Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(10): 23974–23988.
DOI: 10.3934/math.20231222
Received: 01 June 2023
Revised: 18 July 2023
Accepted: 26 July 2023
Published: 07 August 2023

Research article

Asymptotic analysis of stretching modes for a folded plate

Nabil Kerdid*

Department of Mathematics and Statistics, College of Sciences, Imam Mohammad Ibn Saud Islamic University (IMSIU), P. O. Box 90950, Riyadh 11623, KSA

* **Correspondence:** Email: nkerdid@gmail.com.

Abstract: In this paper, we show that the spectral problem associated to stretching modes in a thin folded plate can be derived from the three-dimensional eigenvalue problem of linear elasticity through a rigourous convergence analysis as the thickness of the plate goes to zero. We show, using a nonstandard asymptotic analysis technique, that each stretching frequency of an elastic thin folded plate is the limit of a family of high frequencies of the three-dimensional linearized elasticity system in the folded plate, as the thickness approaches zero.

Keywords: linear elasticity; asymptotic analysis; stretching modes; folded plate; eigenvalue problem
Mathematics Subject Classification: 35C20, 35E20, 74B05, 74G10, 74K20

1. Introduction

The purpose of this article is to show that the two-dimensional eigenvalue problem associated to stretching modes of a clamped thin folded plate can be derived from the standard three-dimensional eigenvalue problem of linear elasticity. This result is obtained using a nonstandard asymptotic analysis technique introduced in Irago et al. [1, 2] to derive the one-dimensional eigenvalue problem governing the classical equations for torsion and stretching vibrations in thin rods.

Let us cite the most important works that deal with the asymptotic analysis of the eigenvalue problem in different thin linear elastic structures: Ciarlet and Kessavan [3] in the case of a clamped plate, Le Dret [4] for a junction between two plates, Bourquin and Ciarlet [5] and Lods [6] for a plate inserted in a three-dimensional body, Kerdid [7, 8] for a single rod and junction between two rods, Jumbo and Mulet [9] and Jumbo et al. [10] for thin elastic rod with non-uniform cross-section, Serpilli and Lenci [11] for laminated beams, Tambača [12] for curved rods, and Qaudiello et al. [13] for a thin T-like shaped structure.

These works are concerned with the convergence of low frequency modes of the three-dimensional linear elasticity and the limit problems obtained are the classical spectral problems associated with the flexural displacement of the structure as the thickness of the body goes to zero. However, the techniques used turned out to be unsuitable for the analysis of the asymptotic behaviour of high frequency modes. In this work, we study the asymptotic behaviour of high frequency modes in a thin multi-structure when its thickness goes to zero. We consider an homogeneous isotropic linearly elastic structure consisting of two plates of thickness ε perpendicular to each other. The plates are assumed to be clamped on parts of the edges of both plates.

The problem of modeling folded plates is a particular case of the more general problem of modeling elastic multi-structures combining plates and rods that are held together by appropriate junctions. The central idea to study these problems consists in scaling each part of the elastic structure independently of the other in such a way that the junction region between the two parts is taken into account in each of the scaled parts. The scaled displacements are then defined on separate domains, but satisfy some compatibility relations. Passing to the limit in these relations yield the limit junction conditions that the limit displacements must satisfy.

The case of low frequency modes in a folded plate was studied in [4]. It has been shown that for each integer $m \geq 1$ the eigenvalues $\lambda_m(\varepsilon)$ of the three-dimensional eigenvalue problem converge as $\varepsilon \to 0$ to the eigenvalues $\lambda_m(0)$ of a well posed 2d-2d eigenvalue problem. The associated eigenvectors converge towards the eigenvectors of the same problem. The limit eigenvectors are of Kirchhoff-Love type in each plate with no stretching components. They are thus determined by pairs $(\zeta_2^{1,m}, \zeta_1^{2,m})$ of functions of in-plane variables of each plate corresponding to the flexural displacements of the plates. However, the case of high frequency modes is more complicated. Indeed, if we fix the index m and we make ε tend to zero, all the sequence $\eta_m(\varepsilon) = \varepsilon^2 \lambda_m(\varepsilon)$ of high frequency modes goes to zero. This comes from the fact that the high frequency modes are concentrated at infinity when ε approaches zero and cannot be obtained using such a passage to the limit. So, the idea in order to characterize this kind of frequencies, consists in associating to each integer $m \geq 1$, a family of index $\{\ell_\varepsilon^m\}_{\varepsilon>0}$ that depend on ε and such that $\ell_\varepsilon^m \to +\infty$ as $\varepsilon \to 0$.

We will thus show that the eigenvalues $\eta_{\ell_\varepsilon^m}(\varepsilon)$ of the three-dimensional elasticity problem converge, as the thickness of the plate goes to zero, towards the eigenfunctions $\eta_m(0)$ of a two-dimensional spectral problem. The associated eigenvectors $(u^{1,\ell_\varepsilon^m}(\varepsilon), u^{2,\ell_\varepsilon^m}(\varepsilon))$ converge towards the eigenvectors $(u^{1,m}(0), u^{2,m}(0))$ of the same problem. The limit eigenvectors are of Kirchhoff-Love type in each plate with no flexural components. That is,

$$u^{1,m}(0)(x) = (\zeta_1^{1,m}(x_1, x_3), 0, \zeta_3^{1,m}(x_1, x_3))$$

and

$$u^{2,m}(0)(x) = (0, \zeta_2^{2,m}(x_2, x_3), \zeta_3^{2,m}(x_2, x_3)),$$

where the pairs $\zeta^{1,m}$ and $\zeta^{2,m}$ correspond to the stretching displacements of the plates and verify the system of classical equations of stretching vibrations.

2. The three-dimensional problem

Given $\varepsilon \in \mathbb{R}, 0 < \varepsilon < 1$, we define

$$\begin{aligned}
\Omega_1^\varepsilon &= \{x \in \mathbb{R}^3 \; ; \; 0 < x_1, x_3 < 1, 0 < x_2 < \varepsilon\}, \\
\Omega_2^\varepsilon &= \{x \in \mathbb{R}^3 \; ; \; 0 < x_2, x_3 < 1, 0 < x_1 < \varepsilon\},
\end{aligned}$$

and we denote by

$$\Omega^\varepsilon = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon,$$

the open set that is assumed to be the reference configuration of the folded plate under consideration. We set

$$\omega_1 = \partial\Omega_1^\varepsilon \cap \{x_2 = 0\}, \quad \omega_2 = \partial\Omega_2^\varepsilon \cap \{x_1 = 0\}. \tag{2.1}$$

The folded plate is assumed to be clamped on its boundaries Γ_1^ε and Γ_2^ε defined by

$$\Gamma_1^\varepsilon = \partial\Omega_1^\varepsilon \cap \{x_1 = 1\}, \quad \Gamma_2^\varepsilon = \partial\Omega_2^\varepsilon \cap \{x_2 = 1\}.$$

We also define the junction region as

$$J^\varepsilon = \Omega_1^\varepsilon \cap \Omega_2^\varepsilon = \{x \in \Omega^\varepsilon; \ \ 0 < x_1, x_2 < \varepsilon\}.$$

The material that constitute the plate is assumed to be homogeneous and isotropic with Young's modulus E and Poisson's ratio ν, all independent of ε. We will also use the Lame's coefficients λ and μ related to E and ν by the formulas:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}. \tag{2.2}$$

In the sequel, we shall use the repeated index convention, Latin indices take their values in the set $\{1, 2, 3\}$, Greek indices with exponent 1 take their values in the set $\{1, 3\}$ and Greek indices with exponent 2 take their values in the set $\{2, 3\}$.

