AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(10): 24153–24161.
DOI: 10.3934/math.20231231
Received: 22 April 2023
Revised: 26 July 2023
Accepted: 26 July 2023
Published: 10 August 2023

Research article

Gradient estimates in generalized Orlicz spaces for quasilinear elliptic equations via extrapolation

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Abstract: The gradient estimates in the generalized Orlicz space for weak solutions of a class of quasilinear elliptic boundary value problems are obtained using the modern technique of extrapolation. The coefficients are assumed to have small BMO seminorms, and the boundary of the domain is sufficiently flat in the sense of Reifenberg. As a corollary, we apply our results to the variable Lebesgue spaces.

Keywords: elliptic PDE of p-Laplacian; Reifenberg flat domain; generalized Orlic space; extrapolation
Mathematics Subject Classification: 35R35, 42B20, 46E30

1. Introduction

The Rubio de Francia extrapolation is a powerful tool to deal with the weighted norm inequalities for operators in harmonic analysis, which was first discovered by Rubio de Francia [24]. We refer to [11] for more information on the history of extrapolation and an extensive bibliography. The extrapolation was explored to prove norm inequalities on Banach spaces, variable Lebesgue spaces [8, 10–12] and generalized Orlicz spaces [14], provided that the maximal operator is bounded on their associate spaces. S. Liang and S. Zheng [25] proved a global C-Z type estimate in the framework of Lorentz spaces for a variable power of the gradients to the zero-Dirichlet problem of general nonlinear elliptic equations with the nonlinearities satisfying Orlicz growth. It is mainly assumed that the variable exponents $p(x)$ satisfy the log-hölder continuity, while the nonlinearity and underlying domain (A, Ω) is (δ, R_0) in $x \in \Omega$. G. Mingione and V. Rădulescu [26] provided an overview of recent results concerning elliptic variational problems with nonstandard growth conditions and related to different kinds of nonuniformly elliptic operators. S. Yang, D. Yang and W.Yuan [27] investigated (weighted) global gradient estimates for Dirichlet boundary value problems of second-order elliptic equations of divergence form with an elliptic symmetric part and a BMO antisymmetric part in Ω, and obtain the global gradient estimate, respectively, in (weighted) Lorentz spaces, (Lorentz-) Morrey spaces, (Musielak-) Orlicz spaces and variable Lebesgue spaces. A. Vitolo [28] considered the Dirichlet problem for partial trace operators which include the smallest and the largest eigenvalue of the Hessian matrix, proved an interior Lipschitz estimate under a non-standard assumption: the solution exists in a larger, unbounded domain, and vanishes at infinity, and extend a few qualitative properties of solutions, known for uniformly elliptic operators, to partial trace operators. On this basis, we apply the extrapolation theorem of Rubio de Francia combined with some standard techniques from the theory of partial differential equations to get the gradient estimates in generalize Orlicz spaces for weak solutions of elliptic equations of p-Laplacian type.

Fix $p \in (1, \infty)$. We then study the following quasilinear boundary value problems of p-Laplacian type

$$\begin{cases} \operatorname{div}\left((A\nabla u \cdot \nabla u)^{\frac{p-2}{2}} A\nabla u\right) = \operatorname{div}\left(|\mathbf{f}|^{p-2}\,\mathbf{f}\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded domain in $\mathbb{R}$ with nonsmooth boundary $\partial\Omega$. Function $\mathbf{f} := (f_1, \ldots, f_m)$ is a given vector valued function at least in $L^p(\Omega, \mathbb{R}^n)$. The coefficient matrix $A := \{a_{ij}(x)\}_{n\times n}$ is a symmetric matrix with measurable entries and satisfies the uniform ellipticity condition

$$\Lambda^{-1}|\xi|^2 \le A(x)\xi \cdot \xi \le \Lambda|\xi|^2, \tag{1.2}$$

for all $x \in \mathbb{R}^n$, almost every $x \in \Omega$ and some positive constant Λ. A solution to Eq (1.1) is understood in the standard weak sense, that is, $u \in W_0^{1,p}(\Omega)$ is a weak solution of Eq (1.1) if for any test function $\phi \in W_0^{1,p}(\Omega)$, it holds

$$\int_\Omega (A\nabla u \cdot \nabla u)^{\frac{p-2}{2}} A\nabla u \cdot \nabla\phi \, dx = \int_\Omega |\mathbf{f}|^{p-2}\,\mathbf{f} \cdot \nabla\phi \, dx.$$

