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Research article

On semi best proximity points for multivalued mappings in quasi metric spaces

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Abstract: Due to the lack of symmetry property for the quasi metrics, we have considered left and right versions of best proximity points of multivalued mappings of quasi metric spaces. Further we consider the problem of existence of semi (left and right) best proximity points of generalized multivalued contractions of quasi metric spaces via various versions of so called $p-$property. Some examples are given to explain the results.

Keywords: best proximity points; left (right) best proximity points; quasi metric; multivalued mappings
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1. Introduction

Best proximity point (BPP) theory provides basic tools to find the approximate solutions of problems in applied mathematics and in nonlinear analysis whose exact solution does not exist. Let $\mathcal{L} : \mathcal{M} \to \mathcal{S}$ be a nonself mapping, where $\mathcal{M}$ and $\mathcal{S}$ two nonempty subsets of a metric space $(\mathcal{G}, d)$. A point $\mathfrak{m} \in \mathcal{M}$ is said to be the exact solution or the fixed point (FP) of $\mathcal{L}$ if $\mathfrak{m} = \mathcal{L}\mathfrak{m}$. This is only possible if $\mathcal{M} \cap \mathcal{L}(\mathcal{M})$ is nonempty, otherwise $\mathcal{L}$ does not have a FP. In this situation, the best way to find a point $\mathfrak{m} \in \mathcal{M}$ such that the distance between $\mathfrak{m}$ and $\mathcal{L}\mathfrak{m}$ is minimum. That is,

$$d(\mathfrak{m}, \mathcal{L}\mathfrak{m}) = d(\mathcal{M}, \mathcal{S}),$$

where

$$d(\mathcal{M}, \mathcal{S}) = \inf_{\substack{a\in\mathcal{M} \\ b\in\mathcal{S}}} d(a, b).$$

Such $\mathfrak{m}$ if exists is called the BPP of $\mathcal{L}$. In case $\mathcal{M} = \mathcal{S} = \mathcal{G}$, then $\mathfrak{m}$ becomes a FP of $\mathcal{L}$. So BPP theory is a generalized structure of FP theory. Probably the first attempt in this connection is due to Ky Fan [12] in 1969, who provided a remarkable result for the existence of BPPs which is given as follows.

Theorem 1. *Let $\mathcal{L} : \mathcal{K} \to \mathcal{G}$ be a continuous mapping, where $\mathcal{K}$ a nonempty compact convex subset of a normed space $\mathcal{G}$. Then there exists $\kappa \in \mathcal{K}$ such that*

$$||\kappa - \mathcal{L}\kappa|| = d(\mathcal{K}, \mathcal{L}\kappa) = \inf\{||\mathcal{L}\kappa - k|| : k \in \mathcal{K}\}.$$

After this, the BPP theory has flourished in one to many directions, for instance, Fallahi et al. [7,8] in 2020 developed best proximity points in partially ordered metric spaces and in $b-$metric spaces endowed with graph, for further details, interested readers can also explore the references [2,11,16,17].

Banach [5] in 1922, provided one of the fundamental results known as Banach contraction principle (BCP) in FP theory. Due to the significance of BCP, it has been generalized in different contexts, for more details, one can see the references [1,6,9].

Wardowski [18] in 2012 generalized the BCP by introducing $\mathcal{F}-$contractions and developed some FP results. After that, many mathematicians generalized $\mathcal{F}-$contractions in one to many directions and contributed for the development of FP theory (for self mappings) as well as for BPP theory (for nonself mappings), for more details, interested readers can explore the references [3,6,10].

Due to the significance of FP theory, it has been further extended to multivalued mappings. In 1969, Nadler [14] developed the multivalued version of BCP. After that a new horizon has opened and a number of mathematicians have contributed for the development of FP theory in this direction [15].

On the other hand, Wilson [19] in 1931 developed a generalized structure of metric space as quasi metric space by relaxing the symmetric condition of metric space. Since quasi metric has asymmetric characteristic in its domain, so it plays a vital role in computer sciences and other disciplines of mathematics which are asymmetric in their structure. Since then, FP theory has flourished in the domain of quasi metric space, for instance [6,13].

Hancer et al. [9] in 2019 generalized $\mathcal{F}-$contractions to multivalued $\mathcal{F}-$ contractions and developed some FP results in quasi metric spaces.

In 2023, Aslantas et al. [4] introduced left (right) best proximity points and developed some best proximity results for proximal contractions.

Motivated by the works of Hancer and Aslantas, in this paper, we have introduced new type of generalized multivalued $\mathcal{F}-$contractions and developed some results for the existence of left (right) BPPs for these contractions in the domain of quasi metric spaces.

Throughout this article, we denote $\mathbb{R}^+$, $\mathbb{N}$, τ_d, $C_d(\mathcal{G}), C_B(\mathcal{G})$ and $2^{\mathcal{G}}\backslash\phi$ respectively as non-negative reals, positive integers, topology induced by quasi metric d, family of all nonempty τ_d-closed subsets of $\mathcal{G}$, closed and bounded subsets of $\mathcal{G}$ and set of nonempty subsets of $\mathcal{G}$.

Definition 1. *[9] Let $\mathcal{G}$ be a nonempty set. A mapping $d : \mathcal{G}\times\mathcal{G} \to \mathbb{R}^+$ is said to be a T_1-quasi metric (shortly $T_1 - Q$ metric) if it satisfies the following axioms:*

($\mathbf{q}_1$) $d(\varrho_1, \varrho_1) = 0$,

($\mathbf{q}_2$) $d(\varrho_1, \varrho_3) \leq d(\varrho_1, \varrho_2) + d(\varrho_2, \varrho_3)$,

(q$_3$) $d(\varrho_1, \varrho_2) = d(\varrho_2, \varrho_1) = 0$ *implies* $\varrho_1 = \varrho_2$,

(q$_4$) $d(\varrho_1, \varrho_2) = 0$ *implies* $\varrho_1 = \varrho_2$,

for all $\varrho_1, \varrho_2, \varrho_3 \in \mathcal{G}$. *In this case, the pair* $(d,\mathcal{G})$ *is called* $\mathcal{Q}$ *metric space* ($\mathcal{Q}$*−MS). If* d *is a* $\mathcal{Q}$ *metric on* $\mathcal{G}$, *then* d^{-1} *defined as*

$$d^{-1}(\varrho_1, \varrho_2) = d(\varrho_2, \varrho_1)$$

is a $\mathcal{Q}$ *metric on* $\mathcal{G}$ *as well. Further,*

$$d^s(\varrho_1, \varrho_2) = \max\left\{d(\varrho_1, \varrho_2), d^{-1}(\varrho_1, \varrho_2)\right\}$$

is a metric on $\mathcal{G}$ *induced by* $\mathcal{Q}$ *metric* d.

