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Research article

Asymptotic stability of a quasi-linear viscoelastic Kirchhoff plate equation with logarithmic source and time delay

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Abstract: In this paper, a quasi-linear viscoelastic Kirchhoff plate equation with logarithmic source and time delay involving free boundary conditions in a bounded domain is considered. The local existence and global existence are proved, respectively. Under the assumptions on a more general type of relaxation functions and suitable conditions on the coefficients between damping term and delay term, an explicit and general decay rate result is established by using the multiplier method and some properties of the convex functions. As the considered assumption here on the kernel is more general than earlier papers, our result improves and generalizes earlier result in the literature.

Keywords: Kirchhoff plate equation; logarithmic source; time delay, general decay; relaxation function
Mathematics Subject Classification: 35B40, 93D15, 93D20

1. Introduction

In this paper, we are concerned with the following quasilinear Kirchhoff plate equation

$$\begin{cases} |y_t|^\rho y_{tt} - \Delta y_{tt} + \Delta^2 y - \int_0^t g(t-s)\Delta^2 y(s)\, ds + a_0 y_t(x,t) & \\ \qquad\qquad\qquad + a_1 y_t(x, t-\tau_0) = ky \ln|y| & \text{in} \quad \Omega \times (0,\infty) \\ y = \partial_\nu y = 0 & \text{on} \quad \Gamma_0 \times (0,\infty) \\ \mathscr{B}_1 y - \mathscr{B}_1 \left\{ \int_0^t g(t-s) y(s)\, ds \right\} = 0 & \text{on} \quad \Gamma_1 \times (0,\infty) \\ \mathscr{B}_2 y - \partial_\nu y_{tt} - \mathscr{B}_2 \left\{ \int_0^t g(t-s) y(s)\, ds \right\} = 0 & \text{on} \quad \Gamma_1 \times (0,\infty) \\ y(0) = y^0, \;\; y_t(0) = y^1 & \text{in} \quad \Omega, \\ y_t(x,t) = j_0(x,t) \text{ for } (x,t) \in \Omega \times (-\tau_0, 0), & \end{cases} \tag{1.1}$$

where Ω is an open set of $\mathbb{R}^2$ with regular boundary $\Gamma = \partial\Omega = \Gamma_0 \cup \Gamma_1$ (class C^4 will be enough) such that $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$, the initial data y^0 and y^1 lie in appropriate Hilbert space, and k is a positive constant. The boundary operators $\mathbf{B}_1$, $\mathbf{B}_2$ are defined by

$$\mathscr{B}_1 y = \Delta y + (1-\mu) B_1 y,$$

$$\mathscr{B}_2 y = \partial_\nu \Delta y + (1-\mu) B_2 y,$$

and

$$B_1 y = 2\nu_1\nu_2 y_{x_1x_2} - \nu_1^2 y_{x_2x_2} - \nu_2^2 y_{x_1x_1},$$
$$B_2 y = \partial_\tau \left((\nu_1^2 - \nu_2^2) y_{x_1x_2} + \nu_1\nu_2 (y_{x_2x_2} - y_{x_1x_1}) \right),$$

where the constant $0 < \mu < \frac{1}{2}$ is the Poisson coefficient, $\nu = (\nu_1, \nu_2)$ is the unit outer normal vector to Γ and $\tau = (-\nu_2, \nu_1)$ is a unit tangent vector.

Model (1.1) describes a viscoelastic Kirchhoff plate with rotational forces, which possesses a rigid surface and whose interiors are somehow permissive to slight deformations, such that the material density varies according to the velocity. This plate is clamped along Γ_0 without bending and twisting moments on Γ_1. The analysis of stability issues for plate models is more challenging due to free boundary conditions and the presence of rotational forces. Moreover, in our case, the logarithmic source term competes with a delay term and the dissipation induced by both viscoelastic and frictional terms. Therefore, it will be interesting to study this interaction.

In the past decades, the non-delayed wave equation under the influence of viscoelastic term has create great interest in the research field of partial differential equations. The well-posedness, stability and blow-up of solutions of such equations have recently been established in many papers. It has been stabilized through various controls, such as internal control damping, boundary control, dynamic boundary conditions, distributed damping, and thermal damping, see [1–12]. In [13], the authors considered a viscoelastic plate equation with p−Laplacian and memory terms

$$u_{tt} + \Delta^2 u - \Delta_p u + \int_0^t g(t-s)\Delta u(s) ds - \Delta u_t + f(u) = 0, \tag{1.2}$$

and the existence of weak and strong solutions was obtained by Faedo-Galerkin approach and the exponential stability was established by assuming that g decays exponentially. In [14], Cavalcanti et al. considered the following wave equation

$$u_{tt} - \kappa_0 \Delta u + \int_0^t \operatorname{div}[a(x) g(t-s) \nabla u(s)] ds + f(u) + b(x) h(u_t) = 0,$$

where frictional damping was also considered. They proved an exponential stability result for g decaying exponentially and h being linear and polynomial stability result for g decaying polynomially and h having a polynomial growth near zero. We mention, in the case where $\kappa_0 = 1$ and $f = h = 0$, that the uniform decay of solutions was obtained in [15]. For viscoelastic Kirchhoff plate equation, in [10], the authors showed exponential and polynomial decay of the energy of the solutions. They considered a relaxation function satisfying

$$-d_0 g(t) \leq g'(t) \leq -d_1 g(t), \quad 0 \leq g''(t) \leq d_2 g(t), \tag{1.3}$$

for some positive constant d_i, $i = 0, 1, 2$. Jorge Silva, Muñoz Rivera and Racke [16] studied the following viscoelastic Kirchhoff plate equation

$$u_{tt} - \sigma(t)\Delta u_{tt} + \Delta^2 u - \mathrm{div} F(\nabla u) - \int_0^t g(t-s)\Delta^2 u(s) ds = 0,$$

and established the general rates of energy decay of the system by assuming that

$$g'(t) \leq -\xi(t) g(t), \tag{1.4}$$

where $\xi(t)$ is a non-increasing positive function satisfying that there exists a constant $\xi_0 > 0$ such that

$$\left| \frac{\xi'(t)}{\xi(t)} \right| \leq \xi_0.$$

Motivated by the work of Lasiecka and Tataru [17], where a wave equation with frictional damping was considered, another step forward was done by considering relaxation functions satisfying

$$g'(t) \leq -H(g(t)),$$

where the function $H > 0$ satisfying $H(0) = H'(0) = 0$, and is a strictly increasing and strictly convex near the origin. This condition was first introduced by Alabau-Boussouira and Cannarsa [18]. It turned out that the convexity properties can be explored for a general class of dissipative systems [19, 20]. We also point out that the importance of the works [19, 20] in which simple sharp optimal or quasi-optimal upper energy decay rates have been established. Since then, the optimal energy decay of numerous related systems are established by the methodology established in [19, 20] (see also [21, 22]).

For the case of viscoelastic plate equation with infinite history, we mention the recent work of Al-Mahdi [23], where the author proved, under a general assumption on the behavior of the relaxation function at infinity and by dropping the boundedness assumption on the history data, a relation between the decay rate of the solution and the growth of the relaxation function at infinity.

Logarithmic nonlinearity usually occurs in expansion cosmology, supersymmetry field theory, quantum mechanics and nuclear physics. Such problems have applications in many branches of physics, such as nuclear physics, optics and geophysics. In [24, 25], the authors studied a relativistic version of logarithmic quantum mechanics in a bounded interval $[a, b]$,

$$u_{tt} - u_{xx} + u = \varepsilon u \ln |u|^2.$$

In [26], the global existence and uniqueness of solutions of a 3-D wave equation with logarithmic nonlinearity was proved. Gorka [27] considered a 1-D case of the model in [26], and proved the global existence of weak solutions. In [28], the authors proved the existence of global classical solutions and also studied the Cauchy problem of the 1-D case of the model. When studying the dynamics of Q-ball in theoretical physics, Hiramatsu et al. [29] introduced the following equation

$$u_{tt} - \Delta u + u + u_t + |u|^2 u = u \ln |u|,$$

and some numerical results are obtained. The global existence of weak solutions was proved by Han [30]. Hu et al. [31] considered the equation

$$u_{tt} - \Delta u + u_t = u \ln |u|^k,$$

and established some energy decay rates. The result was improved in [32]. In [33], the authors considered the plate equation with logarithmic nonlinearity, proved the global existence of solutions and established that the solutions decay exponentially for a suitable initial data. Later, they extended the results to the case of nonlinear damping, see [34]. We also mention the recent work [35], in which the authors studied the global well-posedness of nonlinear fourth order dispersive wave equations with logarithmic source term and subject to nonlinear weak damping and linear strong damping. The general decay of a viscoelastic wave equation with logarithmic source, and of a Balakrishnan-Taylor viscoelastic equation with nonlinear frictional damping and logarithmic source term were proved in [36, 37] and [38], respectively. The energy decay of viscoelastic plate equation with logarithmic nonlinearity was established in [39] by assuming that $g'(t) \leq -\xi(t)g^p(t)$.

The delay effects often appear in many practical problems. However, the delay effects can be generally regarded as a source of instability. In [40], Nicaise and Pignotti established an exponential energy decay of a wave equation with time delay

$$u_{tt} - \Delta u + \mu_1 u_t + \mu_2 u_t(t - \tau) = 0,$$

under the assumption $0 < \mu_2 < \mu_1$. Kirane and Said-Houari [41] studied a viscoelastic wave equation with a delay term in internal feedbacks and obtained a general decay result of the total energy to the system by assuming (1.4) and $\mu_2 \leq \mu_1$. The result was improved in [42]. For the plate equation with time delay term, Park [43] considered

$$u_{tt} + \Delta^2 u - M(\|\nabla u\|^2)\Delta u + \sigma(t) \int_0^t g(t - s)\Delta u(s)ds + a_0 u_t + a_1 u_t(t - \tau) = 0,$$

with Dirichlet-Neumann boundary conditions, and obtained a general decay result of energy under the assumption (1.4). In [44], the author considered a viscoelastic plate equation with a linear time delay term

$$u_{tt} + \Delta^2 u - \int_0^t g(t - s)\Delta^2 u(s)ds + \mu_1 u_t + \mu_2 u_t(t - \tau) = 0.$$

and established the decay property of energy for either $0 < |\mu_2| < \mu_1$ or $\mu_1 = 0$, $0 < |\mu_2| < a$, and assuming that the kernel g satisfies (1.3). For wave equations with time delay and logarithmic source, we can find a recent result in [45].

Inspired by above results, in this paper, we study a quasilinear Kirchhoff plate with time delay and logarithmic source with a wider class of relaxation functions. We first prove the local existence of system (1.1), and then prove the global existence of solutions. We also establish a very general energy decay result of the system by following the general approach in [18]. As the considered assumption here on the kernel is more general than earlier papers, hence our result improves and generalizes earlier result in the literature.

