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Research article

Dynamics of a nonlinear discrete predator-prey system with fear effect

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Abstract: In this paper, we investigate a nonlinear discrete prey-predator system with fear effects. The existence, local stability and boundedness of positive equilibrium point are discussed. Using the center manifold theorem and bifurcation theory, the conditions for the existence of flip bifurcation and Neimark-Sacker bifurcation in the interior of $\mathbb{R}_+^2$ are established. Furthermore, the numerical simulations not only show complex dynamical behaviors, but also verify our analysis results. A feedback control strategy is employed to control bifurcation and chaos in the system.

Keywords: predator-prey system; flip bifurcation; Neimark-Sacker bifurcation; fear factor; chaos
Mathematics Subject Classification: 37G15, 37N25, 39A28, 92D25, 93C10

1. Introduction

Populations in nature are rarely isolated, and intensively interact with others in the biological community. All kinds of organisms are divided into different levels according to their physiological characteristics, food sources, etc. and different levels of populations have a variety of connections. The predator-prey process is the most fundamental, important and universal process in the study of population dynamic behavior. Many researchers have studied the dynamic behavior of many prey-predator systems in ecology and behavioral phenomena between species [1–11]. Some authors have also explored the complexity, stability, and conditional requirements for spatial pattern formation in prey-predator systems [12–14].

Numerous studies have shown that discrete-time systems are more suitable than the continuous system of small populations, and provide valid evidence for these [15–18]. Cheng et al. [19] studied a discrete-time ratio-dependent prey-predator system with Allee effect, and obtained the model with logistic growth function that have somewhat similar bifurcation structures. In the past few years, a large number of research studies have indicated that discrete prey-predator systems have more abundant dynamic behaviors than continuous systems, such as chaos. Scientists have also analyzed the corresponding dynamic behaviors between populations by numerical simulation [20–30].

Qamar Din [31] studied the following discrete-time system:

$$\begin{cases} u_{n+1} = u_n \exp[r(1 - \frac{u_n}{K}) - \frac{\beta v_n}{u_n+\gamma}], \\ v_{n+1} = v_n \exp(1 - \frac{a v_n}{b u_n + c} - d), \end{cases} \tag{1.1}$$

where $r, K, \beta, \gamma, a, b, c, n$ and d are greater than zero, r is the intrinsic growth rate of the prey u population, K denotes environmental carrying capacity of the prey u in a particular habitat, β and a represent the maximum value of the per capita reduction rate of the prey u and predator v, respectively. γ and c indicate the extent to which the environment provides protection to prey and predator. b denotes the quality of food that the prey provides for conversion into predator births, and d measures the death rate of the predator. n stands for time.

In 2016, Wang et al. [21] showed, through experiments, that prey′s fear of predators would lead to a decrease in the birth rate of prey, and $F(k, v) = \frac{1}{1+kv}$ was used to denote the fear factor. Here, k reflects the degree of fear that drives prey anti-predator behavior. In the past, many researchers have only studied the effects of direct killing, no matter how they improve the predator-prey model. In this paper, we combine fear (indirect effects) and investigate the effects of fear on population dynamics.

To study the effects of fear on population dynamics, on the basis of system (1.1), we introduce the fear factor $F(k, v) = \frac{1}{1+kv}$ and the growth rate α of the predator v population, and consider the discrete-time predator-prey system:

$$\begin{cases} u_{n+1} = u_n \exp[\frac{r}{1+kv_n}(1 - \frac{u_n}{K}) - \frac{\beta v_n}{u_n+\gamma}], \\ v_{n+1} = v_n \exp(\alpha - \frac{a v_n}{b u_n + c} - d). \end{cases} \tag{1.2}$$

This article is organized as follows: in Section 2, the existence, stability and boundedness of the system at different equilibrium points are analyzed. In Section 3, we discuss the specific conditions for the existence of Neimark-Sacker bifurcation and flip bifurcation. In Section 4, chaos is controlled by the feedback control method. In Section 5, we carry out numerical simulations, including the bifurcation diagrams, phase portraits and solution diagrams. Finally, a brief conclusion is given in the last section.

2. The properties of equilibrium points

In this section, we consider the discrete-time system (1.2) in the closed first quadrant $\mathbb{R}^2_+$ of the (u, v) plane. We study the existence, stability and boundedness of the equilibrium points by the eigenvalues for the Jacobian matrix of (1.2) at the equilibrium points.

2.1. Existence and stability

To obtain the equilibrium points of (1.2), we calculate the following equations:

$$\begin{cases} u = u \exp[\frac{r}{1+kv}(1 - \frac{u}{K}) - \frac{\beta v}{u+\gamma}], \\ v = v \exp(\alpha - \frac{a v}{b u + c} - d). \end{cases}$$

Through calculation, the following results can be gained directly:

Proposition 1. *(i)* For all parameter values, system (1.2) has two equilibrium points $H_0 = (0,0)$, $H_1 = (K,0)$;

(ii) If $\alpha > d$, then system (1.2) has a boundary positive equilibrium point $H_2 = (0, \frac{(\alpha-d)c}{a})$;

(iii) System (1.2) has a unique positive equilibrium point $H_3 = (u^*, v^*) = (u^*, \frac{1}{a}(\alpha - d)(bu^* + c))$, where $\alpha > d$, and u^* is the only positive solution to the quadratic equation of one variabie

$$C_0u^2 + C_1u + C_2 = 0 \ ,$$

where

$C_0 = \frac{Kk\beta}{a^2}(\alpha - d)^2b^2 + r,$
$C_1 = \frac{2bcKk\beta}{a^2}(\alpha - d)^2 + \frac{bK\beta}{a}(\alpha - d) + r\gamma - rK,$
$C_2 = \frac{Kk\beta}{a^2}(\alpha - d)^2c^2 + \frac{cK\beta}{a}(\alpha - d) - rK\gamma.$

Definition 1. *[11]* Suppose that λ_1 and λ_2 are two roots of the characteristic equation $F(\lambda) = \lambda^2 + M\lambda + N = 0$, where M and N are constants. Then equilibrium point (u,v) is called

(i) sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, and it is locally asymptotically stable;

(ii) source or repeller if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, and it is locally unstable;

(iii) saddle if either ($|\lambda_1| < 1$ and $|\lambda_2| > 1$) or ($|\lambda_1| > 1$ and $|\lambda_2| < 1$);

(iv) non-hyperbolic if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

The Jacobian matrix for equilibrium point $H_0(0,0)$ is:

$$Q_{H_0} = \begin{bmatrix} e^r & 0 \\ 0 & e^{\alpha-d} \end{bmatrix}. \tag{2.1}$$

Then, $\lambda_1 = e^r, \ \lambda_2 = e^{\alpha-d}$. Thus, the following proposition holds.

Proposition 2. Equilibrium point $H_0(0,0)$ is

(i) source and it is locally unstable if $\alpha > d$;

(ii) saddle if $\alpha < d$.

Proof. According to (2.1), the two eigenvalues of (1.2) at the equilibrium point $H_0(0,0)$ are $\lambda_1 = e^r, \ \lambda_2 = e^{\alpha-d}$. If $\alpha - d > 0$ and $r > 0$, then $|\lambda_1| > 1$, $|\lambda_2| > 1$. Thus from Definition 1, $H_0 = (0,0)$ is a source. If $\alpha < d$, then $0 < |\lambda_2| < 1$. $H_0 = (0,0)$ is a saddle. This completes the proof. □

For equilibrium point $H_1 = (K,0)$, the Jacobian matrix is described as follows:

$$Q_{H_1} = \begin{bmatrix} 1-r & -\frac{\beta K}{K+\gamma} \\ 0 & e^{\alpha-d} \end{bmatrix}, \tag{2.2}$$

the corresponding characteristic roots are $\lambda_1 = 1 - r, \ \lambda_2 = e^{\alpha-d}$. Thus, the following proposition holds.

Proposition 3. *The eigenvalues at the boundary equilibrium point $H_1 = (K,0)$ are $\lambda_1 = 1-r, \ \lambda_2 = e^{\alpha-d}$, then*

(i) $H_1 = (K,0)$ is sink, if $0 < r < 2$ and $\alpha - d < 0$;

(ii) $H_1 = (K,0)$ is saddle, if one of the following conditions is true:

(ii-1) $\alpha - d > 0$ and $0 < r < 2$;

(ii-2) $\alpha - d < 0$ and $r > 2$;

(iii) $H_1 = (K,0)$ is non-hyperbolic, if either $r = 2$ or $\alpha - d = 0$;

(iv) $H_1 = (K,0)$ is source, if $r > 2$ and $\alpha - d > 0$.

Proof. According to (2.2), the two eigenvalues of (1.2) at the boundary equilibrium point are $\lambda_1 = 1 - r$, $\lambda_2 = e^{\alpha-d}$. If $\alpha - d$ are greater than zero, then $|\lambda_2| > 1$. Thus from Definition 1, when $|\lambda_1| < 1$, then $0 < r < 2$. Thus, $H_1 = (K, 0)$ is a saddle. Similarly, when $|\lambda_1| > 1$, then $r > 2$, $H_1 = (K, 0)$ is a source. Similarly, we can prove (i), (iii) by the same way. □

For equilibrium point $H_2 = (0, \frac{(\alpha-d)c}{a})$, the Jacobian matrix is evaluated as follows:

$$Q_{H_2} = \begin{bmatrix} \exp[\frac{ar}{a+kc(\alpha-d)} - \frac{\beta(\alpha-d)c}{a\gamma}] & 0 \\ \frac{bc(\alpha-d)^2}{a} & 1-(\alpha-d) \end{bmatrix}, \tag{2.3}$$

the corresponding characteristic roots are $\lambda_1 = \exp[\frac{ar}{a+kc(\alpha-d)} - \frac{\beta(\alpha-d)c}{a\gamma}]$, $\lambda_2 = 1 - (\alpha - d)$. Thus, the following proposition holds.

