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Research article

A new error bound for linear complementarity problems involving B-matrices

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Abstract: In this paper, a new error bound for the linear complementarity problems of B-matrices which is a subclass of the P-matrices is presented. Theoretical analysis and numerical example illustrate that the new error bound improves some existing results.

Keywords: B-matrices; linear complementarity problems; error bounds; diagonally dominant matrices
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1. Introduction and preliminaries

The linear complementarity problem $LCP(A, q)$ has a wide range of applications in the contact problems, the optimal stopping, the free boundary problem for journal bearing, and the network equilibrium problems, etc, for more details see [1–3]. The $LCP(A, q)$ is to find a vector $x \in R^n$ satisfying

$$x \geqslant 0, \qquad Ax + q \geqslant 0, \qquad (Ax + q)^T x = 0, \tag{1.1}$$

where $q \in R^n$ and $A = (a_{ij}) \in R^{n\times n}$. As we all know that a matrix $A = (a_{ij}) \in R^{n\times n}$ is called an P-matrix if all its principal minors are positive [4], and that the $LCP(A, q)$ has a unique solution for any $q \in R^n$ if and only if A is an P-matrix [2]. In [5], Chen and Xiang presented the following error bound of the $LCP(A, q)$ when A is an P-matrix:

$$\|x - x^*\|_\infty \leqslant \max_{d\in[0,1]^n} \|(I - D + DA)^{-1}\|_\infty \|r(x)\|_\infty,$$

where x^* is the solution of the $LCP(A,\ q)$, $r(x) = \min\{x,\ Ax + q\}$, $D = diag(d_1,\ \cdots, d_n)$ $(0 \leqslant d_i \leqslant 1)$, and the min operator $r(x)$ denotes the componentwise minimum of two vectors.

The upper bound of the $\max_{d\in[0,1]^n} \|(I - D + DA)^{-1}\|_\infty$ is related with the special structure of the matrix A which is involved in the $LCP(A, q)$, for details, see [7–9] and references therein. B-matrices which are introduced in [4] is a subclass of P-matrices. M. García-Esnaola and J.M. Peňa in [7] presented the upper bound of the $\max_{d\in[0,1]^n} \|(I-D+DA)^{-1}\|_\infty$ when A is an B-matrix.

Definition 1. [4, Definition 2.1]. *A matrix $A=(a_{ij})\in R^{n\times n}$ is called an B-matrix if for any $i,j\in N=\{1,2,\cdots,n\}$,*

$$\sum_{k\in N} a_{ik} > 0, \qquad \frac{1}{n}(\sum_{k\in N} a_{ik}) > a_{ij}, \quad j\neq i.$$

Theorem 1. [7, Theorem 2.2]. *Let $A=(a_{ij})\in R^{n\times n}$ be an B-matrix with the form $A=B^+ + C$, where*

$$B^+ = (b_{ij}) = \begin{pmatrix} a_{11}-r_1^+ & \cdots & a_{1n}-r_1^+ \\ \vdots & \ddots & \vdots \\ a_{n1}-r_n^+ & \cdots & a_{nn}-r_n^+ \end{pmatrix}, \ C = \begin{pmatrix} r_1^+ & \cdots & r_1^+ \\ \vdots & \ddots & \vdots \\ r_n^+ & \cdots & r_n^+ \end{pmatrix}, \tag{1.2}$$

and $r_i^+ = \max\{0,\ a_{ij}|j\neq i\}$, then

$$\max_{d\in[0,1]^n} \|(I-D+DA)^{-1}\|_\infty \leqslant \frac{n-1}{\min\{\beta,\ 1\}}, \tag{1.3}$$

where $\beta=\min_{i\in N}\{\beta_i\}$ and $\beta_i = b_{ii}-\sum_{j\neq i}|b_{ij}|$.

In 2016, Li et al. improved the previous bound (3) as show below.