The plan of the paper is as follows. In Section 2, we give some preliminaries. In Section 3, we prove the local existence of solutions. The global existence will be proved in Section 4. Section 5 is devoted to the general energy decay.

2. Preliminaries

We let

$$V = \{\phi \in H^1(\Omega) : \phi = 0 \text{ on } \Gamma_0\},\ W = \{\phi \in H^2(\Omega) : \phi = \frac{\partial \phi}{\partial \nu} = 0 \text{ on } \Gamma_0\},$$

$$(\varphi, \phi) = \int_\Omega \varphi(x)\phi(x)dx,\ \|\varphi\|^2 = (\varphi, \varphi),$$

and

$$(\varphi, \phi)_{\Gamma_1} = \int_{\Gamma_1} \varphi(x)\phi(x)d\Gamma,\ \|\varphi\|^2_{\Gamma_1} = (\varphi, \varphi)_{\Gamma_1}.$$

The operator $b(\cdot, \cdot)$ is defined as

$$b(\varphi, \phi) = \int_\Omega \Big(\varphi_{x_1x_1}\phi_{x_1x_1} + \varphi_{x_2x_2}\phi_{x_2x_2} + \mu\left(\varphi_{x_1x_1}\phi_{x_2x_2} + \varphi_{x_2x_2}\phi_{x_1x_1}\right) + 2(1-\mu)\varphi_{x_1x_2}\phi_{x_1x_2}\Big)dx.$$

For $(\varphi, \phi) \in \left(H^4(\Omega) \cap W\right) \times W$, we know

$$\int_\Omega \left(\Delta^2\varphi\right)\phi dx = b(\varphi, \phi) - \left(\mathcal{B}_1\varphi, \frac{\partial \phi}{\partial \nu}\right)_{\Gamma_1} + (\mathcal{B}_2\varphi, \phi)_{\Gamma_1}. \tag{2.1}$$

Due to $\Gamma_0 \neq \emptyset$, it is well known ([5]) that

$$c_1\|\phi\|^2_{H^2(\Omega)} \leq b(\phi, \phi) \leq c_2\|\phi\|^2_{H^2(\Omega)} \quad \text{for some } c_1, c_2 > 0. \tag{2.2}$$

Let C_p, C_{p,Γ_1} and C_s be the imbedding constants with

$$\|\varphi\|^2 \leq C_p b(\varphi, \varphi),\ \ \|\varphi\|^2_{\Gamma_1} \leq C_{p,\Gamma_1} b(\varphi, \varphi) \tag{2.3}$$

and

$$\|\nabla\varphi\|^2 \leq C_s b(\varphi, \varphi), \quad \forall \varphi \in W. \tag{2.4}$$

Assumptions

Throughout this paper, we assume that:

(H_1) The coefficients a_0 and a_1 satisfy

$$0 < |a_1| < a_0. \tag{2.5}$$

(H_2) The kernel $g : [0,\infty) \to (0,\infty)$ is a non-increasing differentiable function, with $1 - \int_0^\infty g(s)ds := g_l > 0$, verifying

$$g'(t) \leq -\zeta(t)G(g(t)) \quad \text{for all } t \geq 0, \tag{2.6}$$

where $G : (0,\infty) \to (0,\infty)$ is a C^1-function, which is either linear or strictly increasing and strictly convex C^2-function on $(0, r_0]$, $r_0 \leq g(0)$, $G(0) = G'(0) = 0$, and ζ is positive, differentiable, and non-increasing.

(H_3) The constant k in (1.1) satisfies

$$0 < k < \frac{2\pi g_l e^3}{C_s}. \tag{2.7}$$

Remark 2.1. *([46]) If G is a strictly increasing and strictly convex C^2 function on $(0, r_0]$, with $G(0) = G'(0) = 0$, then it has an extension $\overline{G}$, which is strictly increasing and strictly convex C^2 function on $(0,\infty)$. For instance, if $G(r_0) = a, G'(r_0) = b, G''(r_0) = c$, we can define $\overline{G}$, for $t > r_0$, by*

$$\overline{G}(t) = \frac{c}{2}t^2 + (b - cr_0)t + \left(a + \frac{c}{2}{r_0}^2 - br_0\right). \tag{2.8}$$

As in [40], we define

$$z(x,\theta,t) = y_t(x, t - \theta\tau_0) \text{ for } (x,\theta,t) \in \Omega \times (0,1) \times (0,T). \tag{2.9}$$

Then, problem (1.1) is equivalent to

$$\begin{cases} |y_t|^\rho y_{tt} - \Delta y_{tt} + \Delta^2 y - \displaystyle\int_0^t g(t-s)\Delta^2 y(s)\, ds + a_0 y_t(x,t) & \\ \qquad\qquad\qquad + a_1 z(x,1,t) = ky\ln|y| & \text{in} \quad \Omega \times (0,\infty) \\ \tau_0 z_t(x,\theta,t) + z_\theta(x,\theta,t) = 0 \ \text{ for } \ (x,\theta,t) \in \Omega \times (0,1) \times (0,T), & \\ y = \partial_\nu y = 0 & \text{on} \quad \Gamma_0 \times (0,\infty) \\ \mathcal{B}_1 y - \mathcal{B}_1 \left\{\displaystyle\int_0^t g(t-s)y(s)\, ds\right\} = 0 & \text{on} \quad \Gamma_1 \times (0,\infty) \\ \mathcal{B}_2 y - \partial_\nu y_{tt} - \mathcal{B}_2 \left\{\displaystyle\int_0^t g(t-s)y(s)\, ds\right\} = 0 & \text{on} \quad \Gamma_1 \times (0,\infty) \\ y(0) = y^0, \ \ y_t(0) = y^1 & \text{in} \quad \Omega, \\ z(x,\theta,0) = j_0(x, -\theta\tau_0) := z_0 & \text{in} \quad \Omega \times (0,1). \end{cases} \tag{2.10}$$

Below, we recall some lemmas that are useful for our work.

Lemma 2.1 ([39]). *Let $\epsilon_0 \in (0,1)$. Then there exists $a_{\epsilon_0} > 0$ such that*

$$s|\ln s| \leq s^2 + a_{\epsilon_0} s^{1-\epsilon_0} \ \textit{for all } s > 0. \tag{2.11}$$

Lemma 2.2. *[47, 48] (Logarithmic Sobolev inequality) Let y be any function in $H^1(\mathbb{R}^2)$ and $a > 0$ be any number. Then*

$$\int_{\mathbb{R}^2} |y|^2 \ln |y| dx \leq \ln(\|y\|)\|y\|^2 + \frac{a^2}{2\pi}\|\nabla y\|^2 - (1 + \ln a)\|y\|^2.$$

For $y \in V$, we can define $y(x) = 0$, for $x \in \mathbb{R}^2 \setminus \Omega$. Then $y \in H^1(\mathbb{R}^2)$, that is to say, for a general domain Ω, we have the following logarithmic Sobolev inequality,

$$\int_{\Omega} |y|^2 \ln |y| dx \leq \ln(\|y\|)\|y\|^2 + \frac{a^2}{2\pi}\|\nabla y\|^2 - (1 + \ln a)\|y\|^2,$$

where y is any function in V and $a > 0$ is any number.

Corollary 2.1. *Let y be any function in W and $a > 0$ be any number. Then*

$$\int_{\Omega} |y|^2 \ln |y| dx \leq \ln(\|y\|)\|y\|^2 + \frac{a^2 C_s}{2\pi} b(y, y) - (1 + \ln a)\|y\|^2. \tag{2.12}$$

Lemma 2.3 (Logarithmic Gronwall inequality [26])**.** *Let $d > 0$ and $\beta \in L^1(0, T; [0, \infty))$. If a function $f : [0, T] \to [1, \infty)$ satisfies*

$$f(t) \leq d\left(1 + \int_0^t \beta(s) f(s) \ln f(s)\, ds\right), \ 0 \leq t \leq T,$$

then

$$f(t) \leq d \exp\left(d \int_0^t \beta(s)\, ds\right), \ 0 \leq t \leq T.$$

We need the following lemma.

Lemma 2.4. *([10]) For any $y \in C^1(0, T; H^2(\Omega))$, we have*

$$\begin{aligned} b\left(\int_0^t g(t-s) y(s) ds, y_t\right) \;=\; & -\frac{1}{2} g(t) b(y, y) - \frac{1}{2}\frac{d}{dt}\left\{(g \circ \partial^2 y)(t) - \left(\int_0^t g(s) ds\right) b(y, y)\right\} \\ & + \frac{1}{2}(g' \circ \partial^2 y)(t), \end{aligned} \tag{2.13}$$

where

$$(g \circ \partial^2 y)(t) = \int_0^t g(t-s) b(y(t) - y(s), y(t) - y(s)) ds.$$

3. Local existence

Definition 3.1. *Let $T > 0$. A pair of functions (y, z) is a weak solution of the problem* (2.10) *if it satisfies:*

$$y \in C([0, T], V) \cap C^1([0, T], W),$$

$$\int_\Omega |y_t|^\rho y_{tt} w\,dx + \int_\Omega \nabla y_{tt} \nabla w\,dx + b(y,w) - \int_0^t g(t-s) b(y(s),w)\,ds + a_0 \int_\Omega y_t w\,dx$$
$$+a_1 \int_\Omega z(x,1,t) w\,dx = k \int_\Omega y \ln|y| w\,dx,$$
$$y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), \quad z(x,0,t) = y_t(x,t)$$

and

$$\int_\Omega \int_0^1 (\tau_0 z_t(x,\theta,t) + z_\theta(x,\theta,t))\, v\, d\theta dx = 0,$$
$$z(x,0) = z_0,$$

for a.e. $t \in [0,T]$ *and all test functions* $w \in W$ *and* $v \in L^2(\Omega \times (0,1))$.