Proposition 4. *The eigenvalues of Q_{H_2} are $\lambda_1 = \exp[\frac{ar}{a+kc(\alpha-d)} - \frac{\beta(\alpha-d)c}{a\gamma}]$ and $\lambda_2 = 1 - (\alpha - d)$, then*

(i) $H_2 = (0, \frac{(\alpha-d)c}{a})$ is sink if $0 < \alpha - d < 2$ and $0 < r < \frac{\beta c(\alpha-d)[a+kc(\alpha-d)]}{a^2\gamma}$;

(ii) $H_2 = (0, \frac{(\alpha-d)c}{a})$ is source if $\alpha - d > 2$ and $r > \frac{\beta c(\alpha-d)[a+kc(\alpha-d)]}{a^2\gamma}$;

(iii) $H_2 = (0, \frac{(\alpha-d)c}{a})$ is saddle if one of the following conditions is true:

(iii-1) $\alpha - d = 2$ and $0 < r < \frac{\beta c(\alpha-d)[a+kc(\alpha-d)]}{a^2\gamma}$;

(iii-2) $0 < \alpha - d < 2$ and $r > \frac{\beta c(\alpha-d)[a+kc(\alpha-d)]}{a^2\gamma}$;

(iv) $H_2 = (0, \frac{(\alpha-d)c}{a})$ is non-hyperbolic if either $\alpha - d = 2$ or $r = \frac{\beta c(\alpha-d)[a+kc(\alpha-d)]}{a^2\gamma}$.

Proof. (i) According to (2.3), the two eigenvalues of (1.2) at the boundary equilibrium point H_2 are $\lambda_1 = \exp[\frac{ar}{a+kc(\alpha-d)} - \frac{\beta(\alpha-d)c}{a\gamma}]$, $\lambda_2 = 1-(\alpha-d)$. $H_2 = (0, \frac{(\alpha-d)c}{a})$ is sink if and only if $|\lambda_1| < 1$ and $|\lambda_2| < 1$. When $|\lambda_1| < 1$, then $0 < \alpha - d < 2$. When $|\lambda_2| < 1$, then $0 < r < \frac{\beta c(\alpha-d)[a+kc(\alpha-d)]}{a^2\gamma}$. In conclusion, $H_2 = (0, \frac{(\alpha-d)c}{a})$ is sink if $0 < \alpha - d < 2$ and $0 < r < \frac{\beta c(\alpha-d)[a+kc(\alpha-d)]}{a^2\gamma}$; Similarly, Proposition 4 (ii)–(iv) can be proved. □

Lemma 1. *[11] Suppose that $F(\lambda) = \lambda^2 - M\lambda + N$, and $F(1) > 0$, λ_1 and λ_2 are roots of $F(\lambda) = 0$. Then the following results hold true:*

(i) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $N < 1$;

(ii) $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) if and only if $F(-1) < 0$;

(iii) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$ and $N > 1$;

(iv) $\lambda_1 = -1$ and $|\lambda_2| \neq 1$ if and only if $F(-1) = 0$ and $N \neq 0, 2$;

(v) λ_1 and λ_2 are complex and $|\lambda_1| = |\lambda_2| = 1$ if and only if $M^2 - 4N < 0$ and $N = 1$.

The Jacobian matrix $Q(u^*, v^*)$ relevant system (1.2) at the positive equilibrium point $H_3(u^*, v^*)$ is as follows:

$$Q_{H_3} = \begin{bmatrix} 1 - \frac{ru^*}{(1+kv^*)K} + \frac{\beta u^* v^*}{(u^*+\gamma)^2} & -(1 - \frac{u^*}{K})\frac{rku^*}{(1+kv^*)^2} - \frac{\beta u^*}{u^*+\gamma} \\ \frac{abu^{*2}}{(bu^*+c)^2} & 1-(\alpha-d) \end{bmatrix}.$$

Then the characteristic equation related to Q_{H_3} is

$$F(\lambda) = \lambda^2 - m(u^*, v^*)\lambda + n(u^*, v^*) = 0,$$

where

$m(u^*, v^*) = trQ_{H_3} = 2 - \frac{ru^*}{(1+kv^*)K} + \frac{\beta u^* v^*}{(u^*+\gamma)^2} - (\alpha - d)$
and
$n(u^*, v^*) = detQ_{H_3} = [1 - \frac{ru^*}{(1+kv^*)K} + \frac{\beta u^* v^*}{(u^*+\gamma)^2}][1 - (\alpha - d)] + [(1 - \frac{u^*}{K})\frac{rku^*}{(1+kv^*)^2} + \frac{\beta u^*}{u^*+\gamma}]\frac{abu^{*2}}{(bu^*+c)^2}$.
Therefore

$$F(1) = 1 - m(u^*, v^*) + n(u^*, v^*), \quad F(-1) = 1 + m(u^*, v^*) + n(u^*, v^*).$$

Using Lemma 1, we have the following proposition:

Proposition 5. *Let $H_3(u^*, v^*)$ be the unique positive equilibrium point of system (1.2), then the following propositions hold:*
(i) $H_3(u^*, v^*)$ is sink if and only if $|m(u^*, v^*)| < 1 + n(u^*, v^*) < 2$;
(ii) $H_3(u^*, v^*)$ is saddle if and only if $m^2(u^*, v^*) > 4n(u^*, v^*)$ and $|m(u^*, v^*)| > |1 + n(u^*, v^*)|$;
(iii) $H_3(u^*, v^*)$ is source if and only if $|n(u^*, v^*)| > 1$ and $|m(u^*, v^*)| < |1 + n(u^*, v^*)|$;
(iv) $H_3(u^*, v^*)$ is non-hyperbolic if and only if $|m(u^*, v^*)| = |1 + n(u^*, v^*)|$ or $n(u^*, v^*) = 1$ and $|m(u^*, v^*)| \leq 2$.

Proof. (i) According to Lemma 1, $H_3(u^*, v^*)$ is a sink point if and only if $F(1) > 0, F(-1) > 0$ and $N < 1$, it can be acquired by calculation $|m(u^*, v^*)| < 1 + n(u^*, v^*) < 2$. Consequently, Proposition 5 (i) holds. Similarly, Proposition 5 (ii)–(iv) can be established. □

2.2. Boundedness

Lemma 2. *[20] Assume that u_t satisfies $u_0 > 0$, and $u_{t+1} \leq u_t \exp[A(1 - Bu_t)]$ for $t \in [t_1, \infty)$, where B is a positive constant. Then $\limsup_{t\to\infty} u_t \leq \frac{1}{AB}\exp(A - 1)$.*

Theorem 1. *Every positive solution $\{(u_n, v_n)\}$ of system (1.2) is uniformly bounded.*

Proof. Suppose that $\{(u_n, v_n)\}$ be an arbitrary positive solution corresponding to system (1.2). Then, by the first part of (1.2), it is known

$$u_{n+1} \leq u_n \exp[\frac{r}{1 + kv_n}(1 - \frac{u_n}{K})] \leq u_n \exp[r(1 - \frac{u_n}{K})]$$

for all $n = 0, 1, 2, \cdots$. Suppose that $u_0 > 0$, then according to Lemma 2, we gain

$$\limsup_{n\to\infty} u_n \leq \frac{K}{r}\exp(r - 1) := M_1.$$

From the second part of (1.2), we acquire

$$\begin{aligned} v_{n+1} &= v_n \exp(\alpha - \frac{av_n}{bu_n + c} - d) \\ &\leq v_n \exp(\alpha - \frac{av_n}{bu_n + c}) \\ &\leq v_n \exp(\alpha - \frac{av_n}{bM_1 + c}). \end{aligned}$$

Assume that $v_0 > 0$, then using Lemma 2, we gain

$$\limsup_{n\to\infty} v_n \leq \frac{bM_1 + c}{a}\exp(\alpha - 1) := M_2.$$

That is to say that $\limsup_{n\to\infty} (u_n, v_n) \leq M$, where $M = \max\{M_1, M_2\}$. This completes the proof. □

3. Bifurcation analysis

3.1. Flip bifurcation

The characteristic equation related to system (1.2) at the positive interior equilibrium point H_3 is

$$F(\lambda) = \lambda^2 - m(u^*, v^*)\lambda + n(u^*, v^*) = 0, \tag{3.1}$$

where

$$\begin{aligned}
&m(u^*, v^*) = 1 - \tfrac{ru^*}{(1+kv^*)K} + \Phi + \Theta,\\
&n(u^*, v^*) = \Theta\left[1 - \tfrac{ru^*}{(1+kv^*)K} + \Phi\right] + \Psi,\\
&\Theta := 1 - (\alpha - d), \quad \Phi := \tfrac{\beta u^* v^*}{(u^*+\gamma)^2},\\
&\Psi := \left[(1 - \tfrac{u^*}{K})\tfrac{rku^*}{(1+kv^*)^2} + \tfrac{\beta u^*}{u^*+\gamma}\right]\tfrac{abu^{*2}}{(bu^*+c)^2}.
\end{aligned}$$

Assume that $m^2(u^*, v^*) > 4n(u^*, v^*)$, that is,

$$\left(1 - \tfrac{ru^*}{(1+kv^*)K} + \Phi + \Theta\right)^2 > 4\Theta\left[1 - \tfrac{ru^*}{(1+kv^*)K} + \Phi\right] + 4\Psi \tag{3.2}$$

and $m(u^*, v^*) + n(u^*, v^*) = -1$, that is to say

$$r = \tfrac{(1+kv^*)K}{u^*(1+\Theta)}\left(2 + 2\Theta + \Phi(1 + \Theta) + \Psi\right) . \tag{3.3}$$

Then eigenvalue of $F(\lambda) = 0$ are $\lambda_1 = -1$ and $\lambda_2 = 2 + \Phi + \Theta - \frac{ru^*}{(1+kv^*)K}$. The condition $|\lambda_2| \neq 1$ indicates that

$$\Theta\left[1 - \tfrac{ru^*}{(1+kv^*)K} + \Phi\right] + \Psi \neq \pm 1 . \tag{3.4}$$

Consider the following set

$$A_1 = \left\{(a, b, c, d, K, r, k, \alpha, \beta, \gamma) \in \mathbb{R}_+^{10} : (3.2),\ (3.3) \text{ and } (3.4) \text{ are satisfied}\right\}.$$

Based on the above analysis, we can obtain that when the parameters change on set A_1, system (1.2) will occur flip bifurcation at $H_3(u^*, v^*)$.

We consider the following system

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \exp\left[\frac{r_1}{1+kv}(1 - \frac{u}{K}) - \frac{\beta v}{u+\gamma}\right] \\ v \exp(\alpha - \frac{av}{bu+c} - d) \end{pmatrix}, \tag{3.5}$$

here $(a, b, c, d, K, r_1, k, \alpha, \beta, \gamma) \in A_1$.