The eigenvalue problem for the folded plate under consideration consists in finding pairs $(\eta^\varepsilon, u^\varepsilon)$ satisfying the classical eigenvalue problem:

$$\begin{cases} -\partial_j \sigma_{ij}^\varepsilon &= \eta^\varepsilon u_i^\varepsilon, & \text{in} \ \ \Omega^\varepsilon, \\ \sigma_{ij}^\varepsilon(\boldsymbol{u}^\varepsilon) &= \lambda e_{pp}^\varepsilon(\boldsymbol{u}^\varepsilon)\delta_{ij} + 2\mu e_{ij}^\varepsilon(\boldsymbol{u}^\varepsilon), & \text{in} \ \ \Omega^\varepsilon, \\ \boldsymbol{u}^\varepsilon &= 0, & \text{on} \ \ \Gamma_1^\varepsilon \cup \Gamma_2^\varepsilon, \\ \sigma^\varepsilon n^\varepsilon &= 0, & \text{on} \ \ \partial\Omega^\varepsilon \backslash \Gamma_1^\varepsilon \cup \Gamma_2^\varepsilon, \end{cases} \tag{2.3}$$

where $\sigma^\varepsilon(\boldsymbol{u}^\varepsilon)$ is the stress tensor, n^ε is the outer unit normal vector to $\partial\Omega^\varepsilon$ and $e^\varepsilon(\boldsymbol{u}^\varepsilon)$ is the linearized strain tensor corresponding to the displacement $\boldsymbol{u}^\varepsilon$:

$$e_{ij}^\varepsilon(\boldsymbol{u}^\varepsilon) = \frac{1}{2}\left(\frac{\partial u_j^\varepsilon}{\partial x_i^\varepsilon} + \frac{\partial u_i^\varepsilon}{\partial x_j^\varepsilon}\right). \tag{2.4}$$

The variational formulation of problem (2.3) is written: Find $(\eta^\varepsilon, u^\varepsilon) \in \mathbb{R} \times V^\varepsilon$ satisfying

$$\int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon(\boldsymbol{u}^\varepsilon) e_{ij}^\varepsilon(\boldsymbol{v}^\varepsilon) dx^\varepsilon = \eta^\varepsilon \int_{\Omega^\varepsilon} u_i^\varepsilon v_i^\varepsilon dx^\varepsilon, \quad \forall v^\varepsilon \in V^\varepsilon, \tag{2.5}$$

where

$$V^\varepsilon = \{v = (v_i) \in [H^1(\Omega^\varepsilon)]^3, \ \ v = 0 \ \text{ on } \Gamma_1^\varepsilon \cup \Gamma_2^\varepsilon\}. \tag{2.6}$$

Thanks to Korn inequality and the clamping conditions, the bilinear form

$$(u^\varepsilon, v^\varepsilon) \in V^\varepsilon \times V^\varepsilon \longmapsto \int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon(\boldsymbol{u}^\varepsilon) e_{ij}^\varepsilon(\boldsymbol{v}^\varepsilon) dx^\varepsilon \tag{2.7}$$

is V^ε-elliptic and then problem (2.5) has a sequence of eigenvalues $(\eta_m^\varepsilon)_{m\geq 1}$ satisfying

$$0 < \eta_1^\varepsilon \leq \eta_2^\varepsilon \leq \eta_3^\varepsilon \leq \cdots \leq \eta_m^\varepsilon \leq \cdots \tag{2.8}$$

with

$$\lim_{m\longrightarrow\infty} \eta_m^\varepsilon = +\infty, \tag{2.9}$$

associated with a family of eigenfunctions $(u^{\varepsilon,m})_{m\geq 1}$, that is

$$\int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon(\boldsymbol{u}^{\varepsilon,m})e_{ij}^\varepsilon(\boldsymbol{v}^\varepsilon)dx^\varepsilon = \eta_m^\varepsilon \int_{\Omega^\varepsilon} u_i^{\varepsilon,m}v_i^\varepsilon dx^\varepsilon, \qquad \forall v^\varepsilon \in V^\varepsilon, \tag{2.10}$$

which can be orthonormalized as

$$\int_{\Omega^\varepsilon} u_i^{\varepsilon,m}u_i^{\varepsilon,n}dx^\varepsilon = \delta_{mn}, \qquad \forall m,n \geq 1 \tag{2.11}$$

and which make a basis in both Hilbert spaces V^ε and $[L^2(\Omega^\varepsilon)]^3$.

These eigenvalues are characterized by

$$\eta_m^\varepsilon = \min_{w_m^\varepsilon\in E_m^\varepsilon} \max_{v^\varepsilon\in w_m^\varepsilon} R^\varepsilon(v^\varepsilon), \tag{2.12}$$

where E_m^ε is the set of all vector spaces of V^ε of dimension m and $R^\varepsilon(v^\varepsilon)$ the Rayleigh quotient defined as

$$R^\varepsilon(v^\varepsilon) = \frac{\displaystyle\int_{\Omega^\varepsilon} \sigma^\varepsilon(\boldsymbol{v}^\varepsilon) : e^\varepsilon(\boldsymbol{v}^\varepsilon)dx^\varepsilon}{\displaystyle\int_{\Omega^\varepsilon} \boldsymbol{v}^\varepsilon.\boldsymbol{v}^\varepsilon dx^\varepsilon}, \qquad \forall v^\varepsilon \in V^\varepsilon\backslash\{0\}. \tag{2.13}$$

3. The scaled problems

To switch to a domain which does not depend on ε, let us define

$$\Omega_1 = \Omega_1^1, \qquad \Omega_2 = \Omega_2^1 \tag{3.1}$$

and

$$\Gamma_1 = \Gamma_1^1, \qquad \Gamma_2 = \Gamma_2^1. \tag{3.2}$$

Then, we introduce the following mapping

$$\begin{aligned} \phi^\varepsilon : \Omega_1 \cup \Omega_2 &\rightarrow \Omega^\varepsilon, \\ x &\mapsto \begin{cases} (x_1, \varepsilon x_2, x_3), & \text{if } x \in \Omega_1, \\ (\varepsilon x_1, x_2, x_3), & \text{if } x \in \Omega_2. \end{cases} \end{aligned}$$

The junction region is the image by ϕ^ε of the open sets

$$J_\varepsilon^1 = \{x \in \Omega_1 : x_1 < \varepsilon\} \quad \text{and} \quad J_\varepsilon^2 = \{x \in \Omega_2 : x_2 < \varepsilon\}. \tag{3.3}$$

We define

$$V = H^1_{\Gamma_1}(\Omega_1, \mathbb{R}^3) \times H^1_{\Gamma_2}(\Omega_2, \mathbb{R}^3) \tag{3.4}$$

and we associate the scaling operator:

$$\begin{array}{rcl} \Phi^\varepsilon : V^\varepsilon & \to & V, \\ v^\varepsilon & \mapsto & ((\varepsilon^{-1} v_1^\varepsilon, v_2^\varepsilon, \varepsilon^{-1} v_3^\varepsilon) \circ \phi^\varepsilon, (v_1^\varepsilon, \varepsilon^{-1} v_2^\varepsilon, \varepsilon^{-1} v_3^\varepsilon) \circ \phi^\varepsilon). \end{array}$$

We note

$$v(\varepsilon) = \Phi^\varepsilon(v^\varepsilon) \tag{3.5}$$

and

$$V(\varepsilon) = \Phi^\varepsilon(V^\varepsilon). \tag{3.6}$$

$V(\varepsilon)$ is the set of pairs $(v^1(\varepsilon), v^2(\varepsilon)) \in V$ satisfying the following relations:

$$\left\{ \begin{array}{rcl} \varepsilon v_1^1(\varepsilon)(\varepsilon x_1, x_2, x_3) & = & v_1^2(\varepsilon)(x_1, \varepsilon x_2, x_3), \\ v_2^1(\varepsilon)(\varepsilon x_1, x_2, x_3) & = & \varepsilon v_2^2(\varepsilon)(x_1, \varepsilon x_2, x_3), \\ v_3^1(\varepsilon)(\varepsilon x_1, x_2, x_3) & = & v_3^2(\varepsilon)(x_1, \varepsilon x_2, x_3). \end{array} \right. \tag{3.7}$$

These relations are called the multidimensional junction relations.

