak{g}}A) = i_{\mathfrak{g}}A$ *and* $i_{\mathfrak{g}}A \subseteq A \subseteq c_{\mathfrak{g}}A$*.*

(vi) A *is* $\mathfrak{g}$*-open if* $A = i_{\mathfrak{g}}A$*,* A *is* $\mathfrak{g}$*-closed if* $A = c_{\mathfrak{g}}A$ *and* $c_{\mathfrak{g}}A = X \setminus (i_{\mathfrak{g}}(X \setminus A))$*.*

Definition 2.3. *[12] In a generalized topological space* $(X, \mathfrak{g})$*. Consider an operator* ψ*:* $X \to 2^{2^X}$ *satisfies* $x \in U$ *for* $U \in \psi(x)$*. Then,* $U \in \psi(x)$ *is known as a generalized neighbourhood of a point* x *in the space* X*, or briefly (GN).*

Furthermore, ψ *is known as a generalized neighbourhood system on a space* X*.*

Definition 2.4. *[12] The collection of all generalized neighbourhood systems on* X *is denoted by* $\Psi(X)$*.*

Definition 2.5. *[1] Suppose that* $X \neq \phi$*. A family* $\mathcal{P}$ *of* 2^X *is named a primal on* X*, when the next hold:*

(i) $X \notin \mathcal{P}$*.*

(ii) For $A, B \subseteq X$ *with* $B \subseteq A$*. If* $A \in \mathcal{P}$*, then* $B \in \mathcal{P}$*.*

(iii) For $A, B \subseteq X$*, then* $A \in \mathcal{P}$ *or* $B \in \mathcal{P}$*, whenever* $A \cap B \in \mathcal{P}$*.*

Definition 2.6. *A primal topological space is defined as the pair* (X, τ) *with a primal* $\mathcal{P}$ *on* X*. Moreover, it is represented by* $(X, \tau, \mathcal{P})$*.*

Corollary 2.1. *[1] Suppose that* $X \neq \phi$*. A family* $\mathcal{P}$ *of* 2^X *is named a primal on* X*, if and only if the next hold:*

(i) $X \notin \mathcal{P}$*.*

(ii) For $A, B \subseteq X$ *with* $B \subseteq A$*. If* $B \notin \mathcal{P}$*, then* $A \notin \mathcal{P}$*.*

(iii) For $A, B \subseteq X$*, then* $A \cap B \notin \mathcal{P}$*, whenever* $A \notin \mathcal{P}$ *and* $B \notin \mathcal{P}$*.*

Theorem 2.1. *[1] If the collection* $\mathcal{G}$ *forms a grill on* X*, then the set* $\{A : A^c \in \mathcal{G}\}$ *is a primal on* X*.*

Theorem 2.2. *[1] The union of two primals on* X *gives a primal on* X*.*

Remark 2.2. *According to [1], the intersection of two primals on* X *need not be a primal on* X*.*

3. Generalized primal topology

In this part, we establish a new category of generalized topology called "generalized primal topology" and define it as:

Definition 3.1. *A generalized primal topology is a generalized topology* $\mathfrak{g}$ *with a primal* $\mathcal{P}$ *on* X*. The triples* $(X, \mathfrak{g}, \mathcal{P})$ *denote the generalized primal topological space.*

This space's elements are identified as $(\mathfrak{g}, \mathcal{P})$*-open sets, and their own complement known as* $(\mathfrak{g}, \mathcal{P})$*-closed sets.*

$C_{(\mathfrak{g},\mathcal{P})}(X)$ denoted the set of all $(\mathfrak{g},\mathcal{P})$-closed sets on X, and $cl_{(\mathfrak{g},\mathcal{P})}(A)$ represents the closure of $A \subseteq X$, which is described as the intersection of all $(\mathfrak{g},\mathcal{P})$-closed sets that contain A.