To study the regularity of solutions of Eq (1.1), it seems to obtain the following implication, for a given function space $\mathcal{F}$,

$$|\mathbf{f}| \in \mathcal{F} \Longrightarrow \nabla u \in \mathcal{F},$$

under minimal conditions on the coefficients and on the boundary of the domain. Let $\mathcal{F} = L^q$. When A is the identity matrix, Dibebedetto and Manfredi [13] and Ivaniec [17] obtained $W^{1,q}$ regularity results. Kinnunen and Zhou [18, 19] extended $W^{1,q}$ regularity results to the case $A \in VMO$ and $\partial\Omega \in C^{1,\alpha}$. When A is (δ, R)-vanishing and Ω is a (δ, R)-Reibenberg flat, Byun, Wang and Zhou [4] considered the global $W^{1,q}$ regularity. Let $\mathcal{F} = L^\phi$ be a Orlicz space. Byun, Yao and Zhou [5] obtained the gradient estimates in Orlicz space for weak solutions of Eq (1.1) with small BMO coefficients A in δ-Reifenberg flat domain. Let $\mathcal{F} = L^q_\omega$ be the weighted Lebesgue space. Mengesha and Phuc [22, 23] obtained a weighted version of gradient estimates for weak solutions of Eq (1.1) with A is (δ, R)-vanishing and Ω is a (δ, R)-Reibenberg flat. The main approach is based on the method of approximation developed by Caffarelli and Peral [6] and makes use of techniques of weak compactness, the Vitali covering lemma and the Hardy-Littlewood maximal function.

The main result of this paper is a global and generalized Orlicz space estimate for the gradient of solutions of (1.1). The estimate generalizes the classical global L^q estimates obtained in [4] and the Orlicz space estimate extended in [5]. We find that the condition in [4, 5] are sufficient to obtain the generalized Orlicz space estimate considered in this work. To that end, we follow [4, 5] to state the conditions on the regularity of coefficients and the boundary of the domain. The coefficients of $A = \{a_{ij}\}$ considered in this note are in the BMO space and their semi-norms are small enough.

Definition 1.1. We say that $A(x)$ satisfies the (δ, R)-BMO condition for some $\delta, R > 0$, if

$$\sup_{0<r\le R}\sup_{x\in\mathbb{R}^n} \fint_{B_r(x)} \left|A(y) - \bar{A}_{B_r(x)}\right| dy \le \delta, \tag{1.3}$$

where $B_r(x) := \{y \in \mathbb{R}^n : |x - y| < r\}$ and

$$\bar{A}_{B_r(x)} := \fint_{B_r(x)} A(y)\, dy.$$

For the domain Ω, at each boundary point and every scale the boundary of the domain is between two hyperplanes separated by a distance which depends on the scale. Precisely, we require that Ω in this paper is a δ-Reifenberg domain.

Definition 1.2. We say that a domain Ω is (δ, R)-Reifenberg flat if for every $x \in \partial\Omega$ and every $r \in (0, R]$, there exists a coordinate system $\{y_1, \ldots, y_n\}$, which depends on r and x, so that $x = 0$ in this coordinated system and

$$B_r(0) \cap \{y_n > \delta r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{y_n > -\delta r\}.$$

The objective of this paper is to discuss the regularity of nonlinear elliptic equations of p-Laplacian type in the genralized Orlicz space. Generalized Orlicz spaces, also known as Musielak-Orlicz spaces, include a number of spaces of interest in harmonic analysis and PDEs as special cases, such as, Lebesgue spaces, classical Orlicz spaces and variable Lebesgue spaces. The monographs [8] and [14] present a framework for the basics of these spaces and some properties in harmonic analysis. Very recently, the study of differential equations related to the generalized Orlicz spaces has attracted many authors with increasing intensity; see, for instance, [1–3, 7, 15, 16, 25–27]. For reader's convenience, we recall some definitions about the generalized Orlicz space.