Definition 2. *[9] A sequence* $\{\mathfrak{I}_\eta\}$ *in* $\mathcal{G}$ *is*

i) $d-$*convergent to* $\mathfrak{I}$ *with respect to* τ_d *that is* $\mathfrak{I}_\eta \to^d \mathfrak{I}$ *if and only if* $d(\mathfrak{I}, \mathfrak{I}_\eta) \to 0$, *as* $\eta \to \infty$.

ii) $d^{-1}-$*convergent to* $\mathfrak{I}$ *with respect to* $\tau_{d^{-1}}$ *that is* $\mathfrak{I}_\eta \to^{d^{-1}} \mathfrak{I}$ *if and only if* $d(\mathfrak{I}_\eta, \mathfrak{I}) \to 0$, *as* $\eta \to \infty$.

iii) *Left* $\mathcal{K}-$*Cauchy if for every* $\epsilon > 0$, *there exists* $\wp \in \mathbb{N}$ *such that*

$$\forall \imath, \jmath, \imath \geq \jmath \geq \wp, d(\mathfrak{I}_\jmath, \mathfrak{I}_\imath) < \epsilon.$$

iv) *Right* $\mathcal{K}-$*Cauchy if for every* $\epsilon > 0$, *there exists* $\wp \in \mathbb{N}$ *such that*

$$\forall\ \imath, \jmath,\ \imath \geq \jmath \geq \wp,\ d(\mathfrak{I}_\imath, \mathfrak{I}_\jmath) < \epsilon.$$

Definition 3. *[9] Let* $(\mathcal{G}, d)$ *be a quasi metric space (shortly as* $\mathcal{Q}$*−MS). Then* $(\mathcal{G}, d)$ *is said to be*

i) *left* $\mathcal{K}-$*complete if every left* $\mathcal{K}-$*Cauchy sequence is* $d-$*convergent,*

ii) *right* $\mathcal{K}-$*complete if every right* $\mathcal{K}-$*Cauchy sequence is* $d-$*convergent.*

Wardowski [18] introduced the family of the functions given in the following definition.

Definition 4. *[18] Let the family* $\mathfrak{F}$ *of all functions* $\mathcal{F} : (0, \infty) \to \mathbb{R}$ *satisfying the following conditions;*

(F$_1$) *For all* $\rho, \sigma \in (0, \infty)$ *such that* $\rho < \sigma$, $\mathcal{F}(\rho) < \mathcal{F}(\sigma)$.

(F$_2$) *For every sequence* $\{\aleph_\eta\}$ *of positive numbers* $\lim_{\eta\to\infty} \aleph_\eta = 0$ *if and only if*

$$\lim_{\eta\to\infty} \mathcal{F}(\aleph_\eta) = -\infty.$$

(F$_3$) *There exists* $\mu \in (0, 1)$ *such that*

$$\lim_{\rho\to 0^+} \rho^\mu \mathcal{F}(\rho) = 0.$$

Definition 5. *[9] Let* $(\mathcal{G}, d)$ *be a* $\mathcal{Q}$*−MS and* $\mathcal{L} : \mathcal{G} \to 2^{\mathcal{G}} \backslash \phi$ *and* $\mathcal{F} \in \mathfrak{F}$, *then* $\mathcal{L}$ *is said to be a multivalued* $\mathcal{F}_d-$*contraction if there exists* $\tau > 0$ *such that for each* $\mathfrak{l}, \mathfrak{m} \in \mathcal{M}$ *with* $d(\mathfrak{l}, \mathfrak{m}) > 0$ *and for each* $\mathfrak{u} \in \mathcal{L}\mathfrak{l}$, *there exists* $\mathfrak{v} \in \mathcal{L}\mathfrak{m}$ *satisfying either* $d(\mathfrak{u}, \mathfrak{v}) = 0$ *or* $d(\mathfrak{u}, \mathfrak{v}) > 0$ *such that*

$$\tau + \mathcal{F}(d(\mathfrak{u}, \mathfrak{v})) \leq \mathcal{F}(d(\mathfrak{l}, \mathfrak{m})).$$

Theorem 2. *[9] Let $(\mathcal{G}, d)$ be a left K−complete $T_1 - Q$−MS, $\mathcal{L} : \mathcal{G} \to C_d(\mathcal{G})$ a multivalued mapping and $\mathcal{F} \in \mathfrak{F}$. If $\mathcal{L}$ is multivalued $\mathcal{F}_d$−contraction, then $\mathcal{L}$ has a fixed point provided that $f(x) = d(x, \mathcal{L}x)$ is lower semicontinuous with respect to τ_d.*

Definition 6. *[10] Let $(\mathcal{G}, d)$ be a metric space. The pair $(\mathcal{M}, \mathcal{S})$ has the p−property if*

$$\begin{cases} d(\mathfrak{m}_1, s_1) = d(\mathcal{M}, \mathcal{S}) \\ d(\mathfrak{m}_2, s_2) = d(\mathcal{M}, \mathcal{S}) \end{cases} \text{implies } d(\mathfrak{m}_1, \mathfrak{m}_2) = d(s_1, s_2),$$

for all $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathcal{M}$, $s_1, s_2 \in \mathcal{S}$.

Define

$$\mathcal{S}_0 = \{s \in \mathcal{S} : d(\mathfrak{m}, s) = d(\mathcal{M}, \mathcal{S}) \text{ for some } \mathfrak{m} \in \mathcal{M}\} \text{ and}$$
$$\mathcal{M}_0 = \{\mathfrak{m} \in \mathcal{M} : d(\mathfrak{m}, s) = d(\mathcal{M}, \mathcal{S}) \text{ for some } s \in \mathcal{S}\}.$$

Aslantas et al. [4] in 2023 introduced left (right) best proximity points of a nonself mapping $\mathcal{L} : \mathcal{M} \to \mathcal{S}$. We define left (right) best proximity points of a multivalued mapping as follows.

Definition 7. *Let $\mathcal{L} : \mathcal{M} \to 2^{\mathcal{S}} \backslash \phi$ be a nonself multivalued mapping, where $\mathcal{M}$ and $\mathcal{S}$ are two nonempty subsets of a Q−MS $(\mathcal{G}, d)$. Then a point $\mathfrak{m} \in \mathcal{M}$ is said to be a*

i) *left BPP (LBPP) if*

$$d(\mathfrak{m}, \mathcal{L}\mathfrak{m}) = d(\mathcal{M}, \mathcal{S}),$$

ii) *right BPP (RBPP) if*

$$d(\mathcal{L}\mathfrak{m}, \mathfrak{m}) = d(\mathcal{S}, \mathcal{M}).$$

Remark 1. *If we replace Q−metric by metric then LBPP is same as RBPP. Note that LBPP with respect to d^{-1} is RBPP with respect to d.*

In this paper, we have introduced p_d−property and p_{d_*}−property as follows.