Let us introduce the scaled bilinear forms

$$\begin{array}{rcl} b_\varepsilon^1(u, v) & = & 2\mu e_{\alpha^1\beta^1}(u) e_{\alpha^1\beta^1}(v) + \lambda e_{\alpha^1\alpha^1}(u) e_{\beta^1\beta^1}(v) \\ & + & \varepsilon^{-2}[4\mu e_{\alpha^1 2}(u) e_{\alpha^1 2}(v) + \lambda(e_{\alpha^1\alpha^1}(u) e_{22}(v) + e_{22}(u) e_{\alpha^1\alpha^1}(v))] \\ & + & \varepsilon^{-4}(\lambda + 2\mu) e_{22}(u) e_{22}(v) \end{array} \tag{3.8}$$

and

$$\begin{array}{rcl} b_\varepsilon^2(u, v) & = & 2\mu e_{\alpha^2\beta^2}(u) e_{\alpha^2\beta^2}(v) + \lambda e_{\alpha^2\alpha^2}(u) e_{\beta^2\beta^2}(v) \\ & + & \varepsilon^{-2}[4\mu e_{\alpha^2 1}(u) e_{\alpha^2 1}(v) + \lambda(e_{\alpha^2\alpha^2}(u) e_{11}(v) + e_{11}(u) e_{\alpha^2\alpha^2}(v))] \\ & + & \varepsilon^{-4}(\lambda + 2\mu) e_{11}(u) e_{11}(v). \end{array} \tag{3.9}$$

Substituting (3.5) in (2.10) we obtain the following scaled variational formulation: Find $(\eta_m(\varepsilon), u^{1,m}(\varepsilon), u^{2,m}(\varepsilon)) \in \mathbb{R} \times V(\varepsilon)$, such that for all $v(\varepsilon) \in V(\varepsilon)$

$$\begin{array}{rl} \displaystyle\int_{\Omega_1} b_\varepsilon^1(u^{1,m}(\varepsilon), v^1(\varepsilon)) dx & + \displaystyle\int_{\Omega_2 \backslash J_\varepsilon^2} b_\varepsilon^2(u^{2,m}(\varepsilon), v^2(\varepsilon)) dx \\ & = \eta_m(\varepsilon) \displaystyle\int_{\Omega_1} [u_{\alpha^1}^{1,m}(\varepsilon) v_{\alpha^1}^1(\varepsilon) + \varepsilon^{-2} u_2^{1,m}(\varepsilon) v_2^1(\varepsilon)] dx \\ & + \eta_m(\varepsilon) \displaystyle\int_{\Omega_2 \backslash J_\varepsilon^2} [u_{\alpha^2}^{2,m}(\varepsilon) v_{\alpha^2}^2(\varepsilon) + \varepsilon^{-2} u_1^{2,m}(\varepsilon) v_1^2(\varepsilon)] dx, \end{array} \tag{3.10}$$

where

$$\eta_m(\varepsilon) = \eta_m^\varepsilon \tag{3.11}$$

and with the normalization condition

$$\int_{\Omega_1} [u_{\alpha^1}^{1,m}(\varepsilon) u_{\alpha^1}^{1,n}(\varepsilon) + \varepsilon^{-2} u_2^{1,m}(\varepsilon) u_2^{1,n}(\varepsilon)] dx + \int_{\Omega_2 \backslash J_\varepsilon^2} [u_{\alpha^2}^{2,m}(\varepsilon) u_{\alpha^2}^{2,n}(\varepsilon) + \varepsilon^{-2} u_1^{2,m}(\varepsilon) u_1^{2,n}(\varepsilon)] dx = \delta_{mn}. \tag{3.12}$$

Finally, we define the space of Kirchhoff-Love on Ω_1 as

$$V_{KL}(\Omega_1) = \left\{ v \in H^1(\Omega_1, \mathbb{R}^3), \quad e_{i2}(v) = 0 \right\}. \tag{3.13}$$

Elements of this space are characterized by

$$\begin{cases} v_{\alpha^1}(x) &= \zeta_{\alpha^1}(x_1, x_3) - (x_2 - \frac{1}{2})\partial_{\alpha^1}\zeta_2(x_1, x_3), \\ v_2(x) &= \zeta_2(x_1, x_3), \end{cases} \tag{3.14}$$

where $\zeta_{\alpha^1} \in H^1(\omega_1)$ and $\zeta_2 \in H^2(\omega_1)$. We define $V_{KL}(\Omega_2)$ by the analogue formulas.

4. Convergence of the high frequencies modes

A first convergence analysis of the three-dimensional linearized elasticity system in a folded thin plate was done in [4]. It has been shown that the eigenvalues $\lambda_m(\varepsilon) = \varepsilon^{-2}\eta_m(\varepsilon)$ associated to low frequencies of the three-dimensional linearized elasticity problem and the corresponding eigenvectors converge towards the eigenvalues and eigenvectors of the 2d-2d classical spectral problem associated with the flexural displacements of the folded plate, as the thickness of the plate goes to zero. To study the convergence of the eigenvalues and eigenvectors associated to high frequencies let us start with the following lemmas:

Lemma 4.1. *There exists an increasing sequence of constants $K_m > 0, m \geq 1$, independent of ε such that*

$$\eta_m(\varepsilon) \leq K_m \varepsilon^2. \tag{4.1}$$

Proof. From Lemma 3.1 in [4] we have, for each integer $m \geq 1$,

$$\varepsilon^{-2}\eta_m(\varepsilon) = \lambda_m(\varepsilon) \leq K_m, \tag{4.2}$$

where K_m is a constant independent of ε which gives directly (4.1). □

Of course, if we fix the index m and we make ε tend to zero, all the sequence of high frequency modes $\eta_m(\varepsilon)$ goes to zero. So, the idea in order to characterize the limit of high frequency modes consists in associating to each integer $m \geq 1$ the family of indices $\{\ell_\varepsilon^m\}_{\varepsilon>0}$ that depend on ε and defined by

$$\ell_\varepsilon^m = \max\{j \in \mathbb{N}^* : \eta_j(\varepsilon) \leq K_m\}. \tag{4.3}$$

It is clear that

$$\lim_{\varepsilon \to 0} \ell_\varepsilon^m = +\infty \tag{4.4}$$

and for $\varepsilon \in (0, 1]$,

$$\eta_{\ell_\varepsilon^m}(\varepsilon) < K_m. \tag{4.5}$$

The family $\{\ell_\varepsilon^m\}$ varies with m and ε and for each $\varepsilon > 0$, $\{\ell_\varepsilon^m\}_{m\geq 1}$ is an increasing subsequence of positive integers satisfying $\ell_\varepsilon^m \geq m, \forall m \geq 1$ and contains the indices of the modes associated to stretching vibrations among all the modes $\{\eta_m(\varepsilon)\}_{m\geq 1}$ of the plate.

To illustrate this idea and show the layout of the family of stretching modes $\{\eta_{\ell_\varepsilon^m}(\varepsilon)\}$ when m and ε vary, let us represent the elements of the family $\{\eta_m(\varepsilon_n)\}$ where $\{\varepsilon_n\}_{n\geq 1}$ is a decreasing sequence converging to 0, in a double-entry table where the elements of the sequences $\{\eta_m(\varepsilon_n)\}_{m\geq 1}$ are arranged in rows while the elements of the sequences $\{\eta_m(\varepsilon_n)\}_{n\geq 1}$ are arranged in columns. Since for each $m \geq 1$ the family $\{\ell^m_{\varepsilon_n}\}_{n\geq 1}$ is increasing, that is $\ell^m_{\varepsilon_n} \geq \ell^m_{\varepsilon_{n'}}$ for $n > n'$, the elements of the sequence $\{\eta_{\ell^m_{\varepsilon_n}}(\varepsilon_n)\}_{n\geq 1}$ corresponding to the modes associated to the stretching vibrations of the folded plate are arranged diagonally.