Definition 1.3. A function $\phi : (0, \infty) \to \mathbb{R}$ is *almost increasing* if there exists a constant $a, (a > 1)$ such that $\phi(s) \le a\phi(t)$ for all $0 < s < t$. *Almost decreasing* is defined analogously.

Definition 1.4. Let $\phi : \Omega \times [0, \infty) \to \mathbb{R}$ and $p,\ q > 0$. We say that ϕ satisfies:
$(\mathrm{Inc})_p$: if $\frac{\phi(x,t)}{t^p}$ is increasing;
$(\mathrm{aInc})_p$: if $\frac{\phi(x,t)}{t^p}$ is almost increasing;
$(\mathrm{Dec})_q$: if $\frac{\phi(x,t)}{t^q}$ is decreasing;
$(\mathrm{aDec})_q$: if $\frac{\phi(x,t)}{t^q}$ is almost decreasing;
all conditions should hold for almost every $x \in \Omega$ and the almost increasing/decreasing constant should be independent of x. We say that ϕ satisfies (aInc), (aDec) if there exists $p > 1$ or $q < \infty$ such that ϕ satisfies $(\mathrm{aInc})_p$ and $(\mathrm{aDec})_q$.

Remark 1.1. If ϕ satisfies $(\mathrm{aInc})_p$ and $(\mathrm{aDec})_q$ for $p,\ q > 0$, then $p \le q$.

Definition 1.5. Let $\phi : [0, \infty) \to [0, \infty]$ be increasing with $\phi(0) = 0$, $\lim_{t\to0^+} \phi(t) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$. We say that such ϕ is a (weak) Φ-function if it satisfies $(\mathrm{aInc})_1$ on $(0, \infty)$. The set of weak Φ-functions is denoted by Φ_w.

Definition 1.6. A function $\varphi : \Omega \times [0,\infty) \to [0,\infty]$ is said to be a (generalized weak) Φ- function, denoted $\varphi \in \Phi_w(\Omega)$, if $x \to \varphi(y, |f(x)|)$ is measurable for every measurable f, $\varphi(y,\cdot)$ is a weak Φ-function for almost every $y \in \Omega$ and φ satisfies $(\text{aInc})_1$.

Definition 1.7. Let $\varphi \in \Phi_w(\Omega)$ and define the semimodular $\varrho_{\varphi(\cdot)}$ for any measurable function f on Ω by

$$\varrho_{\varphi(\cdot)}(f) := \int_\Omega \varphi(x, |f(x)|)dx.$$

The generalized Orlicz space, also called a Musielak–Orlicz space, is defined as the set

$$L^{\varphi(\cdot)}(\Omega) := \left\{ f \text{ measurable on } \Omega : \lim_{\lambda\to 0} \varrho_{\varphi(\cdot)}(\lambda f) = 0 \right\}$$

equipped with the (Luxemburg) norm

$$\|f\|_{L^{\varphi(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0 : \ \varrho_{\varphi(\cdot)}(\frac{f}{\lambda}) \le 1 \right\}.$$

Let $\varphi \in \Phi_w(\Omega)$ and define a left-inverse of it by

$$\varphi^{-1}(x,\tau) := \inf\{t \ge 0 : \ \varphi(x,t) \ge \tau\}.$$

To state our main results, we still recall some conditions for the weak Φ-function.

Definition 1.8. Let $\varphi \in \Phi_w(\Omega)$. We say that φ satisfies:
(A0) There exists $\alpha \in (0,1]$ such that $\alpha < \varphi^{-1}(x,1) \le \frac{1}{\alpha}$ for almost every $x \in \Omega$.
(A1) There exists $\beta \in (0,1)$ such that

$$\beta\varphi^{-1}(x,t) \le \varphi^{-1}(y,t),$$

for every $t \in [1, \frac{1}{|B|}]$, almost every $x, y \in B \cap \Omega$ and every ball B with $|B| \le 1$.
(A2) For every $s > 0$ there exist $\gamma \in (0,1]$ and $h \in L^1(\Omega) \cap L^\infty(\Omega)$ such that

$$\gamma\varphi^{-1}(x,t) \le \varphi^{-1}(y,t)$$

for almost every $x, y \in \Omega$ and every $t \in [h(x) + h(y), s]$.

