Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(10): 24116–24131.
DOI: 10.3934/math.20231229
Received: 14 May 2023
Revised: 17 July 2023
Accepted: 20 July 2023
Published: 09 August 2023

Research article

Uniform boundedness of solutions to linear difference equations with periodic forcing functions

Rinko Miyazaki[1,2], **Dohan Kim**[3,*] **and Jong Son Shin**[1]

[1] Graduate School of Engineering, Shizuoka University, Hamamatsu, Shizuoka 432-8561, Japan
[2] Graduate School of Engineering Sciences, Osaka University, Toyonaka, Osaka 560-8531, Japan
[3] Department of Mathematics, Seoul National University, Gwanak-gu, Seoul 08826, Korea

* **Correspondence:** Email: dhkim@snu.ac.kr; Tel: +821071318995; Fax: +8228874694.

Abstract: In this paper we give criteria on the uniform boundedness of the solutions to linear difference equations (LEs) with periodic forcing functions. First, we give a necessary and sufficient condition that the sequence $\{L^n\}$ of a square matrix L is bounded, from which a criterion on the uniform boundedness of the solutions to LEs is obtained. Second, a criterion on the uniform boundedness of the solutions for LEs with periodic forcing functions is given by applying a certain representation of solutions. In connection with LEs with delay, we give the characteristic equation of a matrix under the commuting condition.

Keywords: uniform boundedness; spectral decomposition; periodic forcing functions; linear difference equation; characteristic equation
Mathematics Subject Classification: Primary: 39A05, 39A06; Secondary: 15A18

1. Introduction

Let $\mathbb{C}$ be the set of all complex numbers and $\mathbb{R}$ the set of all real numbers. We set $\mathbb{N} = \{1, 2, \cdots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$. We denote by $\mathbb{C}^p$ the set of all p dimensional complex column vectors, and by $M_p(\mathbb{C})$ the set of all $p \times p$ complex matrices.

In this paper we consider periodic linear difference equations of the forms

$$x(n+1) = Hx(n), \tag{1.1}$$

$$x(n+1) = Hx(n) + b(n) \tag{1.2}$$

where $n \in \mathbb{Z}$, $H \in M_p(\mathbb{C})$, $x(n) \in \mathbb{C}^p$ and $b(n) \in \mathbb{C}^p$ is a vector valued function with period $\rho \in \mathbb{N}$.

The purpose of this paper is to give criteria of the uniform boundedness of the solutions to the above equations. Criteria of the boundedness of solutions were given in [1–5]. Some related one-dimensional results can be found in [6] (see also the references therein).

First, we give a necessary and sufficient condition for the sequence $\{L^n\}$ of a square matrix L to be bounded, from which the criterion on the uniform boundedness of the solutions to the Eq (1.1) is obtained immediately.

Second, a criterion on the uniform boundedness of the solutions for the Eq (1.2) is given by applying a certain representation of solutions developed in [3]. It seems that its proof is not easy to obtain from the usual representation of the solution by the variation of constants formula.

Finally, in connection with the Eq (1.2) with delay, we give the characteristic equation of a matrix under the commuting condition. In more details, making use of the simultaneous diagonalization theorem under the commuting condition $AB = BA$, we can apply the preceding results to the periodic linear difference equation with delay of the form

$$x(n+1) = Ax(n) + Bx(n-\rho) + f(n), \tag{1.3}$$

where $A,\ B \in M_p(\mathbb{C})$, $x(n) \in \mathbb{C}^p$ and $f(n) \in \mathbb{C}^p$ is a vector-valued function with period $\rho \in \mathbb{N}$. But we only consider the characteristic equation of the matrix M in a reduced equation $y(n+1) = My(n) + g(n)$ derived from the Eq (1.3).

2. Boundedness of the sequence $\{L^n\}$

2.1. *Spectral decomposition theorem*

We define $(n)_k$ as follows.

$$(n)_k = \begin{cases} 1, & (k = 0), \\ n(n-1)(n-2)\cdots(n-k+1), & (k = 1, 2, \cdots, n), \\ 0, & (k = n+1, n+2, \cdots). \end{cases}$$

Denoting by $\binom{n}{k}$ a binomial coefficient, we have

$$\frac{(n)_k}{k!} = \binom{n}{k},\ (n)_n = n! \ \text{ and } \ (n)_k = 0\ (k > n).$$

E or E_p is the identity matrix in $M_p(\mathbb{C})$. We denote by O and 0 the zero matrix and the zero vector, respectively.

Moreover, we denote by $\sigma(L)$ the set of all eigenvalues of a matrix $L \in M_p(\mathbb{C})$ and by $h_\eta(L)$ the index of $\eta \in \sigma(L)$. Then $G_\eta(L) = \mathcal{N}((L - \eta E)^{h_\eta(L)})$ is the generalized eigenspace of $\eta \in \sigma(L)$, where $\mathcal{N}(L) = \{x \in \mathbb{C}^p \ : \ Lx = 0\}$. Clearly, $\mathbb{C}^p$ is decomposed as $\mathbb{C}^p = \bigoplus_{\eta\in\sigma(L)} G_\eta(L)$. We denote by $Q_\eta(L)$ the projection from $\mathbb{C}^p$ to $G_\eta(L)$. Then $Q_\eta^2(L) = Q_\eta(L)$ and $LQ_\eta(L) = Q_\eta(L)L$.

Now, we state the spectral decomposition theorem for the matrix $L \in M_p(\mathbb{C})$, which plays an important role in this paper.

Lemma 1. [1] *Let $\eta \in \sigma(L)$. If $\eta \neq 0$, then*

$$L^n = \sum_{\eta\in\sigma(L)} \sum_{j=0}^{h_\eta(L)-1} \binom{n}{j} \eta^{n-j}(L-\eta E)^j Q_\eta(L), \quad n = 0, 1, 2, \cdots . \tag{2.1}$$

In particular, operating $Q_\eta(L)$ to (2.1), we have

$$L^n Q_\eta(L) = \sum_{j=0}^{h_\eta(L)-1} \binom{n}{j} \eta^{n-j}(L-\eta E)^j Q_\eta(L). \tag{2.2}$$

If $\eta = 0 \in \sigma(L)$, then

$$L^n Q_\eta(L) = \begin{cases} O, & n \geq h_\eta(L), \\ L^n Q_\eta(L), & n \leq h_\eta(L) - 1. \end{cases}$$

2.2. Asymptotic behavior of L^n

We discuss the asymptotic behavior of L^n as $n \to \infty$ using Lemma 1. For $L \in M_p(\mathbb{C})$ we take the operator norm $\|L\| = \sup_{\|x\|\leq 1} \|Lx\|$. Then we have

$$\|Lx\| \leq \|L\|\|x\|.$$

Clearly, if $\lim_{n\to\infty} \|L^n u\| = \infty$ for some $u \in \mathbb{C}^p \setminus \{0\}$, then $\lim_{n\to\infty} \|L^n\| = \infty$.