Consider a perturbation corresponding to system (3.5) as follows:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \exp\left[\frac{(r_1+\overline{r})}{1+kv}(1 - \frac{u}{K}) - \frac{\beta v}{u+\gamma}\right] \\ v \exp(\alpha - \frac{av}{bu+c} - d) \end{pmatrix}, \tag{3.6}$$

where $\overline{r}$ is a small perturbation parameter and $|\overline{r}| \ll 1$.

Let $p = u - u^*$ and $q = v - v^*$. Then we gain

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} f_1(p, q, \overline{r}) \\ f_2(p, q, \overline{r}) \end{pmatrix}, \tag{3.7}$$

where

$$
\begin{aligned}
f_1(p,q,\overline{r}) &= S_{13}p^2 + S_{14}pq + S_{15}q^2 + T_{11}p^3 + T_{12}p^2q + T_{13}pq^2 + T_{14}q^3 + W_1p\overline{r} + W_2q\overline{r} \\
&\quad + W_3\overline{r}^2 + W_4pq\overline{r} + W_5p^2\overline{r} + W_6q^2\overline{r} + W_7p\overline{r}^2 + W_8q\overline{r}^2 + W_9\overline{r}^3 + O((|p|,|q|,|\overline{r}|)^4),\\
f_2(p,q,\overline{r}) &= S_{23}p^2 + S_{24}pq + S_{25}q^2 + T_{21}p^3 + T_{22}p^2q + T_{23}pq^2 + T_{24}q^3 + O((|p|,|q|,|\overline{r}|)^4),\\
S_{11} &= 1 - \frac{r_1u^*}{(1+kv^*)K} + \frac{\beta u^*v^*}{(u^*+\gamma)^2}, \quad S_{12} = -(1-\frac{u^*}{K})\frac{r_1ku^*}{(1+kv^*)^2} - \frac{\beta u^*}{u^*+\gamma},\\
S_{13} &= -\frac{r_1}{K(1+kv^*)} - \frac{r_1\beta u^*v^*}{(u^*+\gamma)^2K(1+kv^*)} + \frac{\beta v^*}{(u^*+\gamma)^2} - \frac{\beta u^*v^*}{(u^*+\gamma)^3} + \frac{r_1^2u^*}{2K^2(1+kv^*)^2}\\
&\quad + \frac{\beta^2u^*v^{*2}}{2(u^*+\gamma)^4},\\
S_{14} &= -\frac{r_1k(K-u^*)}{K(1+kv^*)^2} - \frac{\beta}{u^*+\gamma} - \frac{r_1^2k^2(K-u^*)u^*}{K^2(1+kv^*)^4} - \frac{r_1k\beta(K-u^*)u^*}{K(1+kv^*)^2(u^*+\gamma)^2} - \frac{\beta^2u^*}{(u^*+\gamma)^3}\\
&\quad - \frac{r_1k\beta u^*}{(u^*+\gamma)K(1+kv^*)^2},\\
S_{15} &= \frac{u^*}{2}\left(\frac{r_1k(K-u^*)}{K(1+kv^*)^2} + \frac{\beta}{u^*+\gamma}\right)^2 + \frac{r_1k^2(K-u^*)u^*}{K(1+kv^*)^3},\\
T_{11} &= \frac{r_1^2}{2K^2(1+kv^*)^2} - \frac{r_1\beta v^*}{2K(1+kv^*)(u^*+\gamma)^2} + \frac{\beta^2v^{*2} - 2\beta v^*(\gamma-u^*)}{6(u^*+\gamma)^4} - \frac{r_1^3u^*}{6K^3(1+kv^*)^3}\\
&\quad - \frac{2\beta v^*}{3(u^*+\gamma)^3} - \frac{r_1\beta v^*(u^*-\gamma)}{6K(1+kv^*)(u^*+\gamma)^3} + \frac{\beta^2v^{*2}(\gamma-2u^*-\gamma)}{3(u^*+\gamma)^5} + \frac{r_1^2\beta u^*v^*}{2K^2(1+kv^*)^2(u^*+\gamma)^2}\\
&\quad - \frac{r_1\beta^2u^*v^{*2}}{2K(1+kv^*)(u^*+\gamma)^4} + \frac{\beta^3u^*v^{*3}}{(u^*+\gamma)^6},\\
T_{12} &= \frac{r_1^2k(K-2u^*)}{K^2(1+kv^*)^3} - \frac{r_1k\beta v^*(K-u^*) - 2r_1\beta u^*}{2K(1+kv^*)^2(u^*+\gamma)^2} + \frac{r_1\beta}{K(1+kv^*)(u^*+\gamma)} + \frac{r_1k}{K(1+kv^*)^2}\\
&\quad + \frac{\beta}{2(u^*+\gamma)^2} + \frac{\beta(\gamma-u^*)}{2(1+kv^*)^3} - \frac{r_1k\beta v^*(K-u^*)(\gamma-u^*)}{2K(1+kv^*)^2(u^*+\gamma)^3} - \frac{\beta^2v^*}{2(u^*+\gamma)^3} - \frac{r_1^3k(K-u^*)u^*}{2K^3(1+kv^*)^3}\\
&\quad - \frac{r_1^2\beta u^*}{2K^2(1+kv^*)^2(u^*+\gamma)} + \frac{r_1^2k\beta u^*v^*(K-u^*)}{K^2(1+kv^*)^3(u^*+\gamma)^2} + \frac{r_1\beta^2u^*v^*}{K(1+kv^*)(u^*+\gamma)^3}\\
&\quad - \frac{r_1k\beta^2u^*v^{*2}(K-u^*)}{2K(1+kv^*)^2(u^*+\gamma)^4} - \frac{\beta^3u^*v^{*2}}{2(u^*+\gamma)^5},\\
T_{13} &= \frac{r_1^2k^2(K-u^*)(K-(2+K)u^*)}{2K^2(1+kv^*)^4} + \frac{r_1k\beta(K-3u^*)K(1+kv^*) - 2r_1^2k\beta(K-u^*)u^*}{2K^2(1+kv^*)^3(u^*+\gamma)}\\
&\quad + \frac{\beta^2(u^*+\gamma)^2 + \beta^3u^*v^* - 2\beta^2u^*(u^*+\gamma)}{2(u^*+\gamma)^4} + \frac{r_1k^2(K-2u^*)}{K(1+kv^*)^3} - \frac{r_1^3k^3(K-u^*)^2u^*}{2K^3(1+kv^*)^5}\\
&\quad + \frac{r_1^2k^2\beta(K-u^*)^2u^*v^*}{2K^2(1+kv^*)^4(u^*+\gamma)^2} + \frac{r_1k\beta^2(K-u^*)^2u^*v^*}{2K(1+kv^*)^2(u^*+\gamma)^3} - \frac{r_1k\beta(K-u^*)^2u^*(1+kv^*)}{2K(1+kv^*)^3(u^*+\gamma)^2}\\
&\quad - \frac{r_1\beta^2u^*}{2K(1+kv^*)(u^*+\gamma)^2} - \frac{r_1k\beta(K-u^*)u^*}{2K(1+kv^*)^2(u^*+\gamma)^2},
\end{aligned}
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$$
\begin{aligned}
T_{14} = & -\frac{r_1^3k^3(K-u^*)^3u^*}{6K^3(1+kv^*)^6} - \frac{r_1k\beta^2(K-u^*)u^*}{2K(1+kv^*)^2(u^*+\gamma)^2} - \frac{r_1^2k^2\beta(K-u^*)^2u^*}{2K^2(1+kv^*)^4(u^*+\gamma)} - \frac{\beta^3u^*}{6(u^*+\gamma)^3} \\
& - \frac{r_1^2k^3(K-u^*)^2u^*}{K^2(1+kv^*)^5} - \frac{r_1k^2\beta(K-u^*)u^*}{K(1+kv^*)^3(u^*+\gamma)} - \frac{r_1k^3(K-u^*)u^*}{(1+kv^*)^4}, \\
W_1 = & \frac{K-2u^*}{K(1+kv^*)} + \frac{(K-u^*)u^*}{K(1+kv^*)}\left(\frac{\beta v^*}{(u^*+\gamma)^2} - \frac{r_1}{K(1+kv^*)}\right), \quad W_3 = \frac{(K-u^*)^2u^*}{2K^2(1+kv^*)^2}, \\
W_2 = & \frac{(K-u^*)u^*}{K(1+kv^*)}\left(\frac{-r_1k(K-u^*)}{K(1+kv^*)^2} - \frac{\beta}{u^*+\gamma}\right) - \frac{k(K-u^*)u^*}{K(1+kv^*)^2}, \\
W_4 = & \frac{\beta(K-u^*)u^* - \beta(K-2u^*)(u^*+\gamma)}{K(1+kv^*)(u^*+\gamma)^2} + \frac{r_1^2k(K-u^*)^2u^* - kK^2(K-2u^*)(1+kv^*)^2}{K^3(1+kv^*)^4} \\
& + \frac{3r_1k(K-u^*)u^* - r_1k(K-u^*)^2}{K^2(1+kv^*)^3} - \frac{r_1k\beta(K-u^*)^2u^*v^*}{K^2(1+kv^*)^3(u^*+\gamma)^2} + \frac{r_1\beta(K-u^*)u^*}{K^2(1+kv^*)^2(u^*+\gamma)} \\
& - \frac{\beta^2(K-u^*)u^*v^*}{K(1+kv^*)(u^*+\gamma)^3} - \frac{k\beta(K-u^*)u^*v^*}{K(1+kv^*)^2(u^*+\gamma)^2}, \\
W_5 = & \frac{r_1(2u^*-K)}{K^2(1+kv^*)^2} + \frac{\beta(K-3u^*)v^*}{2K(1+kv^*)(u^*+\gamma)^2} - \frac{1}{K(1+kv^*)} + \frac{\beta(K-u^*)(\gamma-u^*)v^*}{2K(1+kv^*)(u^*+\gamma)^3} \\
& + \frac{r_1^2(K-u^*)u^*}{2K^3(1+kv^*)^3} - \frac{r_1\beta(K-u^*)u^*v^*}{K^2(1+kv^*)^2(u^*+\gamma)^2} + \frac{\beta^2(K-u^*)u^*v^{*2}}{2K(1+kv^*)(u^*+\gamma)^4}, \\
W_6 = & \frac{r_1^2k^2(K-u^*)^3u^*}{2K^3(1+kv^*)^5} + \frac{\beta^2(K-u^*)u^*}{2K(1+kv^*)(u^*+\gamma)^2} + \frac{r_1k\beta(K-u^*)^2u^*}{2K^2(1+kv^*)^3(u^*+\gamma)} + \frac{(K-u^*)u^*}{2K(1+kv^*)} \\
& + \frac{r_1k^2(K-u^*)^2u^*}{K^2(1+kv^*)^4} + \frac{k\beta(K-u^*)u^*}{2K(1+kv^*)^2(u^*+\gamma)} + \frac{k^2(K-u^*)u^*}{K(1+kv^*)^3}, \\
W_7 = & \frac{(K-u^*)^2 - 2(K-u^*)u^*}{2K^2(1+kv^*)^2} - \frac{r_1(K-u^*)^2u^*}{2K^3(1+kv^*)^3} + \frac{\beta(K-u^*)^2u^*v^*}{2K^2(1+kv^*)^2(u^*+\gamma)^2}, \\
W_8 = & \frac{-r_1k(K-u^*)^3u^*}{2K^3(1+kv^*)^4} - \frac{\beta(K-u^*)^2u^*}{2K^2(1+kv^*)^2(u^*+\gamma)} - \frac{k(K-u^*)^2u^*}{K^2(1+kv^*)^3}, \\
W_9 = & \frac{(K-u^*)^3u^*}{6K^3(1+kv^*)^3}, \ S_{21} = \frac{abv^{*2}}{(bu^*+c)^2}, \ S_{22} = 1-(\alpha-d), \ S_{23} = \frac{ab^2v^{*2}(av^*-2bu^*-2c)}{2(bu^*+c)^4}, \\
S_{25} = & \frac{a(av^*-2bu^*-2c)}{2(bu^*+c)^2}, \ T_{21} = \frac{ab^3v^*(6b^2u^{*2} - 6abu^*v^* + a^2v^{*2} + 12bcu^* - 6acv^* + 6c^2)}{6(bu^*+c)^6}, \\
S_{24} = & \frac{abv(2bu^*+2c-av^*)}{(bu^*+c)^3}, \ T_{22} = \frac{-ab^2v^*(4b^2u^{*2} - 5abu^*v^* + a^2v^{*2} + 8bcu^* - 5acv^* + 4c^2)}{2(bu^*+c)^5}, \\
T_{23} = & \frac{ab(2b^2u^{*2} - 4abu^*v^* + a^2v^{*2} + 4bcu^* - 4acv^* + 2c^2)}{2(bu^*+c)^4}, \ T_{24} = \frac{a^2(3bu^*-av^*+3c)}{6(bu^*+c)^3}.
\end{aligned}
$$