Theorem 2. [8, Theorem 4]. *Let $A=(a_{ij})\in R^{n\times n}$ be an B-matrix with the form $A=B^+ + C$, where $B^+=(b_{ij})$ is expressed as* (2). *Then*

$$\max_{d\in[0,1]^n} \|(I-D+DA)^{-1}\|_\infty \leqslant \sum_{i=1}^{n} \frac{n-1}{\min\{\bar\beta_i,\ 1\}} \prod_{j=1}^{i-1}\left(1+\frac{1}{\bar\beta_j}\sum_{k=j+1}^{n}|b_{jk}|\right), \tag{1.4}$$

where $\bar\beta_i = b_{ii}-\sum_{j=i+1}^{n}|b_{ij}|l_i(B^+)$, $l_k(B^+)=\max_{k\leqslant i\leqslant n}\left\{\frac{\sum_{j=k,\neq i}^{n}|b_{ij}|}{|b_{ii}|}\right\}$ and

$$\prod_{j=1}^{i-1}\left(1+\frac{1}{\bar\beta_j}\sum_{k=j+1}^{n}|b_{jk}|\right)=1 \ if\ i=1.$$

In the same year, Li et al. also derived the error bounds for linear complementarity problems of weakly chained diagonally dominant B-matrix, this bound holds for the case that weakly chained diagonally dominant B-matrix is an B-matrix. the error bound as follows.

Theorem 3. [9, Theorem 2]. *Let $A=(a_{ij})\in R^{n\times n}$ be an B-matrix with the form $A=B^+ + C$, where $B^+=(b_{ij})$ is expressed as* (2). *Then*

$$\max_{d\in[0,1]^n} \|(I-D+DA)^{-1}\|_\infty \leqslant \sum_{i=1}^{n}\left(\frac{n-1}{\min\{\widetilde{\beta}_i,\ 1\}}\prod_{j=1}^{i-1}\frac{b_{jj}}{\widetilde{\beta}_j}\right), \tag{1.5}$$

where $\widetilde{\beta}_i = b_{ii}-\sum_{j=i+1}^{n}|b_{ij}|>0$ and $\prod_{j=1}^{i-1}\frac{b_{jj}}{\widetilde{\beta}_j}=1$ if i=1.

Next, we will continue to research the upper bound of the $\max_{d\in[0,1]^n} \|(I-D+DA)^{-1}\|_\infty$ when the matrix A is an B-matrix based on the bound of the infinity norm of the inverse of strictly diagonally dominant M-matrices. We first recall some related definitions.

Definition 2. [4, Corollary 2.7]. *A matrix $A = (a_{ij}) \in R^{n\times n}$ is called an Z-matrix if $a_{ij} \leqslant 0$ for any $i \neq j$.*

Definition 3. [6, Corollary 3]. *A matrix $A = (a_{ij}) \in R^{n\times n}$ is called a strictly diagonally dominant (SDD) matrix if for any $i, j \in N = \{1, 2, \cdots, n\}$,*

$$|a_{ii}| > r_i(A) = \sum_{j=1,\ j\neq i}^{n} |a_{ij}|.$$

Definition 4. [6, Corollary 4]. *A matrix $A = (a_{ij}) \in R^{n\times n}$ is called an M-matrix if A is an Z-matrix and $A^{-1} \geqslant 0$.*

For convenience, we employ the following notations throughout the paper. Let $A = (a_{ij}) \in R^{n\times n}, i, j, k, m, n \in N, i \neq j$, denote

$$d_i = \frac{\sum\limits_{j\in N, j\neq i} |a_{ij}|}{a_{ii}}, \quad u_i = \frac{\sum\limits_{j=i+1}^{n} |a_{ij}|}{a_{ii}},$$

$$r_{ji}^{(1)} = \frac{|a_{ji}|}{|a_{jj}| - \sum\limits_{k\neq j,i}^{n} |a_{jk}|}, \quad r_{ji} = r_{ji}^{(1)},$$

$$r_{ji}^{(m)} = \frac{|a_{ji}|}{(|a_{jj}| - \sum\limits_{k\neq j,i}^{n} |a_{jk}|) r_i^{(1)} r_i^{(2)} \cdots r_i^{(m-1)}}, \quad m = 2, 3, \cdots, n,$$