For $\lambda \in \sigma(L)$, we set $P_\lambda = Q_\lambda(L)$. We also set

$$\sigma_S(L) = \{\eta \in \sigma(L) : |\eta| < 1\}, \ \sigma_U(L) = \{\eta \in \sigma(L) : |\eta| > 1\}$$

and

$$\sigma_N(L) = \{\eta \in \sigma(L) : |\eta| = 1\}.$$

The following lemma, which slightly modifies 1) and 2) of Theorem 6.1 in [3], is the most probably known. We give a proof of it for completeness.

Lemma 2. *Let $\lambda \in \sigma(L)$.*

1) *If $\lambda \in \sigma_S(L)$, then*

$$\|L^n P_\lambda\| \leq (n)_{h_\lambda(L)-1} |\lambda|^{n-h_\lambda(L)} C(\lambda) \|P_\lambda\|, \quad n \geq h_\lambda(L)$$

where $C(\lambda) = h_\lambda(L) \max_{0\leq j\leq h_\lambda(L)-1} \|(L-\lambda E)^j\|$, hence

$$\lim_{n\to\infty} \|L^n P_\lambda\| = 0.$$

2) *If $\lambda \in \sigma_U(L)$, then*

$$\lim_{n\to\infty} \|L^n P_\lambda u\| = \infty$$

for all $u \in \mathbb{C}^p$ satisfying $P_\lambda u \neq 0$.

Proof. 1) Let $\lambda \in \sigma_S(L)$.

(a) The case $\lambda \neq 0$: Since $\lim_{n\to\infty} \frac{(n)_j}{r^n} = 0, (r > 1)$, it follows from Lemma 1 that for a sufficiently large n,

$$\begin{aligned} \|L^n P_\lambda\| &\le \sum_{j=0}^{h_\lambda(L)-1} \left| \binom{n}{j} \lambda^{n-j} \right| \|(L-\lambda E)^j P_\lambda\| \\ &\le \sum_{j=0}^{h_\lambda(L)-1} \frac{(n)_j}{j!} |\lambda|^{n-j} \frac{1}{h_\lambda(L)} C(\lambda) \|P_\lambda\| \\ &\le (n)_{h_\lambda(L)-1} |\lambda|^{n-h_\lambda(L)} C(\lambda) \|P_\lambda\| \ \longrightarrow 0 \text{ as } n \to \infty. \end{aligned}$$

Thus $\lim_{n\to\infty} \|L^n P_\lambda\| = 0$.

(b) The case $\lambda = 0$: Clearly, $\lambda^0 = 1, \lambda^n = 0\ (n \neq 0)$ and hence $L^n P_0 = O$.

Combining (a) and (b), we conclude that $\lim_{n\to\infty} L^n P_\lambda = O$ holds.

2) Let $\lambda \in \sigma_U(L)$.

For every $u \in \mathbb{C}^p$ satisfying $P_\lambda u \neq 0$, there is a d, $1 \le d \le h_\lambda(L)$ such that

$$(L - \lambda E)^{d-1} P_\lambda u \neq 0, \ (L - \lambda E)^d P_\lambda u = 0,$$

and hence,

$$L^n P_\lambda u = \frac{(n)_{d-1}}{(d-1)!} \lambda^{n-d+1} (L - \lambda E)^{d-1} P_\lambda u + o((n)_{d-1} \lambda^n)\ (n \to \infty).$$

Since $|\lambda| > 1$, we have $\lim_{n\to\infty} \|L^n P_\lambda u\| = \infty$. Therefore, the proof is complete. □

The following lemma is certainly known, but we also give a proof.

Lemma 3. *Let $\lambda \in \sigma_N(L)$. If $h_\lambda(L) > 1$, then there exists a $u \in \mathbb{C}^p$ such that $P_\lambda u \neq 0$ and* $\lim_{n\to\infty} \|L^n P_\lambda u\| = \infty$.

Proof. Since $h_\lambda(L) > 1$, there exists a $v \neq 0$ such that

$$(L - \lambda E)v \neq 0, \ \ (L - \lambda E)^2 v = 0.$$

It follows by induction that

$$L^n v = n\lambda^{n-1} L v - (n-1)\lambda^n v, \ \ n = 2, 3, \cdots.$$

Hence we obtain

$$\lim_{n\to\infty} \left\| \frac{L^n v}{n} \right\| = \lim_{n\to\infty} \|Lv - \lambda v + \frac{1}{n} \lambda v\| = \|Lv - \lambda v\| \neq 0.$$

Thus there exists a $u \in \mathbb{C}^p$ such that $v = P_\lambda u$. Therefore, $\lim_{n\to\infty} \|L^n P_\lambda u\| = \infty$ holds. □

Now we consider the case when $\lambda \in \sigma_N(L)$ with $h_\lambda(L) = 1$.

Theroem 1. *Let $L \in M_p(\mathbb{C})$ and $\lambda \in \sigma_N(L)$. Then the following statements are equivalent*:

1) $h_\lambda(L) = 1$.

2) $\|L^n P_\lambda u\| = \|P_\lambda u\|$ *for all $u \in \mathbb{C}^p$ and $n \in \mathbb{N}$.*

3) $\|L^n P_\lambda\| = \|P_\lambda\|$ *for all $n \in \mathbb{N}$.*

Proof. 1) $\Longrightarrow$ 2). For any vector $u \in \mathbb{C}^p$ we have $L^n P_\lambda u = \lambda^n P_\lambda u$ by using (2.2). Thus

$$\|L^n P_\lambda u\| = \|\lambda^n P_\lambda u\| = |\lambda|^n \|P_\lambda u\| = \|P_\lambda u\|.$$

2) $\Longrightarrow$ 1). Assume that 1) does not hold. Then we have $h_\lambda(L) > 1$. It follows from Lemma 3 that there exists a $u \in \mathbb{C}^p$ such that $\lim_{n\to\infty} \|L^n P_\lambda u\| = \infty$. Thus we obtain that $\sup_{n\in\mathbb{N}} \|L^n P_\lambda u\| = \infty$ holds, which contradicts the assertion 2).