We construct a nonsingular matrix D_1 and translate it as follows:

$$
\begin{pmatrix} p \\ q \end{pmatrix} = D_1 \begin{pmatrix} u \\ v \end{pmatrix}, \tag{3.8}
$$

where

$$
D_1 = \begin{pmatrix} S_{12} & S_{12} \\ -1-S_{11} & \lambda_2 - S_{11} \end{pmatrix}.
$$

Taking D_1^{-1} on both sides of Eq (3.8), we obtain

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} g_1(p,q,\overline{r}) \\ g_2(p,q,\overline{r}) \end{pmatrix}, \tag{3.9}$$

where

$$\begin{aligned}
g_1(p,q,\overline{r}) = {} & \frac{[S_{13}(\lambda_2 - S_{11}) - S_{12}S_{23}]p^2}{S_{12}(\lambda_2+1)} + \frac{[S_{14}(\lambda_2 - S_{11}) - S_{12}S_{24}]pq}{S_{12}(\lambda_2+1)} + \frac{W_1(\lambda_2 - S_{11})p\overline{r}}{S_{12}(\lambda_2+1)} \\
& + \frac{[S_{15}(\lambda_2 - S_{11}) - S_{12}S_{25}]q^2}{S_{12}(\lambda_2+1)} + \frac{[T_{11}(\lambda_2 - S_{11}) - S_{12}T_{21}]p^3}{S_{12}(\lambda_2+1)} + \frac{W_2(\lambda_2 - S_{11})q\overline{r}}{S_{12}(\lambda_2+1)} \\
& + \frac{[T_{12}(\lambda_2 - S_{11}) - S_{12}T_{22}]p^2q}{S_{12}(\lambda_2+1)} + \frac{[T_{13}(\lambda_2 - S_{11}) - S_{12}T_{23}]pq^2}{S_{12}(\lambda_2+1)} + \frac{W_3(\lambda_2 - S_{11})\overline{r}^2}{S_{12}(\lambda_2+1)} \\
& + \frac{[T_{14}(\lambda_2 - S_{11}) - S_{12}T_{24}]q^3}{S_{12}(\lambda_2+1)} + \frac{W_4(\lambda_2 - S_{11})pq\overline{r}}{S_{12}(\lambda_2+1)} + \frac{W_5(\lambda_2 - S_{11})p^2\overline{r}}{S_{12}(\lambda_2+1)} \\
& + \frac{W_6(\lambda_2 - S_{11})q^2\overline{r}}{S_{12}(\lambda_2+1)} + \frac{W_7(\lambda_2 - S_{11})p\overline{r}^2}{S_{12}(\lambda_2+1)} + \frac{W_8(\lambda_2 - S_{11})q\overline{r}^2}{S_{12}(\lambda_2+1)} + \frac{W_9(\lambda_2 - S_{11})\overline{r}^3}{S_{12}(\lambda_2+1)} \\
& + O((|p|,|q|,|\overline{r}|)^4), \\
g_2(p,q,\overline{r}) = {} & \frac{[S_{13}(\lambda_2 + S_{11}) + S_{12}S_{23}]p^2}{S_{12}(\lambda_2+1)} + \frac{[S_{14}(\lambda_2 + S_{11}) + S_{12}S_{24}]pq}{S_{12}(\lambda_2+1)} + \frac{W_1(\lambda_2 + S_{11})p\overline{r}}{S_{12}(\lambda_2+1)} \\
& + \frac{[S_{15}(\lambda_2 + S_{11}) + S_{12}S_{25}]q^2}{S_{12}(\lambda_2+1)} + \frac{[T_{11}(\lambda_2 + S_{11}) + S_{12}T_{21}]p^3}{S_{12}(\lambda_2+1)} + \frac{W_2(\lambda_2 + S_{11})q\overline{r}}{S_{12}(\lambda_2+1)} \\
& + \frac{[T_{12}(\lambda_2 + S_{11}) + S_{12}T_{22}]p^2q}{S_{12}(\lambda_2+1)} + \frac{[T_{13}(\lambda_2 + S_{11}) + S_{12}T_{23}]pq^2}{S_{12}(\lambda_2+1)} + \frac{W_3(\lambda_2 + S_{11})\overline{r}^2}{S_{12}(\lambda_2+1)} \\
& + \frac{[T_{14}(\lambda_2 + S_{11}) + S_{12}T_{24}]q^3}{S_{12}(\lambda_2+1)} + \frac{W_4(\lambda_2 + S_{11})pq\overline{r}}{S_{12}(\lambda_2+1)} + \frac{W_5(\lambda_2 + S_{11})p^2\overline{r}}{S_{12}(\lambda_2+1)} \\
& + \frac{W_6(\lambda_2 + S_{11})q^2\overline{r}}{S_{12}(\lambda_2+1)} + \frac{W_7(\lambda_2 + S_{11})p\overline{r}^2}{S_{12}(\lambda_2+1)} + \frac{W_8(\lambda_2 + S_{11})q\overline{r}^2}{S_{12}(\lambda_2+1)} + \frac{W_9(\lambda_2 + S_{11})\overline{r}^3}{S_{12}(\lambda_2+1)} \\
& + O((|p|,|q|,|\overline{r}|)^4), \\
p = {} & S_{12}(u+v), \quad q = (\lambda_2 + S_{11})v - (1 + S_{11})u.
\end{aligned}$$

Applying the center manifold theorem $W^c(0)$ of system (3.9) at the trivial equilibrium point (0, 0) in a limited field of $\overline{r} = 0$. Then there exists a center manifold $W^c(0)$ as follows:

$$W^c(0) = \left\{(p,q,\overline{r}) \in R^3 : q(p,\overline{r}) = e_0\overline{r} + e_1p^2 + e_2p\overline{r} + e_3\overline{r}^2 + O((|p| + |\overline{r}|)^3)\right\}$$

and satisfies

$$H(q(p,\overline{r})) = q\left(-u + g_1(p,q(p,\overline{r}),\overline{r})\right) - \lambda_2 q(p,\overline{r}) - g_2(p,q(p,\overline{r}),\overline{r}) = 0,$$

and we have

$$
\begin{aligned}
e_0 &= 0,\\
e_1 &= \frac{[S_{13}(1+S_{11})+S_{12}S_{23}]S_{12}-[S_{14}(1+S_{11})+S_{12}S_{24}](1+S_{11})}{1-\lambda_2^2}\\
&\quad+\frac{[S_{15}(1+S_{11})+S_{12}S_{25}](1+S_{11})^2}{(1-\lambda_2^2)S_{12}},\\
e_2 &= \frac{[S_{12}W_1-W_2(1+S_{11})](1+S_{11})}{(1-\lambda_2)^2},\\
e_3 &= \frac{W_3(1+S_{11})}{S_{12}(1-\lambda_2)^2}.
\end{aligned}
$$