Theorem 3.1. *Let* $y_0 \in W,\ y_1 \in V$ *and* $z_0 \in L^2(\Omega \times (0,1))$. *Assume that assumptions (H1)-(H3) are true. Then, the system* (2.10) *has a weak solution.*

Proof. By the Faedo-Galerkin approach, we construct approximations of the solution (y,z) as follows. Let $\{w_j\}_{j=1}^\infty$ be a basis of W. Define $W_m = span\{w_1, w_2, ..., w_m\}$ and $v_j(x,0) = w_j(x)$. We can extend $v_j(x,0)$ by $v_j(x,\theta)$ over $L^2(\Omega \times (0,1))$. We denote $V_m = span\{v_1, v_2, ..., v_m\}$ for $m \geq 1$. We define the approximations $y^m \in W_m$ and $z^m \in V_m$ by

$$y^m(x,t) = \sum_{j=1}^m p_j(t) w_j(x), \qquad z^m(x,\theta,t) = \sum_{j=1}^m q_j(t) v_j(x,\theta),$$

and which solve the approximate system

$$\int_\Omega |y_t^m|^\rho y_{tt}^m w\,dx + \int_\Omega \nabla y_{tt}^m \nabla w\,dx + b(y^m,w) - \int_0^t g(t-s) b(y^m(s),w)\,ds + a_0 \int_\Omega y_t^m w\,dx,$$
$$+a_1 \int_\Omega z^m(x,1,t) w\,dx = k \int_\Omega y^m \ln|y^m| w\,dx,\ \forall\ w \in W_m,$$
$$y^m(x,0) = y_0^m(x), \quad y_t^m(x,0) = y_1^m(x), \quad z^m(x,0,t) = y_t^m(x,t), \tag{3.1}$$

$$\int_\Omega \int_0^1 (\tau_0 z_t^m(x,\theta,t) + z_\theta^m(x,\theta,t))\, v\, d\theta dx = 0,\ \forall\ v \in V_m$$
$$z^m(x,0) = z_0^m, \tag{3.2}$$

where

$$y_0^m \to y_0 \ \text{ in } \ W, \ \ y_1^m \to y_1 \ \text{ in } \ V \ \text{ and } \ z_0^m \to z_0 \ \text{ in } \ L^2(\Omega \times (0,1)). \tag{3.3}$$

This leads to a system of ordinary differential equations (ODEs) for unknown functions p_j and q_j. Hence, from the standard theory of system of ODEs, a solution (y^m, z^m) of (3.1)–(3.2) exists, for all $m \geq 1$, on $[0, t_m)$, with $0 < t_m \leq T,\ \forall\ m \geq 1$.

Replacing w by y_t^m in (3.1), integrating by parts over Ω and using Lemma 2.4 to obtain

$$\frac{d}{dt}\Big\{\frac{1}{\rho+2}\|y_t^m\|_{\rho+2}^{\rho+2}+\frac{1}{2}\Big(1-\int_0^t g(s)ds\Big)b(y^m,y^m)+\frac{1}{2}\|\nabla y_t^m\|^2$$
$$+\frac{1}{2}(g\circ\partial^2 y^m)-\frac{k}{2}\int_\Omega |y^m|^2\ln|y^m|dx+\frac{k}{4}\|y^m\|^2\Big\}$$
$$=-a_0\|y_t^m\|^2-a_1\int_\Omega z^m(x,1,t)y_t^m dx+\frac{1}{2}(g'\circ\partial^2 y^m)(t)-\frac{1}{2}g(t)b(y^m,y^m). \tag{3.4}$$

Let ξ be a positive number satisfying

$$|a_1|<\xi<2a_0-|a_1|. \tag{3.5}$$

Replacing v by ξz^m in (3.2), and integrating by parts over $\Omega\times(0,1)$, yields to

$$\frac{\xi\tau_0}{2}\frac{d}{dt}\int_\Omega\int_0^1 |z^m(x,\theta,t)|^2 d\theta dx=-\frac{\xi}{2}\|z^m(1,t)\|^2+\frac{\xi}{2}\|y_t^m\|^2. \tag{3.6}$$

Adding (3.4) and (3.6), we infer that

$$\begin{aligned}\frac{d}{dt}\mathcal{E}^m(t) &= -a_0\|y_t^m\|^2-a_1\int_\Omega z^m(x,1,t)y_t^m dx+\frac{1}{2}(g'\circ\partial^2 y^m)(t)-\frac{1}{2}g(t)b(y^m,y^m)\\ &\quad -\frac{\xi}{2}\|z^m(x,1,t)\|^2+\frac{\xi}{2}\|y_t^m\|^2,\end{aligned}$$

where

$$\begin{aligned}\mathcal{E}^m(t) &= \frac{1}{\rho+2}\|y_t^m\|_{\rho+2}^{\rho+2}+\frac{1}{2}\Big(1-\int_0^t g(s)ds\Big)b(y^m,y^m)+\frac{1}{2}\|\nabla y_t^m\|^2+\frac{1}{2}(g\circ\partial^2 y^m)\\ &\quad -\frac{k}{2}\int_\Omega |y^m|^2\ln|y^m|dx+\frac{k}{4}\|y^m\|^2+\frac{\xi\tau_0}{2}\int_\Omega\int_0^1 |z^m(x,\theta,t)|^2 d\theta dx.\end{aligned} \tag{3.7}$$

By using Young's inequality and (3.5), one gets

$$\frac{d}{dt}\mathcal{E}^m(t)=-\left(a_0-\frac{|a_1|}{2}-\frac{\xi}{2}\right)\|y_t^m\|^2-\left(\frac{\xi}{2}-\frac{|a_1|}{2}\right)\|z^m(x,1,t)\|^2+\frac{1}{2}(g'\circ\partial^2 y^m)(t)-\frac{1}{2}g(t)b(y^m,y^m)\le 0.$$

By integrating the last inequality over $(0,t), 0<t<t_m$, it holds that

$$\mathcal{E}^m(t)+M_1\int_0^t \|y_t^m\|^2 ds+M_2\int_0^t \|z^m(x,1,s)\|^2 ds\le\mathcal{E}^m(0), \tag{3.8}$$

where $M_1=\left(a_0-\frac{|a_1|}{2}-\frac{\xi}{2}\right)$ and $M_2=\left(\frac{\xi}{2}-\frac{|a_1|}{2}\right)$.

By using Lemma 2.2, (3.7) and (3.8), we observe that

$$\frac{2}{\rho+2}\|y_t^m\|_{\rho+2}^{\rho+2}+(g_l-\frac{ka^2C_s}{2\pi})b(y^m,y^m)+\|\nabla y_t^m\|^2+\frac{k}{2}\left(1+2(1+\ln a)\right)\|y^m\|^2$$
$$+(g\circ\partial^2 y^m)+\xi\tau_0\int_\Omega\int_0^1 |z^m(x,\theta,t)|^2 d\theta dx+2M_1\int_0^t \|y_t^m\|^2 ds+2M_2\int_0^t \|z^m(x,1,s)\|^2 ds$$
$$\le 2\mathcal{E}^m(0)+k\|y^m\|^2\ln\|y^m\|^2 \tag{3.9}$$

By choosing

$$e^{-\frac{3}{2}} < a < \sqrt{\frac{2\pi g_l}{kC_s}}, \tag{3.10}$$

one guarantees that

$$g_l - \frac{ka^2 C_s}{2\pi} > 0,$$

and

$$1 + 2(1 + \ln a) > 0.$$

This choice is possible thanks to (H3). Hence, we get

$$\begin{aligned}
&\|y_t^m\|_{\rho+2}^{\rho+2} + b(y^m, y^m) + \|\nabla y_t^m\|^2 + \|y^m\|^2 + (g \circ \partial^2 y^m) + \int_\Omega \int_0^1 |z^m(x, \theta, t)|^2 d\theta dx \\
&+ \int_0^t \|y_t^m\|^2 ds + \int_0^t \|z^m(x, 1, s)\|^2 ds \\
&\leq M_3(1 + \|y^m\|^2 \ln \|y^m\|^2),
\end{aligned} \tag{3.11}$$

where M_3 is a positive constant. On the other hand, we note that

$$y^m(., t) = y^m(., 0) + \int_0^t y_t^m(., s) ds.$$

Applying Cauchy-Schwarz's inequality and (3.11), one deduces that

$$\begin{aligned}
\|y^m(t)\|^2 &\leq 2\|y^m(0)\|^2 + 2T \int_0^t \|y_t^m(s)\|^2 ds \\
&\leq 2\|y^m(0)\|^2 + 2T \int_0^t M_3(1 + \|y^m\|^2 \ln \|y^m\|^2) ds \\
&\leq 2M_4 \left(1 + \int_0^t \|y^m\|^2 \ln \|y^m\|^2 ds\right),
\end{aligned}$$

for some positive constant M_4. Applying the Logarithmic Gronwall inequality to the last inequality, we arrive at

$$\|y^m(t)\|^2 \leq 2M_4 \exp(2M_4 T).$$

By combining the last inequality with (3.11), there exists a constant $M_5 > 0$ such that

$$\begin{aligned}
&\|y_t^m\|_{\rho+2}^{\rho+2} + b(y^m, y^m) + \|\nabla y_t^m\|^2 + \|y^m\|^2 + (g \circ \partial^2 y^m) + \int_\Omega \int_0^1 |z^m(x, \theta, t)|^2 d\theta dx \\
&+ \int_0^t \|y_t^m\|^2 ds + \int_0^t \|z^m(x, 1, s)\|^2 ds \leq M_5.
\end{aligned}$$

This latter implies that

$$\begin{aligned}
&y^m \text{ is uniformly bounded in } L^\infty(0, T; W), \\
&y_t^m \text{ is uniformly bounded in } L^\infty(0, T; V) \cap L^\infty(0, T; L^{\rho+2}(\Omega)),
\end{aligned}$$

$$z^m \text{ is uniformly bounded in } L^\infty(0,T;L^2(\Omega\times(0,1))),$$
$$z^m(1) \text{ is uniformly bounded in } L^2(0,T;L^2(\Omega)).$$

Hence, we can extract subsequence of (y^m) and (z^m), still denoted by (y^m) and (z^m) respectively, such that

$$\begin{aligned}
&y^m \rightharpoonup y \quad \text{weakly star in } L^\infty(0,T;W) \text{ and weakly in } L^2(0,T;W),\\
&y_t^m \rightharpoonup y_t \quad \text{weakly star in } L^\infty(0,T;V)\cap L^\infty(0,T;L^{\rho+2}(\Omega))\\
&\qquad\qquad \text{and weakly in } L^2(0,T;V)\cap L^2(0,T;L^{\rho+2}(\Omega)),\\
&z^m \rightharpoonup z \quad \text{weakly star in } L^\infty(0,T;L^2(\Omega\times(0,1))) \text{ and weakly in } L^2(0,T;L^2(\Omega\times(0,1))),\\
&z^m(1) \rightharpoonup z(1) \quad \text{weakly in } L^2(0,T;L^2(\Omega)).
\end{aligned}$$

