Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(10): 23871–23888.
DOI: 10.3934/math.20231217
Received: 28 April 2023
Revised: 14 June 2023
Accepted: 04 July 2023
Published: 07 August 2023

Research article

On the Perov's type (β, F)-contraction principle and an application to delay integro-differential problem

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Abstract: We present Perov's type (β, F)-contraction principle and examine the fixed points of the self-operators satisfying Perov's type (β, F)-contraction principle in the context of vector-valued b-metrics. A specific instance of the (β, F)-contraction principle is the F-contraction principle. We generalize a number of recent findings that are already in the literature and provide an example to illustrate the hypothesis of the main theorem. We apply the obtained fixed point theorem to show the existence of the solution to the delay integro-differential problem.

Keywords: fixed point; Perov's type (β, F)-contraction operator; β-complete vector-valued b-metric space
Mathematics Subject Classification: 45J05, 47H10, 54H25

1. Introduction

The Banach contraction principle [1] is considered a landmark fixed point theorem in the metric fixed point theory. The main component of the metric fixed-point theory is the exploration of new contraction principles to give fresh and helpful fixed-point theorems. Rakotch [2] was the first mathematician who involved a function instead of a Lipschitz constant in the Banach contraction principle. Then, Boyd and Wong [3] generalized the Rakotch contraction principle. In an effort to give a new contraction principle, Kannan [4] introduced a contraction principle that characterizes the metric completeness, and gave a new direction in metric fixed point theory that led many mathematicians to introduce various contraction principles. Among the classical contraction principles, the most famous are: Meir and Keeler contraction principle [5], Chatterjea contraction principle [6], Riech contraction principle [7], Hardy and Rogers contraction principle [8], Círíc contraction principle [9] and Caristi contraction principle [10].

Recently, Wardowski [11] generalized Banach contraction principle by using an auxiliary nonlinear function $F : (0,\infty) \to (-\infty,\infty)$ that satisfied three conditions. In the literature, this new contraction principle is known as F-contraction principle. This idea proved another milestone in metric fixed point theory. The F-contraction principle has been revisited and generalized in many abstract spaces (see [12–27] and references therein).

On the other hand, metric generalization also has a significant impact on metric fixed point theory. Many mathematicians have contributed in this direction, producing many generalizations of a metric space (see [28]). One of these generalizations was done by Perov [29]. By extending the co-domain of the metric function from $\mathbb{R}$ to $\mathbb{R}^n$, Perov [29] gave a vector version of the metric and hence produced another generalization of the Banach contraction principle. By following the Perov, Boriceanu [30] introduced a vector-valued b-metric and hence extended Perov [29] fixed point result to vector-valued b-metric spaces.

In 2020, Altun et al. [31] presented a vector version of F-contraction principle and obtained an extension of Wardowski [11] fixed point theorem in the vector-valued metric spaces as follow:

Theorem 1.1. *[31] Every self-mapping T on a complete vector-valued metric space (X, d) that satisfies the following inequality:*

$$\mathbf{d}(T(q), T(h)) > \mathbf{0} \Rightarrow \mathbf{I} \oplus \mathbf{F}(\mathbf{d}(T(q), T(h))) \le \mathbf{F}(\mathbf{d}(q, h))\ \forall\, q, h \in X,$$

admits a unique fixed point, provided $\mathbf{F}$ satisfies $(AF_1) - (AF_3)$ and $\mathbf{I} = (\tau_i)_{i=1}^m \ge \mathbf{0}$.

Where the operator $\mathbf{F} : \mathbb{P}^m \to \mathbb{R}^m$ satisfies the following conditions:

(AF_1) $\forall\, Q, W \in \mathbb{P}^m$ with $Q \le W$, we have $\mathbf{F}(Q) \le \mathbf{F}(W)$;

(AF_2) $\forall\, \{\mathbf{v}_n : n \in \mathbb{N}\} \subset \mathbb{P}^m$, we have

$$\lim_{n\to\infty} v_n^{(i)} = 0 \text{ if and only if } \lim_{n\to\infty} u_n^{(i)} = -\infty, \text{ for each } i;$$

(AF_3) $\exists\, \kappa \in (0, 1)$ satisfying $\lim_{v_i \to 0^+} (v_i)^\kappa u_i = 0$.

In this paper, we investigate the possible conditions on $\mathbf{F}$ and T for which the mapping T admits a unique fixed point in a vector-valued b-metric space and in this way, we generalize the Theorem 1.1. Moreover, we introduce the $(\beta, \mathbf{F})$-contraction principle and call it Perov's type $(\beta, \mathbf{F})$-contraction principle. We obtain fixed point theorems on Perov's type $(\beta, \mathbf{F})$-contraction in a vector-valued b-metric space. Illustrative example and an application of the obtained fixed point theorem are given.

2. Preliminaries and related results

In this section we present a summary of prerequisites and notations to be considered in the sequel. Let $\mathbb{R}^m = \{\mathbf{v} = (x_i)_{i=1}^m = (x_1, x_2, \cdots, x_m) \mid \forall i\ \ x_i \in \mathbb{R}\}$ represents all matrices of order $m \times 1$ (will be called vectors), then $(\mathbb{R}^m, \oplus, \odot)$ is a linear space with respect $\oplus$ and $\odot$ defined by

$$\mathbf{v} \oplus \mathbf{w} = (x_i + y_i)_{i=1}^m \text{ for all } \mathbf{v} = (x_i)_{i=1}^m \text{ and } \mathbf{w} = (y_i)_{i=1}^m \in \mathbb{R}^m.$$

$$k \odot \mathbf{v} = (k \cdot x_i)_{i=1}^m \text{ for all } \mathbf{v} = (x_i)_{i=1}^m \in \mathbb{R}^m \text{ and } k \in \mathbb{R}.$$

Note that + and $\cdot$ represent usual addition and multiplication of scalars. By using above operations, we can define difference of vectors as: $\mathbf{v} \ominus \mathbf{w} = \mathbf{v} \oplus (-1) \odot \mathbf{w}$. Define the relations $\preceq$ and $\prec$ on $\mathbb{R}^m$ by

$$\mathbf{v} \preceq \mathbf{w} \Leftrightarrow x_i \le y_i \text{ and } \mathbf{v} \prec \mathbf{w} \Leftrightarrow x_i < y_i; \forall i. \tag{2.1}$$

The relation $\preceq$ defines a partial-order on $\mathbb{R}^m$. Let $\mathbb{P}^m$ denotes the set of positive definite vectors, that is, if $\mathbf{v} = (x_i)_{i=1}^m \succ \mathbf{0}$ (zero vector of order $m \times 1$) and $\mathbf{v} = (x_i)_{i=1}^m \in \mathbb{R}^m$ then $\mathbf{v} = (x_i)_{i=1}^m \in \mathbb{P}^m$. Also let $\mathbb{R}_0^m = \{\mathbf{v} = (x_i)_{i=1}^m = (x_1, x_2, \cdots, x_m) \mid \forall i \ \ x_i \in [0, \infty)\}$. The two vectors are considered equal if their corresponding coordinates are equal.

Definition 2.1. *[32] 1) Let $V = [v_{ij}]$ be an $m \times m$ complex matrix having eigenvalues λ_i, $1 \le i \le n$. Then, the spectral radius $\rho(V)$ of matrix V is defined by $\rho(V) = \max_{1 \le i \le m} |\lambda_i|$.*

2) The matrix V converges to zero, if the sequence $\{V^n; n \in \mathbb{N}\}$ converges to zero matrix O.

Theorem 2.2. [32] Let $\mathbf{V}$ be any complex matrix of order $m \times m$, then $\mathbf{V}$ is convergent if and only if $\rho(V) < 1$.

Perov [29] applied Theorem 2.2 to obtain the following result in the vector-valued metric spaces. It states that:

Theorem 2.3. *[29] Every self-mapping J defined on a complete vector-valued metric space $(X, \mathbf{d})$ satisfying the following inequality:*

$$\mathbf{d}(J(g), J(h)) \preceq A\mathbf{d}(g, h) \, \forall \, (g, h) \in X \times X,$$

admits a unique fixed point provided $\rho(A) < 1$; A is a positive square matrix of order m.