Therefore, we consider the map restricted to the center manifold $W^c(0)$ as below:

$$
G: p \to -p+n_1p^2+n_2p\overline{r}+n_3p^2\overline{r}+n_4p\overline{r}^2+n_5p^3+O((|p|+|\overline{r}|)^4),
$$

where

$$
\begin{aligned}
n_1 =& \frac{[S_{13}(\lambda_2-S_{11})-S_{12}S_{23}]S_{12}}{1+\lambda_2}-\frac{[S_{14}(\lambda_2-S_{11})-S_{12}S_{24}](1+S_{11})}{1+\lambda_2}\\
&+\frac{[S_{15}(\lambda_2-S_{11})-S_{12}S_{25}](1+S_{11})^2}{S_{12}(1+\lambda_2)},\\
n_2 =& \frac{W_1(\lambda_2-S_{11})}{1+\lambda_2}-\frac{W_2(\lambda_2-S_{11})(1+S_{11})}{S_{12}(1+\lambda_2)},\\
n_3 =& \frac{[S_{13}(\lambda_2-S_{11})-S_{12}S_{23}]2e_2S_{12}}{1+\lambda_2}+\frac{[S_{14}(\lambda_2-S_{11})-S_{12}S_{24}](\lambda_2-S_{11})e_2}{1+\lambda_2}\\
&-\frac{2[S_{15}(\lambda_2-S_{11})-S_{12}S_{25}](1+S_{11})(\lambda_2-S_{11})e_2}{S_{12}(1+\lambda_2)}+\frac{W_1(\lambda_2-S_{11})e_1}{1+\lambda_2}\\
&+\frac{(\lambda_2-S_{11})^2e_1}{S_{12}(1+\lambda_2)}+\frac{S_{12}(\lambda_2-S_{11})W_5}{1+\lambda_2}+\frac{W_6(\lambda_2-S_{11})(1+S_{11})^2}{(1+\lambda_2)S_{12}}\\
&-\frac{[S_{14}(\lambda_2-S_{11})-S_{12}S_{24}]e_2(1+S_{11})+W_4(\lambda_2-S_{11})(1+S_{11})}{1+\lambda_2},\\
n_4 =& \frac{[S_{13}(\lambda_2-S_{11})-S_{12}S_{23}]2e_2S_{12}}{1+\lambda_2}+\frac{[S_{14}(\lambda_2-S_{11})-S_{12}S_{24}](\lambda_2-S_{11})e_3}{\lambda_2+1}\\
&-\frac{[S_{14}(\lambda_2-S_{11})-S_{12}S_{24}](1+S_{11})e_3}{\lambda_2+1}+\frac{(\lambda_2-S_{11})(W_1e_2+W_7)}{\lambda_2+1}\\
&-\frac{[S_{15}(\lambda_2-S_{11})-S_{12}S_{25}](1+S_{11})(\lambda_2-S_{11})e_3}{\lambda_2+1}+\frac{(\lambda_2-S_{11})^2W_2e_2}{S_{12}(\lambda_2+1)}\\
&-\frac{W_8(\lambda_2-S_{11})(S_{11}+1)}{S_{12}(\lambda_2+1)},
\end{aligned}
$$

$$
\begin{aligned}
n_5 =& \frac{[T_{11}(\lambda_2 - S_{11}) - S_{12}T_{21}]S_{12}^2}{1+\lambda_2} - \frac{[T_{12}(\lambda_2 - S_{11}) - S_{12}T_{22}](1+S_{11})S_{12}}{\lambda_2+1} \\
&- \frac{2[S_{15}(\lambda_2 - S_{11}) - S_{12}S_{25}](1+S_{11})(\lambda_2 - S_{11})e_1}{(1+\lambda_2)S_{12}} + \frac{[S_{13}(\lambda_2 - S_{11}) - S_{12}S_{23}]2e_2S_{12}}{1+\lambda_2} \\
&- \frac{[S_{14}(\lambda_2 - S_{11}) - S_{12}S_{24}](1+S_{11})e_1}{\lambda_2+1} + \frac{[S_{14}(\lambda_2 - S_{11}) - S_{12}S_{24}](\lambda_2 - S_{11})e_1}{\lambda_2+1} \\
&+ \frac{[T_{13}(\lambda_2 - S_{11}) - S_{12}T_{23}](1+S_{11})^2}{\lambda_2+1} - \frac{[T_{14}(\lambda_2 - S_{11}) - S_{12}T_{24}](1+S_{11})^3}{S_{12}(\lambda_2+1)}.
\end{aligned}
$$

According to flip bifurcation, we define the following two nonzero real numbers δ_1 and δ_2, where

$$
\delta_1 = \left(\frac{\partial^2 G}{\partial p \partial \bar{r}} + \frac{1}{2}\frac{\partial G}{\partial \bar{r}}\frac{\partial^2 G}{\partial p^2} \right)\Bigg|_{(0,0)} = n_2, \quad \delta_2 = \left(\frac{1}{6}\frac{\partial^3 G}{\partial p^3} + \left(\frac{1}{2}\frac{\partial^2 G}{\partial p^2} \right)^2 \right)\Bigg|_{(0,0)} = n_1^2 + n_5.
$$

From the above analysis, we get the following theorem:

Theorem 2. *If $\delta_1 \neq 0, \delta_2 \neq 0$, then system (1.2) passes through a flip bifurcation at the fixed point $H_3(u^*, v^*)$ when the parameter r alters in the small region of r_1. In addition, if $\delta_2 > 0$ (resp., $\delta_2 < 0$), then the period-two orbits that bifurcate from fixed point $H_3(u^*, v^*)$ are stable (resp., unstable).*

3.2. Neimark-Sacker bifurcation

When the parameters change on set A_2, system (1.2) will undergo Neimark-Sacker bifurcation at the unique positive interior equilibrium point $H_3(u^*, v^*)$, where

$$
A_2 = \left\{ (a, b, c, d, K, \alpha, \beta, \gamma, k, r_2) : r_2 = \tfrac{(1+kv^*)K}{u^*\Theta}(\Theta + \Phi\Theta + \Psi - 1), \left| 1 - \tfrac{ru^*}{(1+kv^*)K} + \Phi + \Theta \right| < 2 \right\}.
$$

Consider a perturbation related to system (1.2) as follows:

$$
\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \exp\left[\frac{(r_2+\bar{r})}{1+kv}(1 - \frac{u}{K}) - \frac{\beta v}{u+\gamma} \right] \\ v \exp(\alpha - \frac{av}{bu+c} - d) \end{pmatrix}, \tag{3.10}
$$

where $\bar{r}$ is a limited perturbation parameter and $|\bar{r}| \ll 1$.

The characteristic equation of system (3.10) at $H_3(u^*, v^*)$ is as follows:

$$
\lambda^2 - m(\bar{r})\lambda + n(\bar{r}) = 0,
$$

where

$$
m(\bar{r}) = 1 - \frac{(r_2 + \bar{r})u^*}{(1 + kv^*)K} + \Phi + \Theta
$$

and

$$
n(\bar{r}) = \Theta \left[1 - \frac{(r_2 + \bar{r})u^*}{(1 + kv^*)K} + \Phi \right] + \Psi.
$$

Since parameters $(a, b, c, d, K, \alpha, \beta, \gamma, k, r_2) \in A_2$, the characteristic values of system (3.10) at $H_3(u^*, v^*)$ are a pair of complex conjugate numbers λ and $\bar{\lambda}$ with $|\lambda| = |\bar{\lambda}| = 1$ as follows

$$\lambda,\ \bar{\lambda} = \frac{m(\bar{r}) \pm i\sqrt{4n(\bar{r}) - m^2(\bar{r})}}{2}.$$

Therefore we have

$$|\lambda| = |\bar{\lambda}| = n(\bar{r}))^{1/2}, \quad \left.\frac{d|\lambda|}{d\bar{r}}\right|_{\bar{r}=0} = \left.\frac{d|\bar{\lambda}|}{d\bar{r}}\right|_{\bar{r}=0} = -\frac{\Theta u^*}{2(1+kv^*)\sqrt{n(0)}} < 0.$$

When $\bar{r}$ changes in limited field of $\bar{r} = 0$, then $\lambda,\ \bar{\lambda} = x \pm iy$, where

$$x = \frac{m(0)}{2},\ \ y = \frac{\sqrt{4n(0) - m^2(0)}}{2}.$$

In addition, Neimark-Sacker bifurcation requires that $\bar{r} = 0$, λ^z, $\bar{\lambda}^z \neq 1$ (z=1, 2, 3, 4), which is equivalent to $m(0) \neq$ -2, 0, -1, 2. Because parameters $(a, b, c, d, K, \alpha, \beta, \gamma, k, r_2) \in A_2$, therefore $m(0) \neq$ -2, 2. We only require $m(0) \neq$ 0, -1, so that

$$1 + \Phi + \Theta \neq \frac{r_2 u^*}{(1+kv^*)K},\ \ 1 + \Phi + \Theta \neq \frac{r_2 u^*}{(1+kv^*)K} - 1. \tag{3.11}$$

Let $p = u - u^*$ and $q = v - v^*$.

After the transformation of the equilibrium point $H_3(u^*, v^*)$ of system (3.10) to the origin, we have

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} f_1(p,q) \\ f_2(p,q) \end{pmatrix}, \tag{3.12}$$

where

$$f_1(p,q) = S_{13}p^2 + S_{14}pq + S_{15}q^2 + T_{11}p^3 + T_{12}p^2q + T_{13}pq^2 + T_{14}q^3 + O((|p|,|q|)^4),$$
$$f_2(p,q) = S_{23}p^2 + S_{24}pq + S_{25}q^2 + T_{21}p^3 + T_{22}p^2q + T_{23}pq^2 + T_{24}q^3 + O((|p|,|q|)^4),$$

and $S_{11}, S_{12}, S_{13}, S_{14}, S_{15}, T_{11}, T_{12}, T_{13}, T_{14}, S_{21}, S_{22}, S_{23}, S_{24}, S_{25}, T_{21}, T_{22}, T_{23}, T_{24}$, are given in (3.7) by substituting r_1 for $r_2 + \bar{r}$.

Besides that, we analyse the normal form of system (3.12) when $\bar{r} = 0$.