$$r_i^{(m)} = \max_{j\neq i}\{r_{ji}^{(m)}\}, \quad m = 1, 2, \cdots, n,$$

$$d_{ki}^{(m)} = \frac{|a_{ki}| + \sum\limits_{j\neq k,i} |a_{kj}| r_i^{(1)} r_i^{(2)} \cdots r_i^{(m)}}{|a_{kk}| r_i^{(1)} r_i^{(2)} \cdots r_i^{(m)}}, \quad k \neq i;\ m = 1, 2, \cdots, n,$$

$$q_{ji}^{(m)} = \frac{|a_{ji}| + \sum\limits_{k\neq j,i} |a_{jk}| r_i^{(1)} r_i^{(2)} \cdots r_i^{(m)} d_{ki}^{(m)}}{|a_{jj}|},$$

$$q_i^{(m)} = \max_{j\neq i}\{q_{ji}^{(m)}\}, \quad q^{(m)} = \max_{m\leqslant i\leqslant n}\{q_i^{(m)}\}, \quad m = 1, 2, \cdots, n.$$

2. Main results

In this part, we first give some lemmas which will be used later, then a new upper bound of the $\max_{d\in[0,1]^n} \| (I-D+DA)^{-1} \|_\infty$ when A is an B-matrix will be derived.

Lemma 1. [10, Theorem 5]. *Let $A = (a_{ij}) \in R^{n\times n}$ be an SDD M-matrix. Then*

$$\| A^{-1} \|_\infty \leqslant \max\left\{ \frac{1}{a_{11} - \sum\limits_{j=2}^{n} |a_{1j}| q_{j1}^{(1)}} + \sum_{i=2}^{n} \left[\frac{1}{a_{ii} - \sum\limits_{j=i+1}^{n} |a_{ij}| q_{ji}^{(i)}} \prod_{j=1}^{i-1} \frac{u_j}{(1 - u_j q^{(j)})} \right], \right.$$

$$\left. \frac{q^{(1)}}{a_{11} - \sum\limits_{j=2}^{n} |a_{1j}| q_{j1}^{(1)}} + \sum_{i=2}^{n} \left[\frac{q^{(i)}}{a_{ii} - \sum\limits_{j=i+1}^{n} |a_{ij}| q_{ji}^{(i)}} \prod_{j=1}^{i-1} \frac{1}{(1 - u_j q^{(j)})} \right] \right\}, \tag{2.1}$$

where $\prod\limits_{j=1}^{0} \frac{u_j}{(1-u_j q^{(j)})} = 1$ *and* $\prod\limits_{j=1}^{0} \frac{1}{(1-u_j q^{(j)})} = 1.$

Lemma 2. [9, Lemma 4]. *Let $\theta > 0$ and $\eta \geqslant 0$. Then for any $x \in [0, 1]$,*

$$\frac{1}{1 - x + \theta x} \leqslant \frac{1}{\min\{\theta, 1\}}, \tag{2.2}$$

and

$$\frac{\eta x}{1 - x + \theta x} \leqslant \frac{\eta}{\theta}. \tag{2.3}$$

Lemma 3. [9, Lemma 5]. *Let $A = (a_{ij}) \in R^{n\times n}$ and $a_{ii} > \sum\limits_{k=i+1}^{n} |a_{ik}| (\forall i \in N)$. Then for any $x_i \in [0, 1]$,*

$$\frac{1 - x_i + a_{ii} x_i}{1 - x_i + a_{ii} x_i - \sum\limits_{k=i+1}^{n} |a_{ik}| x_i} \leqslant \frac{a_{ii}}{a_{ii} - \sum\limits_{k=i+1}^{n} |a_{ik}|}. \tag{2.4}$$