1) $\Longrightarrow$ 3). It follows from (2.2) that $L^n P_\lambda = \lambda^n P_\lambda$. Hence $\|L^n P_\lambda\| = \|P_\lambda\|$ holds.

3) $\Longrightarrow$ 2). This is obvious from the property of the operator norm. □

The following result is easily derived from Theorems 1–3.

Proposition 1. *Let $\sigma(L) = \sigma_S(L) \cup \sigma_N(L)$. Then the following statements are equivalent*:

1) $h_\lambda(L) = 1$ *for all* $\lambda \in \sigma_N(L)$.

2) $\sup_{n\in\mathbb{N}} \|L^n\| < \infty$.

3) $\sup_{n\in\mathbb{N}} \|L^n u\| < \infty$ *for all* $u \in \mathbb{C}^p$.

Proof. 1) $\Longleftrightarrow$ 2). Assume that 1) holds. Since $\sigma(L) = \sigma_S(L) \cup \sigma_N(L)$, we have $E = \sum_{\lambda\in\sigma(L)} P_\lambda$. Thus $L^n = \sum_{\lambda\in\sigma(L)} L^n P_\lambda$. It follows from Lemma 2 and Theorem 1 that $\|L^n P_\lambda\| < \infty$ for all $n \in \mathbb{N}$ and all $\lambda \in \sigma(L)$. Therefore, we have

$$\|L^n\| \leq \sum_{\lambda\in\sigma(L)} \|L^n P_\lambda\| < \infty.$$

Conversely, we assume that 2) holds. Since $\sup_{n\in\mathbb{N}} \|L^n\| < \infty$, we have $\|L^n P_\lambda\| \leq \|L^n\|\|P_\lambda\| < \infty$ for $\lambda \in \sigma_N(L)$. On the other hand, if $\lambda \in \sigma_N(L)$ with $h_\lambda(L) > 1$, then $\lim_{n\to\infty} \|L^n P_\lambda\| = \infty$ by Lemma 3, which leads to a contradiction. Hence 1) holds.

2) $\Longrightarrow$ 3) is obvious. 3) $\Longrightarrow$ 2) follows from the principle of uniform boundedness in Functional Analysis ([7, p.249]). □

A spectral radius $r_\sigma(L)$ of a matrix $L \in M_p(\mathbb{C})$ is defined by $r_\sigma(L) = \max\{|\lambda| \ : \ \lambda \in \sigma(L)\}$ and the spectral radius $r_\sigma(L)$ of L is given as follows:

$$r_\sigma(L) = \lim_{n\to\infty} \|L^n\|^{\frac{1}{n}}. \tag{2.3}$$

Clearly, $r_\sigma(L) \leq \|L\|$.

The following results follow from Theorems 1–3.

Lemma 4. *The following statements are equivalent*:

1) $r_\sigma(L) < 1$.

2) $\lim_{n\to\infty} \|L^n P_\lambda\| = 0$ *for all* $\lambda \in \sigma(L)$ *and* $n \in \mathbb{N}$.

3) $\lim_{n\to\infty} \|L^n\| = 0$ *for all* $n \in \mathbb{N}$.

Lemma 5. *Let* $r_\sigma(L) > 1$. *Then there exists a* $\lambda \in \sigma_U(L)$ *such that* $\lim_{n\to\infty} \|L^n P_\lambda u\| = \infty$ *for all* $u \in \mathbb{C}^p$ *satisfying* $P_\lambda u \neq 0$.

The following proposition gives a relationship between the spectral radius of a square matrix L and $\lim_{n\to\infty} L^n$.

Proposition 2. *Let $r_\sigma(L) = 1$. Then the following statements hold.*

(1) $\lim_{n\to\infty} \|L^n P_\lambda\| = 0$ *for all* $\lambda \in \sigma_S(L)$.

(2) $\sup_{n\in\mathbb{N}} \|L^n P_\lambda\| = \|P_\lambda\| < \infty$ *for all* $\lambda \in \sigma_N(L)$ *with* $h_\lambda(L) = 1$.

(3) *If* $\lambda \in \sigma_N(L)$ *with* $h_\lambda(L) > 1$, *then there exists a* $v \in \mathbb{C}^p$ *such that* $P_\lambda v \neq 0$ *and* $\lim_{n\to\infty} \|L^n P_\lambda v\| = \infty$.

2.3. *Uniform boundedness of the solution to the Eq* (1.1)

We denote by $x(n; \tau, w, b(\cdot))$ the solution of the Eq (1.2) through the point $(\tau, w) \in \mathbb{Z} \times \mathbb{C}^p$. Then $x(n; \tau, w) := x(n; \tau, w, 0)$ is the solution of the Eq (1.1) through the point (τ, w).

Definition 1. [8] *The solutions to the Eq* (1.2) *are said to be uniformly bounded if for any* $\alpha > 0$ *there exists a* $\beta(\alpha) > 0$ *such that* $\|x(n; \tau, w, b(\cdot))\| < \beta(\alpha)$ *for all* $(\tau, w) \in \mathbb{Z} \times B_\alpha$ *and* $n \geq \tau$, *where* $B_\alpha = \{w \in \mathbb{C}^p \ : \ \|w\| < \alpha\}$.

The solution of the Eq (1.1) through the point $(\tau, w) \in \mathbb{Z} \times \mathbb{C}^p$ is expressed as $x(n; \tau, w) = H^{n-\tau} w$. Therefore, the following result, which is concerned with [9, Theorem 4.9 and Theorem 4.13], follows immediately from Proposition 1.

Proposition 3. *The solutions to the Eq* (1.1) *are uniformly bounded if and only if every eigenvalue* η *of* H *satisfies either* $|\eta| < 1$ *or* $|\eta| = 1$ *with the index* $h_\eta(H) = 1$.

The following result is easily derived from Proposition 1 and Proposition 3.

Corollary 1. *Let* $\sigma(H) = \sigma_S(H) \cup \sigma_N(H)$. *Then the following statements are equivalent*:

1) $h_\eta(H) = 1$ *for all* $\eta \in \sigma_N(H)$.

2) *All the solutions of the Eq* (1.1) *are bounded.*

3) *The solutions of the Eq* (1.1) *are uniformly bounded.*

3. Uniform boundedness of the solution to the Eq (1.2)

In this section, we give a criterion on the uniform boundedness of the solutions for the Eq (1.2), namely, we state and prove the main result in the paper.