As the modes associated to stretching vibrations are high frequency modes and are concentrated at infinity when ε approaches zero, they can only be reached through such a family of indices:

$$
\begin{array}{cccccccccc}
\eta_1(\varepsilon_1) & \cdots & \eta_{\ell^1_{\varepsilon_1}}(\varepsilon_1) & & \cdots & \eta_{\ell^m_{\varepsilon_1}}(\varepsilon_1) & & \cdots & & \\
\eta_1(\varepsilon_2) & \cdots & & \eta_{\ell^1_{\varepsilon_2}}(\varepsilon_2) & \cdots & & \eta_{\ell^m_{\varepsilon_2}}(\varepsilon_2) & \cdots & & \\
\vdots & & & & \ddots & & & \ddots & & \\
\eta_1(\varepsilon_n) & \cdots & & & & \eta_{\ell^1_{\varepsilon_n}}(\varepsilon_n) & \cdots & \eta_{\ell^m_{\varepsilon_n}}(\varepsilon_n) & \cdots & \\
\downarrow & & & & & & \searrow & & \searrow & \\
0 & \cdots & & & & & \eta_1(0) & \cdots & \eta_m(0) & \cdots
\end{array}
$$

Now, we are able to establish an appropriate bound for the stretching eigenfunctions.

Lemma 4.2. *For each $m \geq 1$, there exists a constant $C_m > 0$ independent of ε, such that*

$$
\begin{aligned}
\| u^{1,\ell^m_\varepsilon}(\varepsilon) \|_{H^1(\Omega_1;\mathbb{R}^3)} &\leq C_m, \\
\| u^{2,\ell^m_\varepsilon}(\varepsilon) \|_{H^1(\Omega_2;\mathbb{R}^3)} &\leq C_m.
\end{aligned} \tag{4.6}
$$

Proof. Let us define the scaled strain tensors $\kappa^{1,\ell^m_\varepsilon}(\varepsilon)$ as

$$
\begin{aligned}
\kappa^{1,\ell^m_\varepsilon}_{\alpha^1\beta^1}(\varepsilon) &= e_{\alpha^1\beta^1}(u^{1,\ell^m_\varepsilon}(\varepsilon)), \\
\kappa^{1,\ell^m_\varepsilon}_{\alpha^1 2}(\varepsilon) &= \varepsilon^{-1} e_{\alpha^1 2}(u^{1,\ell^m_\varepsilon}(\varepsilon)), \\
\kappa^{1,\ell^m_\varepsilon}_{22}(\varepsilon) &= \varepsilon^{-2} e_{22}(u^{1,\ell^m_\varepsilon}(\varepsilon))
\end{aligned} \tag{4.7}
$$

and the similar for $\kappa^{2,\ell^m_\varepsilon}(\varepsilon)$.

Taking

$$
v(\varepsilon) = (u^{1,\ell^m_\varepsilon}(\varepsilon), u^{2,\ell^m_\varepsilon}(\varepsilon))
$$

in (3.10) and using (3.8), (3.9) and (3.12), we obtain

$$
\begin{aligned}
2\mu\|\kappa^{1,\ell^m_\varepsilon}(\varepsilon)\|_{L^2(\Omega_1)^3} + 2\mu\|\kappa^{2,\ell^m_\varepsilon}(\varepsilon)\|_{L^2(\Omega_2)^3} &\leq \int_{\Omega_1} b_\varepsilon(u^{1,\ell^m_\varepsilon}(\varepsilon), u^{1,\ell^m_\varepsilon}(\varepsilon))dx + \int_{\Omega_2\setminus J^2_\varepsilon} b_\varepsilon(u^{2,\ell^m_\varepsilon}(\varepsilon), u^{2,\ell^m_\varepsilon}(\varepsilon))dx \\
&= \eta_{\ell^m_\varepsilon}(\varepsilon) \leq K_m.
\end{aligned}
$$

So, we have

$$
\begin{aligned}
&\|\kappa^{1,\ell^m_\varepsilon}_{ij}(\varepsilon)\|_{L^2(\Omega_1)} \leq C_m, \\
&\|\kappa^{2,\ell^m_\varepsilon}_{ij}(\varepsilon)\|_{L^2(\Omega_2)} \leq C_m,
\end{aligned} \tag{4.8}
$$

and consequently

$$
\|e_{\alpha^1\beta^1}(u^{1,\ell^m_\varepsilon}(\varepsilon))\|_{L^2(\Omega_1)} \leq C_m,
$$

$$\begin{aligned}\|e_{\alpha^1 2}(u^{1,\ell_\varepsilon^m}(\varepsilon))\|_{L^2(\Omega_1)} &\leq C_m\varepsilon \leq C_m, \\ \|e_{22}(u^{1,\ell_\varepsilon^m}(\varepsilon))\|_{L^2(\Omega_1)} &\leq C_m\varepsilon^2 \leq C_m\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\|e_{\alpha^2\beta^2}(u^{2,\ell_\varepsilon^m}(\varepsilon))\|_{L^2(\Omega_2)} &\leq C_m, \\ \|e_{\alpha^2 1}(u^{2,\ell_\varepsilon^m}(\varepsilon))\|_{L^2(\Omega_2)} &\leq C_m\varepsilon \leq C_m, \\ \|e_{11}(u^{2,\ell_\varepsilon^m}(\varepsilon))\|_{L^2(\Omega_2)} &\leq C_m\varepsilon^2 \leq C_m.\end{aligned} \tag{4.10}$$

Therefore, inequality (4.6) is obtained using Korn inequality in $H^1_{\Gamma_1}(\Omega_1;\mathbb{R}^3)$ and $H^1_{\Gamma_2}(\Omega_2;\mathbb{R}^3)$ respectively. □

Now, we are able to pass to the limit in the scaled eigenvalues and eigenvectors.

Lemma 4.3. *For each $m \geq 1$, there exists a subsequence, still denoted ε such that*

$$\eta_{\ell_\varepsilon^m}(\varepsilon) \to \eta_m(0) \tag{4.11}$$

and

$$(u^{1,\ell_\varepsilon^m}(\varepsilon), u^{2,\ell_\varepsilon^m}(\varepsilon)) \rightharpoonup (u^{1,m}(0), u^{2,m}(0)) \quad \text{weakly in } \; V, \tag{4.12}$$

where

$$u^{1,m}_{\alpha^1}(0)(x) = \zeta^{1,m}_{\alpha^1}(x_1,x_3) - (x_2 - 1/2)\partial_{\alpha^1}\zeta^{1,m}_2(x_1,x_3), \tag{4.13}$$

$$u^{1,m}_2(0)(x) = \zeta^{1,m}_2(x_1,x_3) \tag{4.14}$$

and

$$u^{2,m}_{\alpha^2}(0)(x) = \zeta^{2,m}_{\alpha^2}(x_2,x_3) - (x_1 - 1/2)\partial_{\alpha^2}\zeta^{2,m}_1(x_2,x_3), \tag{4.15}$$

$$u^{2,m}_1(0)(x) = \zeta^{2,m}_1(x_2,x_3) \tag{4.16}$$

with

$$(\zeta^{1,m}_{\alpha^1}, \zeta^{2,m}_{\alpha^2}) \quad \in [H^1(\omega_1)]^2 \times [H^1(\omega_2)]^2$$

and

$$(\zeta^{1,m}_2, \zeta^{2,m}_1) \quad \in H^2(\omega_1) \times H^2(\omega_2).$$

Proof. Convergences (4.11) and (4.12) are deduced from (4.5) and (4.6). From (4.9) we have

$$e_{i2}(u^{1,\ell_\varepsilon^m}(\varepsilon)) \to 0 \quad \text{strongly in} \quad \mathrm{L}^2(\Omega_1) \tag{4.17}$$

and since

$$u^{1,\ell_\varepsilon^m}_i(\varepsilon) \rightharpoonup u^{1,m}_i(0) \;\; \text{weakly in} \;\; \mathrm{H}^1(\Omega_1), \tag{4.18}$$

we have

$$e_{i2}(u^{1,\ell_\varepsilon^m}(\varepsilon)) \rightharpoonup e_{i2}(u^{1,m}(0)) \;\; \text{weakly in} \;\; \mathrm{L}^2(\Omega_1). \tag{4.19}$$