By a vector-valued metric, we mean a mapping $d : X \times X \to \mathbb{R}^m$ obeying all the axioms of the metric. The object $d(x, y)$ is an m-tuple. Let

$$\mathbf{v} = (v_i)_{i=1}^m = (v_1, v_2, v_3, \cdots, v_m) \in \mathbb{P}^m,$$

$$\mathbf{v}_n = \left(v_n^{(i)}\right)_{i=1}^m = \left(v_n^{(1)}, v_n^{(2)}, v_n^{(3)}, \cdots, v_n^{(m)}\right) \in \mathbb{P}^m,$$

$$\mathbf{F}(\mathbf{v}) = (u_i)_{i=1}^m = (u_1, u_2, u_3, \cdots, u_m) \in \mathbb{R}^m \text{ and}$$

$$\mathbf{F}(\mathbf{v_n}) = \left(u_n^{(i)}\right)_{i=1}^m = \left(u_n^{(1)}, u_n^{(2)}, u_n^{(3)}, \cdots, u_n^{(m)}\right) \in \mathbb{R}^m,$$

$$(v)_1^m = (v, v, v, \cdots, v) \in \mathbb{R}^m.$$

We organize this paper as follows:

Section 2 contains definition and related properties of the vector-valued b-metric space. Section 3 consists of necessary lemmas and Perov's type (β, F)-contraction principle, related fixed point theorem and an example explaining hypothesis of obtained result. Section 4 consists of an application of the main theorem.

3. The vector-valued b-metric space

In light of the definitions of b-metric and vector-valued metric given by Czerwik [33] and Perov [31] respectively, we proceed with the following definition.

Definition 3.1. (vector-valued b-metric) [30] Let G be a non-empty set. The operator $\mathbf{A} : G \times G \to \mathbb{R}_0^m$ satisfying the axioms $(\mathbf{A}_1) - (\mathbf{A}_3)$ given below is known as a vector-valued b-metric. For all $q, t, g \in G$, we have

$(\mathbf{A}_1)$ $q = t$ if and only if $\mathbf{A}(q, t) = \mathbf{0}$.
$(\mathbf{A}_2)$ $\mathbf{A}(q, t) = \mathbf{A}(t, q)$.
$(\mathbf{A}_3)$ $\mathbf{A}(q, g) \preceq s \odot [\mathbf{A}(q, t) \oplus \mathbf{A}(t, g)]\,; s \geq 1$.

The pair $(G, \mathbf{A}, s)$ represents a vector-valued b-metric-space.

For $s = 1$, every vector-valued b-metric space is a vector-valued metric-space, but this is not true when $s > 1$. Thus, it can be remarked that every vector-valued metric-space is a vector-valued b-metric-space but not conversely.

Example 3.2. Let $G = \mathbb{R}$ and the operator $\mathbf{A} : G \times G \to \mathbb{R}_0^m$ is defined by

$$\mathbf{A}(l, q) = \left(|H|^2, |H|^3, \cdots, |H|^{m+1}\right) \forall l, q \in G,$$

where $H = |l - q|$. Then $(G, \mathbf{A}, s = 2^m)$ is a vector-valued b-metric space. Note that it is not vector-valued-metric space.

Example 3.3. Let G be a non-empty set. Let $d_i : G \times G \to [0, \infty)$ be a b-metric for each i with respective constant $s_i \geq 1$ $(1 \leq i \leq m)$ for each positive integer i. The mapping $\mathbf{A} : G \times G \to \mathbb{R}_0^m$ defined by

$$\mathbf{A}(q, t) = (d_1(q, t), d_2(q, t), \cdots, d_m(q, t)) \text{ for all } q, t \in G$$

defines a vector-valued b-metric on G.

The axioms $(\mathbf{A}_1)$ and $(\mathbf{A}_2)$ hold trivially. For $(\mathbf{A}_3)$, for all $l, q, t \in G$, consider

$$\begin{aligned}
\mathbf{A}(l, t) &= (d_1(l, t), d_2(l, t), \cdots, d_m(l, t)) \\
&\preceq (s_1(d_1(l, q) + d_1(q, t)), s_2(d_2(l, q) + d_2(q, t)), \cdots, s_m(d_m(l, q) + d_m(q, t))) \\
&\preceq s \odot (d_1(l, q) + d_1(q, t), d_2(l, q) + d_2(q, t), \cdots, d_m(l, q) + d_m(q, t)) \\
&= s \odot ((d_1(l, q), d_2(l, q), \cdots, d_m(l, q)) \oplus (d_1(q, t), d_2(q, t), \cdots, d_m(q, t))) \\
&= s \odot (\mathbf{A}(l, q) \oplus \mathbf{A}(q, t))\,; s = \max\{s_1, s_2, \cdots, s_m\}.
\end{aligned}$$

In general, the vector-valued b-metric is discontinuous, so, it requires an auxiliary convergence result to establish fixed point theorems in the vector-valued b-metric spaces. For this purpose we give the following lemma (Lemma 3.4).

Lemma 3.4. *Let $(G, \mathbf{A}, s)$ be a vector-valued b-metric space. If $l^*, g^* \in G$ and $\{l_n\}_{n \in \mathbb{N}}$ is such that $\lim_{n\to\infty} l_n = l^*$, then*

$$\frac{1}{s} \odot \mathbf{A}(l^*, g^*) \preceq \liminf_{n\to\infty} \mathbf{A}(l_n, g^*) \preceq \limsup_{n\to\infty} \mathbf{A}(l_n, g^*) \preceq s \odot \mathbf{A}(l^*, g^*).$$

Proof. By ($\mathbf{A}_3$), we have

$$\begin{aligned} \frac{1}{s}\odot\mathbf{A}(l^*,g^*) &\preceq \mathbf{A}(l^*,l_n)\oplus\mathbf{A}(l_n,g^*) \\ \Rightarrow \frac{1}{s}\odot\mathbf{A}(l^*,g^*)\ominus\mathbf{A}(l_n,l^*) &\preceq \mathbf{A}(l_n,g^*). \end{aligned}$$

Taking lim inf, we have

$$\frac{1}{s}\odot\mathbf{A}(l^*,g^*)\preceq\lim_{n\to\infty}\inf\mathbf{A}(l_n,g^*). \tag{3.1}$$

Again by ($\mathbf{A}_3$), we get

$$\mathbf{A}(l_n,g^*)\preceq s\odot(\mathbf{A}(l_n,l^*)\oplus\mathbf{A}(l^*,g^*)).$$

This implies

$$\lim_{n\to\infty}\sup\mathbf{A}(l_n,g^*)\preceq s\odot\mathbf{A}(l^*,g^*). \tag{3.2}$$

But also, we know that

$$\lim_{n\to\infty}\inf\mathbf{A}(l_n,g^*)\preceq\lim_{n\to\infty}\sup\mathbf{A}(l_n,g^*). \tag{3.3}$$

Combining (3.1)–(3.3), we get required result. □

Apart from Lemma 3.4, to fulfill the objective of this paper, the following compatibility condition is required:

(AF_4): for every positive term sequence $\mathbf{v}_n=\left(x_n^{(i)}\right)_{i=1}^m$, $\exists\,\mathbf{I}=(\tau_i)_{i=1}^m\succeq\mathbf{0}$ satisfying

$$\mathbf{I}\oplus\mathbf{F}(s\mathbf{v}_n)\preceq\mathbf{F}(\mathbf{v}_{n-1}) \text{ implies } \mathbf{I}\oplus\mathbf{F}(s^n\mathbf{v}_n)\preceq\mathbf{F}(s^{n-1}\mathbf{v}_{n-1}).$$

Our findings rely mostly on the class of vector-valued nonlinear functions satisfying (AF_1), (AF_3) and (AF_4) denoted by Π_s^b.