Theroem 2. *The solutions to the Eq* (1.2) *are uniformly bounded if and only if every eigenvalue* η *of* H *satisfies either* $|\eta| < 1$ *or* $|\eta| = 1, \eta^\rho \neq 1$ *with the index* $h_\eta(H) = 1$.

To prove this theorem, we prepare some results and lemmas.

3.1. *A representation of solutions to the Eq* (1.2)

First, we give a representation of solutions to the Eq (1.2), which was given in [3]. Hereafter, we abbreviate $Q_\eta = Q_\eta(H)$. We denote by $x(n; \tau, w, b(\cdot))$ the solution of the Eq (1.2) satisfying the initial condition $x(\tau) = w \in \mathbb{C}^p$, while by $x(n; \tau, w)$ if $b(n) = 0$. Any $n \in \mathbb{N}$ can be written as $n = k(n)\rho + r(n),\ k(n) = \left[\frac{n}{\rho}\right],\ 0 \leq r(n) \leq \rho - 1$, where the symbol $[a]$ stands for the largest integer which is not greater than $a \in \mathbb{R}$. Set $S_n(H) = \sum_{k=0}^{n-1} H^k,\ S_0(H) = O$, and

$$S_n(H, b(\tau + \cdot)) = \sum_{i=0}^{n-1} H^{n-1-i} b(\tau + i),\ S_0(H, b(\tau + \cdot)) = 0.$$

Then the unique solution $x(n;\tau,w,b(\cdot)),\ n \geq \tau$ of the equation (1.2) with $x(\tau) = w$ is expressed as follows:

$$\begin{aligned} x(n;\tau,w,b(\cdot)) &= H^{n-\tau}w + H^{n-\tau-1}b(\tau) + H^{n-\tau-2}b(\tau+1) \\ &\qquad + \cdots + Hb(n-2) + b(n-1) \\ &= H^{n-\tau}w + H^{r(n-\tau)}S_{k(n-\tau)}(H^{\rho})S_{\rho}(H, b(\tau+\cdot)) \\ &\qquad + S_{r(n-\tau)}(H, b(\tau+\cdot)). \end{aligned}$$

To obtain the representation of solutions to the Eq (1.2), we define the characteristic quantities $\gamma_\eta(\tau,w,b(\cdot))$ and $\delta_\eta(\tau,w,b(\cdot))$ as in [3].

For $k,m,n \in \mathbb{N}_0$, $p(k,m,n)$ stands for the set of all finite sequences $\alpha := (\alpha_1,\alpha_2,\cdots,\alpha_k)$, $\alpha_i \in \mathbb{N}_0\ (i = 1,2,\cdots,k)$:

$$\alpha_1 + \alpha_2 + \cdots + \alpha_k = m,\ \ \alpha_1 + 2\alpha_2 + \cdots + k\alpha_k = n.$$

For $k,m \in \mathbb{N}_0$ and $j \in \mathbb{N}$ we define

$$\left\{ \begin{array}{c} k \\ m \end{array} \right\}_j := k! \sum_{\alpha\in p(k,m,k)} \prod_{i=1}^{k} \frac{((j)_i)^{\alpha_i}}{(\alpha_i!)(i!)^{\alpha_i}} \text{ and } \left\{ \begin{array}{c} k \\ 0 \end{array} \right\}_j = \begin{cases} 0 & (k \neq 0) \\ 1 & (k = 0). \end{cases}$$

Let $f^{(k)}(t)$ be the k-th derivative of a function $f(t)$ and $f^{(0)}(t) = f(t)$. If $a(w) = (w-1)^{-1}$, $(w \neq 1)$ and $w = z^{\rho}$, then the k-th derivative of the composite function $c(z) = a(z^{\rho})$ is given as follows:

By using Faá di Bruno's formula [10] the k-th derivative $c^{(k)}(z)$ at η is expressed as

$$\eta^k c^{(k)}(\eta) = \sum_{i=0}^{k} \left\{ \begin{array}{c} k \\ i \end{array} \right\}_{\rho} \eta^{i\rho} a^{(i)}(\eta^{\rho})\ \ (\eta \neq 0,\ \eta^{\rho} \neq 1). \tag{3.1}$$

For an $\eta \in \sigma(H)$ we set

$$Z^0_\eta(H) = \begin{cases} \displaystyle\sum_{i=0}^{h_\eta(H)-1} \frac{1}{i!} c^{(i)}(\eta)(H - \eta E)^i & (\eta \neq 0), \\ \displaystyle -\sum_{i=0}^{[\frac{h_\eta(H)-1}{\rho}]} H^{\rho i} & (\eta = 0). \end{cases}$$

Then we define $Z_\eta(H, b(\tau+\cdot))$ by

$$Z_\eta(H, b(\tau+\cdot)) = Z^0_\eta(H) S_\rho(H, Q_\eta b(\tau+\cdot)).$$

Based on this, we can define the characteristic quantities $\gamma_\eta(\tau,w,b(\cdot))$ and $\delta_\eta(\tau,w,b(\cdot))$ for the Eq (1.2) as follows:

$$\gamma_\eta(\tau,w,b(\cdot)) := \gamma_\eta(\tau,w,b(\cdot);H) = Q_\eta w + Z_\eta(H, b(\tau+\cdot))\ (\eta^{\rho} \neq 1),$$

$$\begin{aligned} \delta_\eta(\tau,w,b(\cdot)) &:= \delta_\eta(\tau,w,b(\cdot);H) \\ &= (H^{\rho} - E)Q_\eta w + S_\rho(H, Q_\eta b(\tau+\cdot)) \end{aligned}$$

$$= \sum_{i=1}^{h_\eta(H)-1} \frac{1}{\eta^i}\binom{\rho}{i}(H-\eta E)^i Q_\eta w + S_\rho(H, Q_\eta b(\tau+\cdot)),\ (\eta^\rho = 1).$$

Furthermore, we set $H_{[k,\eta]} = \frac{1}{k!\eta^k}(H-\eta E)^k,\ (\eta \neq 0)$ and

$$B_\eta(r(n-\tau);\tau, b(\cdot)) = -H^{r(n-\tau)} Z_\eta(H, b(\tau+\cdot)) + S_{r(n-\tau)}(H, Q_\eta b(\tau+\cdot)),\ (\eta^\rho \neq 1),$$

$$B_\eta(r(n-\tau);\tau, w, b(\cdot)) = H^{r(n-\tau)} Q_\eta w + S_{r(n-\tau)}(H, Q_\eta b(\tau+\cdot)),\ (\eta^\rho = 1).$$

Clearly, $B_\eta(r(n);\tau, w, b(\cdot))$ is a function with period ρ.