Thus, $e_{i2}(u^{1,m}(0)) = 0$. Therefore, $u^{1,m}(0) \in V_{KL}(\Omega_1)$ and by the same way we show that $u^{2,m}(0) \in V_{KL}(\Omega_2)$. Consequently, we deduce (4.13)–(4.16) from (3.14). □

As shown in the following lemma, the flexural components of the limit displacements vanish.

Lemma 4.4. *For each $m \geq 1$, if $\eta_m(0) \neq 0$, then*

$$\zeta_2^{1,m} = \zeta_1^{2,m} = 0 \tag{4.20}$$

and the expressions of the limit displacements are reduced to the following form:

$$\begin{aligned} u^{1,m}(0)(x) &= (\zeta_1^{1,m}(x_1, x_3), 0, \zeta_3^{1,m}(x_1, x_3)), \\ u^{2,m}(0)(x) &= (0, \zeta_2^{2,m}(x_2, x_3), \zeta_3^{2,m}(x_2, x_3)) \end{aligned} \tag{4.21}$$

with

$$(\zeta_{\alpha^1}^{1,m}, \zeta_{\alpha^2}^{2,m}) \quad \in [H^1_{\gamma^1}(\omega_1)]^2 \times [H^1_{\gamma^2}(\omega_2)]^2, \tag{4.22}$$

where

$$\gamma^1 = \partial\omega_1 \cap \{x_1 = 1\}, \quad \gamma^2 = \partial\omega_2 \cap \{x_2 = 1\}.$$

Proof. Consider a test-function of the form $v(\varepsilon) = (v^1(\varepsilon), 0)$ such that

$$v^1(\varepsilon) = (0, v_2^1, 0) \text{ with } v_2^1 \in \mathcal{D}(\omega_1). \tag{4.23}$$

It is clear that for ε sufficiently small, this test-function belongs in $V(\varepsilon)$ and we have

$$e_{\alpha^1\beta^1}(v^1(\varepsilon)) = 0, \qquad e_{\alpha^1 2}(v^1(\varepsilon)) = \frac{1}{2}\partial_{\alpha^1} v_2^1 \quad \text{and} \quad e_{22}(v^1(\varepsilon)) = 0. \tag{4.24}$$

Substituting (4.23) and (4.24) in (3.10) and multiplying the equation by ε^2 we have

$$\varepsilon \int_{\Omega_1} 2\mu\kappa_{\alpha^1 2}^{1,\ell_\varepsilon^m}(\varepsilon)\partial_{\alpha^1} v_2^1 dx = \eta_m(\varepsilon)\int_{\Omega_1} u_2^{1,\ell_\varepsilon^m}(\varepsilon) v_2^1 dx. \tag{4.25}$$

Passing to the limit as $\varepsilon \to 0$, we obtain

$$\int_{\omega_1} \zeta_2^{1,m} v_2^1 dx_1 dx_3 = 0, \qquad \forall v_2^1 \in \mathcal{D}(\omega_1).$$

Therefore,

$$\zeta_2^{1,m} = 0.$$

In the same way, we can show that

$$\zeta_1^{2,m} = 0.$$

Relations (4.21) are obtained by substituting (4.20) in (4.13)–(4.16) and the boundary conditions in (4.22) are a direct consequence of passing to the limit in the three-dimensional boundary conditions on Γ_1 and Γ_2. □

Let us identify the limit junction condition satisfied by the stretching components of the limit eigenvectors.

Lemma 4.5. *For each integer $m \geq 1$ we have the following limit junction condition:*

$$\zeta_3^{1,m}(0, x_3) = \zeta_3^{2,m}(0, x_3). \tag{4.26}$$

Proof. To prove (4.26) we use the third relation of the multidimensional junction relations (3.7):

$$u_3^{1,\ell_\varepsilon^m}(\varepsilon)(\varepsilon x_1, x_2, x_3) = u_3^{2,\ell_\varepsilon^m}(\varepsilon)(x_1, \varepsilon x_2, x_3).$$

Integrating both sides, we obtain

$$\begin{aligned} \int_0^1 \int_0^1 u_3^{1,\ell_\varepsilon^m}(\varepsilon)(\varepsilon x_1, x_2, x_3) dx_1 dx_2 &= \int_0^1 \int_0^1 u_3^{2,\ell_\varepsilon^m}(\varepsilon)(x_1, \varepsilon x_2, x_3) dx_2 dx_1, \\ \int_0^1 \frac{1}{\varepsilon} \int_0^\varepsilon u_3^{1,\ell_\varepsilon^m}(\varepsilon)(x_1, x_2, x_3) dx_1 dx_2 &= \int_0^1 \frac{1}{\varepsilon} \int_0^\varepsilon u_3^{2,\ell_\varepsilon^m}(\varepsilon)(x_1, x_2, x_3) dx_2 dx_1. \end{aligned}$$

Let us define

$$T_1^\varepsilon u_3^{1,\ell_\varepsilon^m}(\varepsilon)(x_2, x_3) = \frac{1}{\varepsilon} \int_0^\varepsilon u_3^{1,\ell_\varepsilon^m}(\varepsilon)(x_1, x_2, x_3) dx_1$$

and

$$T_2^\varepsilon u_3^{2,\ell_\varepsilon^m}(\varepsilon)(x_1, x_3) = \frac{1}{\varepsilon} \int_0^\varepsilon u_3^{2,\ell_\varepsilon^m}(\varepsilon)(x_1, x_2, x_3) dx_2.$$

Since

$$u_3^{1,\ell_\varepsilon^m}(\varepsilon) \in H^1(\Omega_1) \hookrightarrow H^1([0,1]; L^2(\omega_2)), \quad T_1^\varepsilon u_3^{1,\ell_\varepsilon^m}(\varepsilon) \in L^2(\omega_2)$$

and we have, for each $m \geq 1$,

$$\begin{aligned} \| T_1^\varepsilon u_3^{1,\ell_\varepsilon^m}(\varepsilon) - u_3^{1,\ell_\varepsilon^m}(\varepsilon)_{|\omega_2} \|^2_{L^2(\omega_2)} &= \| \frac{1}{\varepsilon} \int_0^\varepsilon u_3^{1,\ell_\varepsilon^m}(\varepsilon)(x_1, x_2, x_3) dx_1 - u_3^{1,\ell_\varepsilon^m}(\varepsilon)(0, x_2, x_3) \|^2_{L^2(\omega_2)} \\ &\leq C\varepsilon \| u_3^{1,\ell_\varepsilon^m}(\varepsilon) \|^2_{H^1(\Omega_1)} . \end{aligned}$$