Consider the translation as follows:

$$\begin{pmatrix} p \\ q \end{pmatrix} = D_2 \begin{pmatrix} u \\ v \end{pmatrix},$$

where

$$D_2 = \begin{pmatrix} S_{12} & 0 \\ x - S_{11} & -y \end{pmatrix}.$$

Taking D_2^{-1} on both sides of system (3.12), we acquire

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix},$$

where

$$
\begin{aligned}
f(u,v) &= \frac{S_{13}p^2}{S_{12}} + \frac{S_{14}pq}{S_{12}} + \frac{S_{15}q^2}{S_{12}} + \frac{T_{11}p^3}{S_{12}} + \frac{T_{12}p^2q}{S_{12}} + \frac{T_{13}pq^2}{S_{12}} + \frac{T_{14}q^3}{S_{12}} + O((|p|,|q|)^4),\\
g(u,v) &= \frac{[S_{13}(x-S_{11})-S_{12}S_{23}]p^2}{S_{12}y} + \frac{[S_{14}(x-S_{11})-S_{12}S_{24}]pq}{S_{12}y} + \frac{[S_{15}(x-S_{11})-S_{12}S_{25}]q^2}{S_{12}y}\\
&\quad + \frac{[T_{11}(x-S_{11})-S_{12}T_{21}]p^3}{S_{12}y} + \frac{[T_{12}(x-S_{11})-S_{12}T_{22}]p^2q}{S_{12}y} + \frac{[T_{13}(x-S_{11})-S_{12}T_{23}]pq^2}{S_{12}y}\\
&\quad + \frac{[T_{14}(x-S_{11})-S_{12}T_{24}]q^3}{S_{12}y} + O((|p|,|q|)^4),\\
p &= S_{12}u, \quad q = (x-S_{11})u - yv.
\end{aligned}
$$

System (1.2) occurs the Neimark-Sacker bifurcation if the following quantity ϑ is not zero,

$$
\vartheta = -\operatorname{Re}\left[\frac{(1-2\bar{\lambda})\bar{\lambda}^2}{1-\lambda}\varrho_{11}\varrho_{20}\right] - \frac{1}{2}\|\varrho_{11}\|^2 - \|\varrho_{02}\|^2 + \operatorname{Re}(\bar{\lambda}\varrho_{21}),
$$

where

$$
\begin{aligned}
\varrho_{11} &= \frac{1}{4}\left[(f_{pp}+f_{qq}) + i(g_{pp}+g_{qq})\right],\\
\varrho_{20} &= \frac{1}{8}\left[(f_{pp}-f_{qq}+2g_{pq}) + i(g_{pp}-g_{qq}-2f_{pq})\right],\\
\varrho_{02} &= \frac{1}{8}\left[(f_{pp}-f_{qq}-2g_{pq}) + i(g_{pp}-g_{qq}+2f_{pq})\right],\\
\varrho_{21} &= \frac{1}{16}\left[(f_{ppp}+f_{pqq}+g_{ppq}+g_{qqq}) + i(g_{ppp}+g_{pqq}-f_{ppq}-f_{qqq})\right].
\end{aligned}
$$

If $\vartheta \neq 0$, Neimark-Sacker bifurcation will undergo in system (1.2), and the following theorem holds:

Theorem 3. *System (1.2) undergoes a Neimark-Sacker bifurcation at the positive equilibrium point $H_3(u^*, v^*)$ if conditions (3.11) are satisfied and $\vartheta \neq 0$. In addition, if $\vartheta > 0$ (resp., $\vartheta < 0$), then an repelling (resp., attracting) invariant closed curve bifurcates from fixed point $H_3(u^*, v^*)$ for $r < r_2$ (resp., $r > r_2$).*

4. Chaos control

In this section, we will adopt the feedback control method [26–28] to stabilize the chaotic orbit at an unstable equilibrium point by adding a feedback control term to the system (1.2). Therefore, system (1.2) makes the following form:

$$
\begin{cases}
u_{n+1} = u_n \exp[\frac{r}{1+kv_n}(1-\frac{u_n}{K}) - \frac{\beta v_n}{u_n+\gamma}] - x(u_n,v_n) = f(u_n,v_n),\\
v_{n+1} = v_n \exp(\alpha - \frac{av_n}{bu_n+c} - d) = g(u_n,v_n),
\end{cases}
\tag{4.1}
$$

where $x(u_n, v_n) = h_1(u_n - u^*) + h_2(v_n - v^*)$ is feedback controlling force, h_1 and h_2 are feedback gains, and (u^*, v^*) is the unique positive equilibrium point of (1.2). Furthermore, $f(u^*, v^*) = u^*$, and $g(u^*, v^*) = v^*$.

The Jacobian matrix of system (4.1) at positive equilibrium point (u^*, v^*) is as follows:

$$J(u^*, v^*) = \begin{bmatrix} S_{11} - h_1 & S_{12} - h_2 \\ S_{21} & S_{22} \end{bmatrix},$$

where

$$S_{11} = 1 - \frac{r_1 u^*}{(1+kv^*)K} + \frac{\beta u^* v^*}{(u^*+\gamma)^2}, \quad S_{12} = -(1 - \frac{u^*}{K})\frac{r_1 k u^*}{(1+kv^*)^2} - \frac{\beta u^*}{u^*+\gamma},$$
$$S_{21} = \frac{abv^{*2}}{(bu^*+c)^2}, \quad S_{22} = 1 - (\alpha - d).$$

Thus, the characteristic equation related to $J(u^*, v^*)$ is:

$$\lambda^2 - (S_{11} + S_{22} - h_1)\lambda + (S_{11} - h_1)S_{22} - (S_{12} - h_2)S_{21} = 0. \tag{4.2}$$

Let λ_1 and λ_2 be the eigenvalues of characteristic equation (4.2), then

$$\lambda_1 + \lambda_2 = S_{11} + S_{22} - h_1, \quad \lambda_1\lambda_2 = (S_{11} - h_1)S_{22} - (S_{12} - h_2)S_{21}. \tag{4.3}$$

Next, we must solve equations $\lambda_1 = \pm 1$ and $\lambda_1\lambda_2 = 1$ to gain the critical stability line. At the same time, it also ensures that the absolute value λ_1 and λ_2 are less than one.

Suppose that $\lambda_1\lambda_2 = 1$, then we gain

$$L_1 : S_{11}S_{22} - S_{12}S_{21} - 1 = S_{22}h_1 - S_{21}h_2.$$

Assume that $\lambda_1 = 1$, then we have

$$L_2 : S_{11} + S_{22} - S_{11}S_{22} + S_{12}S_{21} - 1 = (1 - S_{22})h_1 + S_{21}h_2.$$

Assume that $\lambda_1 = -1$, then we obtain

$$L_3 : S_{11} + S_{22} + S_{11}S_{22} - S_{12}S_{21} + 1 = (1 + S_{22})h_1 - S_{21}h_2.$$

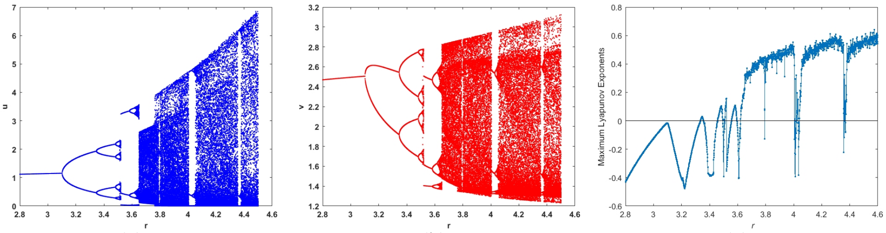
Thus, the stable eigenvalues lie within the triangular area with the boundaries of the straight lines L_1, L_2, L_3. In addition, when the control parameters h_1 and h_2 take values in the triangular region, system (4.1) will not generate chaos.

5. Numerical simulations

In this section, we draw the bifurcation diagrams, phase portraits, solution of the figures and maximum Lyapunov exponents for system (1.2) to verify the above theoretical analysis and show the new interesting complex dynamical behaviors and the stability of the predator-prey system at the equilibrium point by using numerical simulations.

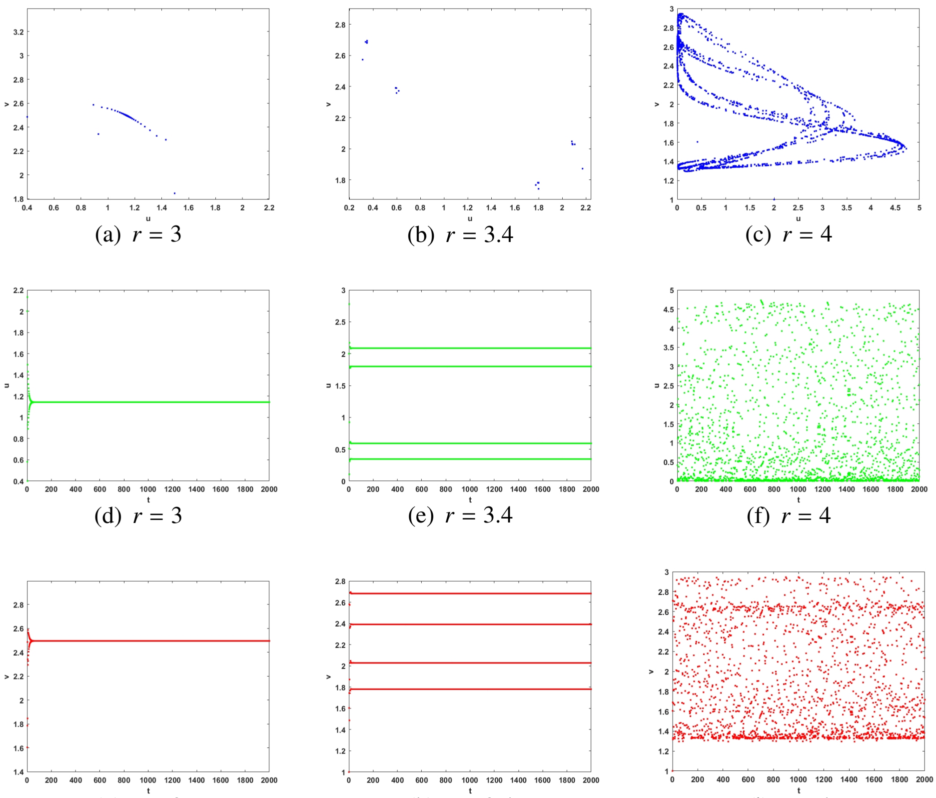
5.1. System (1.2) without fear factor ($k = 0$)

First, in Figure 1, we consider that the fear factor $k = 0$ and take r as the bifurcation parameter to discuss the dynamic behavior of (1.2) at $H_3(u^*, v^*)$. We consider the parameter values as $(a, b, c, d, \alpha, \beta, \gamma, K) = (1.8, 2.8, 3.5, 0.6, 1.27, 1.1, 2.7, 1.5) \in A_1$ with the initial value of (u_0, v_0)=(2, 1) and $r \in [2.8, 4.6]$. Flip bifurcation emerges from the unique positive equilibrium point and loses its stability as r goes through a critical value $r = 3.102$, and it is stable when $r < 3.102$, and when $r > 3.102$, system (1.2) oscillates with periods of $2, 2^2, 2^3, \cdots$. It can be obtained from Figure 1(c) and Figure 2(a–c) that chaos will happen in system (1.2) as the bifurcation parameters r continue to increase.