A representation of solutions to the Eq (1.2) is given by the following lemma.

Lemma 6. [3] *Let $\eta \in \sigma(H)$. Then the component $Q_\eta x(n,\tau,w,b(\cdot))$ of the solution $x(n;\tau,w,b(\cdot))$ to the Eq* (1.2) *is expressed as follows*:

1) *If $\eta^\rho \neq 1$, then*

$$Q_\eta x(n;\tau,w,b(\cdot)) = H^{n-\tau}\gamma_\eta(\tau,w,b(\cdot)) + B_\eta(r(n-\tau);\tau,b(\cdot)).$$

In particular,

$$Q_\eta x(n;\tau,w,b(\cdot)) = \sum_{j=0}^{h_\eta(H)-1}(n-\tau)_j \eta^{n-\tau} H_{[j,\eta]}\gamma_\eta(\tau,w,b(\cdot)) + B_\eta(r(n-\tau);\tau,b(\cdot)),\ (\eta \neq 0).$$

2) *If $\eta^\rho = 1$, then*

$$\begin{aligned} &Q_\eta x(n;\tau,w,b(\cdot)) \\ &= \left(\sum_{j=0}^{h_\eta(H)-1} \frac{\left(\left[\frac{n-\tau}{\rho}\right]\right)_{j+1}}{j+1} \sum_{i=j}^{h_\eta(H)-1} \left\{ \begin{array}{c} i \\ j \end{array} \right\}_\rho H_{[i,\eta]}\right) H^{r(n-\tau)}\delta_\eta(\tau,w,b(\cdot)) + B_\eta(r(n-\tau);\tau,w,b(\cdot)). \end{aligned}$$

3.2. Lemmas

Next, we give some lemmas.

We set $b = \max_{0\le n\le\rho}\|b(n)\|$. Then by the definition of $Z_\eta^0(H)$ there exists a constant $K(\eta) > 0$ such that $\|Z_\eta^0(H)\| \le K(\eta)$. We also set $S(b) = \sum_{k=0}^{\rho}\|H\|^k b$.

Lemma 7. *Let $\eta \in \sigma(H)$. Then the following inequalities hold*:

1) $\max_{0\le n\le\rho}\|S_n(H, Q_\eta b(\tau+\cdot))\| \le S(b)$.
2) $\|Z_\eta(H, b(\tau+\cdot))\| \le K(\eta)S(b)$.
3) $\|\gamma_\eta(\tau,w,b(\cdot))\| \le \|Q_\eta\|\|w\| + K(\eta)S(b),\ (\eta^\rho \neq 1)$.
4) $\|\delta_\eta(\tau,w,b(\cdot))\| \le \|H^\rho - E\|\|Q_\eta\|\|w\| + S(b),\ (\eta^\rho = 1)$.

Proof. The proof follows from the definitions of $S_n(H, Q_\eta b(\tau+\cdot)), Z_\eta(\tau, H, b(\cdot)),\ \gamma_\eta(\tau,w,b(\cdot))$ and $\delta_\eta(\tau,w,b(\cdot))$. □

Lemma 8. *Let $\eta \in \sigma(H)$. Then the following statements hold.*
1) *There exists a $\beta_\eta > 0$ such that*

$$\max_{0\le n\le \rho}\{\|B_\eta(r(n-\tau);\tau, b(\cdot))\|\} < \beta_\eta$$

holds for all $\tau \in \mathbb{Z}$ and $n \ge \tau$.
2) *There exists a $\beta_\eta(\alpha) > 0$ such that*

$$\max_{0\le n\le \rho}\{\|B_\eta(r(n-\tau);\tau, w, b(\cdot))\|\} < \beta_\eta(\alpha)$$

holds for all $\tau \in \mathbb{Z}$, $n \ge \tau$ and $w \in B_\alpha$.

Proof. Note that for any $\eta \in \sigma(H)$, we have, by Lemma 7,

$$\|S_{r(n-\tau)}(H, Q_\eta b(\tau + \cdot))\| \le S(b).$$

1) Let $\eta^\rho \neq 1$. Since $\|Z^0_\eta(H)\| \le K(\eta)$ and $\|Z_\eta(H, b(\tau + \cdot))\| \le K(\eta)S(b)$, we obtain that

$$\begin{aligned}
&\|B_\eta(r(n-\tau);\tau, b(\cdot))\| \\
&\le \|H^{r(n)}\|\|Z_\eta(H, b(\tau + \cdot))\| + \|S_{r(n-\tau)}(H, Q_\eta b(\tau + \cdot))\| \\
&\le K(\eta)\max_{0\le k\le\rho}\|H\|^k S(b) + S(b) \\
&= \left(K(\eta)\max_{0\le k\le\rho}\|H\|^k + 1\right) S(b) =: \beta_\eta.
\end{aligned}$$

2) Let $\eta^\rho = 1$. If $\|w\| < \alpha$, then it follows that for any $n \ge \tau$

$$\begin{aligned}
\|B_\eta(r(n);\tau, w, b(\cdot))\| &\le \|H^{r(n)}\|\|Q_\eta\|\|w\| + \|S_{r(n)}(H, Q_\eta b(\tau + \cdot))\| \\
&\le \max_{0\le k\le\rho-1}\|H\|^k\|Q_\eta\|\alpha + \sum_{k=0}^{\rho-1}\|H\|^k b =: \beta_\eta(\alpha).
\end{aligned}$$

Since the remainder is obvious, the proof is complete. □

Lemma 9. *Let $\eta \in \sigma(H)$.*
1) *If $\eta \in \sigma_S(H)$, then*

$$\|Q_\eta x(n;\tau, w, b(\cdot))\| \le T(\eta)\left(\|Q_\eta\|\|w\| + K(\eta)S(b)\right) + \beta_\eta, \tag{3.2}$$

where $T(\eta) = \max_{\tau\le n<\infty}(n-\tau)_{h_\eta(H)}|\eta|^{n-\tau-h_\eta(H)}C(\eta)$.
2) *If $\eta \in \sigma_U(H)$, then $\lim_{n\to\infty}\|Q_\eta x(n;\tau, w, b(\cdot))\| = \infty$ if $\gamma_\eta(\tau, w, b(\cdot)) \neq 0$.*