Since (y^m) is bounded in $L^\infty(0,T;W)$, then, by the use of the embedding of $W\subset L^\infty(\Omega)(\Omega\subset\mathbb{R}^2)$, we infer that (y^m) is bounded in $L^2(\Omega\times(0,T))$. Likewise, (y_t^m) is bounded in $L^2(\Omega\times(0,T))$. Hence, by the use of the Aubin-Lions Theorem, we get, up to a subsequence, that

$$y^m \to y \text{ strongly in } L^2(\Omega\times(0,T)),$$

and

$$y^m \to y \text{ a.e in } \Omega\times(0,T).$$

Since $s\to ks\ln|s|$ is continuous, it holds that

$$ky^m\ln|y^m| \to ky\ln|y| \text{ a.e in } \Omega\times(0,T).$$

The embedding of W in $L^\infty(\Omega)$ implies that the sequence $(y^m\ln|y^m|)$ is bounded in $L^\infty(\Omega\times(0,T))$. Applying the Lebesgue bounded convergence theorem, we deduce that

$$ky^m\ln|y^m| \to ky\ln|y| \text{ strongly in } L^2(0,T;L^2(\Omega)).$$

The remainder of the proof can be done as in [33, 49], so we skip it. □

4. Global existence

The energy to problem (2.10) is

$$\begin{aligned}
E(t) &= \frac{1}{\rho+2}\|y_t\|_{\rho+2}^{\rho+2} + \frac{1}{2}\Big(1-\int_0^t g(s)ds\Big)b(y,y) + \frac{1}{2}\|\nabla y_t\|^2\\
&\quad + \frac{1}{2}(g\circ\partial^2 y) - \frac{k}{2}\int_\Omega y^2\ln|y|dx + \frac{k}{4}\|y\|^2 + \frac{\xi\tau_0}{2}\int_\Omega\int_0^1 |z(x,\theta,t)|^2 d\theta dx. \qquad (4.1)
\end{aligned}$$

Let

$$I(t) = \Big(1-\int_0^t g(s)ds\Big)b(y,y) - k\int_\Omega y^2\ln|y|dx + \frac{k}{2}\|y\|^2. \qquad (4.2)$$

Then, it holds

$$E(t) = \frac{1}{\rho+2}\|y_t\|_{\rho+2}^{\rho+2} + \frac{1}{2}\|\nabla y_t\|^2 + \frac{1}{2}(g\circ\partial^2 y) + \frac{\xi\tau_0}{2}\int_\Omega\int_0^1 |z(x,\theta,t)|^2 d\theta dx + \frac{1}{2}I(t). \qquad (4.3)$$

Lemma 4.1. *There exists $c_3 > 0$ so that*

$$E'(t) \leq -c_3(\|y_t\|^2 + \|z(1,t)\|^2) + \frac{1}{2}(g' \circ \partial^2 y)(t) - \frac{1}{2}g(t)b(y,y) \leq 0. \tag{4.4}$$

Proof. Multiplying the first equation of (2.10) by y_t, we see

$$\begin{aligned} E'(t) &= -a_0\|y_t\|^2 - a_1\int_\Omega z(x,1,t)y_t dx + \frac{1}{2}(g' \circ \partial^2 y)(t) - \frac{1}{2}g(t)b(y,y) \\ &\quad +\xi\tau_0\int_\Omega\int_0^1 z(x,\theta,t)z_t(x,\theta,t)d\theta dx. \end{aligned}$$

Applying Young's inequality, we get

$$-a_1\int_\Omega z(x,1,t)y_t dx \leq \frac{|a_1|}{2}\|y_t\|^2 + \frac{|a_1|}{2}\|z(1,t)\|^2. \tag{4.5}$$

Using the second equation of (2.10), we have

$$\begin{aligned} &\xi\tau_0\int_\Omega\int_0^1 z(x,\theta,t)z_t(x,\theta,t)d\theta dx \\ &= -\xi\int_\Omega\int_0^1 z(x,\theta,t)z_\theta(x,\theta,t)d\theta dx \\ &= -\frac{\xi}{2}\int_\Omega\int_0^1 \frac{\partial}{\partial\theta}(z(x,\theta,t))^2 d\theta dx \\ &= -\frac{\xi}{2}\int_\Omega (z(x,1,t))^2 dx + \frac{\xi}{2}\int_\Omega (z(x,0,t))^2 dx \\ &= -\frac{\xi}{2}\|z(1,t)\|^2 + \frac{\xi}{2}\|y_t\|^2. \end{aligned} \tag{4.6}$$

So, we find

$$E'(t) \leq -\Big(a_0 - \frac{|a_1|}{2} - \frac{\xi}{2}\Big)\|y_t\|^2 - \Big(\frac{\xi}{2} - \frac{|a_1|}{2}\Big)\|z(1,t)\|^2 + \frac{1}{2}(g' \circ \partial^2 y) - \frac{g(t)}{2}b(y,y). \tag{4.7}$$

By the assumption on a_0, a_1 and the choice of ξ, the proof is completed. □

We define

$$Q(r) = \frac{C_1}{2}r^2 - \frac{k}{2}(\ln r)r^2,$$

where

$$C_1 = \frac{3k}{2} + k\ln a.$$

Then, Q has the maximum value $E_1 = \frac{k}{4}r_*^2$ at $r_* = \exp(\frac{2C_1-k}{2k}) = ae$, Q is increasing on $(0, r_*)$, Q is decreasing on (r_*, ∞), $\lim_{r\to 0^+} Q(r) = 0$ and $\lim_{r\to\infty} Q(r) = -\infty$.

Lemma 4.2. *Assume $\|y_0\| < r_*$ and $0 < E(0) < E_1$. Then,*

$$\|y(t)\| < r_* \text{ and } I(t) > 0 \text{ for all } t \in [0,T). \tag{4.8}$$

Proof. From (2.12) and (3.10), we know

$$\begin{aligned} I(t) &\geq \Big(g_l - \frac{ka^2C_s}{2\pi}\Big)b(y(t),y(t)) + \Big(\frac{k}{2} + k(1+\ln a) - k\ln\|y(t)\|\Big)\|y(t)\|^2 \\ &> \Big(C_1 - k\ln\|y(t)\|\Big)\|y(t)\|^2 \\ &= 2Q(\|y(t)\|). \end{aligned} \tag{4.9}$$

First, we claim $\|y(t)\| < r_*$ for all $t \in [0,T)$. Indeed, from (4.3) and (4.9), we have

$$E(t) \geq \frac{1}{2}I(t) \geq Q(\|y(t)\|). \tag{4.10}$$

Suppose $\|y(t)\| \geq r_*$ for some $0 < t < T$. Since the mapping $t \to \|y(t)\|$ is continuous (as a composition of two continuous functions), then there exists $t_0 \in (0,T)$ such that $\|y(t_0)\| = r_*$. So,

$$E(t_0) \geq Q(\|y(t_0)\|) = Q(r_*) = E_1.$$

This contradicts to $E(t) \leq E(0) < E_1$ for all $t \geq 0$.

Now, we show $I(t) > 0$ for all $t \in [0,T)$. Using (4.9), $\|y(t)\| < r_*$ for all $t \in [0,T)$, and the definition of r_*, we see

$$I(t) > (C_1 - k\ln r_*)\|y\|^2 = \frac{k}{2}\|y\|^2 \geq 0.$$

□

Theorem 4.1 (Global existence)**.** *Assume the conditions of Lemma 4.2 hold. Then the solution* (y,z) *to problem* (1.1) *is global.*

Proof. From (4.3), (4.8) and (4.9), we obtain

$$\|y_t\|_{\rho+2}^{\rho+2} \leq (\rho+2)E(t), \tag{4.11}$$

$$b(y,y) \leq \frac{2\pi}{2\pi g_l - ka^2C_s}I(t) \leq \frac{4\pi}{2\pi g_l - ka^2C_s}E(t), \tag{4.12}$$

$$(g \circ \partial^2 y) \leq 2E(t), \tag{4.13}$$

$$\|\nabla y_t\|^2 \leq 2E(t), \tag{4.14}$$

and

$$\int_\Omega \int_0^1 |z(x,\theta,t)|^2 d\theta dx \leq \frac{2}{\xi\tau_0}E(t). \tag{4.15}$$

From these estimations and (4.4), we get, for some $c_4 > 0$,

$$\|y_t\|_{\rho+2}^{\rho+2} + b(y,y) + (g \circ \partial^2 y) + \|\nabla y_t\|^2 + \int_\Omega \int_0^1 |z(x,\theta,t)|^2 d\theta dx \leq c_4 E(0),$$

which ends the proof. □

5. Stability result

Let $N > 0, N_1 > 0$ and $N_2 > 0$ be constant that will be chosen later, and define

$$L(t) = NE(t) + N_1\Phi(t) + N_2\Psi(t) + \Upsilon(t),$$

$$\Phi(t) = \frac{1}{\rho+1}(|y_t|^\rho y_t, y) + (\nabla y_t, \nabla y)$$

$$\Psi(t) = -\frac{1}{\rho+1}\int_0^t g(t-s)(|y_t|^\rho y_t, y(t)-y(s))ds - \int_0^t g(t-s)(\nabla y_t, \nabla y(t) - \nabla y(s))ds,$$

and

$$\Upsilon(t) = \tau_0 \int_\Omega \int_0^1 e^{-\theta\tau_0}|z(x,\theta,t)|^2 d\theta dx.$$

In the sequel, we denote $c > 0$ and $c_i > 0$ a generic constant different from line to line even in the same line.

Lemma 5.1. *Assume the conditions of Lemma 4.2 hold. Then, $L(t)$ is equivalent to $E(t)$.*

Proof. Using (2.4), (4.12) and (4.4), one sees

$$\begin{aligned}
\|y(t)-y(s)\|_{\rho+2}^{\rho+2} &\le c\|\nabla y(t) - \nabla y(s)\|^{\rho+2} \\
&\le c(\|\nabla y(t)\|^2 + \|\nabla y(s)\|^2)^{\frac{\rho}{2}}\|\nabla y(t) - \nabla y(s)\|^2 \\
&\le cC_s^{\frac{\rho+2}{2}}(b(y(t),y(t)) + b(y(s),y(s)))^{\frac{\rho}{2}} b(y(t)-y(s), y(t)-y(s)) \\
&\le c\Big(\frac{8\pi}{2\pi g_l - ka^2C_s}E(0)\Big)^{\frac{\rho}{2}} b(y(t)-y(s), y(t)-y(s)) \qquad (5.1)
\end{aligned}$$

and

$$\|y(t)\|_{\rho+2}^{\rho+2} \le c\Big(\frac{4\pi}{2\pi g_l - ka^2C_s}E(0)\Big)^{\frac{\rho}{2}} b(y(t),y(t)). \qquad (5.2)$$