Definition 8. *Let $(\mathcal{G}, d)$ be a Q−MS and $\mathcal{M}, \mathcal{S}$ be two nonempty subsets of $\mathcal{G}$, then the pair $(\mathcal{M}, \mathcal{S})$ has*

i) *p_d−property if*

$$\begin{cases} d(\mathfrak{m}_1, s_1) = d(\mathcal{M}, \mathcal{S}) \\ d(\mathfrak{m}_2, s_2) = d(\mathcal{M}, \mathcal{S}) \end{cases} \text{implies } d(\mathfrak{m}_1, \mathfrak{m}_2) = d(s_1, s_2),$$

for all $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathcal{M}$ and $s_1, s_2 \in \mathcal{S}$.

ii) *p_{d_*}−property if*

$$\begin{cases} d(\mathfrak{m}_1, s_1) = d(\mathcal{M}, \mathcal{S}) \\ d(\mathfrak{m}_2, s_2) = d(\mathcal{M}, \mathcal{S}) \end{cases} \text{implies } d(\mathfrak{m}_1, \mathfrak{m}_2) = d(s_2, s_1),$$

for all $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathcal{M}$ and $s_1, s_2 \in \mathcal{S}$.

Remark 2. *If we replace Q−metric by metric then all above properties reduces to p−property.*

In this paper, we have introduced generalized multivalued $\mathcal{F}_d$−contraction and generalized multivalued $\mathcal{F}_{d_*}$−contraction as follows.

Definition 9. *Let* $(\mathcal{G}, d)$ *be a* $\mathcal{Q}$*–MS and* $\mathcal{M}, \mathcal{S}$ *nonempty subsets of* $\mathcal{G}$, $\mathcal{L} : \mathcal{M} \to 2^{\mathcal{S}} \backslash \phi$ *and* $\mathcal{F} \in \mathfrak{F}$ *, then* $\mathcal{L}$ *is said to be*

(1) a generalized multivalued $\mathcal{F}_d$*–contraction if there exists* $\tau > 0$ *such that for each* $\mathfrak{l}, \mathfrak{m} \in \mathcal{M}$ *with* $d(\mathfrak{l}, \mathfrak{m}) > 0$ *and for each* $\mathfrak{u} \in \mathcal{L}\mathfrak{l}$, *there exists* $\mathfrak{v} \in \mathcal{L}\mathfrak{m}$ *satisfying either* $d(\mathfrak{u}, \mathfrak{v}) = 0$ *or*

$$\tau + \mathcal{F}(d(\mathfrak{u}, \mathfrak{v})) \leq \mathcal{F}(M(\mathfrak{l}, \mathfrak{m})), \tag{1.1}$$

(2) a generalized multivalued $\mathcal{F}_{d_*}$*–contraction if there exists* $\tau > 0$ *such that for each* $\mathfrak{l}, \mathfrak{m} \in \mathcal{M}$ *with* $d(\mathfrak{l}, \mathfrak{m}) > 0$ *and for each* $\mathfrak{u} \in \mathcal{L}\mathfrak{l}$, *there exists* $\mathfrak{v} \in \mathcal{L}\mathfrak{m}$ *satisfying either* $d(\mathfrak{v}, \mathfrak{u}) = 0$ *or*

$$\tau + \mathcal{F}(d(\mathfrak{v}, \mathfrak{u})) \leq \mathcal{F}(M(\mathfrak{l}, \mathfrak{m})), \tag{1.2}$$

where

$$M(\mathfrak{l}, \mathfrak{m}) = \max \left\{ \begin{array}{l} d(\mathfrak{l}, \mathfrak{m}), d(\mathfrak{l}, \mathcal{L}\mathfrak{l}) - d(\mathcal{M}, \mathcal{S}), d(\mathfrak{m}, \mathcal{L}\mathfrak{m}) - d(\mathcal{M}, \mathcal{S}), \\ \dfrac{d(\mathfrak{m}, \mathcal{L}\mathfrak{l}) + d(\mathfrak{l}, \mathcal{L}\,\mathfrak{m})}{2} - d(\mathcal{M}, \mathcal{S}) \end{array} \right\}.$$

2. Main results

The following is the first main result.

Theorem 3. *Let* $(\mathcal{G}, d)$ *be a left* $\mathcal{K}$*–complete* $T_1 - \mathcal{Q}$ *metric.* $\mathcal{L} : \mathcal{M} \to C_B(\mathcal{S})$ *be a generalized multivalued* $\mathcal{F}_d$*–contraction satisfying the following axioms:*

i) *For each* $\mathfrak{m} \in \mathcal{M}_0$, *we have* $\mathcal{L}(\mathfrak{m}) \subseteq \mathcal{S}_0$, *and the pair* $(\mathcal{M}, \mathcal{S})$ *satisfies the* p_d*–property.*

ii) *For* $\mathfrak{m}_0, \mathfrak{m}_1 \in \mathcal{M}_0$, *there exists* $s_1 \in \mathcal{L}\mathfrak{m}_0$ *such that*

$$d(\mathfrak{m}_0, \mathfrak{m}_1) > 0 \text{ and } d(\mathfrak{m}_1, s_1) = d(\mathcal{M}, \mathcal{S}).$$

Then $\mathcal{L}$ *has a LBPP provided that the function* $f(\mathfrak{m}) = d(\mathfrak{m}, \mathcal{L}\mathfrak{m})$ *is a lower semicontinuous with respect to* τ_d.

Proof. From the given assumption (ii), we have $\mathfrak{m}_0, \mathfrak{m}_1 \in \mathcal{M}_0$, there exists $s_1 \in \mathcal{L}\mathfrak{m}_0$ such that

$$d(\mathfrak{m}_0, \mathfrak{m}_1) > 0 \text{ and } d(\mathfrak{m}_1, s_1) = d(\mathcal{M}, \mathcal{S}). \tag{2.1}$$