Proof. 1) Let $\eta \in \sigma_S(H)$. Then it follows from Lemma 2 and Lemma 7 that

$$\begin{aligned}
&\|H^{n-\tau}\gamma_\eta(\tau, w, b(\cdot))\| \\
&\le \|H^{n-\tau}Q_\eta\|\|\gamma_\eta(\tau, w, b(\cdot))\| \\
&\le (n-\tau)_{h_\eta(H)}|\eta|^{n-\tau-h_\eta(H)}C(\eta)\|Q_\eta\|\left(\|Q_\eta\|\|w\| + K(\eta)S(b)\right),
\end{aligned}$$

where $C(\eta) = h_\lambda(H) \max_{0\le j\le h_\eta(H)-1} \|(H-\eta E)^j\|$. Since

$$\lim_{n\to\infty}(n-\tau)_{h_\eta(H)}|\eta|^{n-\tau-h_\eta(H)}C(\eta) = 0,$$

we have

$$(n-\tau)_{h_\eta(H)}|\eta|^{n-\tau-h_\eta(H)}C(\eta) \le T(\eta).$$

Thus for all $n \ge \tau$ we obtain

$$\|Q_\eta x(n;\tau,w,b(\cdot))\| \le T(\eta)\Big(\|Q_\eta\|\alpha + K(\eta)S(b)\Big) + \beta_\eta.$$

2) Let $\eta \in \sigma_U(H)$.

If $\gamma_\eta(\tau,w,b(\cdot)) \ne 0$, then there is a $d \ge 1$ such that

$$(H-\eta E)^{d-1}\gamma_\eta(\tau,w,b(\cdot)) \ne 0,\ (H-\eta E)^d\gamma_\eta(\tau,w,b(\cdot)) = 0;$$

It follows from Lemma 6 and Lemma 8 that if $\gamma_\eta(\tau,w,b(\cdot)) \ne 0$, then

$$H^{n-\tau}\gamma_\eta(\tau,w,b(\cdot)) = \binom{n-\tau}{j}Q_\eta\gamma_\eta(\tau,w,b(\cdot)) \ \to \infty \text{ as } n\to\infty.$$

Thus $Q_\eta x(n;\tau,w,b(\cdot)) \to \infty$ as $n \to \infty$. Therefore, the proof is complete. □

3.3. Proof of Theorem 2

We are now in a position to prove Theorem 2.

We set

$$\sigma_{N_0}(H) = \{\eta \in \sigma(H) : |\eta| = 1, \eta^\rho \ne 1\},$$

and

$$\sigma_{N_1}(H) = \{\eta \in \sigma(H) : |\eta| = 1, \eta^\rho = 1\}.$$

Then $\sigma_N(H) = \sigma_{N_0}(H) \cup \sigma_{N_1}(H)$.

For any $(\tau, w) \in \mathbb{Z}\times\mathbb{C}^p$ the component $Q_\eta x(n;\tau,w,b(\cdot))$ of the solution to the Eq (1.2) is given by Lemma 6.

I. First we prove "if" part of Theorem 2.

If $|\eta| < 1$, then (3.2) holds.

Let $\eta \in \sigma_{N_0}(H)$ with $h_\eta(H) = 1$. Then it follows from Lemma 6 that

$$H^{n-\tau}\gamma_\eta(\tau,w,b(\cdot)) = \eta^{n-\tau}\gamma_\eta(\tau,w,b(\cdot)). \tag{3.3}$$

Applying Lemma 7 and Lemma 8, we obtain that if $w \in B_\alpha$, then

$$\|Q_\eta x(n;\tau,w,b(\cdot))\| \le \|Q_\eta\|\alpha + K(\eta)S(b) + \beta_\eta. \tag{3.4}$$

Indeed, we have

$$\begin{aligned}&\|Q_\eta x(n;\tau,w,b(\cdot))\|\\ &\le |\eta|^{n-\tau}\|\gamma_\eta(\tau,w,b(\cdot))\| + \|B_\eta(r(n-\tau);\tau,b(\cdot))\|\end{aligned}$$

$$
\begin{aligned}
&\le \|\gamma_\eta(\tau, w, b(\cdot))\| + \beta_\eta \\
&\le \Big(\|Q_\eta\|\|w\| + K(\eta)S(b)\Big) + \beta_\eta \\
&\le \|Q_\eta\|\alpha + K(\eta)S(b) + \beta_\eta.
\end{aligned}
$$

Since the hypothesis yields

$$\sigma(H) = \sigma_S(H) \cup \sigma_{N_0}(H), \quad \mathbb{C}^p = (\oplus_{\eta\in\sigma_S(H)} G_\eta(H)) \bigoplus (\oplus_{\eta\in\sigma_{N_0}(H)} W_\eta(H)),$$

any vector $w \in \mathbb{C}^p$ can be represented as

$$w = \sum_{\eta\in\sigma_S(H)} Q_\eta w + \sum_{\eta\in\sigma_{N_0}(H)} Q_\eta w.$$

Set

$$\beta(\alpha) = p(T+1)(q\alpha + KS(b)) + 2p\beta,$$

where

$$q = \sum_{\eta\in\sigma_S(H)\cup\sigma_{N_0}(H)} \|Q_\eta\|, \; K = \max_{\eta\in\sigma_S(H)\cup\sigma_{N_0}(H)} K(\eta),$$

and

$$\beta = \max_{\eta\in\sigma_S(H)\cup\sigma_{N_0}(H)} \beta_\eta, \; T = \max_{\eta\in\sigma_S(H)\cup\sigma_{N_0}(H)} L(\eta).$$

If $w \in B_\alpha$, then it follows from Lemma 9 and (3.4) that

$$
\begin{aligned}
\|x(n;\tau,w,b(\cdot))\| &\le \sum_{\eta\in\sigma_S(H)} \|Q_\eta x(n;\tau,w,b(\cdot))\| + \sum_{\eta\in\sigma_N(H)} \|Q_\eta x(n;\tau,w,b(\cdot))\| \\
&\le \sum_{\eta\in\sigma_S(H)} [T(\eta)\Big(\|Q_\eta\|\alpha + K(\eta)S(b)\Big) + \beta_\eta] \\
&\qquad + \sum_{\eta\in\sigma_N(H)} [\|Q_\eta\|\alpha + K(\eta)S(b) + \beta_\eta] \\
&\le \sum_{\eta\in\sigma_S(H)} [T\,(q\alpha + KS(b)) + \beta] \\
&\qquad + \sum_{\eta\in\sigma_{N_0}(H)} [q\alpha + KS(b) + \beta] \\
&= p(T+1)(q\alpha + KS(b)) + 2p\beta = \beta(\alpha).
\end{aligned}
$$

This implies that $x(n;\tau,w,b(\cdot))$ is uniformly bounded.