Example 3.1. *Consider $X = \{a,b,c\}$, $\mathfrak{g} = \{\phi,\{a,b\},\{a,c\},X\}$ and the primal set $\mathcal{P} = \{\phi,\{a\}\}$. Hence, $(X,\mathfrak{g},\mathcal{P})$ is a generalized primal topological space.*

Definition 3.2. *In a generalized primal topological space $(X,\mathfrak{g},\mathcal{P})$. Consider an operator ψ: $X \to 2^{2^X}$ satisfies $x \in U$ for $U \in \psi(x)$. Then, $U \in \psi(x)$ is known as a generalized primal neighbourhood for a point x in the space X.*

Remark 3.1. *ψ is known as a generalized primal neighbourhood system on a space X. The collection of all generalized primal neighbourhood systems on X is denoted by $\Psi(X)$.*

Definition 3.3. *In a generalized primal topological space $(X,\mathfrak{g},\mathcal{P})$. Consider an operator $(.)^\diamond$: $2^X \to 2^X$ defined by*

$$A^\diamond(X,\mathfrak{g},\mathcal{P}) = \{x \in X : A^c \cup U^c \in \mathcal{P}, \forall\ U \in \psi(x)\},$$

where U is a generalized primal neighbourhood of $x \in X$.

Remark 3.2. *In a generalized primal topological space $(X,\mathfrak{g},\mathcal{P})$:*

(i) We cannot say that $A^\diamond \subseteq A$, or $A \subseteq A^\diamond$ the following example shows that.

(ii) With an additional condition, the relationship $A^\diamond \subseteq A$ is always true, which is proved in the following theorems.

Example 3.2. *Consider*

$$X = \{a,b,c\}, \quad \mathfrak{g} = \{\phi,\{a\},\{b\},\{a,b\},X\}$$

and

$$\mathcal{P} = \{\phi,\{a\},\{b\},\{c\},\{a,b\}\}.$$

Let $A = \{a,b\}$. Then, $A^\diamond = \{c\}$. Therefore, $A^\diamond \nsubseteq A$ and $A \nsubseteq A^\diamond$.

Theorem 3.1. *Consider $(X,\mathfrak{g},\mathcal{P})$ as a generalized primal topological space. Hence, $A^\diamond \subseteq A$, whenever A^c is $(\mathfrak{g},\mathcal{P})$-open.*

Proof. Let A^c be $(\mathfrak{g},\mathcal{P})$-open. Let $x \in A^\diamond$ with $x \notin A$. Then, A^c is an open generalized primal neighbourhood of x briefly ($A^c \in \psi(x)$) and $A^c \cup U^c \in \mathcal{P}$, for all $U \in \psi(x)$. Hence,

$$A^c \cup (A^c)^c = X \in \mathcal{P},$$

which is a contradicts with the fact that $X \notin \mathcal{P}$. Therefore, $A^\diamond \subseteq A$.

Theorem 3.2. *Consider $(X,\mathfrak{g},\mathcal{P})$ as a generalized primal topological space. Hence, we have:*

(i) $\phi^\diamond = \phi$.

(ii) $A^\diamond$ is $(\mathfrak{g},\mathcal{P})$-closed, for $A \subseteq X$, i.e., $cl_{(\mathfrak{g},\mathcal{P})}(A^\diamond) = A^\diamond$.

(iii) For $A \subseteq X$, $(A^\diamond)^\diamond \subseteq A^\diamond$.

(iv) $A^\diamond \subseteq B^\diamond$, whenever, $A \subseteq B$ for $A, B \subseteq X$.

(v) For $A, B \subseteq X$, ${}^\diamond A \cup B^\diamond = (A \cup B)^\diamond$.

(vi) For $A, B \subseteq X$, $(A \cap B)^\diamond \subseteq A^\diamond \cap B^\diamond$.

Proof. (i) It is clear that $\phi^c = X$, but $X \notin \mathcal{P}$. Thus, we are done.

(ii) Since $A^\diamond \subseteq cl_{(\mathfrak{g},\mathcal{P})}(A^\diamond)$ always true. We need to show that $cl_{(\mathfrak{g},\mathcal{P})}(A^\diamond) \subseteq A^\diamond$. Consider $x \in cl_{(\mathfrak{g},\mathcal{P})}(A^\diamond)$ and $U \in \psi(x)$. Thus,

$$U \cap A^\diamond \neq \phi \Rightarrow \exists y \in X$$

satisfies $y \in U \wedge y \in A^\diamond$. Then, $\forall\ V \in \psi(y)$ we have

$$V^c \cup A^c \in \mathcal{P} \Rightarrow U^c \cup A^c \in \mathcal{P}.$$

Therefore, $x \in A^\diamond$. Hence, $cl_{(\mathfrak{g},\mathcal{P})}(A^\diamond) \subseteq A^\diamond$.