Lemma 4. [7, Theorem 2.2]. *Let $P = (p_1, p_2, \cdots, p_n)^T e,\ e = (1, 1, \cdots, 1),\ p_1, p_2, \cdots, p_n \geqslant 0$, then*

$$\| (I + P)^{-1} \|_\infty \leqslant n - 1.$$

Theorem 4. *Let $A = (a_{ij}) \in R^{n\times n}$ be an B-matrix with the form $A = B^+ + C$, where $B^+ = (b_{ij})$ is expressed as* (2)*. Then*

$$\max_{d \in [0,1]^n} \|(I - D + DA)^{-1}\|_\infty \leqslant$$

$$\max\left\{ \frac{n-1}{\min\{t_1, 1\}} + \sum_{i=2}^{n} \left[\frac{n-1}{\min\{t_i, 1\}} \prod_{j=1}^{i-1} \frac{\sum\limits_{k=j+1} |b_{jk}|}{\widetilde{t_j}} \right], \right.$$

$$\left. \frac{(n-1) q^{(1)}(B^+)}{\min\{t_1, 1\}} + \sum_{i=2}^{n} \left[\frac{(n-1) q^{(i)}(B^+)}{\min\{t_i, 1\}} \prod_{j=1}^{i-1} \frac{b_{jj}}{\widetilde{t_j}} \right] \right\}, \tag{2.5}$$

where $t_i = b_{ii} - \sum\limits_{j=i+1}^{n} |b_{ij}| q_{ji}^{(i)}(B^+),\ \ \widetilde{t_j} = b_{jj} - \sum\limits_{k=j+1}^{n} |b_{jk}| q^{(j)}(B^+).$

Proof. Let $A_D = I - D + DA$. Then

$$A_D = I - D + DA = I - D + D(B^+ + C) = B_D^+ + C_D,$$

where $B_D^+ = I - D + DB^+$ and $C_D = DC$. By Theorem 2.2 in [7], we can get that B_D^+ is an SDD M-matrix with positive diagonal elements, by Lemma 4, then

$$\| A_D^{-1} \|_\infty \leqslant \| (I + (B_D^+)^{-1} C_D)^{-1} \|_\infty \| (B_D^+)^{-1} \|_\infty \leqslant (n-1) \| (B_D^+)^{-1} \|_\infty . \tag{2.6}$$

Next, we will give an upper bound for $\| (B_D^+)^{-1} \|_\infty$. By Lemma 1, we have

$$\begin{aligned} &\| (B_D^+)^{-1} \|_\infty \leqslant \\ &\max\left\{ \frac{1}{1 - d_1 + b_{11}d_1 - \sum\limits_{j=2}^{n} |b_{1j}| d_1 q_{j1}^{(1)}(B_D^+)} + \right. \\ &\sum_{i=2}^{n} \left[\frac{1}{1 - d_i + b_{ii}d_i - \sum\limits_{j=i+1}^{n} |b_{ij}| d_i q_{ji}^{(i)}(B_D^+)} \prod_{j=1}^{i-1} \frac{u_j(B_D^+)}{1 - u_j(B_D^+) q^{(j)}(B_D^+)} \right], \\ &\frac{q^{(1)}(B_D^+)}{1 - d_1 + b_{11}d_1 - \sum\limits_{j=2}^{n} |b_{1j}| d_1 q_{j1}^{(1)}(B_D^+)} + \\ &\left. \sum_{i=2}^{n} \left[\frac{q^{(i)}(B_D^+)}{1 - d_i + b_{ii}d_i - \sum\limits_{j=i+1}^{n} |b_{ij}| d_i q_{ji}^{(i)}(B_D^+)} \prod_{j=1}^{i-1} \frac{1}{1 - u_j(B_D^+) q^{(j)}(B_D^+)} \right] \right\}. \end{aligned} \tag{2.7}$$

By Lemma 2, we can easily deduce the following results for any $i, j, k, m, n \in N$,

$$r_{ji}^{(1)}(B_D^+) = \frac{|b_{ji}| d_j}{1 - d_j + b_{jj}d_j - \sum\limits_{k \neq j,i}^{n} |b_{jk}| d_j} \leqslant \frac{|b_{ji}|}{b_{jj} - \sum\limits_{k \neq j,i}^{n} |b_{jk}|} = r_{ji}^{(1)}(B^+), \tag{2.8}$$