For $\mathfrak{m}_1 \in \mathcal{M}_0$, pick $s_2 \in \mathcal{L}\mathfrak{m}_1 \subseteq \mathcal{S}_0$ that is $s_2 \in \mathcal{S}_0$, it implies that there exists $\mathfrak{m}_2 \in \mathcal{M}_0$ such that

$$d(\mathfrak{m}_2, s_2) = d(\mathcal{M}, \mathcal{S}), \tag{2.2}$$

by p_d–property, (2.1) and (2.2) imply

$$d(\mathfrak{m}_1, \mathfrak{m}_2) = d(s_1, s_2). \tag{2.3}$$

If $d(s_1, s_2) = 0$ then $s_1 = s_2$ and so from (2.1) we have

$$d(\mathfrak{m}_1, \mathcal{L}\mathfrak{m}_1) \leq d(\mathfrak{m}_1, s_2) = d(\mathcal{M}, \mathcal{S}) \leq d(\mathfrak{m}_1, \mathcal{L}\mathfrak{m}_1),$$

that is,

$$d(\mathfrak{m}_1, \mathcal{L}\mathfrak{m}_1) = d(\mathcal{M}, \mathcal{S}),$$

it implies $\mathfrak{m}_1$ is the LBPP of $\mathcal{L}$ and the theorem completes. So, let

$$d(s_1, s_2) > 0,$$

then

$$\begin{aligned}
\tau + \mathcal{F}(d(s_1, s_2)) &\leq \mathcal{F}(M(\mathfrak{m}_0, \mathfrak{m}_1)) \\
&= \mathcal{F}\left(\max\left\{\begin{array}{l} d(\mathfrak{m}_0, \mathfrak{m}_1), d(\mathfrak{m}_0, \mathcal{L}\mathfrak{m}_0) - d(\mathcal{M}, \mathcal{S}), \\ d(\mathfrak{m}_1, \mathcal{L}\mathfrak{m}_1) - d(\mathcal{M}, \mathcal{S}), \\ \dfrac{d(\mathfrak{m}_1, \mathcal{L}\mathfrak{m}_0) + d(\mathfrak{m}_0, \mathcal{L}\mathfrak{m}_1)}{2} - d(\mathcal{M}, \mathcal{S}) \end{array}\right\}\right) \\
&\leq \mathcal{F}\left(\max\left\{\begin{array}{l} d(\mathfrak{m}_0, \mathfrak{m}_1), d(\mathfrak{m}_0, s_1) - d(\mathcal{M}, \mathcal{S}), \\ d(\mathfrak{m}_1, s_2) - d(\mathcal{M}, \mathcal{S}), \\ \dfrac{d(\mathfrak{m}_1, s_1) + d(\mathfrak{m}_0, s_2) - 2d(\mathcal{M}, \mathcal{S})}{2} \end{array}\right\}\right) \\
&\leq \mathcal{F}\left(\max\left\{\begin{array}{l} d(\mathfrak{m}_0, \mathfrak{m}_1), \\ d(\mathfrak{m}_0, \mathfrak{m}_1) + d(\mathfrak{m}_1, s_1) - d(\mathcal{M}, \mathcal{S}), \\ d(\mathfrak{m}_1, \mathfrak{m}_2) + d(\mathfrak{m}_2, s_2) - d(\mathcal{M}, \mathcal{S}), \\ \dfrac{d(\mathfrak{m}_0, \mathfrak{m}_1) + d(\mathfrak{m}_1, \mathfrak{m}_2)}{2} \end{array}\right\}\right) \\
&\leq \mathcal{F}\left(\max\left\{\begin{array}{l} d(\mathfrak{m}_0, \mathfrak{m}_1), d(\mathfrak{m}_0, \mathfrak{m}_1), d(\mathfrak{m}_1, \mathfrak{m}_2), \\ \dfrac{d(\mathfrak{m}_0, \mathfrak{m}_1) + d(\mathfrak{m}_1, \mathfrak{m}_2)}{2} \end{array}\right\}\right) \\
&\leq \mathcal{F}\left(\max\left\{d(\mathfrak{m}_0, \mathfrak{m}_1), d(\mathfrak{m}_1, \mathfrak{m}_2)\right\}\right),
\end{aligned}$$

if

$$\max\{d(\mathfrak{m}_0, \mathfrak{m}_1), d(\mathfrak{m}_1, \mathfrak{m}_2)\} = d(\mathfrak{m}_1, \mathfrak{m}_2),$$

then

$$\tau + \mathcal{F}(d(s_1, s_2)) \leq \mathcal{F}(d(\mathfrak{m}_1, \mathfrak{m}_2)).$$

From (2.3) we have

$$\mathcal{F}(d(\mathfrak{m}_1, \mathfrak{m}_2)) = \mathcal{F}(d(s_1, s_2)),$$

so we get

$$\tau + \mathcal{F}(d(\mathfrak{m}_1, \mathfrak{m}_2)) \leq \mathcal{F}(d(\mathfrak{m}_1, \mathfrak{m}_2)),$$

implies $\tau \leq 0$, a contradiction. Hence

$$\max\{d(\mathfrak{m}_0, \mathfrak{m}_1), d(\mathfrak{m}_1, \mathfrak{m}_2)\} = d(\mathfrak{m}_0, \mathfrak{m}_1).$$

Thus we have

$$\mathcal{F}(d(s_1, s_2)) \leq \mathcal{F}(d(\mathfrak{m}_0, \mathfrak{m}_1)) - \tau. \tag{2.4}$$

From (2.3) and (2.4) we get

$$\mathcal{F}(d(\mathfrak{m}_1, \mathfrak{m}_2)) \leq \mathcal{F}(d(\mathfrak{m}_0, \mathfrak{m}_1)) - \tau.$$

Now for $s_3 \in \mathcal{L}\mathfrak{m}_2 \subseteq \mathcal{S}_0$, that is $s_3 \in \mathcal{S}_0$ it implies there exists $\mathfrak{m}_3 \in \mathcal{M}_0$ such that

$$d(\mathfrak{m}_3, s_3) = d(\mathcal{M}, \mathcal{S}), \tag{2.5}$$

by p_d−property (2.2) and (2.5) imply

$$d(\mathfrak{m}_2, \mathfrak{m}_3) = d(s_2, s_3). \tag{2.6}$$

If $d(s_2, s_3) = 0$ then $\mathfrak{m}_2$ is the LBPP of $\mathcal{L}$ and the proof is completed. So, let

$$d(s_2, s_3) > 0,$$

then

$$\begin{aligned}
\tau + \mathcal{F}(d(s_2, s_3)) &\leq \mathcal{F}(M(\mathfrak{m}_1, \mathfrak{m}_2)) \\
&\leq \mathcal{F}\left(\max\left\{\begin{array}{l} d(\mathfrak{m}_1, \mathfrak{m}_2), d(\mathfrak{m}_1, \mathcal{L}\mathfrak{m}_1) - d(\mathcal{M}, \mathcal{S}), \\ d(\mathfrak{m}_2, \mathcal{L}\mathfrak{m}_2) - d(\mathcal{M}, \mathcal{S}), \\ \dfrac{d(\mathfrak{m}_2, \mathcal{L}\mathfrak{m}_1) + d(\mathfrak{m}_1, \mathcal{L}\mathfrak{m}_2)}{2} - d(\mathcal{M}, \mathcal{S}) \end{array}\right\}\right) \\
&\leq \mathcal{F}\left(\max\left\{d(\mathfrak{m}_1, \mathfrak{m}_2), d(\mathfrak{m}_2, \mathfrak{m}_3)\right\}\right),
\end{aligned}$$

if

$$\max\left\{d(\mathfrak{m}_1, \mathfrak{m}_2), d(\mathfrak{m}_2, \mathfrak{m}_3)\right\} = d(\mathfrak{m}_2, \mathfrak{m}_3),$$

then

$$\tau + \mathcal{F}(d(s_2, s_3)) \leq \mathcal{F}(d(\mathfrak{m}_2, \mathfrak{m}_3).$$