Remark 3.5. The collection of vector-valued nonlinear functions Π_s^b is non-empty.
Let $\mathbf{F}:\mathbb{P}^m\to\mathbb{R}^m$ be defined by $\mathbf{F}((x_i)_{i=1}^m)=(\ln(x_i+1))_{i=1}^m$ for all $\mathbf{v}\in\mathbb{P}^m$, then ($AF_1$) and ($AF_3$) are obvious.
We establish (AF_4):
Let $\mathbf{I}\oplus\mathbf{F}(s\mathbf{v}_n)\preceq\mathbf{F}(\mathbf{v}_{n-1})$, then for m-tuple $\mathbf{I}=\left(\ln(s^{n-1}),\ln(s^{n-1}),\cdots,\ln(s^{n-1})\right)=\left(\ln(s^{n-1})\right)_1^m$, we have

$$\begin{aligned} \left(\ln(s^{n-1})\right)_1^m &\oplus \mathbf{F}\left(s\left(x_n^{(i)}\right)_{i=1}^m\right)\preceq\mathbf{F}\left(\left(x_{n-1}^{(i)}\right)\right)_{i=1}^m \\ \left(\ln(s^{n-1})\right)_1^m &\oplus \left(\ln\left(sx_n^{(i)}+1\right)\right)_{i=1}^m\preceq\left(\ln\left(x_{n-1}^{(i)}+1\right)\right)_{i=1}^m \\ &\Rightarrow \left(\ln\left(s^nx_n^{(i)}+s^{n-1}\right)\right)_{i=1}^m\preceq\left(\ln\left(x_{n-1}^{(i)}+1\right)\right)_{i=1}^m \\ &\Rightarrow \ln\left(s^nx_n^{(i)}+s^{n-1}\right)\le\ln\left(x_{n-1}^{(i)}+1\right) \text{ for each } i \\ &\Rightarrow s^nx_n^{(i)}\le x_{n-1}^{(i)}+1-s^{n-1} \text{ for each } i. \end{aligned}$$

Now consider

$$\mathbf{I}\oplus\mathbf{F}(s^n\mathbf{v}_n) = \left(\ln(s^{n-1})\right)_1^m\oplus\left(\ln(s^nx_n^{(i)}+1)\right)_{i=1}^m$$

$$\begin{aligned}
&\preceq \left(\ln(s^{n-1})\right)_1^m \oplus \left(\ln(x_{n-1}^{(i)} + 1 - s^{n-1} + 1)\right)_{i=1}^m \\
&= \left(\ln(s^{n-1}x_{n-1}^{(i)} - s^{2n-2} + 2s^{n-1})\right)_{i=1}^m = \left(\ln(s^{n-1}x_{n-1}^{(i)} + s^{n-1}(2 - s^{n-1}))\right)_{i=1}^m \\
&\preceq \left(\ln(s^{n-1}x_{n-1}^{(i)} + 1)\right)_{i=1}^m = \mathbf{F}\left(\left(s^{n-1}x_{n-1}^{(i)}\right)_{i=1}^m\right) = \mathbf{F}\left(s^{n-1}\mathbf{v}_{n-1}\right).
\end{aligned}$$

Hence, $\mathbf{F} \in \Pi_s^b$.

Example 3.6. *Let* $\mathbf{F} : \mathbb{P}^m \to \mathbb{R}^m$ *be defined by*

(a) $\mathbf{F}(\mathbf{v}) = (\ln(x_i))_{i=1}^m$;
(b) $\mathbf{F}(\mathbf{v}) = (x_i + \ln(x_i))_{i=1}^m$;
(c) $\mathbf{F}(\mathbf{v}) = \ln(x_i^2 + x_i)_{i=1}^m$;
(d) $\mathbf{F}(\mathbf{v}) = \left(-\frac{1}{\sqrt{x_i}}\right)_{i=1}^m$;
(e) $\mathbf{F}(\mathbf{v}) = \left(x_i^a\right)_{i=1}^m$; $a > 0$;
(f) $\mathbf{F}(\mathbf{v}) = (\ln(x_i + 1))_{i=1}^m$.

Among these definitions, definition (a)–(d) satisfy (AF_1)–(AF_3) *and definition (e), (f) belong to the family* Π_s^b.

Define $\mathbf{F} : \mathbb{R}_+^2 \to \mathbb{R}^2$ by $\mathbf{F}((g_1, g_2)) = \left(g_1^t, \ln(g_2 + 1)\right)$; $t > 0$ then $\mathbf{F} \in \Pi_s^b$.

The following lemma explains the reasons to omit axiom (AF_2).

Lemma 3.7. *Let* $\mathbf{F}$ *satisfies* (AF_1) *and* $\{\mathbf{v}_n\}_{n\in\mathbb{N}} \subset \mathbb{P}^m$ *is a decreasing sequence satisfying* $\lim_{n\to\infty} u_n^{(i)} = -\infty$, *then* $\lim_{n\to\infty} v_n^{(i)} = 0$ *for each* $i \in \{1, 2, \cdots, m\}$.

Proof. We note that for each $i \in \{1, 2, \cdots, m\}$, $\{v_n^{(i)}\}_{n\in\mathbb{N}}$ is bounded below and decreasing sequence of real numbers, so it is convergent. Let $\lim_{n\to\infty} v_n^{(i)} = \zeta \geq 0$ for each i. Suppose on contrary $\zeta > 0$. Since, $v_n^{(i)} \geq \zeta$ for each i, therefore, $u_n^{(i)} \geq F(\zeta)$. Thus, $F(\zeta) \leq \lim_{n\to\infty} u_n^{(i)} = -\infty$, a contradiction. Hence, $\lim_{n\to\infty} v_n^{(i)} = 0$. □

4. Fixed points of $(\beta, \mathbf{F})$-contractions

Recently, Altun et al. [31] obtained an existence theorem involving vector-valued nonlinear function and explained it through various nontrivial examples. We will introduce and investigate the notion of $(\beta, \mathbf{F})$-contractions where the function $\mathbf{F}$ is taken from Π_s^b and β is defined below.

Definition 4.1. Let there exists $\mathbf{F} \in \Pi_s^b$ and $\mathbf{I} \succ \mathbf{0}$, the mapping $T : (G, \mathbf{A}, s) \to (G, \mathbf{A}, s)$ is said to be a $(s, \mathbf{F})$-contraction, if it satisfies the following inequality:

$$\mathbf{A}(T(l), T(q)) \succ \mathbf{0} \Rightarrow \ \mathbf{I} \oplus \mathbf{F}(s \odot \mathbf{A}(T(l), T(q)) \preceq \mathbf{F}(\mathbf{A}(l, q)), \text{ for all } l, q \in G. \tag{4.1}$$

Remark 4.2. Note that for $s = 1$, Definition 4.1 is identical to Perov's type F-contraction introduced by Altun et al. [31]. Thus, class of $(s, \mathbf{F})$-contractions (defined in Definition 4.1) is more wider as compared to that of Perov's type F-contraction introduced by Altun et al. [31]. Now we explain inequality (4.1) by the following example (Example 4.3).