$$r_{ji}^{(2)}(B_D^+) = \frac{|b_{ji}| d_j}{(1 - d_j + b_{jj}d_j - \sum\limits_{k \neq j,i} |b_{jk}| d_j) \max\limits_{j \neq i} \left\{ \frac{|b_{ji}| d_j}{1 - d_j + b_{jj}d_j - \sum\limits_{k \neq j,i} |b_{jk}| d_j} \right\}} \leqslant 1, \tag{2.9}$$

$$r_{ji}^{(2)}(B^+) = \frac{|b_{ji}|}{(|b_{jj}| - \sum\limits_{k \neq j,i} |b_{jk}|) \max\limits_{j \neq i} \left\{ \frac{|b_{ji}|}{|b_{jj}| - \sum\limits_{k \neq j,i} |b_{jk}|} \right\}} \leqslant 1. \tag{2.10}$$

By (13)–(15), we have

$$r_i^{(1)}(B_D^+) = \max_{j \neq i} \{ r_{ji}^{(1)}(B_D^+) \} \leqslant \max_{j \neq i} \{ r_{ji}^{(1)}(B^+) \} = r_i^{(1)}(B^+), \tag{2.11}$$

$$r_i^{(2)}(B_D^+) = \max_{j \neq i} \{ r_{ji}^{(2)}(B_D^+) \} = 1, \quad r_i^{(2)}(B^+) = \max_{j \neq i} \{ r_{ji}^{(2)}(B^+) \} = 1. \tag{2.12}$$

By analogy, we can get that for any $m \geqslant 2$,

$$r_i^{(m)} = \max_{j \neq i} \{ r_{ji}^{(m)} \} = 1, \tag{2.13}$$

therefore

$$r_i^{(1)}(B_D^+)r_i^{(2)}(B_D^+)\cdots r_i^{(m)}(B_D^+) \leqslant r_i^{(1)}(B^+)r_i^{(2)}(B^+)\cdots r_i^{(m)}(B^+). \tag{2.14}$$

Because of

$$d_{ki}^{(m)}(B_D^+) = \frac{|b_{ki}|d_k + \sum\limits_{j\neq k,i} |b_{kj}|d_k r_i^{(1)}(B_D^+)r_i^{(2)}(B_D^+)\cdots r_i^{(m)}(B_D^+)}{(1-d_k+b_{kk}d_k)r_i^{(1)}(B_D^+)r_i^{(2)}(B_D^+)\cdots r_i^{(m)}(B_D^+)}, \tag{2.15}$$

and

$$d_{ki}^{(m)}(B^+) = \frac{|b_{ki}| + \sum\limits_{j\neq k,i} |b_{kj}| r_i^{(1)}(B^+)r_i^{(2)}(B^+)\cdots r_i^{(m)}(B^+)}{|b_{kk}|r_i^{(1)}(B^+)r_i^{(2)}(B^+)\cdots r_i^{(m)}(B^+)}. \tag{2.16}$$

By (20), (21) and Lemma 2, we get

$$\begin{aligned} r_i^{(1)}(B_D^+)r_i^{(2)}(B_D^+)\cdots r_i^{(m)}(B_D^+)d_{ki}^{(m)}(B_D^+) &= \frac{|b_{ki}|d_k + \sum\limits_{j\neq k,i} |b_{kj}|d_k r_i^{(1)}(B_D^+)r_i^{(2)}(B_D^+)\cdots r_i^{(m)}(B_D^+)}{(1-d_k+b_{kk}d_k)} \\ &\leqslant \frac{|b_{ki}| + \sum\limits_{j\neq k,i} |b_{kj}| r_i^{(1)}(B_D^+)r_i^{(2)}(B_D^+)\cdots r_i^{(m)}(B_D^+)}{|b_{kk}|} \\ &\leqslant \frac{|b_{ki}| + \sum\limits_{j\neq k,i} |b_{kj}| r_i^{(1)}(B^+)r_i^{(2)}(B^+)\cdots r_i^{(m)}(B^+)}{|b_{kk}|} \\ &= r_i^{(1)}(B^+)r_i^{(2)}(B^+)\cdots r_i^{(m)}(B^+)d_{ki}^{(m)}(B^+). \end{aligned} \tag{2.17}$$