2. Main results

Now we state our theorem as follows.

Theorem 2.1. *Given that φ is a weak Φ-function that satisfies assumptions (A0)–(A2) and (aDec) and $(\text{aInc})_{p_0}$ with $p_0 > p > 1$, there exists a small $\delta = \delta(n, p, \varphi, \Lambda) > 0$ such that if A is uniformly elliptic and (δ, R)-vanishing, Ω is (δ, R)-Reifenberg flat and $|\mathbf{f}| \in L^\varphi(\Omega)$, then the unique weak solution $u \in W_0^{1,p}(\Omega)$ of the problem* (1.1) *satisfies*

$$\|\nabla u\|_{L^\varphi(\Omega)} \le C \|\mathbf{f}\|_{L^\varphi(\Omega)},$$

where $C (C < \infty)$ is independent of u .

Let $p(\cdot) : \Omega \to [0,\infty)$ be a measurable function. Then the variable Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf\left\{\lambda \in (0,\infty) : \int_\Omega [|f(x)|/\lambda]^{p(x)}\,dx \le 1\right\} < \infty.$$

Recall some well-known concepts from variable exponent spaces. Define

$$p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) \quad \text{and} \quad p_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x).$$

The measurable function $\frac{1}{p}$ is said to be log-Hölder continuous, $\frac{1}{p} \in C^{\log}$, if there exists a positive constant C such that, for any distinct $x,\ y \in \Omega$,

$$\left|\frac{1}{p(x)} - \frac{1}{p(y)}\right| \le \frac{C}{\log(e + 1/|x-y|)}.$$

Nekvinda's decay condition for variable exponent spaces can be stated as follows: $1 \in L^{s(\cdot)}(\Omega)$, with $\frac{1}{s(x)} = \frac{1}{p(x)} - \frac{1}{p_\infty}$ and $p_\infty \in [1,\infty]$. Equivalently, this means that there exists $c > 0$ such that

$$\int_{p(x)\neq p_\infty} c^{\frac{1}{\frac{1}{p(x)}-\frac{1}{p_\infty}}}\,dx < \infty.$$

From [14, Lemma 7.1.1, Proposition 7.1.2 and Proposition 7.1.3], we know that if $p_- > 0$, $\frac{1}{p}$ is log-Hölder continuous and p satisfies Nekvinda's decays condition, then $t^{p(x)}$ satisfies $(\text{aInc})_{p_-}$, $(\text{aDec})_{p_+}$, (A0), (A1) and (A2). Thus, we have

Corollary 2.1. *Let $p(\cdot) : \Omega \to [0,\infty)$ be a measurable function. Suppose that $\frac{1}{p}$ is log-Hölder continuous and p satisfies Nekvinda's decays condition with $p_- > p$. Then there exists a small $\delta = \delta(n,p,\varphi,\Lambda) > 0$ such that if A is uniformly elliptic and (δ,R)-vanishing, Ω is (δ,R)-Reifenberg flat and $|\mathbf{f}| \in L^{p(\cdot)}(\Omega)$, then the unique weak solution $u \in W_0^{1,p}(\Omega)$ of the problem* (1.1) *satisfies*

$$\|\nabla u\|_{L^{p(\cdot)}(\Omega)} \le C\,\|\mathbf{f}\|_{L^{p(\cdot)}(\Omega)}.$$

Our proof is based on an extrapolation on the generalized Orlicz space and an weighted regularity estimate for solutions of Eq (1.1) which is obtained in [23]. We begin with recalling the definition of Muckenhoupt weights. As in [21], a nonnegative function $\omega \in L^1_{\text{loc}}(\mathbb{R}^n)$ is called an A_s weight, $1 < s < \infty$, if there exists a positive constant C such that for all balls B

$$\left(\fint_B \omega(x)\,dx\right)\left(\fint_B \omega(x)^{1-s'}\,dx\right)^{s-1} \le C.$$