Applying Young's inequality, (5.2), (5.1), (2.4), we have

$$\begin{aligned}
|\Phi(t)| &\le \frac{1}{\rho+2}\|y_t\|_{\rho+2}^{\rho+2} + \frac{1}{(\rho+1)(\rho+2)}\|y\|_{\rho+2}^{\rho+2} + \frac{1}{2}\|\nabla y_t\|^2 + \frac{1}{2}\|\nabla y\|^2 \\
&\le \frac{1}{\rho+2}\|y_t\|_{\rho+2}^{\rho+2} + \frac{c}{(\rho+1)(\rho+2)}b(y,y) + \frac{1}{2}\|\nabla y_t\|^2 + \frac{C_s}{2}b(y,y) \qquad (5.3)
\end{aligned}$$

and

$$\begin{aligned}
|\Psi(t)| &\le \frac{1}{\rho+2}\|y_t\|_{\rho+2}^{\rho+2} + c\Big\|\int_0^t g(t-s)|y(t)-y(s)|ds\Big\|_{\rho+2}^{\rho+2} \\
&\quad + \frac{1}{2}\|\nabla y_t\|^2 + \frac{1}{2}\Big\|\int_0^t g(t-s)|\nabla y(t) - \nabla y(s)|ds\Big\|^2 \\
&\le \frac{1}{\rho+2}\|y_t\|_{\rho+2}^{\rho+2} + c\Big(1-g_l\Big)^{\rho+1}\int_0^t g(t-s)\|y(t)-y(s)\|_{\rho+2}^{\rho+2} ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\|\nabla y_t\|^2 + \frac{1-g_l}{2}\int_0^t g(t-s)\|\nabla y(t)-\nabla y(s)\|^2 ds\\
\leq\ & \frac{1}{\rho+2}\|y_t\|_{\rho+2}^{\rho+2} + c\Big(1-g_l\Big)^{\rho+1}\Big(\frac{8\pi}{2\pi g_l - ka^2 C_s}E(0)\Big)^{\frac{\rho}{2}}(g\circ\partial^2 y)\\
&+\frac{1}{2}\|\nabla y_t\|^2 + \frac{(1-g_l)C_s}{2}(g\circ\partial^2 y). \qquad (5.4)
\end{aligned}
$$

So, we find

$$
\begin{aligned}
|L(t)-NE(t)| \ \leq\ & \Big(\frac{N_1+N_2}{\rho+2}\Big)\|y_t\|_{\rho+2}^{\rho+2} + cN_1 b(y,y) + \Big(\frac{N_1+N_2}{2}\Big)\|\nabla y_t\|^2\\
&+cN_2(g\circ\partial^2 y) + \tau_0\int_\Omega\int_0^1 |z(x,\theta,t)|^2 d\theta dx\\
\leq\ & cE(t).
\end{aligned}
$$

Taking N large enough, we obtain the desired result. □

Lemma 5.2. *The function* Φ *satisfies*

$$
\begin{aligned}
\Phi'(t) \ \leq\ & \frac{1}{\rho+1}\|y_t\|_{\rho+2}^{\rho+2} + \|\nabla y_t\|^2 + k\int_\Omega y^2\ln|y|dx - \frac{g_l}{4}b(y,y)\\
&+c(g\circ\partial^2 y) + ca_0^2\|y_t\|^2 + ca_1^2\|z(1,t)\|^2. \qquad (5.5)
\end{aligned}
$$

Proof. From (2.10), we get

$$
\begin{aligned}
\Phi'(t) \ =\ & \frac{1}{\rho+1}\|y_t\|_{\rho+2}^{\rho+2} + \|\nabla y_t\|^2 - \Big(1-\int_0^t g(s)ds\Big)b(y,y) + k\int_\Omega y^2\ln|y|dx\\
&-a_0(y_t,y) - a_1(z(1,t),y) + \int_0^t g(t-s)b(y(s)-y(t),y(t))ds. \qquad (5.6)
\end{aligned}
$$

Using (2.4), we see that

$$
-a_0(y_t,y) - a_1(z(1,t),y) \leq \frac{g_l}{4}b(y,y) + ca_0^2\|y_t\|^2 + ca_1^2\|z(1,t)\|^2.
$$

From (2.2), we observe that

$$
\begin{aligned}
& b\Big(\int_0^t g(t-s)(y(s)-y(t))ds, \int_0^t g(t-s)(y(s)-y(t))ds\Big)dx\\
&\leq c_2\Big\|\int_0^t g(t-s)(y(t)-y(s))ds\Big\|_{H^2(\Omega)}^2\\
&\leq c_2(1-g_l)\int_0^t g(t-s)\|y(t)-y(s)\|_{H^2(\Omega)}^2 ds\\
&\leq \frac{c_2(1-g_l)}{c_1}(g\circ\partial^2 y).
\end{aligned}
$$

From this latter, we infer that

$$\begin{aligned}
&\int_0^t g(t-s)b(y(s)-y(t),y)ds\\
&\le \frac{g_l}{2}b(y,y)+\frac{1}{2g_l}b(\int_0^t g(t-s)(y(s)-y(t)ds,\int_0^t g(t-s)(y(s)-y(t))ds\\
&\le \frac{g_l}{2}b(y,y)+\frac{c_2(1-g_l)}{2g_lc_1}(g\circ\partial^2 y).
\end{aligned}$$

Substituting these above estimations into (5.6), we obtain (5.5). □

Lemma 5.3. *For any* $0<\eta<1$, *the function* Ψ *satisfies*

$$\begin{aligned}
\Psi'(t) \;\le\; & -\Big\{\frac{1}{\rho+1}\Big(\int_0^t g(s)ds\Big)-\eta\Big\}\|y_t\|_{\rho+2}^{\rho+2}-\Big(\int_0^t g(s)ds-\eta\Big)\|\nabla y_t\|^2\\
&+2\eta b(y(t),y(t))+\eta\|y_t\|^2+\eta\|z(1,t)\|^2\\
&+\Big(c(\eta)+\frac{g(0)C_s}{4\eta}\Big)(-g'\circ\partial^2 y)+\Big(c+c(\eta)\Big)(g\circ\partial^2 y)\\
&+c(\eta,\epsilon_0)(g\circ\partial^2 y)^{\frac{1}{1+\epsilon_0}}. \qquad (5.7)
\end{aligned}$$

Proof. From (2.10), we get

$$\begin{aligned}
\Psi'(t) \;=\; & -\frac{1}{\rho+1}\Big(\int_0^t g(s)ds\Big)\|y_t\|_{\rho+2}^{\rho+2}-\Big(\int_0^t g(s)ds\Big)\|\nabla y_t\|^2\\
&-\frac{1}{\rho+1}\int_0^t g'(t-s)(|y_t|^\rho y_t,y(t)-y(s))ds\\
&-\int_0^t g'(t-s)(\nabla y_t,\nabla y(t)-\nabla y(s))ds\\
&+b\Big(\int_0^t g(t-s)(y(t)-y(s))ds,\int_0^t g(t-s)(y(t)-y(s))ds\Big)\\
&+\Big(1-\int_0^t g(s)ds\Big)\int_0^t g(t-s)b(y(t),y(t)-y(s))ds\\
&+a_0\int_0^t g(t-s)(y_t,y(t)-y(s))ds+a_1\int_0^t g(t-s)(z(1,t),y(t)-y(s))ds\\
&-\int_0^t g(t-s)(ky\ln|y|,y(t)-y(s))dx\\
:=& -\frac{1}{\rho+1}\Big(\int_0^t g(s)ds\Big)\|y_t\|_{\rho+2}^{\rho+2}-\Big(\int_0^t g(s)ds\Big)\|\nabla y_t\|^2+\sum_{i=3}^9 D_i. \qquad (5.8)
\end{aligned}$$

Using (5.1), we have

$$\begin{aligned}
D_3 \;=\; & -\frac{1}{\rho+1}\int_0^t g'(t-s)(|y_t|^\rho y_t,y(t)-y(s))ds\\
\le\; & \eta\|y_t\|_{\rho+2}^{\rho+2}+c(\eta)\Big\|\int_0^t g'(t-s)(y(t)-y(s))ds\Big\|_{\rho+2}^{\rho+2}
\end{aligned}$$

$$
\begin{aligned}
&\leq \eta\|y_t\|_{\rho+2}^{\rho+2} - c(\eta)(g(0))^{\rho+1}\int_0^t g'(t-s)\|y(t)-y(s)\|_{\rho+2}^{\rho+2}ds \\
&\leq \eta\|y_t\|_{\rho+2}^{\rho+2} - c(\eta)(g(0))^{\rho+1}\Big(\frac{8\pi}{2\pi g_l - ka^2C_s}E(0)\Big)^{\frac{\rho}{2}}\int_0^t g'(t-s)b(y(t)-y(s),y(t)-y(s))ds \\
&= \eta\|y_t\|_{\rho+2}^{\rho+2} + c(\eta)(-g'\circ\partial^2 y). \qquad (5.9)
\end{aligned}
$$

Making use of (2.4), we obtain

$$
\begin{aligned}
D_4 &= -\int_0^t g'(t-s)(\nabla y_t, \nabla y(t)-\nabla y(s))ds \\
&\leq \eta\|\nabla y_t\|^2 + \frac{1}{4\eta}\Big\|\int_0^t g'(t-s)(\nabla y(t)-\nabla y(s))ds\Big\|^2 \\
&\leq \eta\|\nabla y_t\|^2 - \frac{g(0)}{4\eta}\int_0^t g'(t-s)\|\nabla y(t)-\nabla y(s)\|^2 ds \\
&\leq \eta\|\nabla y_t\|^2 - \frac{g(0)C_s}{4\eta}\int_0^t g'(t-s)b(y(t)-y(s),y(t)-y(s))ds \\
&= \eta\|\nabla y_t\|^2 + \frac{g(0)C_s}{4\eta}(-g'\circ\partial^2 y). \qquad (5.10)
\end{aligned}
$$

From (2.2), we find

$$
\begin{aligned}
D_5 &= b\Big(\int_0^t g(t-s)y(t)-y(s)ds, \int_0^t g(t-s)(y(t)-y(s))ds\Big) \\
&\leq c_2\Big\|\int_0^t g(t-s)(y(t)-y(s))ds\Big\|_{H^2(\Omega)}^2 \\
&\leq c_2\Big(\int_0^t g(s)ds\Big)\int_0^t g(t-s)\|y(t)-y(s)\|_{H^2(\Omega)}^2 ds \\
&\leq \frac{c_2}{c_1}\Big(1-g_l\Big)\int_0^t g(t-s)b(y(t)-y(s),y(t)-y(s))ds \\
&= c(g\circ\partial^2 y). \qquad (5.11)
\end{aligned}
$$

Using the inequality $b(u,v) \leq \eta b(u,u) + \frac{1}{4\eta}b(v,v)$, we get

$$
\begin{aligned}
D_6 &= \Big(1-\int_0^t g(s)ds\Big)\int_0^t g(t-s)b(y(t),y(t)-y(s))ds \\
&\leq \eta b(y(t),y(t)) + c(\eta)(g\circ\partial^2 y) \qquad (5.12)
\end{aligned}
$$

and

$$
\begin{aligned}
D_7 &= a_0\int_0^t g(t-s)(y_t, y(t)-y(s))ds \\
&\leq \eta\|y_t\|^2 + c(\eta)(g\circ\partial^2 y). \qquad (5.13)
\end{aligned}
$$