(iii) From Theorem 3.1 and (ii) we can conclusion that $(A^\diamond)^\diamond \subseteq A^\diamond$, that is $A^\diamond$ is $(\mathfrak{g},\mathcal{P})$-closed in X implies $(A^\diamond)^c$ is $(\mathfrak{g},\mathcal{P})$-open.

(iv) Suppose that $A \subseteq B$ and $x \in A^\diamond$. Then, for all $U \in \psi(x)$ we have $A^c \cup U^c \in \mathcal{P}$. Hence, $B^c \cup U^c \in \mathcal{P}$. Therefore, $x \in B^\diamond$.

(v) From (iv) we have $A^\diamond \subseteq (A \cup B)^\diamond$ and $B^\diamond \subseteq (A \cup B)^\diamond$. Then,

$$A^\diamond \cup B^\diamond \subseteq (A \cup B)^\diamond.$$

To prove the opposite direction, consider $x \notin A^\diamond \cup B^\diamond$ implies $x \notin A^\diamond$ and $x \notin B^\diamond$. Then, there exists $U; V \in \psi(x)$ satisfies $A^c \cup U^c \notin \mathcal{P}$ and $B^c \cup V^c \notin \mathcal{P}$. Consider a set $W = U \cap V$, then $W \in \psi(x)$ such that $A^c \cup W^c \notin \mathcal{P}$ and $B^c \cup W^c \notin \mathcal{P}$. Then,

$$(A \cup B)^c \cup W^c = (A^c \cup W^c) \cap (B^c \cup W^c) \notin \mathcal{P}.$$

Hence, $x \notin (A \cup B)^\diamond$. Therefore,

$$(A \cup B)^\diamond \subseteq A^\diamond \cup B^\diamond.$$

(vi) The proof follows in the same way.

Remark 3.3. *In a generalized primal topological space $(X, \mathfrak{g}, \mathcal{P})$, the opposite inclusion of the Theorem 3.2 (vi) is not always true; the following example shows that.*

Example 3.3. *Consider*

$$X = \{a, b, c\}, \quad \mathfrak{g} = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$$

and

$$\mathcal{P} = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}\}.$$

Let $A = \{a, b\}$ and $B = \{c\}$. Then,

$$A^\diamond = \{c\} = B^\diamond.$$

Thus, $A^\diamond \cap B^\diamond = \{c\}$ and $(A \cap B)^\diamond = \phi$. Therefore,

$$A^\diamond \cap B^\diamond \nsubseteq (A \cap B)^\diamond.$$

Theorem 3.3. *Consider $(X, \mathfrak{g}, \mathcal{P})$ as a generalized primal topological space. Hence, for two subsets A and B of X, $A \cap B^\diamond \subseteq (A \cap B)^\diamond$, whenever A is $(\mathfrak{g}, \mathcal{P})$-open.*

Proof. If A is $(\mathfrak{g}, \mathcal{P})$-open and $x \in A \cap B^{\diamond}$, then $x \in A$ and $x \in B^{\diamond}$ implies for all

$$U \in \psi(x) : B^c \cap U^c \in \mathcal{P}.$$

Thus,

$$(A \cap B)^c \cup U^c = B^c \cup (A \cap U)^c \in \mathcal{P}.$$

Hence, x belongs to $(A \cap B)^{\diamond}$. Therefore, we are done.

Theorem 3.4. *Consider $(X, \mathfrak{g}, \mathcal{P})$ as a generalized primal topological space. Suppose $C_{(\mathfrak{g},\mathcal{P})}(X) \setminus \{X\}$ is a primal on X. Then, for all $(\mathfrak{g}, \mathcal{P})$-open sets U, we have $U \subseteq U^{\diamond}$.*

Proof. If $U = \phi$, then by Theorem 3.2 (i) we have $U^{\diamond} = \phi$, hence $U \subseteq U^{\diamond}$. By hypotheses, if $C_{(\mathfrak{g},\mathcal{P})}(X) \setminus \{X\}$ is primal, then $X^{\diamond} = X$ since $X^c = \phi$. From Theorem 3.3, we get

$$U = U \cap X^{\diamond} \subseteq (U \cap X)^{\diamond} = U^{\diamond},$$

$\forall (\mathfrak{g}, \mathcal{P})$-open set U. Therefore, we are done.