Since $u_3^{1,\ell_\varepsilon^m}(\varepsilon)_{|\omega_2} \rightharpoonup u_3^{1,m}(0)_{|\omega_2}$ in $H^{\frac{1}{2}}(\omega_2)$ sense, we deduce that

$$T_1^\varepsilon u_3^{1,\ell_\varepsilon^m}(\varepsilon) \to u_3^{1,m}(0)_{|\omega_2} \text{ strongly in } L^2(\omega_2).$$

By the same argument, we show that

$$T_2^\varepsilon u_3^{2,\ell_\varepsilon^m}(\varepsilon) \to u_3^{2,m}(0)_{|\omega_1} \text{ strongly in } L^2(\omega_1).$$

Now, passing to the limit in the relation

$$\int_0^1 T_1^\varepsilon u_3^{1,\ell_\varepsilon^m}(\varepsilon) dx_2 = \int_0^1 T_2^\varepsilon u_3^{2,\ell_\varepsilon^m}(\varepsilon) dx_1,$$

we obtain

$$\int_0^1 u_3^{1,m}(0)(0, x_2, x_3) dx_2 = \int_0^1 u_3^{2,m}(0)(x_1, 0, x_3) dx_1$$

which gives by (4.21)

$$\zeta_3^{1,m}(0, x_3) = \zeta_3^{2,m}(0, x_3).$$

□

We can now characterize the limit space of stretching displacements as

$$\mathcal{V}_\ell = \left\{(\zeta^{1,m}_{\alpha^1}, \zeta^{2,m}_{\alpha^2}) \in [H^1_{\gamma^1}(\omega_1)]^2 \times [H^1_{\gamma^2}(\omega_2)]^2 : \zeta^{1,m}_3(0,x_3) = \zeta^{2,m}_3(0,x_3)\right\}. \tag{4.27}$$

In order to pass to the limit in the scaled variational formulation, we need the following convergence result regarding the scaled strain tensors.

Lemma 4.6. *For each $m \geq 1$, there exists a subsequence, still denoted ε, such that*

$$\kappa^{1,\ell^m_\varepsilon}_{ij}(\varepsilon) \rightharpoonup \kappa^{1,m}_{ij}(0) \quad \text{weakly in} \quad \mathrm{L}^2(\Omega_1) \tag{4.28}$$

and

$$\kappa^{2,\ell^m_\varepsilon}_{ij}(\varepsilon) \rightharpoonup \kappa^{2,m}_{ij}(0) \quad \text{weakly in} \quad \mathrm{L}^2(\Omega_2), \tag{4.29}$$

with

$$\begin{aligned} \kappa^{1,m}_{\alpha^1\beta^1}(0) &= e_{\alpha^1\beta^1}(\zeta^{1,m}), \\ \kappa^{2,m}_{\alpha^2\beta^2}(0) &= e_{\alpha^2\beta^2}(\zeta^{2,m}), \end{aligned} \tag{4.30}$$

$$\kappa^{1,m}_{\alpha^1 2}(0) = \kappa^{2,m}_{\alpha^2 1}(0) = 0 \tag{4.31}$$

and

$$\kappa^{1,m}_{22}(0) = \frac{-\lambda}{\lambda+2\mu}\kappa^{1,m}_{\alpha^1\alpha^1}(0), \tag{4.32}$$

$$\kappa^{2,m}_{11}(0) = \frac{-\lambda}{\lambda+2\mu}\kappa^{2,m}_{\alpha^2\alpha^2}(0). \tag{4.33}$$

Proof. Convergence (4.28) and (4.29) come from (4.8) and we have from (4.12)

$$\kappa^{1,\ell^m_\varepsilon}_{\alpha^1\beta^1}(\varepsilon) \rightharpoonup e_{\alpha^1\beta^1}(u^{1,m}(0)) \quad \text{weakly in} \quad L^2(\Omega_1), \tag{4.34}$$

$$\kappa^{2,\ell^m_\varepsilon}_{\alpha^2\beta^2}(\varepsilon) \rightharpoonup e_{\alpha^2\beta^2}(u^{2,m}(0)) \quad \text{weakly in} \quad L^2(\Omega_2). \tag{4.35}$$

Using (4.21) we deduce (4.30).

Consider a test-function of the form $v(\varepsilon) = (v^1(\varepsilon), 0)$ with

$$v^1(\varepsilon) = (v^1_1, v^1_2, v^1_3) \ \text{ and } \ v^1_i \in \mathcal{D}(\Omega_1). \tag{4.36}$$

It is clear that for ε sufficiently small, this test-function belongs in $V(\varepsilon)$.

Choosing $v^1_2 = 0$ and substituting in (3.10), we obtain

$$\begin{aligned} \varepsilon \int_{\Omega_1} \left[4\mu\kappa^{1,\ell^m_\varepsilon}_{\alpha^1\beta^1}(\varepsilon) e_{\alpha^1\beta^1}(v^1(\varepsilon)) + \kappa^{1,\ell^m_\varepsilon}_{\alpha^1\alpha^1}(\varepsilon) e_{\beta^1\beta^1}(v^1(\varepsilon))\right] dx \\ +4\mu \int_{\Omega_1} \kappa^{1,\ell^m_\varepsilon}_{\alpha^1 2}(\varepsilon) e_{\alpha^1 2}(v^1(\varepsilon)) dx + \varepsilon\lambda \int_{\Omega_1} \kappa^{1,\ell^m_\varepsilon}_{22}(\varepsilon) e_{\alpha^1\alpha^1}(v^1(\varepsilon)) dx \\ = \varepsilon \eta_{\ell^m_\varepsilon}(\varepsilon) \int_{\Omega_1} u^{\ell^m_\varepsilon}_{\alpha^1}(\varepsilon) v^1_{\alpha^1} dx. \end{aligned} \tag{4.37}$$

Passing to the limit as $\varepsilon \to 0$, we obtain

$$4\mu \int_{\Omega_1} \kappa^{1,m}_{\alpha^1 2}(0)\partial_2 v^1_{\alpha^1} dx = 0, \qquad \forall v^1_{\alpha^1} \in \mathcal{D}(\Omega_1),$$

which has as unique solution (see [4])

$$\kappa^{1,m}_{\alpha^1 2}(0) = 0.$$

In a similar way, we show that

$$\kappa^{2,m}_{\alpha^2 1}(0) = 0.$$

Choosing $v^1_{\alpha^1} = 0$, substituting in (3.10) and multiplying the equation by ε^2, we obtain

$$\begin{aligned}\varepsilon \int_{\Omega_1} 2\mu\kappa^{1,\ell^m_\varepsilon}_{\alpha^1 2}(\varepsilon)\partial_{\alpha^1} v^1_2 dx \quad &+ \quad \lambda \int_{\Omega_1} \kappa^{1,\ell^m_\varepsilon}_{\alpha^1\alpha^1}(\varepsilon)\partial_2 v^1_2 dx + (\lambda + 2\mu) \int_{\Omega_1} \kappa^{1,\ell^m_\varepsilon}_{22}(\varepsilon)\partial_2 v^1_2 dx \\ &= \quad \eta_{\ell^m_\varepsilon}(\varepsilon) \int_{\Omega_1} u^{1,\ell^m_\varepsilon}_2(\varepsilon) v^1_2 dx\end{aligned}$$

and by passing to the limit as $\varepsilon \to 0$

$$\int_{\Omega_1} \left[\lambda\kappa^{1,m}_{\alpha^1\alpha^1}(0) + (\lambda + 2\mu)\kappa^{1,m}_{22}(0)\right] \partial_2 v^1_2 dx = 0, \quad \forall v^1_2 \in \mathcal{D}(\Omega_1).$$