Example 4.3. Let $G = \{l_n = 2^{\frac{n}{2}}n | n \in \mathbb{N}\}$. Define $\mathbf{A}\colon G \times G \to \mathbb{P}^m$ by $\mathbf{A}(l,q) = \left(|l-q|^2\right)_1^m$, then $(G, \mathbf{A}, s = 2)$ is a vector-valued b-metric space. Define the mapping $\phi : G \to G$ by

$$\phi(l) = \begin{cases} 2^{\frac{n-1}{2}}(n-1) & \text{if } l = l_n; \\ l_0 & \text{if } l = l_0. \end{cases}$$

Take $(1)_1^m = \mathbf{I} \succ \mathbf{0}$ and define $\mathbf{F} : \mathbb{P}^m \to \mathbb{R}^m$ by $\mathbf{F}\left((g_i)_{i=1}^m\right) = (g_i)_{i=1}^m$. Then for every $l, q \in G$ such that $\phi(l) \neq \phi(q)$, we have

$$\mathbf{F}\left(2 \odot \mathbf{A}(\phi(l), \phi(q))\right) \ominus \mathbf{F}\left(\mathbf{A}(l,q)\right) \preceq \ominus \mathbf{I}.$$

Indeed for $l = l_{n+k}$ and $q = l_n$, consider

$$\begin{aligned} 2 \odot \mathbf{A}(\phi(l_{n+k}), \phi(l_n)) &\ominus \mathbf{A}(l_{n+k}, l_n) \\ &= \left(\left(2^{\frac{n+k}{2}}(n+k-1) - 2^{\frac{n}{2}}(n-1)\right)^2 - \left(2^{\frac{n+k}{2}}(n+k) - 2^{\frac{n}{2}}(n)\right)^2\right)_1^m \\ &= \left(2^n\left(1 - 2^{\frac{k}{2}}\right)\left(2^{\frac{k}{2}}(2n+2k-1) - (2n-1)\right)\right)_1^m \\ &\preceq (-1)_1^m = \ominus (1)_1^m. \end{aligned}$$

Also we see that $\mathbf{F} \in \Pi_s^b$. Indeed, for $\mathbf{F}\left((g_i)_{i=1}^m\right) = (g_i)_{i=1}^m$, axioms (AF_1) and (AF_3) hold. For axiom (AF_4), we proceed as follow: let $\mathbf{I} \oplus \mathbf{F}\left(s \odot \left(g_n^{(i)}\right)_{i=1}^m\right) \preceq \mathbf{F}\left(\left(g_{n-1}^{(i)}\right)\right)_{i=1}^m$ that is $1 + \ell g_n^{(i)} \leq g_{n-1}^{(i)}$ for each $i \in \{1, 2, \cdots, m\}$. Now consider

$$\begin{aligned} \mathbf{I} \oplus \mathbf{F}\left(\ell^n \odot \left(g_n^{(i)}\right)_{i=1}^m\right) &= \mathbf{I} \oplus \ell^n \odot \left(g_n^{(i)}\right)_{i=1}^m \\ &= \mathbf{I} \oplus \ell^{n-1} \odot \left(\ell g_n^{(i)}\right)_{i=1}^m \preceq \mathbf{I} \oplus \ell^{n-1} \odot \left(g_{n-1}^{(i)} - 1\right)_{i=1}^m \\ &= 1 + \ell^{n-1} g_{n-1}^{(i)} - \ell^{n-1} = 1 - \ell^{n-1} + \ell^{n-1} g_{n-1}^{(i)} \text{ for each } i \\ &\preceq \left(\ell^{n-1} g_{n-1}^{(i)}\right)_{i=1}^m = \mathbf{F}\left(\ell^{n-1} \odot \left(g_{n-1}^{(i)}\right)_{i=1}^m\right). \end{aligned}$$

This shows that for $\mathbf{I} = (1)_1^m$, ϕ is an $\mathbf{F}$-contraction.

Remark 4.4. We observe that the function α_s (defined in [34]) is superficial because we can always have a function $\beta : G \times G \to [0, \infty)$ defined by $\beta(l,q) = \dfrac{\alpha_s(l,q)}{s^2}$ with following properties:

(1) (ϕ is β-admissible)

$$\beta(l,q) \geq 1 \text{ implies } \beta(\phi(l), \phi(q)) \geq 1 \text{ for all } l, q \in G,$$

(2) the α_s-completeness implies β-completeness and vice versa.

Definition 4.5. Let G be a non-empty set and $\beta : G \times G \to [0, \infty)$. The function $\phi : G \to G$ is said to be β-admissible if

$$\beta(l,q) \geq 1 \text{ implies } \beta(\phi(l), \phi(l)) \geq 1 \text{ for all } l, q \in G \text{ and}$$

triangular β-admissible if in addition β follows:

$$\beta(l,j) \geq 1, \quad \beta(j,q) \geq 1, \text{ imply } \beta(l,q) \geq 1.$$

Definition 4.6. Let $(G, \mathbf{A}, s)$ be a vector-valued b-metric space and let $\beta : G \times G \to [0, \infty)$. Let $l \in G$ and sequence $\{l_n\} \subseteq G$. A mapping $q : G \to G$ is a β-continuous at $l = l_0$, if whenever,

$$\lim_{n\to\infty} \mathbf{A}(l_n, l) = \mathbf{0} \text{ and } \beta(l_n, l_{n+1}) \geq 1 \text{ we have } \lim_{n\to\infty} \mathbf{A}(q(l_n), q(l)) = \mathbf{0}.$$

Example 4.7. Let $G = [0, \infty)$ and define $\mathbf{A} : G \times G \to \mathbb{R}_0^m$ by

$$\mathbf{A}(l, q) = \left(|H|^2, |H|^3, \cdots, |H|^{m+1}\right) \forall\, l, q \in G,$$

where $H = |l - q|$, and let $q : G \to G$ be defined by

$$q(l) = \begin{cases} \sin(\pi l) & \text{if } l \in [0, 1]; \\ \cos(\pi l) + 2 & \text{if } l \in (1, \infty), \end{cases} \quad \beta(l, q) = \begin{cases} l + q + 1 & \text{if } l, q \in [0, 1]; \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, q is not continuous at $l_0 = 1$, however, q is a β-continuous mapping at this point. Indeed, the assumption $\lim_{n\to\infty} \mathbf{A}(l_n, l_0) = \mathbf{0}$ leads us to choose $l_n = 1 - \frac{1}{n} \subseteq [0, 1]$ and $\beta(l_n, l_{n+1}) \geq 1$ directs to choose $[0, 1]$ as domain of mapping q. Thus,

$$\lim_{n\to\infty} |q(l_n) - q(l)|^i = \lim_{n\to\infty} \left(\sin\left(\pi\left(1 - \frac{1}{n}\right)\right)\right)^i = 0 \text{ for each } i; \quad 2 \leq i \leq m + 1.$$

Hence $\lim_{n\to\infty} \mathbf{A}(q(l_n), q(l)) = \mathbf{0}$.

Definition 4.8. If an arbitrary Cauchy sequence $\{l_n\} \subseteq G$ satisfying $\beta(l_n, l_{n+1}) \geq 1$ converges in G, the space $(G, \mathbf{A}, s)$ is called β-complete.

Remark 4.9. Every complete vector-valued b-metric space is a β-complete vector-valued b-metric space but not conversely.

Look at the following example.

Example 4.10. Let $G = (0, \infty)$ and define the vector-valued b-metric $\mathbf{A} : G \times G \to \mathbb{R}_0^m$ by

$$\mathbf{A}(l, q) = \left(|H|^2, |H|^3, \cdots, |H|^{m+1}\right) \text{ for all } l, q \in G,$$

where $H = |l - q|$. Define $\beta\colon G \times G \to [0, \infty)$ by

$$\beta(l, q) = \begin{cases} l^2 + q^2 & \text{if } l, q \in [2, 5]; \\ 0 & \text{if not in } [2, 5]. \end{cases}$$

We observe that the space $(G, \mathbf{A}, s)$ is not a complete but it satisfies β-completeness criteria. Indeed, if $\{l_n\}$ is a Cauchy sequence in G such that $\beta(l_n, l_{n+1}) \geq 1$, for all $n \in \mathbb{N}$, then $l_n \in [2, 5]$. Since $[2, 5]$ is a closed subset of $\mathbb{R}$, so, there exists $l \in [2, 5]$ such that $\mathbf{A}(l_n, l) \to \mathbf{0}$ as $n \to \infty$. Hence $(G, \mathbf{A}, 2^m)$ is a β-complete vector-valued b-metric space.

Definition 4.11. If an arbitrary sequence $\{l_n\} \subset G$ satisfies the condition:

$$\beta(l_n, l_{n+1}) \geq 1 \text{ and } \mathbf{A}(l_n, l) \to \mathbf{0} \Rightarrow \beta(l_n, l) \geq 1,$$

$\forall\, n \in \mathbb{N}$. Then the space $(G, \mathbf{A}, s)$ is known as β -regular space.