Moreover, we have

$$\begin{aligned} D_8 &= a_1 \int_0^t g(t-s)(z(1,t), y(t)-y(s))ds \\ &\le \eta\|z(1,t)\|^2 + c(\eta)(g\circ\partial^2 y). \end{aligned} \tag{5.14}$$

Using (2.11) and the relation $\frac{1+\epsilon_0}{2} + \frac{1-\epsilon_0}{2} = 1$, it holds that

$$\begin{aligned} D_9 &= -\int_0^t g(t-s)(ky\ln|y|, y(t)-y(s))ds \\ &\le k\int_0^t g(t-s)(y^2 + a_{\epsilon_0} y^{1-\epsilon_0}, y(t)-y(s))ds \\ &\le k\Big(y^2, \int_0^t g(t-s)(y(t)-y(s))ds\Big) + ka_{\epsilon_0}\Big(y^{1-\epsilon_0}, \int_0^t g(t-s)(y(t)-y(s))ds\Big) \\ &\le \delta_1\|y\|_4^4 + c(\delta_1)(g\circ\partial^2 y) + \delta_1\|y\|^2 + c(\delta_1,\epsilon_0)\Big\|\int_0^t g(t-s)(y(t)-y(s))ds\Big\|_{\frac{2}{1+\epsilon_0}}^{\frac{2}{1+\epsilon_0}} \\ &\le c\delta_1 b(y,y) + c(\delta_1)(g\circ\partial^2 y) + c(\delta_1,\epsilon_0)(g\circ\partial^2 y)^{\frac{1}{1+\epsilon_0}}, \end{aligned} \tag{5.15}$$

where we used the fact that $\|y\|_4^2 \le cb(y,y)$. Thus, we obtain

$$\begin{aligned} \Psi'(t) \le\ & -\Big\{\frac{1}{\rho+1}\Big(\int_0^t g(s)ds\Big) - \eta\Big\}\|y_t\|_{\rho+2}^{\rho+2} - \Big(\int_0^t g(s)ds - \eta\Big)\|\nabla y_t\|^2 \\ & + \Big(\eta + c\delta_1\Big)b(y(t),y(t)) + \eta\|y_t\|^2 + \eta\|z(1,t)\|^2 \\ & + \Big(c(\eta) + \frac{g(0)C_s}{4\eta}\Big)(-g'\circ\partial^2 y) + \Big(c + c(\eta) + c(\delta_1)\Big)(g\circ\partial^2 y) \\ & + c(\delta_1,\epsilon_0)(g\circ\partial^2 y)^{\frac{1}{1+\epsilon_0}}. \end{aligned} \tag{5.16}$$

Taking $\delta_1 > 0$ so that $c\delta_1 = \eta$, we have (5.7).

□

Lemma 5.4. *The function Υ satisfies*

$$\Upsilon'(t) \le \|y_t\|^2 - \tau_0 e^{-\tau_0}\int_\Omega\int_0^1 |z(x,\theta,t)|^2 d\theta dx. \tag{5.17}$$

Proof. Using (2.10) and $z(x,0,t) = y_t(x,t)$, we get

$$\begin{aligned} \Upsilon'(t) &= 2\tau_0\int_\Omega\int_0^1 e^{-\theta\tau_0} z(x,\theta,t)z_t(x,\theta,t)d\theta dx \\ &= -\int_\Omega\int_0^1 e^{-\theta\tau_0}\frac{\partial}{\partial\theta}|z(x,\theta,t)|^2 d\theta dx \\ &= -\int_\Omega e^{-\tau_0}|z(x,1,t)|^2 dx + \int_\Omega |z(x,0,t)|^2 dx - \tau_0\int_\Omega\int_0^1 e^{-\theta\tau_0}|z(x,\theta,t)|^2 d\theta dx \\ &\le \|y_t\|^2 - \tau_0 e^{-\tau_0}\int_\Omega\int_0^1 |z(x,\theta,t)|^2 d\theta dx. \end{aligned}$$

□

Lemma 5.5. *Let $t_0 > 0$. Then, there exists $\chi > 0$ and $c_5 > 0$ satisfying*

$$L'(t) \leq -\chi E(t) + c_5(g \circ \partial^2 y) + c(\epsilon_0)(g \circ \partial^2 y)^{\frac{1}{1+\epsilon_0}} \quad \textit{for } t \geq t_0. \tag{5.18}$$

Proof. Summarizing (4.4), (5.5), (5.7) and (5.17), we obtain

$$\begin{aligned} L'(t) \leq & -\Big\{N_2\Big(\frac{1}{\rho+1}\int_0^t g(s)ds - \eta\Big) - \frac{N_1}{\rho+1}\Big\}\|y_t\|_{\rho+2}^{\rho+2} \\ & -\Big\{\frac{N_1 g_l}{4} - 2N_2\eta\Big\}b(y,y) + N_1 k\int_\Omega y^2 \ln|y|dx \\ & +\Big\{N_1 c + N_2\Big(c + c(\eta)\Big)\Big\}(g \circ \partial^2 y) \\ & +\Big\{\frac{N}{2} - N_2\Big(c(\eta) + \frac{g(0)C_s}{4\eta}\Big)\Big\}(g' \circ \partial^2 y) \\ & -\Big\{Nc - N_1 c a_0^2 - N_2\eta - 1\Big\}\|y_t\|^2 - \Big\{Nc - N_1 c a_1^2 - N_2\eta\Big\}\|z(1,t)\|^2 \\ & -\Big\{N_2\Big(\int_0^t g(s)ds - \eta\Big) - N_1\Big\}\|\nabla y_t\|^2 \\ & +N_2 c(\eta,\epsilon_0)(g \circ \partial^2 y)^{\frac{1}{1+\epsilon_0}} - \tau_0 e^{-\tau_0}\int_\Omega\int_0^1 |z(x,\theta,t)|^2 d\theta dx. \end{aligned} \tag{5.19}$$

By (H_2), for any $t_0 > 0$, we get

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds := g_0 \quad \text{for } t \geq t_0. \tag{5.20}$$

Applying this, (2.3) and (2.12) to (5.19), we deduce, for $0 < \chi < 2N_1$,

$$\begin{aligned} L'(t) \leq & -\chi E(t) - \Big\{\frac{N_2 g_0 - N_1}{\rho+1} - N_2\eta - \frac{\chi}{\rho+2}\Big\}\|y_t\|_{\rho+2}^{\rho+2} \\ & -\Big\{\frac{N_1 g_l}{4} - 2N_2\eta - \frac{\chi(1-g_0)}{2} - \Big(N_1 - \frac{\chi}{2}\Big)\frac{ka^2 C_s}{2\pi} - \frac{\chi k C_p}{4}\Big\}b(y,y) \\ & +\Big\{N_1 c + N_2\Big(c + c(\eta)\Big) + \frac{\chi}{2}\Big\}(g \circ \partial^2 y) \\ & +\Big\{\frac{N}{2} - N_2\Big(c(\eta) + \frac{g(0)C_s}{4\eta}\Big)\Big\}(g' \circ \partial^2 y) \\ & -\Big\{Nc - N_1 c a_0^2 - N_2\eta - 1\Big\}\|y_t\|^2 - \Big\{Nc - N_1 c a_1^2 - N_2\eta\Big\}\|z(1,t)\|^2 \\ & -\Big\{N_2(g_0 - \eta) - N_1 - \frac{\chi}{2}\Big\}\|\nabla y_t\|^2 + N_2 c(\eta,\epsilon_0)(g \circ \partial^2 y)^{\frac{1}{1+\epsilon_0}} \\ & -\Big(\tau_0 e^{-\tau_0} - \frac{\chi\xi\tau_0}{2}\Big)\int_\Omega\int_0^1 |z(x,\theta,t)|^2 d\theta dx \\ & +k\Big\{N_1 - \frac{\chi}{2}\Big\}\Big(\ln\|y\| - 1 - \ln a)\Big)\|y\|^2. \end{aligned} \tag{5.21}$$

First, we pick N_2 satisfying

$$N_2 > \frac{2N_1}{g_0}.$$

Second, we select $\eta > 0$ properly so that

$$0 < \eta < \frac{g_0}{2(\rho+1)} \text{ and } \frac{N_1 g_l}{4} - 2N_2\eta > 0.$$

Then, we get

$$N_2(g_0 - \eta(\rho+1)) - N_1 > \frac{N_2 g_0}{2} - N_1 > 0, \tag{5.22}$$

and

$$N_2(g_0 - \eta) - N_1 > N_2(g_0 - \eta(\rho+1)) - N_1 > 0. \tag{5.23}$$

Third, we take $N > 0$ suitably large so that

$$\frac{N}{2} - N_2\Big(c(\eta) + \frac{g(0)C_s}{4\eta}\Big) > 0,$$

$$Nc - N_1 c a_0^2 - N_2\eta - 1 > 0,$$

and

$$Nc - N_1 c a_1^2 - N_2\eta > 0.$$

Finally, we choose χ and k small enough to get

$$\begin{aligned} L'(t) \quad \leq \quad & -\chi E(t) + c_5(g \circ \partial^2 y) + c(\epsilon_0)(g \circ \partial^2 y)^{\frac{1}{1+\epsilon_0}} \\ & +k\Big\{N_1 - \frac{\chi}{2}\Big\}\Big(\ln\|y\| - 1 - \ln a\Big)\|y\|^2. \end{aligned}$$

From (4.8), we obtain

$$\ln\|y\| - 1 - \ln a < \ln(r^*) - 1 - \ln a = 0, \tag{5.24}$$

which completes the proof. □

Now, we define

$$\lambda_0(t) = \frac{q}{t}\int_0^t b(y(t) - y(t-s), y(t) - y(t-s))ds, \tag{5.25}$$

where $q > 0$.

Lemma 5.6. *Let (H_1) and (H_2) hold. Then, there exists $q > 0$ such that*

$$(g \circ \partial^2 y)(t) \leq \frac{t}{q}\overline{G}^{-1}\Big(\frac{q}{t\zeta(t)}(-g' \circ \partial^2 y)(t)\Big), \tag{5.26}$$

where $\overline{G}$ is an extension of G.