From (2.6), we have

$$\mathcal{F}(d(\mathfrak{m}_2, \mathfrak{m}_3)) = \mathcal{F}(d(s_2, s_3)). \tag{2.7}$$

So we get

$$\tau + \mathcal{F}(d(\mathfrak{m}_2, \mathfrak{m}_3)) \leq \mathcal{F}(d(\mathfrak{m}_2, \mathfrak{m}_3),$$

implies $\tau \leq 0$, a contradiction, hence

$$\mathcal{F}(d(s_2, s_3)) \leq \mathcal{F}(d(\mathfrak{m}_1, \mathfrak{m}_2)) - \tau. \tag{2.8}$$

Using (2.7) in (2.8), we have

$$\mathcal{F}(d(\mathfrak{m}_2, \mathfrak{m}_3)) \leq \mathcal{F}(d(\mathfrak{m}_1, \mathfrak{m}_2)) - \tau,$$

continuing in this way, we get $\mathfrak{m}_{\eta-1} \in \mathcal{M}_0$, $s_\eta \in \mathcal{S}_0$ and $\mathfrak{m}_\eta \in \mathcal{M}_0$ such that

$$d(\mathfrak{m}_\eta, s_\eta) = d(\mathcal{M}, \mathcal{S}), \tag{2.9}$$

similarly for $\mathfrak{m}_\eta \in \mathcal{M}_0$, $s_{\eta+1} \in \mathcal{S}_0$, and there exists $\mathfrak{m}_{\eta+1} \in \mathcal{M}_0$ such that

$$d(\mathfrak{m}_{\eta+1}, s_{\eta+1}) = d(\mathcal{M}, \mathcal{S}), \tag{2.10}$$

from (2.9) and (2.10) and by p_d−property, we have

$$d(\mathfrak{m}_\eta, \mathfrak{m}_{\eta+1}) = d(s_\eta, s_{\eta+1}). \tag{2.11}$$

If $d(s_\eta, s_{\eta+1}) = 0$ for some η, it implies $s_\eta = s_{\eta+1}$, then we have $\mathfrak{m}_\eta$ is the LBPP. So suppose $d(s_\eta, s_{\eta+1}) > 0$, then

$$\begin{aligned}
\tau + \mathcal{F}(d(s_\eta, s_{\eta+1})) &\leq \mathcal{F}\left(M(\mathfrak{m}_{\eta-1}, \mathfrak{m}_\eta)\right) \\
&\leq \mathcal{F}\left(\max\left\{\begin{array}{l} d(\mathfrak{m}_{\eta-1}, \mathfrak{m}_\eta), d(\mathfrak{m}_{\eta-1}, \mathcal{L}\mathfrak{m}_{\eta-1}) - d(\mathcal{M}, \mathcal{S}), \\ d(\mathfrak{m}_\eta, \mathcal{L}\mathfrak{m}_\eta) - d(\mathcal{M}, \mathcal{S}), \\ \frac{d(\mathfrak{m}_\eta, \mathcal{L}\mathfrak{m}_{\eta-1}) + d(\mathfrak{m}_{\eta-1}, \mathfrak{m}_\eta) - 2d(\mathcal{M}, \mathcal{S})}{2} \end{array}\right\}\right) \\
&\leq \mathcal{F}\left(\max\left\{\begin{array}{l} d(\mathfrak{m}_{\eta-1}, \mathfrak{m}_\eta), d(\mathfrak{m}_\eta, \mathfrak{m}_{\eta+1}), \\ \frac{d(\mathfrak{m}_{\eta-1}, \mathfrak{m}_\eta) + d(\mathfrak{m}_\eta, \mathfrak{m}_{\eta+1})}{2} \end{array}\right\}\right) \\
&\leq \mathcal{F}\left(\max\left\{d(\mathfrak{m}_{\eta-1}, \mathfrak{m}_\eta), d(\mathfrak{m}_\eta, \mathfrak{m}_{\eta+1})\right\}\right),
\end{aligned}$$

if

$$\max\left(\left\{d(\mathfrak{m}_{\eta-1}, \mathfrak{m}_\eta), d(\mathfrak{m}_\eta, \mathfrak{m}_{\eta+1})\right\}\right) = d(\mathfrak{m}_\eta, \mathfrak{m}_{\eta+1}),$$

then we get a contradiction. Hence we have

$$\mathcal{F}\left(d(s_\eta, s_{\eta+1})\right) \leq \mathcal{F}\left(M(\mathfrak{m}_{\eta-1}, \mathfrak{m}_\eta)\right) - \tau, \tag{2.12}$$

from (2.11) and (2.12)

$$\begin{aligned}
\mathcal{F}\left(d(\mathfrak{m}_\eta, \mathfrak{m}_{\eta+1})\right) &\leq \mathcal{F}\left(M(\mathfrak{m}_{\eta-1}, \mathfrak{m}_\eta)\right) - \tau \\
&\leq \mathcal{F}\left(M(\mathfrak{m}_{\eta-2}, \mathfrak{m}_{\eta-1})\right) - 2\tau \\
&\quad . \\
&\quad . \\
&\quad . \\
&\leq \mathcal{F}\left(M(\mathfrak{m}_0, \mathfrak{m}_1)\right) - \eta\tau,
\end{aligned} \tag{2.13}$$

for all $\eta \in \mathbb{N}$. From (2.13), we get

$$\lim_{\eta\to\infty} \mathcal{F}\left(d(\mathfrak{m}_\eta, \mathfrak{m}_{\eta+1})\right) = -\infty.$$

Now from (F_2), we have

$$\lim_{\eta\to\infty} d(\mathfrak{m}_\eta, \mathfrak{m}_{\eta+1}) = 0.$$

From (F_3) there exists $k \in (0, 1)$ such that

$$\lim_{\eta\to\infty} d(\mathfrak{m}_\eta, \mathfrak{m}_{\eta+1})^k \mathcal{F} d(\mathfrak{m}_\eta, \mathfrak{m}_{\eta+1}) = 0,$$

then by (2.13), the following hold for all $\eta \in \mathbb{N}$,

$$d(\mathfrak{m}_\eta, \mathfrak{m}_{\eta+1})^k \left(\mathcal{F} d(\mathfrak{m}_\eta, \mathfrak{m}_{\eta+1}) - \mathcal{F} d(\mathfrak{m}_0, \mathfrak{m}_1)\right) \leq -d(\mathfrak{m}_\eta, \mathfrak{m}_{\eta+1})^k \eta\tau.$$

letting $\eta \to \infty$ we get

$$\lim_{\eta\to\infty} \eta d(\mathfrak{m}_\eta, \mathfrak{m}_{\eta+1})^k = 0, \tag{2.14}$$

it implies there exists $\eta_1 \in \mathbb{N}$ such that

$$\eta d(\mathfrak{m}_\eta, \mathfrak{m}_{\eta+1})^k \leq 1,$$

for all $\eta \geq \eta_1$. It implies

$$d(\mathfrak{m}_\eta, \mathfrak{m}_{\eta+1})^k \leq \frac{1}{\eta}, \tag{2.15}$$

for all $\eta \geq \eta_1$.