Lemma 3.1. *Consider $(X, \mathfrak{g}, \mathcal{P})$ as a generalized primal topological space. Hence, $A^{\diamond} = \phi$, whenever $A^c \subseteq X$ is not a primal.*

Proof. Let $x \in A^{\diamond}$. Then, $A^c \cup U^c \in \mathcal{P}$ for all $U \in \psi(x)$. Since A^c is not primal, we get a contradiction. Therefore, $A^{\diamond} = \phi$.

Theorem 3.5. *Consider $(X, \mathfrak{g}, \mathcal{P})$ as a generalized primal topological space. Then,*

$$A^{\diamond} \setminus B^{\diamond} = (A \setminus B)^{\diamond} \setminus B^{\diamond},$$

for two subsets A and B of X.

Proof. We can represent $A^{\diamond}$ as

$$A^{\diamond} = [(A \setminus B) \cup (A \cap B)]^{\diamond}.$$

From Theorem 3.2 (v), we get

$$A^{\diamond} = (A \setminus B)^{\diamond} \cup (A \cap B)^{\diamond} \subseteq (A \setminus B)^{\diamond} \cup (A^{\diamond} \cap B^{\diamond}).$$

This implies

$$A^{\diamond} \subseteq (A \setminus B)^{\diamond} \cup B^{\diamond}.$$

Thus,

$$A^{\diamond} \setminus B^{\diamond} \subseteq (A \setminus B)^{\diamond} \setminus B^{\diamond}.$$

On the other hand, since $(A \setminus B) \subseteq A$, $(A \setminus B)^{\diamond} \subseteq A^{\diamond}$ from Theorem 3.2 (iv). Hence,

$$(A \setminus B)^{\diamond} \setminus B^{\diamond} \subseteq A^{\diamond} \setminus B^{\diamond}.$$

Therefore, we are done.

Corollary 3.1. *Consider $(X, \mathfrak{g}, \mathcal{P})$ as a generalized primal topological space. If $A, B \subseteq X$ and B^c is not a primal, then*

$$(A \cup B)^{\diamond} = A^{\diamond} = (A \setminus B)^{\diamond}.$$

Proof. From Lemma 3.1 we get $B^\diamond = \phi$. Thus,

$$(A \cup B)^\diamond = A^\diamond \cup B^\diamond = A^\diamond \cup \phi = A^\diamond.$$

Also, by Theorem 3.5, $(A \setminus B)^\diamond = A^\diamond$.

Definition 3.4. *In a generalized primal topological space* $(X, \mathfrak{g}, \mathcal{P})$*. Consider an operator* $cl^\diamond$*:* $2^X \to 2^X$ *defined by* $cl^\diamond(A) = A \cup A^\diamond$*, for a subset A of X.*

Remark 3.4. *It is shown by the following theorem that the map* $cl^\diamond$ *is a Kuratowskis closure operator.*

Theorem 3.6. *Consider* $(X, \mathfrak{g}, \mathcal{P})$ *as a generalized primal topological space. Hence, we have:*

(i) $cl^\diamond(\phi) = \phi$*.*

(ii) For a subset A of X, $A \subseteq cl^\diamond(A)$*.*

(iii) $cl^\diamond(cl^\diamond(A)) = cl^\diamond(A)$*.*

(iv) $cl^\diamond(A) \subseteq cl^\diamond(B)$*, whenever* $A \subseteq B$ *for* $A, B \subseteq X$*.*

(v) For $A, B \subseteq X$*,* $cl^\diamond(A) \cup cl^\diamond(B) = cl^\diamond(A \cup B)$*.*

Proof. (i) Since $\phi^\diamond = \phi$,

$$cl^\diamond(\phi) = \phi \cup \phi^\diamond = \phi.$$

(ii) From the definition $cl^\diamond(A) = A \cup A^\diamond$, we have $A \subseteq cl^\diamond(A)$.