Let $G = [2, 5]$ and define vector-valued b-metric as in Example 3.2 and β as in Example 4.10. Let $l_n = 2 + \frac{3}{n}$ be n^{th} term of a sequence in G. Then $(G, \mathbf{A}, s)$ is a β-regular space.

Definition 4.12. Let there exists $\mathbf{F} \in \Pi_s^b$ and $\mathbf{I} \succ \mathbf{0}$ $(G, \mathbf{A}, s)$, the mapping $T : G \to G$ is said to be a $(\beta, \mathbf{F})$-contraction, if it satisfies the following inequality:

$$\mathbf{A}(T(l), T(q)) \succ \mathbf{0}\, \forall l, q \in G,\ \beta(l, q) \geq 1 \text{ imply}$$

$$\mathbf{I} \oplus \mathbf{F}(s\beta(l, q) \odot \mathbf{A}(T(l), T(q)) \preceq \mathbf{F}(\mathbf{A}(l, q)). \tag{4.2}$$

Remark 4.13. Every $\mathbf{F}$-contraction is $(\beta, \mathbf{F})$-contraction but not conversely. For $\beta(l, q) = 1$, we have $(s, \mathbf{F})$-contraction

Suzuki [35] established the following lemma.

Lemma 4.14. [35] If there is a number $C > 0$ such that the sequence $\{x_n\} \subset (G, d)$ satisfies the inequality:

$$d(x_n, x_{n+1}) \leq Cn^{-\nu} \text{ for every } \nu > 1 + \log_2 s.$$

Then $\{x_n\}$ is a Cauchy sequence.

Now we have another lemma that extends the Lemma 4.14.

Lemma 4.15. *If there is a number $C > 0$ such that the sequence $\{x_n\} \subset (G, \mathbf{A}, s)$ satisfies the inequality:*

$$\mathbf{A}(x_n, x_{n+1}) \preceq (Cn^{-\nu})_1^m \textit{ for every } \nu > 1 + \log_2 s \textit{ and for every positive integer } n.$$

Then $\{x_n\}$ is a Cauchy-sequence.

Proof. Let G be any non-empty set and $s = \max\{s_i : \quad 1 \leq i \leq m\}$. Let $d_i : G \times G \to [0, \infty)$ be a b-metric for every $i \in \{1, 2, 3, \cdots, m\}$and $s_i \geq 1$. Define the vector-valued b-metric $\mathbf{A}$ by

$$\mathbf{A}(q, t) = (d_i(q, t))_{i=1}^m \text{ for all } q, t \in G.$$

Let $\{x_n\}$ be a sequence in G and assume that

$$\mathbf{A}(x_n, x_{n+1}) \preceq (Cn^{-\nu})_1^m \text{ for every } \nu > 1 + \log_2 s \text{ and for every } n \in \mathbb{N}.$$

Then, by definition of partial order $\preceq$ defined by (2.1), we have for each i

$$d_i(x_n, x_{n+1}) \leq Cn^{-\nu} \text{ for every } \nu > 1 + \log_2 s_i \text{ and for every } n \in \mathbb{N}.$$

Since Lemma 4.14 does not depend on a particular b-metric, therefore, Lemma 4.14 can be applied for each d_i $(1 \leq i \leq m)$. Thus, $\{x_n\}$ is a Cauchy sequence with respect to every d_i $(1 \leq i \leq m)$. Thus,

$$d_i(x_n, x_m) \to 0 \text{ as } n, m \to \infty \text{ for each } i.$$

This leads us to write that

$$\mathbf{A}(x_n, x_m) \to (0, 0, \cdots, 0) = O \text{ as } n, m \to \infty.$$

Hence $\{x_n\}$ is a Cauchy-sequence in $(G, \mathbf{A}, s)$. □

Now we have an analogue of Lemma 4.15 subject to $\mathbf{F}$ contraction.

Lemma 4.16. *Let $\{D_n\}$ be a sequence in $\mathbb{P}^m$ where $D_n := (j_n^{(i)})_{i=1}^m$. Assume that there exist a mapping $\mathbf{F} : \mathbb{P}^m \to \mathbb{R}^m$, $\mathbf{I} = (\tau_i)_{i=1}^m \succ O$ and $k \in (0, \ell) : \ell = 1/1 + \log_2 s$ satisfying (AF_3) and the following:*

$$n \odot \mathbf{I} \oplus \mathbf{F}\,(s^n \odot D_n) \preceq \mathbf{F}\,(D_0)\,. \tag{4.3}$$

Then $D_n \preceq \left(Cn^{-\frac{1}{k}}\right)_1^m$.

Proof. The inequality (4.3) implies $\lim_{n\to\infty} \mathbf{F}\,(s^n \odot D_n) = (-\infty)_1^m$ and by Lemma 3.7, we get $\lim_{n\to\infty} s^n \odot D_n = \mathbf{0}$. By (AF_3),

$$\lim_{n\to\infty} (s^n j_n^{(i)})^k \vartheta_n^{(i)} = 0 \text{ for each } i\,; \quad \mathbf{F}(s^n \odot D_n) := (\vartheta_n^{(i)})_{i=1}^m \in \mathbb{R}^m.$$

By (4.3), we also have the following information for each i.

$$\left(s^n j_n^{(i)}\right)^k \vartheta_n^{(i)} - (s^n j_n^{(i)})^k \vartheta_0^{(i)} \le -\left(s^n j_n^{(i)}\right)^k n\tau_i \le 0. \tag{4.4}$$

As $n \to \infty$ in (4.4), we have

$$\lim_{n\to\infty} n\left(s^n j_n^{(i)}\right)^k = 0 \text{ for each } i.$$

Equivalently there exists a positive integer N_1 such that $n\left(s^n j_n^{(i)}\right)^k \le 1$ for $n \ge N_1$. It then follows for each i that

$$s^n j_n^{(i)} \le \frac{1}{n^{\frac{1}{k}}} \Rightarrow j_n^{(i)} \le \frac{1}{s^n} n^{-\frac{1}{k}} \le \frac{1}{s} n^{-\frac{1}{k}}.$$

This implies $j_n^{(i)} \le Cn^{-\frac{1}{k}}$ for $n \ge N_1$ and for each i, where $C = s^{-1}$. Hence $D_n \preceq \left(Cn^{-\frac{1}{k}}\right)_1^m$. □

Now we give main theorem of this paper.

Theorem 4.17. *Let there exists $l_0 \in G$ such that $\beta(l_0, f(l_0)) \ge 1$, then, every β-admissible $(\beta, \mathbf{F})$-contraction $f : G \to G$ defined on the β-complete vector-valued b-metric space $(G, \mathbf{A}, s)$ admits a fixed point provided it is β-continuous on $(G, \mathbf{A}, s)$ or $(G, \mathbf{A}, s)$ is a β-regular space with additional assumption that $\mathbf{F}$ is continuous.*

Proof. (a) Let $l_0 \in G$ be as assumed and construct a Picard iterative-sequence $\{l_n\}$ of points in G such that, $l_1 = f(l_0)$, $l_2 = f(r_1)$ and generally $l_n = f(l_{n-1})$. Given $\beta(l_0, f(l_0)) = \beta(l_0, l_1) \ge 1$ and f is β-admissible, so, $\beta(f(l_0), f(l_1)) \ge 1$ *i.e* $\beta(l_1, l_2) \ge 1$. This leads to a general formula $\beta(l_n), l_{n+1}) \ge 1$ for all non-negative integers n. If $\mathbf{A}(l_n, l_{n+1}) = 0$, then $l_n = l_{n+1} = f(l_n)$ as required. (Step 1) Let $\mathbf{A}_n = \mathbf{A}(l_n, l_{n+1}) \succ \mathbf{0}$ and since $\beta(l_n, l_{n+1}) \ge 1$ for all non-negative integers n, so, by contractive condition (4.1), we get

$$\begin{aligned} \mathbf{F}\,(s \odot \mathbf{A}\,(f(l_{n-1}), f(l_n))) &\preceq \mathbf{F}\,(s\beta(l_{n-1}, l_n) \odot \mathbf{A}(f(l_{n-1}), f(l_n))) \\ &\preceq \mathbf{F}\,(\mathbf{A}_{n-1}) \ominus \mathbf{I}. \end{aligned}$$