By (22) and Lemma 2, we have

$$\begin{aligned} q_{ji}^{(m)}(B_D^+) &= \frac{|b_{ji}|d_j + \sum\limits_{k\neq j,i} |b_{jk}|d_j r_i^{(1)}(B_D^+)r_i^{(2)}(B_D^+)\cdots r_i^{(m)}(B_D^+)d_{ki}^{(m)}(B_D^+)}{1-d_j+b_{jj}d_j} \\ &\leqslant \frac{|b_{ji}| + \sum\limits_{k\neq j,i} |b_{jk}| r_i^{(1)}(B^+)r_i^{(2)}(B^+)\cdots r_i^{(m)}(B^+)d_{ki}^{(m)}(B^+)}{|b_{jj}|} \\ &= q_{ji}^{(m)}(B^+). \end{aligned} \tag{2.18}$$

By (23), we have

$$q_i^{(m)}(B_D^+) = \max_{j\neq i}\{q_{ji}^{(m)}(B_D^+)\} \leqslant \max_{j\neq i}\{q_{ji}^{(m)}(B^+)\} = q_i^{(m)}(B^+),\ m = 1,2,\cdots,n,$$

$$q^{(m)}(B_D^+) = \max_{m\leqslant i\leqslant n}\{q_i^{(m)}(B_D^+)\} \leqslant \max_{m\leqslant i\leqslant n}\{q_i^{(m)}(B^+)\} = q^{(m)}(B^+),\ m = 1,2,\cdots,n.$$

By Lemmas 2 and 3, we obtain

$$\begin{aligned} \frac{1}{1-d_i+b_{ii}d_i - \sum\limits_{j=i+1}^{n} |b_{ij}|d_i q_{ji}^{(i)}(B_D^+)} &\leqslant \frac{1}{\min\{b_{ii} - \sum\limits_{j=i+1}^{n} |b_{ij}| q_{ji}^{(i)}(B^+), 1\}} \\ &\leqslant \frac{1}{\min\{t_i, 1\}}, \end{aligned} \tag{2.19}$$

$$\frac{u_i(B_D^+)}{1-u_i(B_D^+)q^{(i)}(B_D^+)} = \frac{\sum\limits_{j=i+1}^{n}|b_{ij}|d_i}{1-d_i+b_{ii}d_i-\sum\limits_{j=i+1}^{n}|b_{ij}|d_iq^{(i)}(B_D^+)}$$
$$\leqslant \frac{\sum\limits_{j=i+1}^{n}|b_{ij}|}{b_{ii}-\sum\limits_{j=i+1}^{n}|b_{ij}|q^{(i)}(B^+)}$$
$$= \frac{\sum\limits_{j=i+1}^{n}|b_{ij}|}{\widetilde{t_i}}, \tag{2.20}$$

and

$$\frac{1}{1-u_i(B_D^+)q^{(i)}(B_D^+)} = \frac{1-d_i+b_{ii}d_i}{1-d_i+b_{ii}d_i-\sum\limits_{j=i+1}^{n}|b_{ij}|d_iq^{(i)}(B_D^+)}$$
$$\leqslant \frac{b_{ii}}{b_{ii}-\sum\limits_{j=i+1}^{n}|b_{ij}|q^{(i)}(B^+)}$$
$$= \frac{b_{ii}}{\widetilde{t_i}}. \tag{2.21}$$

By (24)–(26), we have

$$\|(B_D^+)^{-1}\|_\infty \leqslant \max\left\{\frac{1}{\min\{t_1,1\}}+\sum_{i=2}^{n}\left[\frac{1}{\min\{t_i,1\}}\prod_{j=1}^{i-1}\frac{\sum\limits_{k=j+1}^{n}|b_{jk}|}{\widetilde{t_j}}\right],\right.$$
$$\left.\frac{q^{(1)}(B^+)}{\min\{t_1,1\}}+\sum_{i=2}^{n}\left[\frac{q^{(i)}(B^+)}{\min\{t_i,1\}}\prod_{j=1}^{i-1}\frac{b_{jj}}{\widetilde{t_j}}\right]\right\}. \tag{2.22}$$

Therefore, (10) holds from (11) and (27).