Consequently we have

$$\lambda\kappa^{1,m}_{\alpha^1\alpha^1}(0) + (\lambda + 2\mu)\kappa^{1,m}_{22}(0) = 0,$$

which gives (4.32). Similarly, we show (4.33). □

Theorem 4.7. *The limits $\eta_m(0)$, $\zeta^{1,m}_{\alpha^1}$, $\zeta^{2,m}_{\alpha^2}$ satisfy, for all $(\xi^1_{\alpha^1}, \xi^2_{\alpha^2}) \in \mathcal{V}_\ell$*

$$\begin{aligned}&\frac{E}{1+\nu} \int_{\omega_1} e_{\alpha^1\beta^1}(\zeta^{1,m}) e_{\alpha^1\beta^1}(\xi^1) dx_1 dx_3 + \frac{E}{1+\nu} \int_{\omega_2} e_{\alpha^2\beta^2}(\zeta^{2,m}) e_{\alpha^2\beta^2}(\xi^2) dx_2 dx_3 \\ &+ \frac{E\nu}{1-\nu^2} \int_{\omega_1} e_{\alpha^1\alpha^1}(\zeta^{1,m}) e_{\beta^1\beta^1}(\xi^1) dx_1 dx_3 + \frac{E\nu}{1-\nu^2} \int_{\omega_2} e_{\alpha^2\alpha^2}(\zeta^{2,m}) e_{\beta^2\beta^2}(\xi^2) dx_2 dx_3 \\ &= \eta_m(0) \int_{\omega_1} \zeta^{1,m}_{\alpha^1} \xi^1_{\alpha^1} dx_1 dx_3 + \eta_m(0) \int_{\omega_2} \zeta^{2,m}_{\alpha^2} \xi^2_{\alpha^2} dx_2 dx_3. \end{aligned} \tag{4.38}$$

Proof. Let $v = (\xi^1_{\alpha^1}, \xi^2_{\alpha^2})$ belong in $\mathcal{V}_\ell$, and (v^1, v^2) the corresponding Kirchhoff-Love displacements:

$$\begin{cases} v^1(x) & = (\zeta^1_1(x_1, x_3), 0, \zeta^1_3(x_1, x_3)), \\ v^2(x) & = (0, \zeta^2_2(x_2, x_3), \zeta^2_3(x_2, x_3)). \end{cases} \tag{4.39}$$

We construct an approximation $(v^1(\varepsilon), v^2(\varepsilon))$ of (v^1, v^2) as

$$v^1(\varepsilon) = \begin{cases} (\xi^1_1(0, x_3), \varepsilon\xi^2_2(0, x_3), \xi^1_3(0, x_3)), & \text{for } 0 \le x_1 < \varepsilon, \\ (\xi^1_1(2(x_1 - \varepsilon), x_3), \varepsilon\xi^2_2(2(x_1 - \varepsilon), x_3), \xi^1_3(2(x_1 - \varepsilon), x_3)), & \text{for } \varepsilon \le x_1 < 2\varepsilon, \\ (\xi^1_1(x_1, x_3), \varepsilon\xi^2_2(x_1, x_3), \xi^1_3(x_1, x_3)), & \text{for } x_1 \ge 2\varepsilon, \end{cases} \tag{4.40}$$

and

$$v^2(\varepsilon) = \begin{cases} (\varepsilon\xi_1^1(0, x_3), \xi_2^2(0, x_3), \xi_3^2(0, x_3)), & \text{for } 0 \le x_2 < \varepsilon, \\ (\varepsilon\xi_1^1(2(x_2 - \varepsilon), x_3), \xi_2^2(2(x_2 - \varepsilon), x_3), \xi_3^2(2(x_2 - \varepsilon), x_3)), & \text{for } \varepsilon \le x_2 < 2\varepsilon, \\ (\varepsilon\xi_1^1(x_2, x_3), \xi_2^2(x_2, x_3), \xi_3^2(x_2, x_3)), & \text{for } x_2 \ge 2\varepsilon. \end{cases} \tag{4.41}$$

We verify that $v(\varepsilon) = (v^1(\varepsilon), v^2(\varepsilon))$ belongs in $V(\varepsilon)$ and satisfies

$$e_{22}(v^1(\varepsilon)) = e_{11}(v^2(\varepsilon)) = 0 \tag{4.42}$$

and the following properties:

$$\begin{cases} (v^1(\varepsilon), v^1(\varepsilon)) \to (v^1, v^1), & \text{strongly in } \mathrm{L}^2(\Omega), \\ \varepsilon^{-1}e_{12}(v^1(\varepsilon)) \to \frac{1}{2}\partial_2\xi_2^2, & \text{strongly in } \mathrm{L}^2(\Omega_1), \\ \varepsilon^{-1}e_{12}(v^2(\varepsilon)) \to \frac{1}{2}\partial_1\xi_1^1, & \text{strongly in } \mathrm{L}^2(\Omega_2), \\ \varepsilon^{-1}e_{32}(v^1(\varepsilon)) \to \frac{1}{2}\partial_3\xi_2^2, & \text{strongly in } \mathrm{L}^2(\Omega_1), \\ \varepsilon^{-1}e_{31}(v^2(\varepsilon)) \to \frac{1}{2}\partial_3\xi_1^1, & \text{strongly in } \mathrm{L}^2(\Omega_2), \\ e_{\alpha^1\beta^1}(v^1(\varepsilon)) \to e_{\alpha^1\beta^1}(v^1), & \text{strongly in } \mathrm{L}^2(\Omega_1), \\ e_{\alpha^2\beta^2}(v^1(\varepsilon)) \to e_{\alpha^2\beta^2}(v^2), & \text{strongly in } \mathrm{L}^2(\Omega_2). \end{cases} \tag{4.43}$$

Passing to the limit in (3.10) when $\varepsilon \to 0$ and using convergences (4.28)–(4.33) and (4.43) we obtain

$$\begin{aligned} &2\mu \int_{\omega_1} \kappa^{1,m}_{\alpha^1\beta^1}(0) e_{\alpha^1\beta^1}(\xi^1) dx_1 dx_3 + 2\mu \int_{\omega_2} \kappa^{2,m}_{\alpha^2\beta^2}(0) e_{\alpha^2\beta^2}(\xi^2) dx_2 dx_3 \\ &+ \frac{2\mu\lambda}{\lambda + 2\mu} \int_{\omega_1} \kappa^{1,m}_{\alpha^1\alpha^1}(0) e_{\beta^1\beta^1}(\xi^1) dx_1 dx_3 + \frac{2\mu\lambda}{\lambda + 2\mu} \int_{\omega_2} \kappa^{2,m}_{\alpha^2\alpha^2}(0) e_{\beta^2\beta^2}(\xi^2) dx_2 dx_3 \\ &= \eta_m(0) \int_{\omega_1} u^{1,m}_{\alpha^1}(0) \xi^1_{\alpha^1} dx_1 dx_3 + \eta_m(0) \int_{\omega_2} u^{2,m}_{\alpha^2}(0) \xi^2_{\alpha^2} dx_2 dx_3. \end{aligned}$$

Replacing $\kappa^{1,m}_{\alpha^1\beta^1}(0)$ and $\kappa^{2,m}_{\alpha^2\beta^2}(0)$ by their expressions (4.30), we obtain

$$\begin{aligned} &2\mu \int_{\omega_1} e_{\alpha^1\beta^1}(\zeta^{1,m}) e_{\alpha^1\beta^1}(\xi^1) dx_1 dx_3 + 2\mu \int_{\omega_2} e_{\alpha^2\beta^2}(\zeta^{2,m}) e_{\alpha^2\beta^2}(\xi^2) dx_2 dx_3 \\ &+ \frac{2\mu\lambda}{\lambda + 2\mu} \int_{\omega_1} e_{\alpha^1\alpha^1}(\zeta^{1,m}) e_{\beta^1\beta^1}(\xi^1) dx_1 dx_3 + \frac{2\mu\lambda}{\lambda + 2\mu} \int_{\omega_2} e_{\alpha^2\alpha^2}(\zeta^{2,m}) e_{\beta^2\beta^2}(\xi^2) dx_2 dx_3 \\ &= \eta_m(0) \int_{\omega_1} \zeta^{1,m}_{\alpha^1} \xi^1_{\alpha^1} dx_1 dx_3 + \eta_m(0) \int_{\omega_2} \zeta^{2,m}_{\alpha^2} \xi^2_{\alpha^2} dx_2 dx_3. \end{aligned}$$