Remark 5.1. *For the proof of the above lemma, we need the folowing Jensen's inequality: Assume F is a concave function on $[a,b]$, $f : \Omega \to [a,b]$ and g are in $L^1(\Omega)$, with $g(x) \geq 0$ and $\int_\Omega g(x)\,dx = m > 0$, then*

$$\frac{1}{m}\int_\Omega F[f(x)]g(x)\,dx \leq F\Big[\frac{1}{m}\int_\Omega f(x)g(x)\,dx\Big].$$

Proof. From (4.4) and (4.12), we get

$$\begin{aligned}
\lambda_0(t) &= \frac{q}{t}\int_0^t b(y(t)-y(t-s), y(t)-y(t-s))ds \\
&\le \frac{2q}{t}\int_0^t \Big(b(y(t),y(t)) + b(y(t-s),y(t-s))\Big)ds \\
&\le \frac{8\pi q}{t(2\pi g_l - ka^2C_s)}\int_0^t (E(t)+E(t-s))ds \\
&\le \frac{16\pi q}{t(2\pi g_l - ka^2C_s)}\int_0^t E(0)ds \\
&\le \frac{16\pi q E(0)}{2\pi g_l - ka^2C_s}. \qquad (5.27)
\end{aligned}$$

With suitable choice of q, we can have

$$\lambda_0(t) < 1 \text{ for } t > 0. \qquad (5.28)$$

By using (2.6), (5.28), the relation $G(\kappa s) \le \kappa G(s)$ for $0 \le \kappa \le 1$ and $0 < s \le r_1$, and Jensen's inequality, we find

$$\begin{aligned}
(-g' \circ \partial^2 y)(t) &= -\int_0^t g'(t-s)b(y(t)-y(s), y(t)-y(s))ds \\
&= -\int_0^t g'(s)b(y(t)-y(t-s), y(t)-y(t-s))ds \\
&= -\frac{1}{q\lambda_0(t)}\int_0^t \lambda_0(t)g'(s)qb(y(t)-y(t-s), y(t)-y(t-s))ds \\
&\ge \frac{1}{q\lambda_0(t)}\int_0^t \lambda_0(t)\zeta(t)G(g(s))qb(y(t)-y(t-s), y(t)-y(t-s))ds \\
&\ge \frac{\zeta(t)}{q\lambda_0(t)}\int_0^t G(\lambda_0(t)g(s))qb(y(t)-y(t-s), y(t)-y(t-s))ds \\
&\ge \frac{t\zeta(t)}{q}G\Big(\frac{q}{t}\int_0^t g(s)b(y(t)-y(t-s), y(t)-y(t-s))ds\Big) \\
&= \frac{t\zeta(t)}{q}\overline{G}\Big(\frac{q}{t}\int_0^t g(s)b(y(t)-y(t-s), y(t)-y(t-s))ds\Big) \\
&= \frac{t\zeta(t)}{q}\overline{G}\Big(\frac{q}{t}\int_0^t g(t-s)b(y(t)-y(s), y(t)-y(s))ds\Big) \\
&= \frac{t\zeta(t)}{q}\overline{G}\Big(\frac{q}{t}(g\circ \partial^2 y)(t)\Big), \qquad (5.29)
\end{aligned}$$

which yields (5.26). □

Theorem 5.1. *Suppose that the conditions of Lemma 4.2 hold. Then, there exist constants $C_0 > 0$, $c_0 > 0$ and $\varepsilon > 0$ satisfying*

$$E(t) \le \frac{C_0}{\Big(1+\int_{t_0}^t \zeta^{1+\epsilon_0}(s)ds\Big)^{\frac{1}{\epsilon_0}}} \quad \textit{if } G \textit{ is linear} \qquad (5.30)$$

and

$$E(t) \leq C_0 t^{\frac{1}{1+\epsilon_0}} K_2^{-1}\Big(\frac{c_0}{t^{\frac{1}{1+\epsilon_0}} \int_{t_1}^{t} \zeta(s)ds}\Big) \text{ if } G \text{ is nonlinear} \tag{5.31}$$

for $t > t_1 = \max\{t_0, 1\}$, *where* $K_2(s) = sK'(\varepsilon s)$, $K = \Big((\overline{G}^{-1})^{\frac{1}{1+\epsilon_0}}\Big)^{-1}$.

Proof. Due to (4.4) and (4.13), we observe

$$(g \circ \partial^2 y)(t) = (g \circ \partial^2 y)^{\frac{\epsilon_0}{1+\epsilon_0}} (g \circ \partial^2 y)^{\frac{1}{1+\epsilon_0}} \leq c(g \circ \partial^2 y)^{\frac{1}{1+\epsilon_0}}. \tag{5.32}$$

Case 1: G is linear

From (2.6), (4.4), (5.18) and (5.26), we infer that

$$\begin{aligned}
\zeta(t)L'(t) &\leq -\chi\zeta(t)E(t) + c_5\zeta(t)(g \circ \partial^2 y) + c(\epsilon_0)\zeta(t)(g \circ \partial^2 y)^{\frac{1}{1+\epsilon_0}} \\
&\leq -\chi\zeta(t)E(t) + c_5(-g' \circ \partial^2 y)(t) + c(\epsilon_0)\zeta(0)^{\frac{\epsilon_0}{1+\epsilon_0}}\{\zeta(t)(g \circ \partial^2 y)\}^{\frac{1}{1+\epsilon_0}} \\
&\leq -\chi\zeta(t)E(t) - 2c_5E'(t) + c(\epsilon_0)\{(-g' \circ \partial^2 y)\}^{\frac{1}{1+\epsilon_0}} \\
&\leq -\chi\zeta(t)E(t) - 2c_5E'(t) + c(\epsilon_0)\{-E'(t)\}^{\frac{1}{1+\epsilon_0}}, \ t \geq t_0,
\end{aligned} \tag{5.33}$$

where we used that $\zeta(t) \leq \zeta(t-s)$ in the third inequality.

Applying Young's inequality, we get

$$\begin{aligned}
&\zeta^{1+\epsilon_0}(t)E^{\epsilon_0}(t)L'(t) \\
&\leq -\chi\zeta^{1+\epsilon_0}(t)E^{1+\epsilon_0}(t) - 2c_5\zeta^{\epsilon_0}(t)E^{\epsilon_0}(t)E'(t) + c(\epsilon_0)\zeta^{\epsilon_0}(t)E^{\epsilon_0}(t)(-E'(t))^{\frac{1}{1+\epsilon_0}} \\
&\leq -\chi\zeta^{1+\epsilon_0}(t)E^{1+\epsilon_0}(t) - 2c_5\zeta^{\epsilon_0}(0)E^{\epsilon_0}(0)E'(t) + c(\epsilon_0)\zeta^{\epsilon_0}(t)E^{\epsilon_0}(t)(-E'(t))^{\frac{1}{1+\epsilon_0}} \\
&\leq -\chi\zeta^{1+\epsilon_0}(t)E^{1+\epsilon_0}(t) - 2c_5\zeta^{\epsilon_0}(0)E^{\epsilon_0}(0)E'(t) \\
&\quad + c(\epsilon_0)\Big\{\delta_2\zeta^{1+\epsilon_0}(t)E^{1+\epsilon_0}(t) + c(\delta_2)(-E'(t))\Big\} \\
&\leq -\Big(\chi - c(\epsilon_0)\delta_2\Big)\zeta^{1+\epsilon_0}(t)E^{1+\epsilon_0}(t) - \Big(2c_5\zeta^{\epsilon_0}(0)E^{\epsilon_0}(0) + c(\epsilon_0)c(\delta_2)\Big)E'(t)
\end{aligned} \tag{5.34}$$

for $\delta_2 > 0$.

Taking δ_2 small enough, we have

$$\zeta^{1+\epsilon_0}(t)E^{\epsilon_0}(t)L'(t) \leq -c_6\zeta^{1+\epsilon_0}(t)E^{1+\epsilon_0}(t) - c_7E'(t). \tag{5.35}$$

Thus, we find

$$\begin{aligned}
&(\zeta^{1+\epsilon_0}(t)E^{\epsilon_0}(t)L(t) + c_7E(t))' \\
&= (1+\epsilon_0)(\zeta(t)E(t))^{\epsilon_0}\zeta'(t)L(t) + \epsilon_0\zeta^{1+\epsilon_0}(t)E^{\epsilon_0-1}(t)E'(t)L(t) + \zeta^{1+\epsilon_0}(t)E^{\epsilon_0}(t)L'(t) + c_7E'(t) \\
&\leq -c_6\zeta^{1+\epsilon_0}(t)E^{1+\epsilon_0}(t), \ t \geq t_0.
\end{aligned} \tag{5.36}$$

Let

$$\mathcal{L}_1(t) = \zeta^{1+\epsilon_0}(t)E^{\epsilon_0}(t)L(t) + c_7E(t),$$

then

$$\mathcal{L}_1(t) \sim E(t).$$

This latter and (5.36) yield to

$$\mathcal{L}_1'(t) \leq -c_8\zeta^{1+\epsilon_0}(t)\mathcal{L}_1^{1+\epsilon_0}(t),\ t \geq t_0, \tag{5.37}$$

which gives (5.30).

Case 2 : G is nonlinear

Using (5.18), (5.26), (5.32), and the facts that $\overline{G}^{-1}$ is increasing, and $t^{\frac{1}{1+\epsilon_0}} \leq t$ for $t > 1$, we derive

$$\begin{aligned}
L'(t) &\leq -\chi E(t) + c_5(g \circ \partial^2 y) + c(\epsilon_0)(g \circ \partial^2 y)^{\frac{1}{1+\epsilon_0}} \\
&\leq -\chi E(t) + c_5(g \circ \partial^2 y)^{\frac{1}{1+\epsilon_0}} + c(\epsilon_0)(g \circ \partial^2 y)^{\frac{1}{1+\epsilon_0}} \\
&\leq -\chi E(t) + (c_5 + c(\epsilon_0))\Big\{\frac{t}{q}\overline{G}^{-1}\Big(\frac{q}{t\zeta(t)}(-g' \circ \partial^2 y)(t)\Big)\Big\}^{\frac{1}{1+\epsilon_0}} \\
&\leq -\chi E(t) + (c_5 + c(\epsilon_0))t^{\frac{1}{1+\epsilon_0}}\Big\{\overline{G}^{-1}\Big(\frac{q}{t\zeta(t)}(-g' \circ \partial^2 y)(t)\Big)\Big\}^{\frac{1}{1+\epsilon_0}} \\
&\leq -\chi E(t) + (c_5 + c(\epsilon_0))t^{\frac{1}{1+\epsilon_0}}\Big\{\overline{G}^{-1}\Big(\frac{q}{t^{\frac{1}{1+\epsilon_0}}\zeta(t)}(-g' \circ \partial^2 y)(t)\Big)\Big\}^{\frac{1}{1+\epsilon_0}} \\
&= -\chi E(t) + (c_5 + c(\epsilon_0))t^{\frac{1}{1+\epsilon_0}}K^{-1}(\alpha(t)),\ t \geq t_1,
\end{aligned} \tag{5.38}$$

where

$$t_1 = \max\{t_0, 1\},\ \alpha(t) = \frac{q}{t^{\frac{1}{1+\epsilon_0}}\zeta(t)}(-g' \circ \partial^2 y)(t),\ K^{-1} = (\overline{G}^{-1})^{\frac{1}{1+\epsilon_0}}. \tag{5.39}$$

We define, for $t \geq t_1$,

$$L_1(t) = K'\Big(\frac{\varepsilon}{t^{\frac{1}{1+\epsilon_0}}}\mathcal{E}(t)\Big)L(t) + E(t), \tag{5.40}$$

where $\varepsilon < r_0$ and $\mathcal{E}(t) = \frac{E(t)}{E(0)}$, then we note $\frac{\varepsilon}{t^{\frac{1}{1+\epsilon_0}}}\mathcal{E}(t) < r_0$ and $L_1(t) \sim E(t)$.