II. Next, we prove "only if" part of Theorem 2.

It suffices to prove that if the solutions of the Eq (1.2) are uniformly bounded, then

$$\mathbb{C}^p = \oplus_{\eta\in\sigma(H)} G_\eta(H) = (\oplus_{\eta\in\sigma_S(H)} G_\eta(H)) \bigoplus (\oplus_{\eta\in\sigma_{N_0}(H)} W_\eta(H)).$$

Since $\sigma(H) = \sigma_S(H) \cup \sigma_U(H) \cup \sigma_{N_0}(H) \cup \sigma_{N_1}(H)$, any vector $w \in \mathbb{C}^p$ can be represented as

$$w = \sum_{\eta\in\sigma_S(H)} Q_\eta w + \sum_{\eta\in\sigma_U(H)} Q_\eta w + \sum_{\eta\in\sigma_{N_0}(H)} Q_\eta w + \sum_{\eta\in\sigma_{N_1}(H)} Q_\eta w.$$

The uniform boundedness of the solutions is equivalent to the uniform boundedness of the components $Q_\eta x(n;\tau,w,b(\cdot))$ of solutions for every $\eta \in \sigma(H)$.

(1) The case $\eta \in \sigma_U(H)$: It follows from Lemma 9 that $Q_\eta x(n;\tau,w,b(\cdot)) \to \infty$ as $n \to \infty$ if $\gamma_\eta(\tau,w,b(\cdot)) \neq 0$, which is a contradiction. If $\gamma_\eta(\tau,w,b(\cdot)) = 0$, then $Q_\eta w = -Z_\eta(H, b(\tau+\cdot))$. Thus $Q_\eta x(n;\tau,w,b(\cdot))$ is bounded, but it is not uniformly bounded, which is a contradiction. Therefore, $\sigma_U(H) = \emptyset$, hence $\oplus_{\eta\in\sigma_U(H)} G_\eta(H) = \{0\}$.

(2) The case $\eta \in \sigma_S(H)$: It follows from Lemma 9 that

$$\sum_{\eta\in\sigma_S(H)} Q_\eta x(n;\tau,w,b(\cdot))$$

is uniformly bounded.

(3) The case $\eta \in \sigma_{N_0}(H)$: It suffices to prove $G_\eta(H) = W_\eta(H)$. Assume $h_\eta(H) \geq 2$. By Lemma 6 we have

$$H^{n-\tau}\gamma_\eta(\tau,w,b(\cdot)) = \sum_{j=0}^{h_\eta(H)-1}(n-\tau)_j \eta^{n-\tau} H_{[j,\eta]}\gamma_\eta(\tau,w,b(\cdot)) \to \infty \text{ as } n\to\infty,$$

which is a contradiction. Also, $h_\eta(H) = 0$ yields a contradiction. If $h_\eta(H) = 1$, then (3.3) holds, which is uniformly bounded.

(4) The case $\eta \in \sigma_{N_1}(H)$: Assume $G_\eta(H) = W_\eta(H)$. Then using the same argument as in (1), we have $G_\eta(H) = \{0\}$. Indeed, we obtain from Lemma 6

$$\begin{aligned}
&Q_\eta x(n;\tau,w,b(\cdot)) - B_\eta(r(n-\tau);\tau,w,b(\cdot)) \\
&= \left(\sum_{j=0}^{h_\eta(H)-1} \frac{\left(\left[\frac{n-\tau}{\rho}\right]\right)_{j+1}}{j+1} \sum_{i=j}^{h_\eta(H)-1} \left\{ \begin{matrix} i \\ j \end{matrix} \right\}_\rho H_{[i,\eta]}\right) H^{r(n-\tau)}\delta_\eta(\tau,w,b(\cdot)) \\
&= \left[\frac{n-\tau}{\rho}\right] H^{r(n-\tau)}\delta_\eta(\tau,w,b(\cdot)) \ \to \infty \text{ as } n\to\infty.
\end{aligned}$$

It follows from Lemma 8 that $Q_\eta x(n;\tau,w,b(\cdot)) \to \infty$ as $n \to \infty$. This is a contradiction. Therefore, the proof is complete. □

4. A characteristic equation

Making use of the simultaneous diagonalization theorem under the commuting condition $AB = BA$, we can apply the preceding results to the Eq (1.3). But we only consider the characteristic equation of the matrix M in a reduced equation

$$y(n+1) = My(n) + g(n), \tag{4.1}$$

derived from the Eq (1.3). Indeed, making a change of variables $y_i(n) = x(n-\rho+i)$, $i \in \{0,1,\cdots,\rho\}$, we have

$$\begin{aligned}
y_{i-1}(n+1) &= x(n-\rho+i) = y_i(n),\ i = 1,2,\cdots,\rho, \\
y_\rho(n+1) &= x(n+1) = Ay_\rho(n) + By_0(n) + f(n).
\end{aligned}$$

Therefore, the Eq (1.3) is transformed to the Eq (4.1), where

$$y(n)=\begin{pmatrix} y_0(n)\\ y_1(n)\\ \vdots\\ y_{\rho-1}(n)\\ y_\rho(n)\end{pmatrix},\ M=\begin{pmatrix} O & E & O & \dots & O\\ O & O & E & \dots & O\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ O & O & O & \dots & E\\ B & O & O & \dots & A\end{pmatrix},\ g(n)=\begin{pmatrix} 0\\ 0\\ \vdots\\ 0\\ f(n)\end{pmatrix},\ E=E_p.$$

We denote by $M_{mn}(\mathbb{C})$ the set of all $m\times n$ complex matrices.

First, we give the characteristic equation $\det(zE-M)=0$ of the matrix M in the Eq (4.1) in the following proposition.