Now to show $\{\mathfrak{m}_\eta\}$ is a left $\mathcal{K}-$Cauchy sequence in $\mathcal{M}$, consider $m, \eta \in \mathbb{N}$ such that $m > \eta \geq \eta_1$, then by triangular inequality and by (2.15), we get

$$d(\mathfrak{m}_\eta, \mathfrak{m}_m) \leq d(\mathfrak{m}_\eta, \mathfrak{m}_{\eta+1}) + d(\mathfrak{m}_{\eta+1}, \mathfrak{m}_{\eta+2}) + \cdots + d(\mathfrak{m}_{m-1}, \mathfrak{m}_m),$$

that is

$$d(\mathfrak{m}_\eta, \mathfrak{m}_m) \leq \sum_{i=\eta}^{m-1} d(\mathfrak{m}_i, \mathfrak{m}_{i+1}) \leq \sum_{i=\eta}^{\infty} d(\mathfrak{m}_i, \mathfrak{m}_{i+1}) \leq \sum_{i=\eta}^{\infty} \frac{1}{i^{\frac{1}{k}}}.$$

For any given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(\mathfrak{m}_\eta, \mathfrak{m}_m) \leq \sum_{i=\eta}^{\infty} \frac{1}{i^{\frac{1}{k}}} < \varepsilon, \text{ for all } \eta \geq N.$$

Hence $\{\mathfrak{m}_\eta\}$ is a left $\mathcal{K}-$Cauchy sequence in $\mathcal{M}$. Similarly, we can prove that $\{s_\eta\}$ is a left $\mathcal{K}-$Cauchy sequence in $\mathcal{S}$. So there exists $z \in \mathcal{M}$ such that $d(z, \mathfrak{m}_\eta) \to 0$ as $\eta \to \infty$. On the other hand, since $s_{\eta+1} \in \mathcal{L}\mathfrak{m}_\eta$, so we get

$$\begin{aligned} d(\mathfrak{m}_\eta, \mathcal{L}\mathfrak{m}_\eta) &\leq d(\mathfrak{m}_\eta, s_{\eta+1}) \\ &\leq d(\mathfrak{m}_\eta, \mathfrak{m}_{\eta+1}) + d(\mathfrak{m}_{\eta+1}, s_{\eta+1}) \\ &\leq d(\mathfrak{m}_\eta, \mathfrak{m}_{\eta+1}) + d(\mathcal{M}, \mathcal{S}), \end{aligned}$$

letting $\eta \to \infty$, we get

$$\lim_{\eta\to\infty} d(\mathfrak{m}_\eta, \mathcal{L}\mathfrak{m}_\eta) = (\mathcal{M}, \mathcal{S}).$$

Since f is lower semi-continuous with respect to τ_d, so

$$d(z, \mathcal{L}z) = f(z) = \liminf_{\eta\to\infty}(f(\mathfrak{m}_\eta)) = \liminf_{\eta\to\infty} d(\mathfrak{m}_\eta, \mathcal{L}\mathfrak{m}_\eta).$$

that is,

$$d(z, \mathcal{L}z) = d(\mathcal{M}, \mathcal{S}).$$

Hence z is the LBPP of $\mathcal{L}$. This completes the proof. □

Example 1. *Let* $\mathcal{G} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$, $\mathcal{M} = \left\{1, \frac{1}{2}\right\}$, $\mathcal{S} = \left\{\frac{1}{4}, \frac{1}{5}\right\}$. *Define $\mathcal{Q}-$metric as follows*

$$d(\mathfrak{m}, s) = \left\{ \begin{array}{l} \mathfrak{m} - s \textit{ if } \mathfrak{m} \geq s \\ \frac{1}{3} \textit{ if } \mathfrak{m} < s \end{array} \right\}.$$

Then $d(\mathcal{M},\mathcal{S}) = \frac{1}{4}$, $\mathcal{M}_0 = \left\{\frac{1}{2}\right\}$, $\mathcal{S}_0 = \left\{\frac{1}{4}\right\}$. *Define* $\mathcal{L} : \mathcal{M} \to 2^{\mathcal{S}} \backslash \phi$ *as follows*

$$\mathcal{L}(1) = \left\{\frac{1}{4}, \frac{1}{5}\right\}, \text{ and } \mathcal{L}\left(\frac{1}{2}\right) = \left\{\frac{1}{4}\right\}.$$

Take $\mathcal{F}(\alpha) = \ln(\alpha)$ *and* $\tau = 0.01$. *Now we check* $\mathcal{L}$ *is* $\mathcal{F}_d$*−contraction. For this we discuss all possible cases.*

Case 1: $\mathfrak{m}_1 = 1$, $\mathfrak{m}_2 = \frac{1}{2}$, $d\left(1, \frac{1}{2}\right) > 0$ *implies for* $u = \frac{1}{5} \in \mathcal{L}(1)$ *there exists* $v = \frac{1}{4} \in \mathcal{L}\left(\frac{1}{2}\right)$ *such that*

$$d(u, v) = \frac{1}{3} > 0,$$

it implies $\tau + \mathcal{F}(d(u, v)) = -1.089 < -0.69 = \mathcal{F}(M(\mathfrak{m}_1, \mathfrak{m}_2))$.

Case 2: $\mathfrak{m}_1 = \frac{1}{2}$, $\mathfrak{m}_2 = 1$, $d\left(\frac{1}{2}, 1\right) > 0$ *implies for* $u = \frac{1}{4} \in \mathcal{L}\left(\frac{1}{2}\right)$ *there exists* $v = \frac{1}{5} \in \mathcal{L}(1)$ *such that*

$$d(u, v) = \frac{1}{20} > 0,$$

it implies $\tau + \mathcal{F}(d(u, v)) = -2.9857 < -1.0986 = \mathcal{F}(M(\mathfrak{m}_1, \mathfrak{m}_2))$.