This implies

$$\mathbf{I} \oplus \mathbf{F}\,(s \odot \mathbf{A}_n) \preceq \mathbf{F}\,(\mathbf{A}_{n-1})\,. \tag{4.5}$$

Due to (AF_4), inequality (4.5) implies

$$\mathbf{I} \oplus \mathbf{F}\left(s^n \odot \mathbf{A}_n\right) \preceq \mathbf{F}\left(s^{n-1} \odot \mathbf{A}_{n-1}\right). \tag{4.6}$$

(Step 2) Let $\mathbf{A}_{n-1} = \mathbf{A}(l_{n-1}, l_n) \succ \mathbf{0}$ and since $\beta(l_{n-1}, l_n) \geq 1$ for all positive integers n, so, by contractive condition (4.1), we get

$$\begin{aligned} \mathbf{F}\left(s \odot \mathbf{A}\left(f(l_{n-2}), f(l_{n-1})\right)\right) &\preceq \mathbf{F}\left(s\beta(l_{n-2}, l_{n-1}) \odot \mathbf{A}(f(l_{n-2}), f(l_{n-1}))\right) \\ &\preceq \mathbf{F}\left(\mathbf{A}_{n-2}\right) \ominus \mathbf{I}. \end{aligned}$$

This implies

$$\mathbf{I} \oplus \mathbf{F}\left(s \odot \mathbf{A}_{n-1}\right) \preceq \mathbf{F}\left(\mathbf{A}_{n-2}\right). \tag{4.7}$$

The condition (AF_4) in association with (4.7) implies

$$\mathbf{I} \oplus \mathbf{F}\left(s^{n-1} \odot \mathbf{A}_{n-1}\right) \preceq \mathbf{F}\left(s^{n-2} \odot \mathbf{A}_{n-2}\right).$$

Thus, inequality (4.6) leads to have a new inequality:

$$\mathbf{F}\left(s^n \odot \mathbf{A}_n\right) \preceq \mathbf{F}\left(s^{n-2} \odot \mathbf{A}_{n-2}\right) \ominus 2 \odot \mathbf{I}.$$

Finally, Step n provides the following inequality:

$$\mathbf{F}\left(s^n \odot \mathbf{A}_n\right) \preceq \mathbf{F}\left(\mathbf{A}_0\right) \ominus n \odot \mathbf{I}, \text{ for all positive integers } n. \tag{4.8}$$

By Lemma 4.16, $\{\mathbf{A}_n\} \preceq \left(Cn^{-\frac{1}{k}}\right)_1^m$. Since $\frac{1}{k} > 1 + \log_2 s$, so, by Lemma 4.14, $\{l_n\}$ is a Cauchy sequence in β-complete vector-valued b-metric space $(G, \mathbf{A}, s)$. Thus, there exists (say) $l^* \in G$ such that $\lim_{n\to\infty} \mathbf{A}(l_n, l^*) = \mathbf{0}$. As, $\beta(l_n, l_{n+1}) \geq 1$ for all non-negative integers, so, by the β-continuity of f, we have

$$\lim_{n\to\infty} \mathbf{A}(f(l_n), f(l^*)) = \lim_{n\to\infty} \mathbf{A}(l_{n+1}, f(l^*)) = \mathbf{0_f}.$$

Since $(G, \mathbf{A}, s)$ is not continuous in general, by Lemma 3.4, we have

$$\frac{1}{s} \odot \mathbf{A}(l^*, f(l^*)) \preceq \liminf_{n\to\infty} \mathbf{A}(l_{n+1}, f(l^*)) = \lim_{n\to\infty} \mathbf{A}(l_{n+1}, f(l^*)) = \mathbf{0_f}.$$

This implies that $\mathbf{A}(l^*, f(l^*)) = \mathbf{0_f}$ and by axiom $(\mathbf{A}_1)$ of vector-valued b-metric, we get $l^* = f(l^*)$. Hence, f admits a fixed point l^*.

(b) **Case 1.** If there exists a subsequence $\{l_{n_i}\}$ of $\{l_n\}$ such that $l_{n_i} = f(l^*)$ for all positive integers i, then, $l^* = \lim_{i\to\infty} l_{n_i} = \lim_{i\to\infty} f(l^*) = f(l^*)$. As required.

Case 2. Let there is no such subsequence of $\{l_n\}$ as in Case 1. We have proved in part (a) that $\lim_{n\to\infty} \mathbf{A}(l_n, l^*) = \mathbf{0}$, equivalently, $\exists\, K_0 \in \mathbb{N}$ such that for all $n \geq K_0$, $\mathbf{A}(f(l_n), l^*) \preceq \mathbf{A}(l^*, f(l^*))$. Since $(G, \mathbf{A}, s)$ is β-regular, thus, $\beta(l_n, l^*) \geq 1$. By contractive condition (4.1) and monotonicity of $\mathbf{F}$, we have

$$\begin{aligned} \mathbf{I} \oplus \mathbf{F}(s \odot \mathbf{A}(l_{n+1}, f(l^*))) &\preceq \mathbf{I} \oplus \mathbf{F}(s\beta(l_n, l^*) \odot \mathbf{A}(f(l_n), f(l^*))) \\ &\preceq \mathbf{F}\left(\mathbf{A}(l_n, l^*)\right) \preceq \mathbf{F}\left(\mathbf{A}(l^*, f(l^*))\right). \end{aligned} \tag{4.9}$$

We are looking for $\mathbf{A}(l^*, f(l^*)) = \mathbf{0}$, aiming at contradiction, suppose on contrary that $\mathbf{A}(l^*, f(l^*)) \succ \mathbf{0}$. By Lemma 3.4, monotonicity and continuity of $\mathbf{F}$, we have

$$\begin{aligned} \mathbf{I} \oplus \mathbf{F}(\mathbf{A}(l^*, f(l^*))) &\preceq \mathbf{I} \oplus \mathbf{F}(s \odot \lim_{n\to\infty} \inf \mathbf{A}(l_{n+1}, f(l^*)) \\ &= \mathbf{I} \oplus \lim_{n\to\infty} \inf \mathbf{F}(s \odot \mathbf{A}(l_{n+1}, f(l^*)) \\ &\preceq \mathbf{F}(\mathbf{A}(l^*, f(l^*))) \text{ by (4.9).} \end{aligned}$$

This is a contradiction to $\mathbf{I} \succ \mathbf{0}$. Thus, $\mathbf{A}(l^*, f(l^*)) = \mathbf{0}$. Finally, by axiom $\mathbf{A}_1$, we obtain $l^* = f(l^*)$. □

Remark 4.18. Additionally, if g^* is also a fixed point of f such that $\beta(l^*, g^*) \geq 1$, then f admits a unique fixed point.

Now we explain the hypothesis of the above theorem with an example.