Theorem 5. *Let $A=(a_{ij})\in R^{n\times n}$ be an B-matrix with the form $A=B^++C$, where $B^+=(b_{ij})$ is expressed as* (2)*. Then*

$$\max\left\{\frac{n-1}{\min\{t_1,1\}}+\sum_{i=2}^{n}\left[\frac{n-1}{\min\{t_i,1\}}\prod_{j=1}^{i-1}\frac{\sum\limits_{k=j+1}|b_{jk}|}{\widetilde{t_j}}\right],\right.$$
$$\left.\frac{(n-1)q^{(1)}(B^+)}{\min\{t_1,1\}}+\sum_{i=2}^{n}\left[\frac{(n-1)q^{(i)}(B^+)}{\min\{t_i,1\}}\prod_{j=1}^{i-1}\frac{b_{jj}}{\widetilde{t_j}}\right]\right\}$$
$$\leqslant \sum_{i=1}^{n}\frac{n-1}{\min\{\bar\beta_i,\ 1\}}\prod_{j=1}^{i-1}\left(1+\frac{1}{\bar\beta_j}\sum_{k=j+1}^{n}|b_{jk}|\right)$$

$$\leqslant \sum_{i=1}^{n}\left(\frac{n-1}{\min\{\widetilde{\beta}_i,\ 1\}}\prod_{j=1}^{i-1}\frac{b_{jj}}{\widetilde{\beta}_j}\right),\tag{2.23}$$

where $\bar{\beta}_i$ *and* $\widetilde{\beta}_i$ *are defined as* (4)*,* (5)*.*

Proof. Since $B^+ = (b_{ij})$ is an SDD matrix with positive diagonal elements, by Theorem 6 in [10], we have

$$q_{ji}^{(i)}(B^+) \leqslant q^{(i)}(B^+) \leqslant l_i(B^+) < 1.\tag{2.24}$$

Notice that

$$\widetilde{\beta}_i = b_{ii} - \sum_{j=i+1}^{n}|b_{ij}|,\ \bar{\beta}_i = b_{ii} - \sum_{j=i+1}^{n}|b_{ij}|l_i(B^+),$$

$$t_i = b_{ii} - \sum_{j=i+1}^{n}|b_{ij}|q_{ji}^{(i)}(B^+),\ \widetilde{t}_j = b_{jj} - \sum_{k=j+1}^{n}|b_{jk}|q^{(j)}(B^+),$$

then

$$\frac{q^{(i)}(B^+)}{\min\{t_i, 1\}} \leqslant \frac{1}{\min\{t_i, 1\}} \leqslant \frac{1}{\min\{\bar{\beta}_i, 1\}} \leqslant \frac{1}{\min\{\widetilde{\beta}_i, 1\}},\tag{2.25}$$

$$\frac{\sum\limits_{k=j+1}|b_{jk}|}{\widetilde{t}_j} \leqslant \frac{\sum\limits_{k=j+1}|b_{jk}|}{\bar{\beta}_j} < 1 + \frac{1}{\bar{\beta}_j}\sum_{k=j+1}^{n}|b_{jk}|,\tag{2.26}$$

$$\begin{aligned}\frac{b_{jj}}{\widetilde{t}_j} &= \frac{b_{jj} - \sum\limits_{k=j+1}^{n}|b_{jk}|q^{(j)}(B^+) + \sum\limits_{k=j+1}^{n}|b_{jk}|q^{(j)}(B^+)}{\widetilde{t}_j}\\ &= 1 + \frac{\sum\limits_{k=j+1}^{n}|b_{jk}|q^{(j)}(B^+)}{\widetilde{t}_j}\\ &\leqslant 1 + \frac{1}{\bar{\beta}_j}\sum_{k=j+1}^{n}|b_{jk}|,\end{aligned}\tag{2.27}$$

and

$$1 + \frac{1}{\bar{\beta}_j}\sum_{k=j+1}^{n}|b_{jk}| \leqslant 1 + \frac{1}{\widetilde{\beta}_j}\sum_{k=j+1}^{n}|b_{jk}| = \frac{1}{\widetilde{\beta}_j}\left(\widetilde{\beta}_j + \sum_{k=j+1}^{n}|b_{jk}|\right) = \frac{b_{jj}}{\widetilde{\beta}_j}.\tag{2.28}$$

By (30)–(33), we can obtain (28).