Proposition 4. *The characteristic equation of M is given by*

$$\det(zE-M)=\det(z^{\rho+1}E-z^{\rho}A-B)=0. \tag{4.2}$$

Proof. Let $z\neq 0$. Then we have

$$\begin{aligned}\det(zE-M)&=\det\begin{pmatrix} zE & -E & & & & \\ & zE & -E & & 0 & \\ & & & \ddots & & \\ & 0 & & & -E & \\ & & & & zE & -E\\ -B & & & & & zE-A\end{pmatrix}\\ &=\det(zE)\det\left(\begin{pmatrix} zE & -E & & & & \\ & zE & -E & & 0 & \\ & & & \ddots & & \\ & 0 & & & -E & \\ & & & & zE & -E\\ & & & & & zE-A\end{pmatrix}\right.\\ &\qquad\left.-\begin{pmatrix} O\\ \vdots\\ O\\ -B\end{pmatrix}(zE)^{-1}\begin{pmatrix}-E & O & \dots & O\end{pmatrix}\right)\\ &=z^{p}\det\begin{pmatrix} zE & -E & & & & \\ & zE & -E & & 0 & \\ & & & \ddots & & \\ & 0 & & & -E & \\ & & & & zE & -E\\ -\frac{1}{z}B & & & & & zE-A\end{pmatrix}.\end{aligned}$$

Repeating this procedure we have

$$\det(zE-M) = z^{2p}\det\begin{pmatrix} zE & -E & & & & \\ & zE & -E & & \mathbf{0} & \\ & & & \ddots & & \\ & \mathbf{0} & & & -E & \\ & & & & zE & -E \\ -\frac{1}{z^2}B & & & & & zE-A \end{pmatrix}$$

$$\vdots$$

$$\begin{aligned} &= z^{(\rho-1)p}\det\begin{pmatrix} zE & -E \\ -\frac{1}{z^{\rho-1}}B & zE-A \end{pmatrix} \\ &= z^{(\rho-1)p}\det(zE)\det\left((zE-A)-\left(-\frac{1}{z^{\rho-1}}B(zE)^{-1}(-E)\right)\right) \\ &= z^{\rho p}\det\left(zE-A-\frac{1}{z^{\rho}}B\right) \\ &= \det\left(z^{\rho+1}E-z^{\rho}A-B\right). \end{aligned}$$

Thus the characteristic equation of M becomes

$$\det(z^{\rho+1}E-z^{\rho}A-B)=0.$$

If $z=0$, then $\det(zE-M)=\det(-M)=\det(-B)$. □

Corollary 2. *The matrix M is nonsingular if and only if the matrix B is nonsingular.*

Next, we find the eigenvalues of M under the commuting condition (C): $AB=BA$.

Definition 2. *Let two matrices A and B be semisimple matrices in $M_p(\mathbb{C})$. Then two matrices A and B are said to be simultaneously diagonalizable if there exists a nonsingular matrix $P\in M_p(\mathbb{C})$ such that $P^{-1}AP, P^{-1}BP$ are diagonal matrices.*

Let $\alpha_1,\dots,\alpha_p$ (not necessarily distinct) be all the eigenvalues of A and $\mu_1,\dots,\mu_p$ (not necessarily distinct) all the eigenvalues of B. By the assumption (C) the simultaneous diagonalization theorem holds, that is, there exists a nonsingular matrix $P\in M_p(\mathbb{C})$ such that

$$P^{-1}AP = D_A = \begin{pmatrix} \alpha_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_p \end{pmatrix} \text{and } P^{-1}BP = D_B = \begin{pmatrix} \mu_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mu_p \end{pmatrix}. \tag{4.3}$$

Proposition 5. *Let A, B be semisimple matrices in $M_p(\mathbb{C})$ and satisfy the commuting condition* (C)*. Then*

$$\det(zI-M)=\prod_{i=1}^{p}(z^{\rho+1}-\alpha_i z^{\rho}-\mu_i).$$

Proof. Using Proposition 4, we obtain

$$\begin{aligned}\det(zE - M) &= \det(z^\rho(zE - A) - B)\\ &= \det(z^\rho(zE - PD_AP^{-1}) - PD_BP^{-1})\\ &= \det(P(z^\rho(zE - D_A) - D_B)P^{-1})\\ &= \det(z^\rho(zE - D_A) - D_B)\\ &= \prod_{i=1}^{p}(z^{\rho+1} - \alpha_i z^\rho - \mu_i).\end{aligned}$$

Therefore, the proof is complete. □

5. Conclusions

We have given a criterion on the uniform boundedness of the solutions to linear difference equations (LEs) with periodic forcing functions. In particular, we have shown a subtle difference on the uniform boundedness of the solutions between the nonhomogenuous equation (1.2) and the corresponding homogenuous equation (1.1).

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

Acknowledgments

Dohan Kim was supported by the National Research Foundation of Korea(NRF) grant funded by the Korean Government(MSIT) (No. 2021R1A2C1092945).

Conflict of interest

All authors declare no conflicts of interest that could affect the publication of this paper.

References

1. J. Kato, T. Naito, J. S. Shin, A characterization of solutions in linear differential equations with periodic forcing functions, *J. Differ. Equ. Appl.*, **11** (2005), 1–19. http://dx.doi.org/10.1080/10236190412331317120
2. T. Naito, J. S. Shin, On periodicizing functions, *Bull. Korean Math. Soc.*, **43** (2006), 253–263. http://dx.doi.org/10.4134/BKMS.2006.43.2.253
3. T. Naito, P. H. T. Ngoc, J. S. Shin, Representations and asymptotic behavior to periodic linear difference equations, *Funkc. Ekvacioj*, **51** (2008), 55–80. http://dx.doi.org/10.1619/fesi.51.55
4. T. Naito, P. H. T. Ngoc, J. S. Shin, Floquet representations and asymptotic behavior of solutions to periodic linear difference equations, *Hiroshima Math. J.*, **38** (2008), 135–154.

5. J. S. Shin, T. Naito, Representations of solutions, translation formulae and asymptotic behaviors in discrete and periodic continuous linear systems, *Hiroshima Math. J.*, **44** (2014), 75–126. http://dx.doi.org/10.32917/hmj/1395061558

6. S. Stevic, Bounded and periodic solutions to the linear first-order difference equation on the integer domain, *Adv. Differ. Equ.*, **2017** (2017), 283. http://dx.doi.org/10.1186/s13662-017-1350-8

7. E. Kreyszig, *Introductory functional analysis with applications*, New York: John Wiley & Sons, 1991.

8. T. Yoshizawa, *Stability theory by Liapunov's second method*, Tokyo, 1966.

9. S. Elaydi, *An introduction to difference equations*, 3 Eds., New York: Springer-Verlag, 2005.

10. F. Faá di Bruno, Note sur une nouvelle formule de calcul differentiel, *Quart. J. Pure Appl. Math.*, **1** (1857), 359–360.

AIMS Press