Example 4.19. Let $G = [0, \infty)$ and define the mapping $\mathbf{A} : G \times G \to \mathbb{R}_0^m$ by

$$\mathbf{A}(l, q) = \left(|H|^2, |H|^3, \cdots, |H|^{m+1}\right) \text{ for all } l, q \in G,$$

where $H = |l - q|$. Define $\beta \colon G \times G \to [0, \infty)$ by

$$\beta(l, q) = \begin{cases} K & \text{iff } l \geq q \quad (K \in [1, \infty)); \\ 0 & \text{otherwise.} \end{cases}$$

so that $(G, \mathbf{A}, s)$ is a β-complete vector-valued b-metric space with $s = 2^m$. Define the mapping $f \colon G \to G$, for all $l \in G$ by

$$f(l) = \ln\left(1 + \frac{l}{6}\right). \text{ Then,}$$

f is β-continuous self-mapping: Indeed, consider the sequence $l_n = \frac{K}{n^2}$ for all positive integers n. As, $\frac{K}{n^2} \geq \frac{K}{(n+1)^2}$, so, $\beta(l_n, l_{n+1}) \geq 1$ and $\lim_{n\to\infty} \mathbf{A}(l_n, l) = \mathbf{0}$ implies $(l^2, l^3, \cdots, l^{m+1}) = \mathbf{0}$. This is true for $l = 0$.

$$\text{Now } \lim_{n\to\infty} \mathbf{A}(f(l_n), f(l)) = \lim_{n\to\infty} \left(\left(\ln\left(1 + \frac{K}{6n^2}\right)\right)^i\right)_{i=2}^{m+1} = \mathbf{0}_f.$$

Thus, whenever $\beta(l_n, l_{n+1}) \geq 1$ and $\lim_{n\to\infty} \mathbf{A}(l_n, l) = \mathbf{0}$, we have $\lim_{n\to\infty} \mathbf{A}(f(l_n), f(l)) = \mathbf{0}_f$.
f is β-admissible mapping: Indeed, let $\beta(l, q) \geq 1$, then, $l \geq q$, thus, we have $\ln\left(1 + \frac{l}{6}\right) \geq \ln\left(1 + \frac{q}{6}\right)$ *i.e* $\beta(f(l), f(q)) \geq 1$. Also let $l_0 = 1$, $f(l_0) = \ln\left(\frac{7}{6}\right)$. As, $l_0 \geq f(l_0)$, so, $\beta(l_0, f(l_0)) \geq 1$.

Finally, for each $l, q \in G$ with $l \geq q$ and choosing K such that $\ell > \frac{sK}{6^i}$ $(2 \leq i \leq m+1)$, we have for $\Delta_f = |f(l) - f(q)|$

$$\begin{aligned} \left(s\beta(l, q)|\Delta_f|^i + 1\right)_{i=2}^{m+1} &= \left(sK\left|\ln\left(1 + \frac{l}{6}\right) - \ln\left(1 + \frac{q}{6}\right)\right|^i + 1\right)_{i=2}^{m+1} \\ &\leq \left(sK\left(\frac{l}{6} - \frac{q}{6}\right)^i + 1\right)_{i=2}^{m+1} = \left(\frac{sK}{6^i}(l - q)^i + 1\right)_{i=2}^{m+1}. \end{aligned}$$

$$\text{Thus, } s\beta(l, q)|\Delta_f|^i + 1 \leq \frac{sK}{6^i}(l - q)^i + 1 \text{ for each } i.$$

$$\text{This implies, } \ln\left(\frac{6^i}{sK}\right) + \ln\left(s\beta(l,q)|\Delta_f|^i + 1\right) \leq \ln\left((l-q)^i + 1\right) \text{ for each } i.$$

$$\text{So that, } \left(\ln\left(\frac{6^i}{sK}\right)\right)_{i=2}^{m+1} \oplus \left(\ln\left(s\beta(l,q)|\Delta_f|^i + 1\right)\right)_{i=2}^{m+1} \preceq \left(\ln(l-q)^i + 1\right)_{i=2}^{m+1}.$$

Let $\mathbf{F} : \mathbb{R}_+^m \to \mathbb{R}^m$ be defined by $\mathbf{F}(v) = (\ln(x_i + 1))_{i=1}^m$, for all $v = (x_i)_{i=2}^{m+1} \in \mathbb{R}_+^m$, then $\mathbf{F} \in \Pi_s^b$ (as shown in Section 2). Thus, for all $l, q \in G$ such that $\mathbf{A}(f(l), f(q)) \succ \mathbf{0}$, $\mathbf{I} = \left(\ln\left(\frac{6^i}{sK}\right)\right)_{i=2}^{m+1}$ we obtain

$$\mathbf{I} \oplus \mathbf{F}\left(s\beta(l,q) \odot \mathbf{A}(f(l), g(q))\right) \preceq \mathbf{F}\left(\mathbf{A}(l,q)\right).$$

Note that f has a unique fixed point $l = 0$.

Corollary 4.20. *Let $\mathbf{A} : G \times G \to \mathbb{R}_0^4$ and $(G, \mathbf{A}, s)$ be a β-complete vector-valued b-metric space. If $f : G \to G$ be a continuous self-mapping satisfying the inequality:*

$$s^3 \odot \mathbf{A}(f(l), f(q))) \preceq Q \odot \mathbf{A}(l,q), \tag{4.10}$$

$\forall\ l, q \in G$, $\tau > 0$ be such that $e^{-\tau} \in \left(0, \frac{1}{1+\log_2 s}\right)$. Then f has a unique fixed point in G.

Proof. Define $\mathbf{F} : \mathbb{R}_+^4 \to \mathbb{R}^4$ by $\mathbf{F}(\mathbf{v}) = (\ln(v_i))_{i=1}^4$ and mapping β by $\beta(l,q) = s^2$ $\forall\ l, q \in G$ in the proof of Theorem 4.17 and $Q = \begin{pmatrix} e^{-\tau} & 0 & 0 & 0 \\ 0 & e^{-\tau} & 0 & 0 \\ 0 & 0 & e^{-\tau} & 0 \\ 0 & 0 & 0 & e^{-\tau} \end{pmatrix}$. Then the inequality (4.10) reduces to (4.1) and f has a unique fixed point in G. □

Note that for $s = 1$, Corollary 4.20 represents the Perov's fixed point theorem [29].

5. Application of Theorem 4.17

Perov [29], presented some applications of his fixed point theorem to Cauchy problems for the system of ordinary differential equations and respectively, to the boundary value problems. Here we will see that Theorem 4.17 is also applicable to the following delay integro-differential problem:

$$l(t) = \int_{t-L}^{t} f(h, l(h), l'(h))\, dh. \tag{5.1}$$

The Eq (5.1) generalizes the following delay-integral-equation:

$$l(t) = \int_{t-L}^{t} f(h, l(h))\, dh. \tag{5.2}$$

The model for the spread of a few infectious diseases with a seasonally variable contact rate is represented by the Eq (5.2), where

(a) $l(t)$: The prevalence of infection at time t in the population.
(b) $0 < L$: The amount of time a person can still spread disease.

(c) $l'(t)$: The current rate of infectivity.
(d) $f(t, l(t), l'(t))$: The rate of newly acquired infections per unit of time.

Now we look for existence and uniqueness of the positive, periodic solution to (5.1) by the application of Theorem 4.17.

Let $\exists\, p > 0$ and $f \in C(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})$ satisfying

$$f(t + p, l, q) = f(t, l, q)\ \forall\, (t, l, q) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}.$$

Let us define the functional spaces by

$$\begin{aligned} \mathcal{F}(p) &= \{l \in C^1(\mathbb{R}) : \quad l(t + p) = l(t) \quad t \in \mathbb{R}\}. \\ \mathcal{F}_+(p) &= \{l \in \mathcal{F}(p) : \quad l(t) \geq 0 \quad t \in \mathbb{R}\}. \end{aligned}$$

Let $V = \mathcal{F}_+(p) \times \mathcal{F}(p)$, define a metric $\mathbf{A} : V \times V \to \mathbb{R}^2$ by

$$\mathbf{A}((l_1, q_1), (l_2, q_2)) = (||l_1 - l_2||^2, ||q_1 - q_2||^2),$$

where, $||\mathbf{l}|| = \max\{|l(t)| : \quad t \in [0, p], \quad \mathbf{l} \in \mathcal{F}(p)\}$. The function $\beta : V \times V \to [1, \infty)$ defined by $\beta(l, q) = K^2$ for all $l, q \in V$. Then $(V, \mathbf{A}, s)$ is a β-complete vector-valued b-metric space. Now we develop the structure to apply Theorem 4.17. Let $g(t) = l'(t)$, then, we have

$$g(t) = f(t, l(t), g(t)) - f(t - L, l(t - L), g(t - L)).$$

Thus, Eq (5.1) can be written as:

$$\begin{cases} l(t) = \int_{t-L}^{t} f(h, l(h), g(h))\, dh \\ \\ g(t) = f(t, l(t), g(t)) - f(t - L, l(t - L), g(t - L)). \end{cases}$$