We also say that a nonnegative function ω satisfies the A_1 condition if there exists a constant C such that for all balls B

$$\fint_B \omega(x)\,dx \le C \inf_{x\in B}\omega(x).$$

Theorem 2.2. *Let $\mathcal{F}$ be a given family of pairs (f, g) of non-negative and not identically zero measurable functions on $\mathbb{R}^n$. Suppose that for any $p \in [1, \infty)$ and all $\omega \in A_p(\mathbb{R}^n)$,*

$$\|f\|_{L^p(\omega)} \leq C_{n,p,[\omega]_{A_p}} \|g\|_{L^p(\omega)}, \quad (f, g) \in \mathcal{F}. \tag{2.1}$$

Suppose φ is a weak Φ-function that satisfies assumptions (A0)-(A2) and (aDec). If $p > 1$, then we also assume (aInc). Then

$$\|f\|_{L^{\varphi(\cdot)}} \leq C\|g\|_{L^{\varphi(\cdot)}}, \quad (f, g) \in \mathcal{F}. \tag{2.2}$$

Remark 2.1. Theorem 2.2 is the so-called Rubio de Francia extrapolation theorem in generalized Orlicz spaces which was obtained in [9, Corollary 4.10]. There is, however, a subtle difference between Theorem 2.2 and [9, Corollary 4.10]: in the latter both the hypothesis and the conclusions are assumed to hold for all pairs $(f, g) \in \mathcal{F}$ for which the left-hand sides are finite. Here we do not make such assumptions, in particular, we do have that the infiniteness of the left-hand side will imply that of the right-hand side. This formulation is more convenient for our purposes, and its proof becomes a simple consequence of [9, Corollary 4.10].

Remark 2.2. Carefully reading the proof of [9, Theorem 4.5 and Corollary 4.10], we can replace all $\omega \in A_p$ with all $\omega \in A_1$ in [9, Corollary 4.10]. So do in Theorem 2.2.

Proof. We employ the idea and statements from [20, Lemma 3.3]. For completeness, we give the details. Given a family $\mathcal{F}$ as in the statement and an arbitrary large number $N > 0$, we consider the new family

$$\mathcal{F}_N := \left\{(f_N, g) : \ (f, g) \in \mathcal{F}, \ f_N = f\mathbf{1}_{\{x \in B(0,N): f(x) \leq N\}}\right\}.$$

Notice that, for any $p \in (0, \infty)$ and $\omega \in A_p(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |f(x)|^p \omega(x)\, dx \leq N^p \omega\left(B(0, N)\right) < +\infty. \tag{2.3}$$

By [9, Lemma 3.2], for all $r \in (0, \infty)$,

$$\int_{\mathbb{R}^n} \varphi\left(x, |f_N(x)|^r\right)\, dx = \int_{B(0,N)} \varphi\left(x, N^r\right)\, dx < +\infty. \tag{2.4}$$

From (2.1) and the fact that $f_N \leq f$, we clearly obtain that the same estimate holds for every pair in $\mathcal{F}_N$ (with a constant uniform on N) with a left-hand side that is always finite by (2.3). We can apply [9, Lemma 3.2] to $\mathcal{F}_N$ to conclude that (2.2) holds for all pairs $(f_N, g) \in \mathcal{F}_N$ (with a constant uniform on N), since the left-hand side is always finite by (2.4). Then we invoke the Fatou property [14, Lemma 3.3.8] to complete the proof of Theorem 2.2. □

The following global regularity estimates for solutions to the quasilinear elliptic boundary value problems (1.1) are the special cases of [23, Theorem 2.1].