Equation (4.38) is obtained using relations (2.2). □

Proposition 4.8. *For each $m \ge 1$, the whole family $(\eta_{\ell^m_\varepsilon}(\varepsilon))_{\varepsilon>0}$ converges as $\varepsilon \to 0$. In addition, if $\eta_m(0)$ is a simple eigenvalue of* (4.38)*, then $\eta_{\ell^m_\varepsilon}(\varepsilon)$ is also a simple eigenvalue of* (3.10) *for $\varepsilon < \varepsilon_0$ small enough.*

Proof. See [3]. □

Proposition 4.9. *For each $m \geq 1$, the limit eigensolutions $(\eta_m(0), \zeta^{1,m}, \zeta^{2,m})$ verify the system of classical equations of stretching vibrations:*

$$\left\{\begin{aligned} &\frac{E}{2(1-\nu^2)}\left[2\frac{\partial^2\zeta_1^{1,m}}{\partial x_1^2} + (1-\nu)\frac{\partial^2\zeta_1^{1,m}}{\partial x_3^2} + (1+\nu)\frac{\partial^2\zeta_3^{1,m}}{\partial x_1\partial x_3}\right] = \eta_m(0)\zeta_1^{1,m} \quad in\ \omega_1,\\ &\frac{E}{2(1-\nu^2)}\left[2\frac{\partial^2\zeta_3^{1,m}}{\partial x_3^2} + (1-\nu)\frac{\partial^2\zeta_3^{1,m}}{\partial x_1^2} + (1+\nu)\frac{\partial^2\zeta_1^{1,m}}{\partial x_1\partial x_3}\right] = \eta_m(0)\zeta_3^{1,m} \quad in\ \omega_1,\end{aligned}\right. \tag{4.44}$$

$$\left\{\begin{aligned} &\frac{E}{2(1-\nu^2)}\left[2\frac{\partial^2\zeta_2^{2,m}}{\partial x_2^2} + (1-\nu)\frac{\partial^2\zeta_2^{2,m}}{\partial x_3^2} + (1+\nu)\frac{\partial^2\zeta_3^{2,m}}{\partial x_2\partial x_3}\right] = \eta_m(0)\zeta_2^{2,m} \quad in\ \omega_2,\\ &\frac{E}{2(1-\nu^2)}\left[2\frac{\partial^2\zeta_3^{2,m}}{\partial x_3^2} + (1-\nu)\frac{\partial^2\zeta_2^{2,m}}{\partial x_2^2} + (1+\nu)\frac{\partial^2\zeta_2^{2,m}}{\partial x_2\partial x_3}\right] = \eta_m(0)\zeta_3^{2,m} \quad in\ \omega_2,\end{aligned}\right. \tag{4.45}$$

with the boundary conditions

$$\zeta_1^{1,m} = 0 \quad on\ \gamma^1, \tag{4.46}$$

$$\zeta_2^{2,m} = 0 \quad on\ \gamma^2 \tag{4.47}$$

and the junction relation

$$\zeta_3^{1,m}(0, x_3) = \zeta_3^{2,m}(0, x_3). \tag{4.48}$$

Proof. Equations (4.44) and (4.45) are obtained by carefully performing an integration by parts in the left side of Eq (4.38) while conditions (4.46)–(4.48) are a direct consequence of the characterization of the elements of $\mathcal{V}_\ell$. □

5. Conclusions

In this work, we proved that the standard spectral problem associated to stretching modes in a linear elastic folded plate can be derived mathematically from the standard three-dimensional eigenvalue problem of linear elasticity through a non-standard asymptotic analysis technique. We showed that each stretching frequency of an elastic thin folded plate is the limit of a family of high frequencies of the three-dimensional elastic model of the plate, as the thickness approaches zero. The techniques used can be adapted to study a wide variety of problems of modeling vibrations for thin structures and junction between different thin structures.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author gratefully acknowledges the financial support of the Deanship of research of Imam Mohammad Ibn Saud Islamic University (IMSIU).

Conflict of interest

The author declares no conflict of interest.

References

1. H. Irago, N. Kerdid, J. M. Viaño, Asymptotic analysis of high frequency modes in thin rods, *C. R. Acad. Sci. Ser. I Math.*, **326** (1998), 1255–1260. https://doi.org/10.1016/S0764-4442(98)80238-3
2. H. Irago, N. Kerdid, J. M. Viaño, Asymptotic analysis of torsional and stretching modes of thin rods, *Quart. Appl. Math.*, **58** (2000), 495–510. http://doi.org/10.1090/QAM/1770651
3. P. G. Ciarlet, S. Kesavan, Two-dimensional approximations of three-dimensional eigenvalue problems in plate theory, *Comput. Methods Appl. Mech. Eng.*, **26** (1981), 145–172. http://doi.org/10.1016/0045-7825(81)90091-8
4. H. Le Dret, Vibrations of a folded plate, *ESAIM: Math. Modell. Numer. Anal.*, **24** (1990), 501–521. https://doi.org/10.1051/m2an/1990240405011
5. F. Bourquin, P. G. Ciarlet, Modeling and justification of eigenvalue problems for junctions between elastic structures, *J. Funct. Anal.*, **87** (1989), 392–427. https://doi.org/10.1016/0022-1236(89)90017-7
6. V. Lods, Modeling and justification of an eigenvalue problem for a plate inserted in a three-dimensional support, *ESAIM: Math. Modell. Numer. Anal.*, **30** (1996), 413–444. http://doi.org/10.1051/m2an/1996300404131
7. N. Kerdid, Asymptotic behavior of the eigenvalue problem in a thin linearly elastic clamped rod when its thickness tends to zero, *C. R. Acad. Sci. Ser. I Math.*, **316** (1993), 755–758.
8. N. Kerdid, Modeling the vibrations of a multi-rod structure, *ESAIM: Math. Modell. Numer. Anal.*, **31** (1997), 891–925. https://doi.org/10.1051/m2an/1997310708911
9. S. Jimbo, A. R. Mulet, Asymptotic behavior of eigenfrequencies of a thin elastic rod with non-uniform cross-section, *J. Math. Soc. Jpn*, **72** (2020), 119–154. https://doi.org/10.2969/jmsj/81198119
10. S. Jimbo, E. Ushikoshi, H. Yoshihara, Asymptotic behavior of the eigenfrequencies of a thin elastic rod with non-uniform cross-section of extremely oblate shape, *Calculus Var. Partial Differ. Equations*, **62** (2023), 11. https://doi.org/10.1007/s00526-022-02325-1
11. M. Serpilli, S. Lenci, Asymptotic modelling of the linear dynamics of laminated beams, *Int. J. Solids Struct.*, **49** (2012), 1147–1157. http://doi.org/10.1016/j.ijsolstr.2012.01.012
12. J. Tambača, One-dimensional approximations of the eigenvalue problem of curved rods, *Math. Methods Appl. Sci.*, **24** (2001), 927–948. https://doi.org/10.1002/mma.249
13. A. Gaudiello, D. Gómez, M. E. Pérez-Martínez, Asymptotic analysis of the high frequencies for the Laplace operator in a thin T-like shaped structure, *J. Math. Pures Appl.*, **134** (2020), 299–327. http://doi.org/10.1016/j.matpur.2019.06.005

AIMS Press