Since $K', K'' > 0$ on $(0, r_0]$, we have, for $t \geq t_1$,

$$\begin{aligned}
L_1'(t) &= K''\Big(\frac{\varepsilon}{t^{\frac{1}{1+\epsilon_0}}}\mathcal{E}(t)\Big)\Big(\frac{-\varepsilon}{(1+\epsilon_0)t^{\frac{2+\epsilon_0}{1+\epsilon_0}}}\mathcal{E}(t) + \frac{\varepsilon}{t^{\frac{1}{1+\epsilon_0}}}\mathcal{E}'(t)\Big)L(t) \\
&\quad + K'\Big(\frac{\varepsilon}{t^{\frac{1}{1+\epsilon_0}}}\mathcal{E}(t)\Big)L'(t) + E'(t) \\
&\leq -\chi K'\Big(\frac{\varepsilon}{t^{\frac{1}{1+\epsilon_0}}}\mathcal{E}(t)\Big)E(t) \\
&\quad + (c_5 + c(\epsilon_0))t^{\frac{1}{1+\epsilon_0}}K'\Big(\frac{\varepsilon}{t^{\frac{1}{1+\epsilon_0}}}\mathcal{E}(t)\Big)K^{-1}(\alpha(t)) + E'(t).
\end{aligned} \tag{5.41}$$

One knows the convex function K satisfies

$$\phi\varphi \leq K^*(\phi) + K(\varphi) \quad \text{for } \phi \in (0, K'(r_0)],\ \varphi \in (0, r_0] \tag{5.42}$$

and

$$K^*(\phi) = \phi(K')^{-1}(\phi) - K((K')^{-1}(\phi)) \quad \text{for } \phi \in (0, K'(r_0)], \tag{5.43}$$

where K^* is the conjugate function of K.

Applying (5.42) and (5.43) with $\phi = K'\Big(\frac{\varepsilon}{t^{\frac{1}{1+\epsilon_0}}}\mathcal{E}(t)\Big)$ and $\varphi = K^{-1}(\alpha(t))$ to (5.41), we get

$$\begin{aligned}
L_1'(t) &\leq -\chi K'\Big(\frac{\varepsilon}{t^{\frac{1}{1+\epsilon_0}}}\mathcal{E}(t)\Big)E(t) + (c_5 + c(\epsilon_0))t^{\frac{1}{1+\epsilon_0}}K'\Big(\frac{\varepsilon}{t^{\frac{1}{1+\epsilon_0}}}\mathcal{E}(t)\Big)K^{-1}(\alpha(t)) + E'(t)\\
&\leq -\chi K'\Big(\frac{\varepsilon}{t^{\frac{1}{1+\epsilon_0}}}\mathcal{E}(t)\Big)E(t) + (c_5 + c(\epsilon_0))t^{\frac{1}{1+\epsilon_0}}\alpha(t)\\
&\quad +(c_5 + c(\epsilon_0))t^{\frac{1}{1+\epsilon_0}} \times\\
&\qquad \Big\{K'\Big(\frac{\varepsilon}{t^{\frac{1}{1+\epsilon_0}}}\mathcal{E}(t)\Big)(K')^{-1}\Big(K'\Big(\frac{\varepsilon}{t^{\frac{1}{1+\epsilon_0}}}\mathcal{E}(t)\Big)\Big) - K\Big((K')^{-1}\Big(K'\Big(\frac{\varepsilon}{t^{\frac{1}{1+\epsilon_0}}}\mathcal{E}(t)\Big)\Big)\Big)\Big\}\\
&\leq -\chi E(0)K'\Big(\frac{\varepsilon}{t^{\frac{1}{1+\epsilon_0}}}\mathcal{E}(t)\Big)\mathcal{E}(t) + (c_5 + c(\epsilon_0))t^{\frac{1}{1+\epsilon_0}}\alpha(t)\\
&\quad +(c_5 + c(\epsilon_0))\varepsilon K'\Big(\frac{\varepsilon}{t^{\frac{1}{1+\epsilon_0}}}\mathcal{E}(t)\Big)\mathcal{E}(t)\\
&= -\Big(\chi E(0) - (c_5 + c(\epsilon_0)\varepsilon)\Big)K'\Big(\frac{\varepsilon}{t^{\frac{1}{1+\epsilon_0}}}\mathcal{E}(t)\Big)\mathcal{E}(t) + (c_5 + c(\epsilon_0))t^{\frac{1}{1+\epsilon_0}}\alpha(t), \qquad (5.44)
\end{aligned}$$

for $t \geq t_1$. Taking $\varepsilon > 0$ small enough and using (5.39), we have

$$L_1'(t) \leq -c_9 K'\Big(\frac{\varepsilon}{t^{\frac{1}{1+\epsilon_0}}}\mathcal{E}(t)\Big)\mathcal{E}(t) + \frac{c_{10}}{\zeta(t)}(-g' \circ \partial^2 y)(t), \tag{5.45}$$

for $t \geq t_1$. Now, we set

$$L_2(t) = \zeta(t)L_1(t) + 2c_{10}E(t) \ \text{ for } t \geq t_1, \tag{5.46}$$

then

$$L_2(t) \sim E(t).$$

This latter and (5.45) give us

$$\begin{aligned}
L_2'(t) &= \zeta'(t)L_1(t) + \zeta(t)L_1'(t) + 2c_{10}E'(t)\\
&\leq -c_9\zeta(t)K'\Big(\frac{\varepsilon}{t^{\frac{1}{1+\epsilon_0}}}\mathcal{E}(t)\Big)\mathcal{E}(t) + c_{10}(-g' \circ \partial^2 y)(t) + 2c_{10}E'(t)\\
&\leq -c_9\zeta(t)K'\Big(\frac{\varepsilon}{t^{\frac{1}{1+\epsilon_0}}}\mathcal{E}(t)\Big)\mathcal{E}(t) - 2c_{10}E'(t) + 2c_{10}E'(t)\\
&= -c_9\zeta(t)K'\Big(\frac{\varepsilon}{t^{\frac{1}{1+\epsilon_0}}}\mathcal{E}(t)\Big)\mathcal{E}(t), \ t \geq t_1, \qquad (5.47)
\end{aligned}$$

which gives

$$\int_{t_1}^{t} \zeta(s)K'\Big(\frac{\varepsilon}{s^{\frac{1}{1+\epsilon_0}}}\mathcal{E}(s)\Big)\mathcal{E}(s)ds \leq \frac{L_2(t_1)}{c_9}. \tag{5.48}$$

Since

$$\frac{d}{dt}\Big(K'\Big(\frac{\varepsilon}{t^{\frac{1}{1+\epsilon_0}}}\mathcal{E}(t)\Big)\mathcal{E}(t)\Big) \leq 0,$$

we obtain

$$K'\Big(\frac{\varepsilon}{t^{\frac{1}{1+\epsilon_0}}}\mathcal{E}(t)\Big)\mathcal{E}(t)\int_{t_1}^{t}\zeta(s)ds \le \frac{L_2(t_1)}{c_9}, \tag{5.49}$$

and consequently,

$$\frac{\mathcal{E}(t)}{t^{\frac{1}{1+\epsilon_0}}}K'\Big(\frac{\varepsilon}{t^{\frac{1}{1+\epsilon_0}}}\mathcal{E}(t)\Big)\int_{t_1}^{t}\zeta(s)ds \le \frac{c_0}{t^{\frac{1}{1+\epsilon_0}}}. \tag{5.50}$$

By the definition of K_2, we observe that

$$K_2\Big(\frac{\mathcal{E}(t)}{t^{\frac{1}{1+\epsilon_0}}}\Big)\int_{t_1}^{t}\zeta(s)ds \le \frac{c_0}{t^{\frac{1}{1+\epsilon_0}}}, \tag{5.51}$$

and hence

$$\mathcal{E}(t) \le t^{\frac{1}{1+\epsilon_0}}K_2^{-1}\Big(\frac{c_0}{t^{\frac{1}{1+\epsilon_0}}\displaystyle\int_{t_1}^{t}\zeta(s)ds}\Big),\ t \ge t_1. \tag{5.52}$$

This proves (5.31). □

Examples 5.1. *The following two examples illustrate our results:*
1) *G is linear*

Let $g(t) = ae^{-b(1+t)}$, where $b > 0$ and $a > 0$ is small enough so that (H_2) is satisfied, then $g'(t) = -\xi(t)G(g(t))$ where $G(t) = t$ and $\xi(t) = b$. Therefore, we can use (5.30) *to obtain*

$$E(t) \le \frac{C_0}{(1+t)^{\frac{1}{\epsilon_0}}}.$$

2) *G is non-linear*

Let $g(t) = \frac{a}{(1+t)^q}$, where $q > 1+\epsilon_0$ and a is chosen so that hypothesis (H_2) remains valid. Then

$$g'(t) = -bG(g(t)), \quad \text{with} \quad G(s) = s^{\frac{q+1}{q}},$$

where b is a fixed constant.

Since $K(s) = s^{\frac{(\epsilon_0+1)(q+1)}{q}}$. Then, (5.31) *gives, $\forall t \ge t_1$*

$$E(t) \le \frac{C_0}{t^{\frac{q-1-\epsilon_0}{(1+\epsilon_0)^2(q+1)}}}.$$

6. Conclusions

This paper focuses on the existence and the asymptotic stability of solutions for a quasi-linear Kirchhoff plate equations in a bounded domain of $\mathbb{R}^2$, subject to viscoelastic and frictional dissipative terms and with the presence of rotational forces, delay and logarithmic source terms. This equation describes the motion of a plate, which is clamped along one portion of its boundary and has free vibrations on the other portion of the boundary.

As future works, we can change the type of damping by considering, for example, Balakrishnan-Taylor damping (of the form $(\nabla y, \nabla y_t)\Delta y$) or strong damping (of the form $\Delta^2 y_t$).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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