Lemma 2.1. *Let $1 < p < q < \infty$ and let $\omega \in A_1$ weight. Then there exist positive constant C and δ such that the following holds. For a given vector field $\mathbf{f} \in L^q_\omega(\Omega, \mathbb{R}^n)$, the boundary valued problem* (1.1) *in a (δ, R)-Reifenberg flat domain Ω, with A satisfying* (1.2) *and the (δ, R)-BMO condition for some $R > 0$, has a unique weak solution $u \in W^{1,p}_0(\Omega)$ satisfying $\nabla u \in L^q_\omega(\Omega, \mathbb{R}^n)$ with the estimate*

$$\|\nabla u\|_{L^q_\omega(\Omega)} \leq C\|\mathbf{f}\|_{L^q_\omega(\Omega)}.$$

Here the constant C and δ depend only on $n, p, q, R, \Lambda, \Omega$ and $[\omega]_1$.

Remark 2.3. We remark that in fact Mengesha and N.C. Phuc in [23] proved more general results, that is, the weight functions ω can belong to $A_{q/p}$. Lemma 2.1 is sufficient for our proof.

Remark 2.4. The unweighted version of Lemma 2.1 was proved in [4, Theorem 1.8].

Proof of Theorem 2.1. Since $\mathbf{f} \in L^{\varphi}(\Omega, \mathbb{R}^n)$, from [14, Corollary 3.7.9], we have $\mathbf{f} \in L^{p_0}(\Omega, \mathbb{R}^n) \subset L^p(\Omega, \mathbb{R}^n)$. Then by [4, Theorem 1.8], a unique weak solution $u \in W_0^{1,p}(\Omega)$ exists for (1.1). Now we claim that there exists a weight $\omega \in A_1$ such that $\mathbf{f} \in L^q_\omega(\Omega, \mathbb{R}^n)$. Indeed, from [21, (2.1.6) and Theorem 7.2.7], we deduce that, for any $x \in \mathbb{R}^n$, $\mathcal{M}(\mathbf{1}_{B(\vec{0}_n,1)})(x) \sim (|x|+1)^{-n}$ and for any $\varepsilon \in (0,1)$, $[\mathcal{M}(\mathbf{1}_{B(0,1)})(x)]^{\epsilon} \in A_1$, which, together Hölder's inequality, implies that, for any $q \in (p, p_0)$,

$$\|\mathbf{f}\|_{L^q_\omega(\Omega)} = \left[\int_\Omega |\mathbf{f}(x)|^q [\mathcal{M}(\mathbf{1}_{B(0,1)})(x)]^{\varepsilon}\, dx\right]^{\frac{1}{q}} \leq C\|\mathbf{f}\|_{L^q(\Omega)} \leq C\|\mathbf{f}\|_{L^{p_0}(\Omega)}|\Omega|^{1-\frac{q}{p_0}} < \infty.$$

By this and Lemma 2.1, for all $\omega \in A_1$, if $\|\mathbf{f}\|_{L^q_\omega(\Omega)} < +\infty$,

$$\|\nabla u\|_{L^q_\omega(\Omega)} \leq C\|\mathbf{f}\|_{L^q_\omega(\Omega)}.$$

By Remark 2.2, we conclude that

$$\|\nabla u\|_{L^{\varphi}(\Omega)} \leq C\|\mathbf{f}\|_{L^{\varphi}(\Omega)}.$$

Thus, we finish the proof of Theorem 2.1. □

3. Conclusions

We apply the extrapolation theorem of Rubio de Francia combined with some standard techniques from the theory of partial differential equations to get the gradient estimates in generalize Orlicz spaces for weak solutions of elliptic equations of p-Laplacian type. The estimate generalizes the classical global L^q estimates obtained in [4] and the Orlicz space estimate extended in [5].To that end, we follow [4,5] to state the conditions on the regularity of coefficients and the boundary of the domain.The coefficients are assumed to have small BMO seminorms, and the boundary of the domain is sufficiently flat in the sense of Reifenberg. As a corollary, we apply our results to the variable Lebesgue spaces.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

We are greatly indebted to the anonymous referee for helpful comments and stimulating hints. This project was supported Higher Education Innovation Fund project of Gansu Province (Grant No.2021A-193).

Conflict of interest

The authors declare that they have no competing interests.

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