(iii) From (ii) $A \subseteq cl^\diamond(A)$ implies

$$cl^\diamond(A) \subseteq cl^\diamond(cl^\diamond(A)).$$

Conversely, since $(A^\diamond)^\diamond \subseteq A^\diamond$ by Theorem 3.2 (iii). Then,

$$cl^\diamond(cl^\diamond(A)) = cl^\diamond(A) \cup (cl^\diamond(A))^\diamond.$$

Thus,

$$cl^\diamond(cl^\diamond(A)) = cl^\diamond(A) \cup (A \cup A^\diamond)^\diamond.$$

Hence,

$$cl^\diamond(cl^\diamond(A)) = cl^\diamond(A) \cup A^\diamond \cup (A^\diamond)^\diamond \subseteq cl^\diamond(A) \cup A^\diamond \cup A^\diamond = cl^\diamond(A).$$

Therefore, we are done.

(iv) If $A \subseteq B$, then from Theorem 3.2 (iv) we get $A^\diamond \subseteq B^\diamond$. Hence,

$$A \cup A^\diamond \subseteq B \cup B^\diamond$$

implies $cl^\diamond(A) \subseteq cl^\diamond(B)$.

(v) Since

$$(A \cup B)^\diamond = A^\diamond \cup B^\diamond$$

is holded and $cl^\diamond(A) = A \cup A^\diamond$. Then,

$$cl^\diamond(A) \cup cl^\diamond(B) = (A \cup A^\diamond) \cup (B \cup B^\diamond).$$

Thus,

$$cl^\diamond(A) \cup cl^\diamond(B) = (A \cup B) \cup (A^\diamond \cup B^\diamond) = cl^\diamond(A \cup B).$$

4. New forms of generalized primal topology

Throughout this section, we will define a new kind of generalized primal topology on X that is stronger than the previous structure that is described in Definition 3.1.

Definition 4.1. *In a generalized primal topological space* $(X, \mathfrak{g}, \mathcal{P})$ *the collection*

$$\mathfrak{g}^{\diamond} = \{A \subseteq X : cl^{\diamond}(A^c) = A^c\}$$

is a generalized primal topology on X induced by an operator $cl^{\diamond}$.

Proposition 4.1. *Consider* $(X, \mathfrak{g}, \mathcal{P})$ *as a generalized primal topological space. Hence,* $\mathfrak{g}$ *is weaker than* $\mathfrak{g}^{\diamond}$.

Proof. Suppose that $A \in \mathfrak{g}$. Then, A^c is $(\mathfrak{g}, \mathcal{P})$-closed in X implies $(A^c)^{\diamond} \subseteq A^c$. Hence

$$cl^{\diamond}(A^c) = A^c \cup (A^c)^{\diamond} \subseteq A^c.$$

But $A^c \subseteq cl^{\diamond}(A^c)$. Then, $cl^{\diamond}(A^c) = A^c$. Therefore, $A \in \mathfrak{g}^{\diamond}$.

Theorem 4.1. *Consider* $(X, \mathfrak{g}, \mathcal{P})$ *and* $(X, \mathfrak{g}, \mathcal{Q})$ *as two generalized primal topological spaces. If* $\mathcal{P} \subseteq \mathcal{Q}$, *then* $\mathfrak{g}^{\diamond}$ *induced by primal* $\mathcal{P}$ *is finer than* $\mathfrak{g}^{\diamond}$ *induced by primal* $\mathcal{Q}$.

Proof. Let $A \in \mathfrak{g}^{\diamond}$ induced by primal $\mathcal{Q}$. Then, $A^c \cup (A^c)^{\diamond}$ with respect to $\mathcal{Q}$ equal to A^c implies $(A^c)^{\diamond}$ with respect to $\mathcal{Q} \subseteq A^c$. Let $x \notin A^c$. Then, $x \notin (A^c)^{\diamond}$ with respect to $\mathcal{Q}$ implies $\exists U \in \psi(x)$ such that

$$U^c \cup (A^c)^c = U^c \cup A \notin \mathcal{Q},$$

but $\mathcal{P} \subseteq \mathcal{Q}$. Then, $U^c \cup A \notin \mathcal{P}$. Hence, $x \notin (A^c)^{\diamond}$ with respect to $\mathcal{P}$. Thus, $(A^c)^{\diamond}$ with respect to $\mathcal{P} \subseteq A^c$ implies

$$cl^{\diamond}(A^c) = A^c \cup (A^c)^{\diamond}$$

with respect to $\mathcal{P}$ equal to A^c. Therefore, $A \in \mathfrak{g}^{\diamond}$ induced by $\mathcal{P}$.