Let $\mathbb{Y} : V \to C(\mathbb{R}) \times C(\mathbb{R})$ be a mapping defined by

$$\mathbb{Y}(l, q) = (\mathbb{Y}_1(l, q), \mathbb{Y}_2(l, q) \text{ for all } (l, q) \in V,$$

where $\mathbb{Y}_1$ and $\mathbb{Y}_2$ are defined by the following matrix equation:

$$\begin{pmatrix} \mathbb{Y}_1(l, q)(t) \\ \\ \mathbb{Y}_2(l, q)(t) \end{pmatrix} = \begin{pmatrix} \int_{t-L}^{t} f(h, l(h), q(h))\, dh \\ \\ f(t, l(t), q(t)) - f(t - L, l(t - L), q(t - L)) \end{pmatrix}.$$

We need to assume the following conditions about function f:

(I_1) $f \in C(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})$ and there exists $N_1, N_2 \geq 0$ such that

$$N_1 \leq f(t, l, q) \leq N_2 \text{ for all } (t, l, q) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R};$$

(I_2) $f \in C(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})$ follows the equation:

$$f(t + p, l, q) = f(t, l, q); \quad p > 0;$$

(I_3) there exist $\omega, K > 0$ so that $\forall t \in \mathbb{R}, l(t), u(t) \in \mathbb{R}_+$ and $q(t), \epsilon(t) \in \mathbb{R}$

$$|f(t, l(t), q(t)) - f(t, u(t), \epsilon(t))| \leq \frac{e^{-\omega}}{sK} |l(t) - u(t)|,$$

(I_4) there exists $l_0 \in V$ such that $\beta(l_0, \mathbb{Y}(l_0)) = K^2$.

By the assumption of I_1, we infer that

$$N_1 L \leq \mathbb{Y}_1(l, q)(t) \leq N_2 L \text{ for all } t \in \mathbb{R} \text{ and } (l, q) \in V.$$

Theorem 5.1. *Let the function* $f \in C(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})$ *satisfies conditions* $(I_1) - (I_4)$ *and* $\frac{e^{-2\omega}}{s^2 K^2} < \frac{1}{4+L^2}$*, then, Eq (5.1) admits a solution in* $\mathcal{F}_+(p)$.

Proof. We note that the conditions (1) and (2) of Theorem 4.17 can be verified by using (I_4) and continuity of f respectively. In the following, we prove the contractive condition (4.2). By definition,

$$\begin{aligned}
\mathbb{Y}_1(l, q)(t + p) &= \int_{t+p-L}^{t+p} f(h, l(h), q(h))\, dh \\
&= \int_{t-L}^{t} f(u - p, l(u - p), q(u - p))\, du \\
&= \int_{t-L}^{t} f(u - p + p, l(u - p + p), q(u - p + p))\, du \\
&= \int_{t-L}^{t} f(u, l(u), q(u))\, du \\
&= \mathbb{Y}_1(l, q)(t) \text{ for all } t \in \mathbb{R}, (l, q) \in F.
\end{aligned}$$

This shows $\mathbb{Y}_1(V) \subseteq \mathcal{F}_+(p)$. Similarly, we have $\mathbb{Y}_2(V) \subseteq \mathcal{F}_+(p)$. Let $(l_1, q_1), (l_2, q_2) \in V$ and consider

$$\begin{aligned}
& |\mathbb{Y}_1(l_1, q_1)(t) - \mathbb{Y}_1(l_2, q_2)(t)|^2 \\
= & \left| \int_{t-L}^{t} f(h, l_1(h), q_1(h))\, dh - \int_{t-L}^{t} f(h, l_2(h), q_2(h))\, dh \right|^2 \\
\leq & \left(\int_{t-L}^{t} |f(h, l_1(h), q_1(h)) - f(h, l_2(h), q_2(h))|\, dh \right)^2 \\
\leq & \left(\int_{t-L}^{t} \left(\frac{e^{-\omega}}{sK} (|l_1(h) - l_2(h)|) \right) dh \right)^2 \\
\leq & \frac{e^{-2\omega} L^2}{s^2 K^2} \|l_1 - l_2\|^2
\end{aligned}$$

and

$$\begin{aligned}
& |\mathbb{Y}_2(l_1, q_1)(t) - \mathbb{Y}_2(l_2, q_2)(t)|^2 \\
= & \left| \begin{array}{l} f(t, l_1(t), q_1(t)) - f(t - L, l_1(t - L), q_1(t - L)) - \\ f(t, l_2(t), q_2(t)) + f(t - L, l_2(t - L), q_2(t - L)) \end{array} \right|^2 \\
\leq & \left(\begin{array}{l} |f(t, l_1(t), q_1(t)) - f(t, l_2(t), q_2(t))| + \\ |f(t - L, l_1(t - L), q_1(t - L)) - f(t - L, l_2(t - L), q_2(t - L))| \end{array} \right)^2
\end{aligned}$$

$$\leq \left(\frac{e^{-\omega}}{sK}\left(|l_1(t)-l_2(t)|\right)+\frac{e^{-\omega}}{sK}\left(|l_1(t-L)-l_2(t-L)|\right)\right)^2$$
$$\leq \frac{4e^{-2\omega}}{s^2K^2}\|l_1-l_2\|^2.$$

Consequently, we obtain the following matrix inequality:

$$\begin{pmatrix}\|\mathbb{Y}_1(l_1,q_1)-\mathbb{Y}_1(l_2,q_2)\|^2\\ \|\mathbb{Y}_2(l_1,q_1)-\mathbb{Y}_2(l_2,q_2)\|^2\end{pmatrix} \preceq \begin{pmatrix}\frac{e^{-2\omega}L^2}{s^2K^2}\|l_1-l_2\|^2\\ \frac{4e^{-2\omega}}{s^2K^2}\|l_1-l_2\|^2\end{pmatrix} = \begin{pmatrix}\frac{e^{-2\omega}L^2}{s^2K^2} & 0\\ 0 & \frac{4e^{-2\omega}}{s^2K^2}\end{pmatrix}\begin{pmatrix}\|l_1-l_2\|^2\\ \|l_1-l_2\|^2\end{pmatrix}.$$

Now define the mappings $\mathbf{F}:\mathbb{R}^2_+\to\mathbb{R}^2$ by $\mathbf{F}(l,q)=(\ln(l),\ln(q))$ and $\mathbf{I}=\left(\frac{s}{e^{-2\omega}L^2},\frac{s}{4e^{-2\omega}}\right)$, we obtain

$$\mathbf{I}\oplus\mathbf{F}\left(s\beta((l_1,q_1),(l_2,q_2))\odot\mathbf{A}(\mathbb{Y}(l_1,q_1),\mathbb{Y}(l_2,q_2))\right)\leq\mathbf{F}\left(\mathbf{A}((l_1,q_1),(l_2,q_2))\right).$$

Finally, keeping in mind the definition of mapping β and above inequality, we say that the mapping $\mathbb{Y}$ satisfies all the requirements of Theorem 4.17 and hence, admits a fixed point in its domain. Consequently, the Eq (5.1) has a positive, periodic solution. □

6. Conclusions

The findings and analyses discussed here could inspire further investigation into this topic by interested academics. A fundamental finding in vector-valued b-metric space, the main result (Theorem 4.17), concerns F-contraction in vector-valued b-metric spaces. In order to demonstrate the presence of solutions to various linear and nonlinear equations reflecting models of the associated real-world issues, the application approach is also discussed.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgment

This project is funded by King Saud University, Riyadh, Saudi Arabia. Research Supporting Project number (RSP2023R167), King Saud University, Riyadh, Saudi Arabia.

Conflict of interest

The authors declare that they have no competing interests.

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