From above all cases it is clear that $\mathcal{L}$ *is* $\mathcal{F}_d$*−contraction. So all axioms of Theorem 3 hold. There exists* $\mathfrak{m} = \frac{1}{2} \in \mathcal{M}$ *such that*

$$d\left(\frac{1}{2}, \mathcal{L}\left(\frac{1}{2}\right)\right) = \frac{1}{4}.$$

Hence $\mathfrak{m} = \frac{1}{2}$ *is a LBPP of* $\mathcal{L}$.

Remark 3. *If we replace d by* d^{-1} *in the Definition (1.1) then we can get the following result for the existence of RBPP.*

Theorem 4. *Let* $(\mathcal{G}, d)$ *be a right* $\mathcal{K}$*−complete* $T_1 - \mathcal{Q}$ *metric space.* $\mathcal{L} : \mathcal{M} \to C_B(\mathcal{S})$ *be a generalized* $\mathcal{F}_{d^{-1}}$*−contraction satisfying the following axioms:*

i) *For each* $\mathfrak{m} \in \mathcal{M}_0$ *we have* $\mathcal{L}(\mathfrak{m}) \subseteq \mathcal{S}_0$, *and the pair* $(\mathcal{S}, \mathcal{M})$ *satisfies the* $p_{d^{-1}}$*−property.*

ii) *For* $\mathfrak{m}_0, \mathfrak{m}_1 \in \mathcal{M}_0$ *there exists* $s_1 \in \mathcal{L}\mathfrak{m}_0$ *such that*

$$d^{-1}(\mathfrak{m}_0, \mathfrak{m}_1) > 0 \text{ and } d(s_1, \mathfrak{m}_1) = d(\mathcal{S}, \mathcal{M}).$$

Then $\mathcal{L}$ *has a RBPP provided that the function* $f(\mathfrak{m}) = d(\mathcal{L}\mathfrak{m}, \mathfrak{m})$ *is a lower semicontinuous with respect to* $\tau_{d^{-1}}$.

Proof. As $(\mathcal{G}, d)$ is a right $\mathcal{K}$−complete, so $(\mathcal{G}, d^{-1})$ is a left $\mathcal{K}$−complete. Further, the pair $(\mathcal{S}, \mathcal{M})$ has the $p_{d^{-1}}$−property, implies that $(\mathcal{M}, \mathcal{S})$ has p_d−property. Result follows from Theorem 3. □

Example 2. *If in the Example 1, we replace $\mathcal{Q}$–metric d by*

$$d(\mathfrak{m}, s) = \left\{ \begin{array}{l} s - \mathfrak{m} \text{ if } s \geq \mathfrak{m}, \\ \frac{1}{3} \text{ if } s < \mathfrak{m}, \end{array} \right\}$$

then we get $\mathfrak{m} = \frac{1}{2}$ *as a RBPP.*

Now we derive LBPP and RBPP results using $\mathcal{F}_{d_*}$–contraction as follows:

Theorem 5. *Let $(\mathcal{G}, d)$ be a left $\mathcal{K}$–complete $T_1 - \mathcal{Q}$ metric. $\mathcal{L} : \mathcal{M} \to C_B(\mathcal{S})$ be a generalized multivalued $\mathcal{F}_{d_*}$–contraction satisfying the following axioms:*

i) *For each $\mathfrak{m} \in \mathcal{M}_0$ we have $\mathcal{L}(\mathfrak{m}) \subseteq \mathcal{S}_0$, and the pair $(\mathcal{M}, \mathcal{S})$ satisfies the p_{d_*}–property.*

ii) *For $\mathfrak{m}_0, \mathfrak{m}_1 \in \mathcal{M}_0$ there exists $s_1 \in \mathcal{L}\mathfrak{m}_0$ such that*

$$d(\mathfrak{m}_0, \mathfrak{m}_1) > 0 \text{ and } d(\mathfrak{m}_1, s_1) = d(\mathcal{M}, \mathcal{S}).$$

Then $\mathcal{L}$ has a LBPP provided that the function $f(\mathfrak{m}) = d(\mathfrak{m}, \mathcal{L}\mathfrak{m})$ is a lower semicontinuous with respect to τ_d.

Proof. Using p_{d_*}–property the result follows on the similar lines as in Theorem 3. □

Example 3. *Let* $\mathcal{G} = \left\{\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12}, \frac{1}{15}\right\}$, $\mathcal{M} = \left\{\frac{1}{3}, \frac{1}{6}\right\}$, $\mathcal{S} = \left\{\frac{1}{12}, \frac{1}{15}\right\}$. *Define $\mathcal{Q}$–metric as follows*

$$d(\mathfrak{m}, s) = \left\{ \begin{array}{l} \mathfrak{m} - s \text{ if } \mathfrak{m} \geq s, \\ \frac{1}{9} \text{ if } \mathfrak{m} < s \end{array} \right\}.$$

Then $d(\mathcal{M}, \mathcal{S}) = \frac{1}{12}$, $\mathcal{M}_0 = \left\{\frac{1}{6}\right\}$, $\mathcal{S}_0 = \left\{\frac{1}{12}\right\}$ *Define* $\mathcal{L} : \mathcal{M} \to 2^{\mathcal{S}} \backslash \phi$ *as follows*

$$\mathcal{L}(\frac{1}{3}) = \left\{\frac{1}{12}\right\}, \text{ and } \mathcal{L}\left(\frac{1}{6}\right) = \left\{\frac{1}{12}, \frac{1}{15}\right\}.$$

Take $\mathcal{F}(\alpha) = \ln(\alpha)$ *and* $\tau = 0.1$. *Now we check $\mathcal{L}$ is $\mathcal{F}_{d_*}$–contraction. For this we discuss all possible cases.*

Case 1: $\mathfrak{m}_1 = \frac{1}{3}$, $\mathfrak{m}_2 = \frac{1}{6}$, $d\left(\frac{1}{3}, \frac{1}{6}\right) = \frac{1}{6} > 0$ *implies for* $u = \frac{1}{12} \in \mathcal{L}(\frac{1}{3})$ *there exists* $v = \frac{1}{15} \in \mathcal{L}\left(\frac{1}{6}\right)$ *such that*

$$d(v, u) = \frac{1}{9} > 0,$$

it implies $\tau + \mathcal{F}(d(v, u)) = -2.0972 < -1.7917 = \mathcal{F}(M(\mathfrak{m}_1, \mathfrak{m}_2))$.

Case 2: $\mathfrak{m}_1 = \frac{1}{6}$, $\mathfrak{m}_2 = \frac{1}{3}$, $d\left(\frac{1}{6}, \frac{1}{3}\right) = \frac{1}{9} > 0$ *implies for* $u = \frac{1}{15} \in \mathcal{L}\left(\frac{1}{6}\right)$ *there exists* $v = \frac{1}{12} \in \mathcal{L}(\frac{1}{3})$ *such that*

$$d(v, u) = \frac{1}{60} > 0,$$

it implies $\tau + \mathcal{F}(d(v,u)) = -3.9943 < -1.7917 = \mathcal{F}(M(\mathfrak{m}_1, \mathfrak{m}_2))$.