Theorem 4.2. *Consider* $(X, \mathfrak{g}, \mathcal{P})$ *as a generalized primal topological space. Hence,* $A \in \mathfrak{g}^{\diamond} \Leftrightarrow$ *for all* $x \in A$, $\exists U \in \psi(x)$ *satisfies* $(U^c \cup A) \notin \mathcal{P}$.

Proof. Let $A \in \mathfrak{g}^{\diamond}$. Then,

$$A \in \mathfrak{g}^{\diamond} \Leftrightarrow cl^{\diamond}(A^c) = A^c.$$

Thus,

$$A \in \mathfrak{g}^{\diamond} \Leftrightarrow A^c \cup (A^c)^{\diamond} = A^c.$$

However, $(A^c)^{\diamond} \subseteq A^c$. Then,

$$A \in \mathfrak{g}^{\diamond} \Leftrightarrow A \subseteq ((A^c)^{\diamond})^c.$$

Thus,

$$A \in \mathfrak{g}^{\diamond} \Leftrightarrow x \notin (A^c)^{\diamond},\ \forall x \in A.$$

Therefore,

$$A \in \mathfrak{g}^{\diamond} \Leftrightarrow U^c \cup (A^c)^c = U^c \cup A \notin \mathcal{P},$$

for some $U \in \psi(x)$.

Theorem 4.3. *For a generalized primal topological space* $(X, \mathfrak{g}, \mathcal{P})$ *the collection*

$$\mathcal{B}_{\mathcal{P}} = \{G \cap P : G \in \mathfrak{g} \text{ and } P \notin \mathcal{P}\}$$

is a base for $\mathfrak{g}^\diamond$ *on X.*

Proof. Consider $B \in \mathcal{B}_{\mathcal{P}}$. Then, there exists $G \in \mathfrak{g}$ and $P \notin \mathcal{P}$ such that $B = G \cap P$. Since $\mathfrak{g} \subseteq \mathfrak{g}^\diamond$, then $G \in \mathfrak{g}^\diamond$. Since $cl^\diamond(P^c) = P^c$, then $P \in \mathfrak{g}^\diamond$. Hence, $B \in \mathfrak{g}^\diamond$. Consequently, $\mathcal{B}_{\mathcal{P}} \subseteq \mathfrak{g}^\diamond$. Let $A \in \mathfrak{g}^\diamond$ and $x \in A$. Hence, by Theorem 4.2, $\exists U \in \psi(x)$ satisfies $U^c \cup A \notin \mathcal{P}$. Consider

$$B = U \cap (U^c \cup A).$$

Then, $B \in \mathcal{B}_{\mathcal{P}}$ such that $x \in B \subseteq A$.

Definition 4.2. *In a generalized primal topological space* $(X, \mathfrak{g}, \mathcal{P})$. *Consider an operator* Φ: $2^X \to 2^X$ *identify as*

$$\Phi(A) = \{x \in X : \exists\, U \in \psi(x) \text{ satisfay } (U \setminus A)^c \notin \mathcal{P}\},$$

for $A \subseteq X$.

The following theorem defines the relationship between the previously investigated maps $(.)^\diamond$ and (Φ).

Theorem 4.4. *Consider* $(X, \mathfrak{g}, \mathcal{P})$ *as a generalized primal topological space. Hence,*

$$\Phi(A) = X \setminus (X \setminus A)^\diamond,$$

for a subset A of X.