(a) (b) (c)

Figure 1. (a,b) Bifurcation diagram of system (1.2) with $r \in [2.8, 4.6]$, $a = 1.8, b = 2.8, c = 3.5, d = 0.6, \alpha = 1.27, \beta = 1.1, \gamma = 2.7, k = 0, K = 1.5$ the initial value is (u_0, v_0)=(2, 1). (c) Maximum Lyapunov exponents corresponding to (a,b).



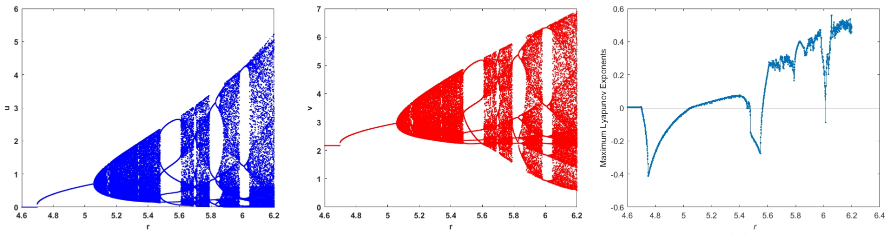
(a) $r = 3$ (b) $r = 3.4$ (c) $r = 4$

(d) $r = 3$ (e) $r = 3.4$ (f) $r = 4$

(g) $r = 3$ (h) $r = 3.4$ (i) $r = 4$

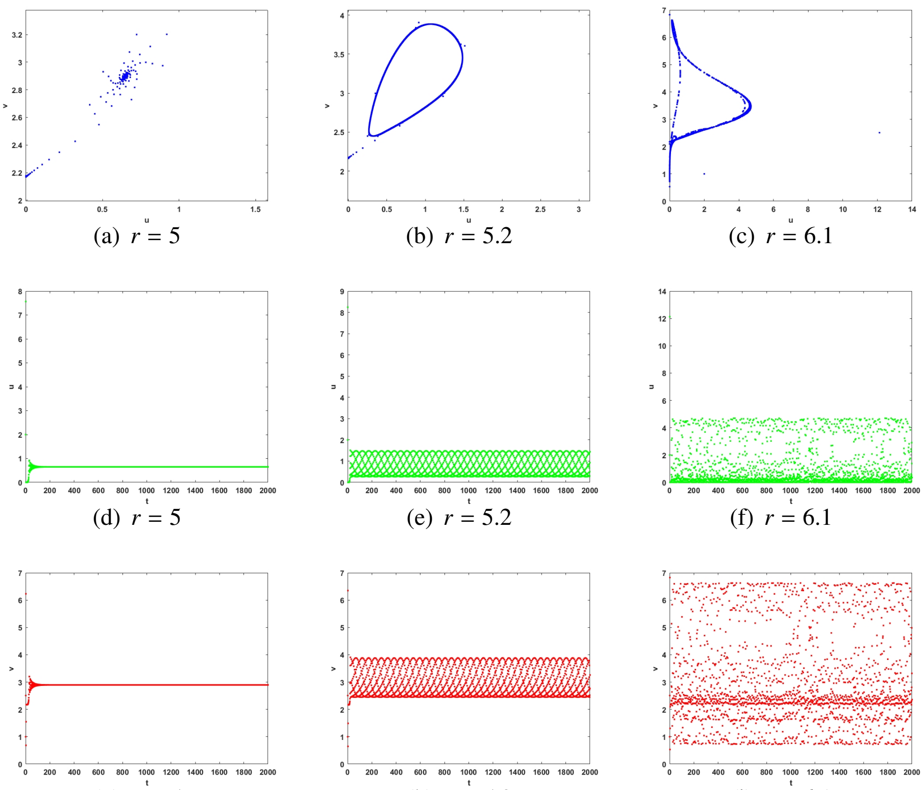
Figure 2. Phase portraits and solution portraits for various values of r corresponding to Figure 1.

In Figure 3, taking $(a, b, c, d, \alpha, \beta, \gamma, K) = (1.7, 1.6, 3.1, 0.01, 1.2, 2.6, 1.2, 3.5) \in A_2$ with the initial value of (u_0, v_0)=(1, 2) and $r \in [4.6, 6.2]$. Neimark-Sacker bifurcation emerges from the unique positive equilibrium point and loses its stability as r goes through a critical value $r = 5.05$. It can be seen from Figure 3(a) that when $r < 4.7$, the equilibrium point of (1.2) with respect to prey does not exist. Not only does the system have an attractive-invariant loop and periodic solutions, but it also exhibits dynamical chaos as the bifurcation parameters r continue to increase. Figure 3(c) is the maximum Lyapunov exponent diagram related to Figure 3(a,b). It can be seen from the MLE that system (1.2) will appear chaotic. By observing the Figure 4(a–c), it can be found that when $r > 5.05$, a limit cycle, a periodic window and chaos appear in system (1.2).



(a) (b) (c)

Figure 3. (a,b) Bifurcation diagram of system (1.2) with $r \in [4.6, 6.2]$, $a = 1.7, b = 1.6, c = 3.1, d = 0.01, \alpha = 1.2, \beta = 2.6, \gamma = 1.2, k = 0, K = 3.5$ the initial value is (u_0, v_0)=(1, 2). (c) Maximum Lyapunov exponents related to (a, b).



(a) $r = 5$ (b) $r = 5.2$ (c) $r = 6.1$

(d) $r = 5$ (e) $r = 5.2$ (f) $r = 6.1$

(g) $r = 5$ (h) $r = 5.2$ (i) $r = 6.1$

Figure 4. Phase portraits and solution portraits for various values of r corresponding to Figure 3.

5.2. System (1.2) with fear factor ($k > 0$)

In Figure 5, we consider that the fear factor $k > 0$. Taking $(a, b, c, d, \alpha, \beta, \gamma, k, K) = (2, 2.8, 3.5, 0.6, 1.1, 1.1, 3, 0.5, 2) \in A_1$ with the initial value of (u_0, v_0)=(2, 1) and $r \in [5.2, 7]$. Flip bifurcation appears from the unique positive equilibrium point and loses its stability as r goes through a critical value $r = 5.63$, and it is stable when $r < 5.63$ and when $r > 5.63$, system (1.2) oscillates with periods of $2, 2^2, 2^3, \cdots$. It can be acquired from Figure 5(c) that chaos will happen in system (1.2) as the bifurcation parameters r continue to increase.

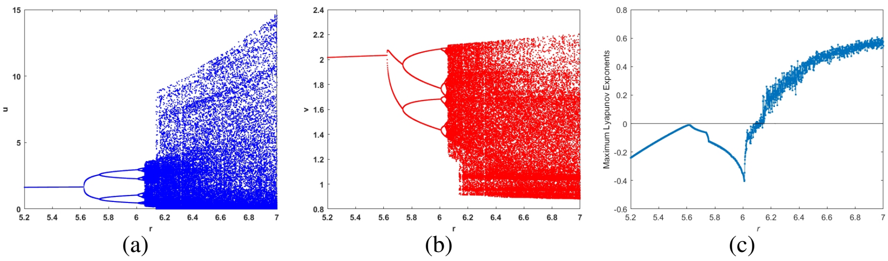


Figure 5. (a,b) Bifurcation diagram of system (1.2) with $r \in [5.2, 7]$, $a = 2, b = 2.8, c = 3.5, d = 0.6, \alpha = 1.1, \beta = 1.1, \gamma = 3, k = 0.5, K = 2$ the initial value is (u_0, v_0)=(2, 1). (c) Maximum Lyapunov exponents related to (a, b).

In Figure 6, taking $(a, b, c, d, \alpha, \beta, \gamma, k, K) = (1.7, 1.6, 3.1, 0.01, 1.2, 2.6, 1.2, 0.3, 3.5) \in A_2$ with the initial value of (u_0, v_0)=(1, 2) and $r \in [8, 11]$. Neimark-Sacker bifurcation emerges from the unique positive equilibrium point and loses its stability as r goes through a critical value $r = 8.63$. We notice that the equilibrium point of (1.2) is stable for $r < 8.63$, loses its stability at $r = 8.63$ and not only a limit cycle but also periodic solution emerge when the bifurcation parameter $r > 8.63$. Other than that, the value of the MLE related to (1.2) is greater than 0 as r continues to increase, and thus chaos will occur, i.e., the solution of (1.2) is arbitrarily periodic.

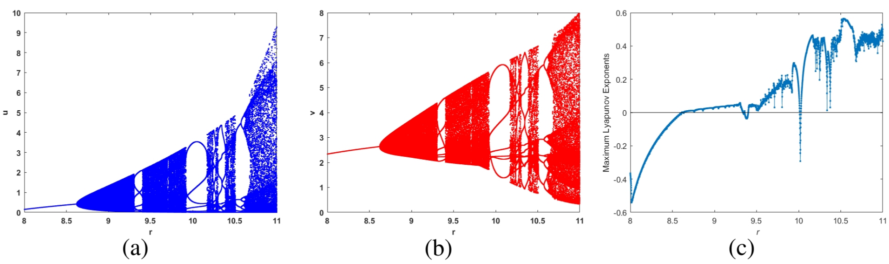
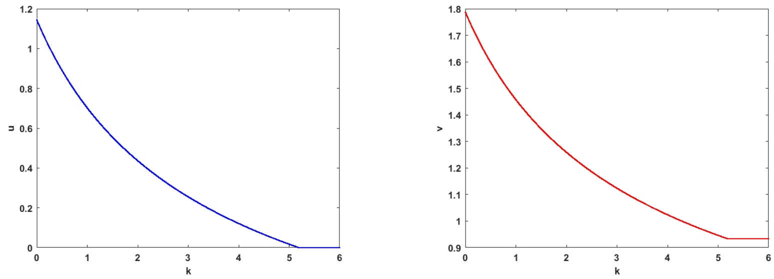


Figure 6. (a,b) Bifurcation diagram relevent u and v in system (1.2) with $r \in [8, 11]$, $a = 1.7, b = 1.6, c = 3.1, d = 0.01, \alpha = 1.2, \beta = 2.6, \gamma = 1.2, k = 0.3, K = 3.5$ the initial value is (u_0, v_0)=(1, 2). (c) Maximum Lyapunov exponents related to (a, b).