From above all cases it is clear that $\mathcal{L}$ *is* $\mathcal{F}_{d_*}$*−contraction. So all axioms of Theorem 5. There exists* $\mathfrak{m} = \frac{1}{6} \in \mathcal{M}$ *such that*

$$d\left(\frac{1}{6}, \mathcal{L}\left(\frac{1}{6}\right)\right) = \frac{1}{12}.$$

Hence $\mathfrak{m} = \frac{1}{6}$ *is a LBPP of* $\mathcal{L}$.

Remark 4. *If we replace* d *by* d^{-1} *in the Definition (1.2) then we have the following result for the existence of RBPP.*

Theorem 6. *Let* $(\mathcal{G}, d)$ *be a right* $\mathcal{K}$*−complete* $T_1 - Q$ *metric space.* $\mathcal{L} : \mathcal{M} \to C_B(\mathcal{S})$ *be a generalized* $\mathcal{F}_{d_*^{-1}}$*−contraction satisfying the following axioms:*

i) *For each* $\mathfrak{m} \in \mathcal{M}_0$ *we have* $\mathcal{L}(\mathfrak{m}) \subseteq \mathcal{S}_0$*, and the pair* $(\mathcal{S}, \mathcal{M})$ *satisfies the* $p_{d_*^{-1}}$*−property.*

ii) *For* $\mathfrak{m}_0, \mathfrak{m}_1 \in \mathcal{M}_0$ *there exists* $s_1 \in \mathcal{L}\mathfrak{m}_0$ *such that*

$$d^{-1}(\mathfrak{m}_1, \mathfrak{m}_0) > 0 \text{ and } d(s_1, \mathfrak{m}_1) = d(\mathcal{S}, \mathcal{M}).$$

Then $\mathcal{L}$ *has a RBPP provided that the function* $f(\mathfrak{m}) = d(\mathcal{L}\mathfrak{m}, \mathfrak{m})$ *is a lower semicontinuous with respect to* $\tau_{d^{-1}}$.

Proof. Proof follows the similar lines as in Theorem 4. □

Example 4. *If in the Example 3, we replace* Q*−metric* d *by*

$$d(\mathfrak{m}, s) = \begin{Bmatrix} s - \mathfrak{m} \text{ if } s \geq \mathfrak{m}, \\ \frac{1}{9} \text{ if } s < \mathfrak{m} \end{Bmatrix},$$

then we get $\mathfrak{m} = \frac{1}{6}$ *as a RBPP of* $\mathcal{L}$.

Now we derive some results of best proximity points and fixed points from our main results.

Corollary 1. *Let* $(\mathcal{G}, d)$ *be a complete metric space.* $\mathcal{L} : \mathcal{M} \to C_B(\mathcal{S})$ *be a generalized multivalued* $\mathcal{F}_d$*−contraction satisfying the following axioms:*

i) *For each* $\mathfrak{m} \in \mathcal{M}_0$ *we have* $\mathcal{L}(\mathfrak{m}) \subseteq \mathcal{S}_0$*, and the pair* $(\mathcal{M}, \mathcal{S})$ *satisfies the* p_d*−property.*

ii) *For* $\mathfrak{m}_0, \mathfrak{m}_1 \in \mathcal{M}_0$ *there exists* $s_1 \in \mathcal{L}\mathfrak{m}_0$ *such that*

$$d(\mathfrak{m}_0, \mathfrak{m}_1) > 0 \text{ and } d(\mathfrak{m}_1, s_1) = d(\mathcal{M}, \mathcal{S}).$$

Then $\mathcal{L}$ *has a BPP provided that the function* $f(\mathfrak{m}) = d(\mathfrak{m}, \mathcal{L}\mathfrak{m})$ *is a lower semicontinuous with respect to* τ_d.

Corollary 2. *Let $(\mathcal{G}, d)$ be a complete metric space, $\mathcal{L} : \mathcal{M} \to C_B(\mathcal{S})$ a generalized multivalued $\mathcal{F}_{d_*}-$contraction satisfying the following axioms:*

i) *For each $\mathfrak{m} \in \mathcal{M}_0$ we have $\mathcal{L}(\mathfrak{m}) \subseteq \mathcal{S}_0$, and the pair $(\mathcal{M}, \mathcal{S})$ satisfies the $p_{d_*}-$property.*

ii) *For $\mathfrak{m}_0, \mathfrak{m}_1 \in \mathcal{M}_0$ there exists $s_1 \in \mathcal{L}\mathfrak{m}_0$ such that*

$$d(\mathfrak{m}_0, \mathfrak{m}_1) > 0 \text{ and } d(s_1, \mathfrak{m}_1) = d(\mathcal{S}, \mathcal{M}).$$

Then $\mathcal{L}$ has a BPP provided that the function $f(\mathfrak{m}) = d(\mathfrak{m}, \mathcal{L}\mathfrak{m})$ is a lower semicontinuous with respect to τ_d.

Corollary 3. *Let $(\mathcal{G}, d)$ be a left $\mathcal{K}-$complete $T_1 - Q$ metric. $\mathcal{L} : \mathcal{G} \to C_B(\mathcal{G})$ be a generalized multivalued $\mathcal{F}_d-$contraction then $\mathcal{L}$ has a fixed point provided that $f(g) = d(g, \mathcal{L}g)$ is lower semicontinuous with respect to τ_d.*

Corollary 4. *Let $(\mathcal{G}, d)$ be a right $\mathcal{K}-$complete $T_1 - Q$ metric. $\mathcal{L} : \mathcal{G} \to C_B(\mathcal{G})$ be a generalized $\mathcal{F}_{d_*}-$contraction then $\mathcal{L}$ has a fixed point provided that $f(g) = d(g, \mathcal{L}g)$ is lower semicontinuous with respect to τ_d.*

Remark 5. *If we replace generalized multivalued $\mathcal{F}_d-$contraction by multivalued $\mathcal{F}_d-$contraction, then Theorem 2 becomes the corollary of Corollary 3.*

3. Conclusions

In this paper, we obtained left and right best proximity points of generalized multivalued $\mathcal{F}-$contractions of quasi metric spaces. On quasi metric spaces, due to the lack of symmetry property, left and right versions of best proximity points and $p-$properties have been introduced and all these versions reduce to their metric analogues. In the literature, there are not many instances of best proximity point theorems in quasi metric spaces and the results in this article will open up more directions for further research in the best proximity point theory of asymmetric distance spaces, for instance we can develop semi best proximity points in partially ordered quasi metric spaces and quasi metric spaces endowed with a graphical structure.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they do not have any conflict of interests regarding this paper.

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