Proof. Suppose that $x \in \Phi(A)$, then $\exists U \in \psi(x)$ such that $(U \setminus A)^c \notin \mathcal{P}$. Hence,

$$(U \cap (X \setminus A))^c = (U^c \cup A) \notin \mathcal{P}.$$

Thus, $x \notin (X \setminus A)^\diamond$ implies

$$x \in X \setminus (X \setminus A)^\diamond.$$

Hence,

$$\Phi(A) \subseteq X \setminus (X \setminus A)^\diamond.$$

Conversely, suppose that

$$x \in X \setminus (X \setminus A)^\diamond$$

implies $x \notin (X \setminus A)^\diamond$. Then, $\exists U \in \psi(x)$ we have

$$U^c \cup (X \setminus A)^c = (U \setminus A)^c \notin \mathcal{P}.$$

Then, $x \in \Phi(A)$. Hence,

$$X \setminus (X \setminus A)^\diamond \subseteq \Phi(A).$$

Therefore, we are done.

Corollary 4.1. *Consider* $(X, \mathfrak{g}, \mathcal{P})$ *as a generalized primal topological space. Hence,* $\Phi(A)$ *is* $(\mathfrak{g}, \mathcal{P})$-*open.*

Proof. The proof follows from Theorem 3.2 (ii) and Theorem 4.4.

Remark 4.1. *Unlike the map $cl^\diamond$, the map Φ does not satisfy the four Kuratowskis closure operator conditions, and this is proven in the following theory.*

Theorem 4.5. *Consider $(X, \mathfrak{g}, \mathcal{P})$ as a generalized primal topological space. Hence, the next hold:*
(i) $\Phi(A) \subseteq \Phi(B)$, whenever $A \subseteq B$.
(ii) If $U \in \mathfrak{g}^\diamond$, then $U \subseteq \Phi(U)$.
(iii) $\Phi(A) \subseteq \Phi(\Phi(A))$.
(iv) $\Phi(A) = \Phi(\Phi(A)) \Leftrightarrow (X \setminus A)^\diamond = ((X \setminus A)^\diamond)^\diamond$.

Proof. (i) Let $x \in \Phi(A)$. Then, $(U \setminus A)^c \notin \mathcal{P}$, but $A \subseteq B$. Then, $(U \setminus B)^c \notin \mathcal{P}$. Hence, $x \in \Phi(B)$. Therefore, $\Phi(A) \subseteq \Phi(B)$.
(ii) If $U \in \mathfrak{g}^\diamond$, then

$$cl^\diamond(X \setminus U) = X \setminus U$$

implies

$$(X \setminus U) \cup (X \setminus U)^\diamond = X \setminus U.$$

Then, $(X \setminus U)^\diamond$ subset of $X \setminus U$. Hence,

$$U \subseteq X \setminus (X \setminus U)^\diamond = \Phi(U).$$

(iii) Since $\Phi(A)$ is $(\mathfrak{g}, \mathcal{P})$-open and $\mathfrak{g} \subseteq \mathfrak{g}^\diamond$,

$$\Phi(A) \in \mathfrak{g}^\diamond \Rightarrow \Phi(A) \subseteq \Phi(\Phi(A))$$

from (ii).
(iv) Let $\Phi(A) = \Phi(\Phi(A))$. Then

$$\Phi(\Phi(A)) = \Phi(X \setminus (X \setminus A)^\diamond) = X \setminus (X \setminus (X \setminus (X \setminus A)^\diamond))^\diamond.$$

Hence, we have

$$\Phi(\Phi(A)) = X \setminus ((X \setminus A)^\diamond)^\diamond.$$

Since $\Phi(A) = \Phi(\Phi(A))$,

$$(X \setminus A)^\diamond = ((X \setminus A)^\diamond)^\diamond.$$

Conversely, let

$$(X \setminus A)^\diamond = ((X \setminus A)^\diamond)^\diamond.$$

From (iii) we get $\Phi(A) \subseteq \Phi(\Phi(A))$. Let $x \in \Phi(\Phi(A))$. Then, x belongs to $X \setminus ((X \setminus A)^\diamond)^\diamond$ implies x belongs to $X \setminus (X \setminus A)^\diamond$. Then, $x \in \Phi(A)$. Hence, $\Phi(\Phi(A)) \subseteq \Phi(A)$. Therefore, $\Phi(A) = \Phi(\Phi(A))$.

Extra properties of the function Φ will be studied in the next theorem.

Theorem 4.6. *Consider $(X, \mathfrak{g}, \mathcal{P})$ as a generalized primal topological space. Hence, the next hold:*
(i) $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$, for $A, B \subseteq X$.
(ii) For $A \subseteq X$, $\Phi(A) = X \setminus X^\diamond$, whenever $A^c \notin \mathcal{P}$.