Theoretical analysis shows that the new upper bound (10) of the $\max\limits_{d\in[0,1]^n}\|(I-D+DA)^{-1}\|_\infty$ when the matrix A is an B-matrix is better than bounds (4) and (5) which all be provided by Li in [8,9]. Next, we further illustrate the effectiveness of the result through numerical example.

Example 1. Consider the family of B-matrices in [8]:

$$A_k = \begin{pmatrix} 1.5 & 0.5 & 0.4 & 0.5 \\ -0.1 & 1.7 & 0.7 & 0.6 \\ 0.8 & -0.1\frac{k}{k+1} & 1.8 & 0.7 \\ 0 & 0.7 & 0.8 & 1.8 \end{pmatrix},$$

where $k \geqslant 1$. Then $A_k = B_k^+ + C_k$, where

$$B_k^+ = \begin{pmatrix} 1 & 0 & -0.1 & 0 \\ -0.8 & 1 & 0 & -0.1 \\ 0 & -0.1\frac{k}{k+1} - 0.8 & 1 & -0.1 \\ -0.8 & -0.1 & 0 & 1 \end{pmatrix}.$$

By Theorem 1, we have

$$\max_{d\in[0,1]^4} \| (I - D + DA_k)^{-1} \|_\infty \leqslant \frac{4-1}{\min\{\beta,\ 1\}} = 30(k+1),$$

it is obvious to see that $30(k+1) \to +\infty$, when $k \to +\infty$. bound (3) in Theorem 1 can be arbitrarily large.

By Theorem 2, we have

$$\max_{d\in[0,1]^4} \| (I - D + DA_k)^{-1} \|_\infty \leqslant \frac{2.97(90k+91)(190k+192) + 6.24(100k+101)^2}{0.99(90k+91)^2},$$

when $k = 1$,

$$\max_{d\in[0,1]^4} \| (I - D + DA_k)^{-1} \|_\infty \leqslant 14.1044,$$

when $k = 2$,

$$\max_{d\in[0,1]^4} \| (I - D + DA_k)^{-1} \|_\infty \leqslant 14.1079.$$

By Theorem 3, for each $k \in N = \{1, 2, \cdots, n\}$, we have

$$\max_{d\in[0,1]^4} \| (I - D + DA_k)^{-1} \|_\infty \leqslant 15.2675.$$

By Theorem 4, we have

$$\max_{d\in[0,1]^4} \| (I - D + DA_k)^{-1} \|_\infty \leqslant \frac{26.73k + 23.79}{8.2k + 8.28} + \frac{27(k+1)(89.1k+79.3)}{(8.838k+8.846)(81.09k+82.07)} + \frac{243(k+1)(9k+8)(9k+10)}{0.9911(k+2)(81.09k+82.07)^2} + \frac{243(k+1)^2(9k+8)(9k+10)}{(81.09k+82.07)^2(0.919k^2+2.838k+1.92)},$$

when $k = 1$,

$$\max_{d\in[0,1]^4} \| (I - D + DA_k)^{-1} \|_\infty \leqslant 10.2779,$$

when $k = 2$,

$$\max_{d\in[0,1]^4} \| (I - D + DA_k)^{-1} \|_\infty \leqslant 10.9614.$$

The numerical example above further illustrate that the bound (10) in Theorem 4 is better than those bounds in Theorems 1–3.

3. Conclusions

In this paper, we present a new upper bound for $\max_{d\in[0,1]^n} \|(I - D + DA)^{-1}\|_\infty$ when A is an B-matrix, theoretical analysis and numerical example illustrate that the new estimation improves the bounds obtained in [7–9].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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