In Figure 7, taking $(a, b, c, d, \alpha, \beta, \gamma, r, K) = (1.5, 2.8, 3.5, 0.6, 1, 1.1, 3, 2, 1.5)$ with the initial value of (u_0, v_0)=(2, 1) and $k \in [0, 6]$, k is a bifurcation parameter. At this time, the bifurcation phenomenon of (1.2) will not occur. The population density of prey and predator will continue to decrease and tend to 0 with the increase of fear factor k. It is important to note that the cost of fear does not lead to the extinction of predators, but rather to the extinction of prey.

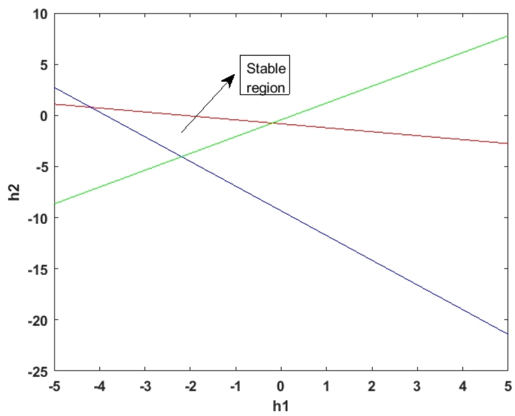


(a) Bifurcation diagram for u (b) Bifurcation diagram for v

Figure 7. Bifurcation diagram of system (1.2) with $k \in [0, 6]$, $a = 1.5, b = 2.8, c = 3.5, d = 0.6, \alpha = 1, \beta = 1.1, \gamma = 3, K = 1.5, r = 2$ the initial value is (u_0, v_0)=(2, 1).

5.3. *Controlling chaos*

In Figure 8, when the parameter value is $(a, b, c, d, \alpha, \beta, \gamma, k, r, K) = (1.7, 1.6, 3.1, 0.01, 1.2, 2.6, 1.2, 0.3, 10.8, 3.5)$ with the initial value of (u_0, v_0)=(1, 2). In Figure 6(c) when the bifurcation parameter $r = 10.8$, system (1.2) will produce chaos. When the h_1 and h_2 are controlled in the triangular region surrounded by three straight lines L_1, L_2, and L_3, the chaos generated by system (4.1) will be controlled near the equilibrium point and become an asymptotically stable state.



(a)

Figure 8. The bounded region for the eigenvalues of the controlled system (4.1) in the (h_1, h_2) plane.

6. Conclusions

Studies have shown that discrete systems have richer and more complex dynamic behaviors than continuous systems. Hence, on the basis of previous research work, this paper discusses the stability, bifurcation and chaos control of a nonlinear discrete prey-predator system with fear effect. Based on the results of the study, we can draw the following conclusions:

(a) System (1.2) has four equilibrium points, where the stable equilibrium point is positive, and depicts the coexistence of prey and predators.

(b) System (1.2) has flip bifurcation and Neimark-Sacker bifurcation happen at the positive interior equilibrium point when r alters in A_1 and A_2 small fields. (see Figures 1, 3, 5, 6). We can also observe the orbits of periods 2, 4, and 8 periodic windows of flip bifurcation.

(c) When $k = 0$, system (1.2) at the positive equilibrium point will generate the Neimark-Sacker bifurcation, flip bifurcation and chaos as the bifurcation parameters r continue to increase.

(d) When the fear parameter k is greater, both predators and prey populations decrease. It is important to note that the cost of fear does not lead to the extinction of predators, but rather to the extinction of prey. (see Figure 7).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

1. A. A. Berryman, The origins and evolution of predator-prey theory, *Ecology*, **73** (1992), 1530–1535. http://dx.doi.org/10.2307/1940005

2. H. Freedman, *Deterministic Mathematical Models in Population Ecology*, Edmonton: HIFR Consulting Ltd, 1980. http://dx.doi.org/10.2307/2975858

3. M. Sen, M. Banerjee, A. Morozov, Bifurcation analysis of a ratio-dependent prey-predator model with the Allee effect, *Ecol. Compl.*, **11** (2012), 12–27. http://dx.doi.org/10.1016/j.ecocom.2012.01.002

4. R. E. Kooij, A. Zegeling, A predator-prey model with Ivlev's functional response, *J. Math. Anal. Appl.*, **198** (1996), 473–489. http://dx.doi.org/10.1016/j.chaos.2004.07.018

5. R. Lopez-Ruiz, R. Fournier-Prunaret, Indirect Allee effect, bistability and chaotic oscillations in a predator-prey discrete model of logistic type, *Chaos Solitons Fract.*, **24** (2005), 85–101. http://dx.doi.org/10.1016/j.chaos.2004.07.018

6. J. D. Murray, *Mathematical Biology*, 2^{nd} edtion, Berlin: Springer-Verlag, 1993. http://dx.doi.org/10.1137/1032093

7. W. Ma, Y. Takeuchi, Stability analysis on a predator-prey system with distributed delays, *J. Comput. Appl. Math.*, **88** (1998), 79–94. http://dx.doi.org/10.1016/S0377-0427(97)00203-3

8. S. Sinha, O. Misra, J. Dhar, Modelling a predator-prey system with infected prey in polluted environment, *Appl. Math. Model.*, **34** (2010), 1861–1872. http://dx.doi.org/10.1016/j.apm.2009.10.003

9. M. Fan, K. Wang, Periodic solutions of a discrete time non-autonomous ratio-dependent predator-prey system, *Math. Comput. Model.*, **35** (2002), 951–961. http://dx.doi.org/10.1016/s0895-7177(02)00062-6

10. D. Summers, J. G. Cranford, B. P. Healey, Chaos in periodically forced discrete-time ecosystem models, *Chaos Solitons Fract.*, **11** (2000), 2331–2342. http://dx.doi.org/10.1016/S0960-0779(99)00154-X

11. L. Yuan, Q. Yang, Bifurcation, invariant curve and hybrid control in a discrete-time predator-prey system, *Appl. Math. Model.*, **39** (2015), 2345–2362. http://dx.doi.org/10.1016/j.apm.2014.10.040

12. G. Q. Sun, J. Zhang, L. P. Song, Z. Jin, B. L. Li, Pattern formation of a spatial predator-prey system, *Appl. Math. Comput.*, **218** (2012), 11151–11162. http://dx.doi.org/10.1016/j.amc.2012.04.071

13. B. Dubey, B. Das, J. Hussain, A predator-prey interaction model with self and cross-diffusion, *Ecol. Model.*, **141** (2001), 67–76. http://dx.doi.org/10.1016/S0304-3800(01)00255-1

14. G. Q. Sun, Z. Jin, L. Li, M. Haque, B. L. Li, Spatial patterns of a predator-prey model with cross diffusion, *Nonlinear Dynam.*, **69** (2012), 1631–1638. http://dx.doi.org/10.1007/s11071-012-0374-6

15. C. Robinson, *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos*, Florida: CRC Press, 1998.

16. K. L. Edelstein, *Mathematical Model in Biology*, New York: McGraw-Hill, 1988.

17. N. Hiroyuki, B. Yoshikazu, *Introduction to Chaos, Physics and Mathematics of Chaotic Phenomena*, Bristol: Institute of Physics Publishing, 1999. http://dx.doi.org/10.1201/9780429187001

18. H. F. Huo, W. T. Li, Existence and global stability of periodic solutions of a discrete predator-prey system with delays, *Appl. Math. Comput.*, **153** (2004), 337–351. http://dx.doi.org/10.1016/S0096-3003(03)00635.0

19. L. F. Cheng, H. G. Cao, Bifurcation analysis of a discrete-time ratio-dependent predator-prey model with Allee Effect, *Commun. Nonlinear Sci. Numer. Simul.*, **38** (2016), 288–302. http://dx.doi.org/10.1016/j.cnsns.2016.02.038

20. X. Yang, Uniform persistence and periodic solutions for a discrete predator-prey system with delays, *Math. Anal. Appl.*, **316** (2006), 161–177. http://dx.doi.org/10.1016/j.jmaa.2005.04.036

21. X. Wang, L. Zanette, X. Zou, Modelling the fear effect in predator-prey interactions, *Math. Biol.*, **73** (2016), 1179–1204. http://dx.doi.org/10.1007/s00285-016-0989-1

22. M. X. Chen, R. C. Wu, Steady state bifurcation in Previte-Hoffman model, *Int. J. Bifur. Chaos*, **33** (2023), 2350020. http://dx.doi.org/10.1142/S0218127423500207

23. S. Gakkhar, A. Singh, Complex dynamics in a prey-predator system with multiple delays, *Commun. Nonlinear Sci. Numer. Simul.*, **17** (2012), 914–929. http://dx.doi.org/10.1016/j.cnsns.2011.05.047

24. M. X. Chen, R. C. Wu, X. H. Wang, Non-constant steady states and Hopf bifurcation of a species interaction model, *Commun. Nonlinear Sci. Numer. Simul.*, **116** (2023), 106846. http://dx.doi.org/10.1016/j.cnsns.2022.106846

25. X. L. Han, C. Y. Lei, Bifurcation and turing instability analysis for a space-and time-discrete predator-prey system with Smith growth function, *Chaos Solitons Fract.*, **166** (2023), 112920. http://dx.doi.org/10.1016/j.chaos.2022.112910

26. S. Lynch, *Dynamical Systems with Applications Using Mathematica*, Boston: Birkhauser, 2007. http://dx.doi.org/10.1007/978-0-8176-4586-1

27. S. Elaydi, *An Introduction to Difference Equations*, 3^{rd} edition, New York: Springer-Verlag, 2005. http://dx.doi.org/10.1007/0-387-27602-5

28. G. Chen, X. Dong, *From Chaos to Order: Perspectives, Methodologies, and Applications*, Singapore: World Scientific, 1998. http://dx.doi.org/10.1142/3033

29. C. Y. Lei, X. L. Han, W. Wang, Bifurcation analysis and chaos control of a discrete-time prey-predator model with fear factor, *Math. Biosci. Eng.*, **19** (2022), 6659–6679. http://dx.doi.org/10.3934/mbe.2022313

30. X. X. Du, X. L. Han, C. Y. Lei, Behavior analysis of a class of discrete-time dynamical system with capture rate, *Mathematics*, **10** (2022), 2410. http://dx.doi.org/10.3390/math10142410

31. Q. Din, Complexity and chaos control in a discrete-time prey-predator model, *Commun. Nonlinear Sci. Numer. Simul.*, **49** (2017), 113–134. http://dx.doi.org/10.1016/j.cnsns.2017.01.025