Proof. (i) It is clear that $\Phi(A \cap B) \subseteq \Phi(A)$ and $\Phi(A \cap B) \subseteq \Phi(B)$. Then,

$$\Phi(A \cap B) \subseteq \Phi(A) \cap \Phi(B).$$

Conversely, suppose that $x \in \Phi(A) \cap \Phi(B)$. Implies there exists $U; V \in \psi(x)$ satisfies $(U \setminus A)^c \notin \mathcal{P}$ and $(V \setminus B)^c \notin \mathcal{P}$. Consider

$$W = U \cap V \in \psi(x)$$

implies $(W \setminus A)^c \notin \mathcal{P}$ and $(W \setminus B)^c \notin \mathcal{P}$. Hence,

$$(W \setminus (A \cap B))^c = (W \setminus A)^c \cap (W \setminus B)^c \notin \mathcal{P}.$$

Then, x belongs to $\Phi(A \cap B)$. Hence,

$$\Phi(A) \cap \Phi(B) \subseteq \Phi(A \cap B).$$

(ii) By using Corollary 3.1, we have $X^\diamond = (X \setminus A)^\diamond$. Hence,

$$\Phi(A) = X \setminus (X \setminus A)^\diamond = X \setminus X^\diamond.$$

Theorem 4.7. *For a generalized primal topological space $(X, \mathfrak{g}, \mathcal{P})$. Consider the collection*

$$\sigma = \{A \subseteq X : A \subseteq \Phi(A)\},$$

thus σ is a generalized primal topology induced by an operator Φ. Moreover, $\sigma = \mathfrak{g}^\diamond$.

Proof. It is clear that $\phi \subseteq \Phi(\phi)$. Then, $\phi \in \sigma$. If

$$\{A_\alpha : \alpha \in \Lambda\} \subseteq \sigma,$$

then for each $\alpha \in \Lambda$ we have

$$A_\alpha \subseteq \Phi(A_\alpha) \subseteq \Phi(\bigcup_{\alpha\in\Lambda} A_\alpha).$$

Hence,

$$\bigcup_{\alpha\in\Lambda} A_\alpha \subseteq \Phi(\bigcup_{\alpha\in\Lambda} A_\alpha).$$

Thus, σ is a generalized primal topology on X induced by Φ.

Let $U \in \mathfrak{g}^\diamond$ and $x \in U$. Then, from Theorem 4.3, there exists $V \in \psi(x)$ and $P \notin \mathcal{P}$ satisfy

$$x \in V \cap P \subseteq U.$$

Since $P \subseteq (V \setminus U)^c$, then $(V \setminus U)^c \notin \mathcal{P}$. Hence, $x \in \Phi(U)$ implies $U \in \Phi(U)$. Therefore, $\mathfrak{g}^\diamond \subseteq \sigma$. On the other hand let $A \in \sigma$, then $A \subseteq \Phi(A)$ implies

$$A \subseteq X \setminus (X \setminus A)^\diamond$$

and

$$(X \setminus A)^\diamond \subseteq X \setminus A.$$

Thus, $X \setminus A$ is $(\mathfrak{g}, \mathcal{P})$-closed. Then, $A \in \mathfrak{g}^\diamond$. Therefore, $\sigma \subseteq \mathfrak{g}^\diamond$.

5. Conclusions

Coinciding with the great spread of many literatures in various fields important mathematical structures appeared in the theory of topology coinciding with this scientific revolution. For example, the concept of generalized topology appeared, which is based on the concept of "generalized open set" and this space was more generalized than the topological space as the intersection condition was neglected. This space has been extensively studied; much literature has been written about it, and many properties and theories have been studied about it.

In this paper, we have made a new contribution to the field of generalized topology by studying the concept of primal generalized topology, which has a lot of interesting properties. The results obtained in this paper are preliminary. Future research could give more insights by exploring further properties of generalized primal topology. This work opens up the door for possible contributions to this trend by combining primal structures with generalized structures in the theory of generalized topology. If possible, we are looking forward to connecting this notion with some ideas like supra-topology and infra-topology.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Conflicts of interest

The authors declare no conflicts of interest.

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