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Research article

Fractional resolvent family generated by normal operators

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Abstract: The main focus of this paper is on the relationship between the spectrum of generators and the regularity of the fractional resolvent family. We will give a counter-example to show that the point-spectral mapping theorem is not valid for $\{S_\alpha(t)\}$ if $\alpha \neq 1$; and we show that if $\{S_\alpha(t)\}$ is stable, then we can determine the decay rate by $\sigma(A)$ and some examples are given; we also prove that $S_\alpha(t)x$ has a continuous derivative of order $\alpha\beta > 0$ if and only if $x \in D(I - A)^\beta$. The main method we used here is the resolution of identity corresponding to a normal operator A and spectral measure integral.

Keywords: fractional resolvent family; normal operator; spectral mapping theorem, stable resolvent
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1. Introduction

Let $\alpha \in (0, 2]$, an α-time resolvent family $S_\alpha(t)$ gives the solution to the α-order Cauchy problems

$$\begin{aligned} D_t^\alpha u(t) &= Au(t),\ t > 0, \\ u(0) &= x, \quad (\text{ in addtion } u'(0) = 0 \text{ if } \alpha > 1) \end{aligned}$$

by $u(t) = S_\alpha(t)x$, where D_t^α is the Caputo derivative of order α. See [28] for definitions and properties of fractional derivatives and fractional differential equations. This definition of fractional resolvent calculus has many applications, such as [16, 24] for abstract Cauchy problems and [30, 31] for engineering applications, and some applications on numerical simulations are given in [32, 33].

The main results of this article are the following three aspects: First, we give a counter-example of the point-spectral mapping theorem; second, we give the constant estimate of decay estimate of fractional resolvent family, and some examples are given at the end of this section. Third, we prove that $D_t^{\alpha\beta} S_\alpha(t)$ exist and continuous iff $x \in D((I - A)^\beta)$.

The main method we used in this paper is the resolution of identity corresponding to a normal operator A. By [25, Theorem 13.33], every normal operator A has a unique spectral decomposition and

$$A = \int_{\sigma(A)} \lambda dP(\lambda). \tag{1.1}$$

If A generates a bounded fractional resolvent family $\{S_\alpha(t)\}$, by using the properties of $P(\lambda)$ and $E_\alpha(t)$ we can represent $\{S_\alpha(t)\}$ as follows:

$$S_\alpha(t) = \int_{\sigma(A)} E_\alpha(\lambda t^\alpha) dP(\lambda). \tag{1.2}$$

It should be noticed that the above representation is a special case of functional calculus [15, 25], which has been widely used to deal with semigroup problems, such as decay estimate, continuation, approximation, and resolvent representation, etc. More details about these topics can be found in [3, 6, 10–13]. The main question addressed in this article are: How to use this integral representation to show the relationship between the spectrum of the generator and the regularity of the fractional resolvent family, and the main results can be summarized as follows.

The first one is the spectral mapping theorem of the resolvent family. In [8, Section 4.3], the authors discuss this problem in semigroup sense in detail and give a large number of examples to show that the conditions of theorems are optimal in some cases. In [19], authors proved that the spectral inclusion theorem is valid for the fractional resolvent family.

Theorem 1.1. [19, Theorem 3.2] *Suppose that there is an α-times resolvent family $\{S_\alpha(t)\}$ for A, where $\alpha \in (0, 2]$, then*
(1) $E_\alpha(t^\alpha \sigma(A)) \subseteq \sigma(S_\alpha(t))$.
(2) $E_\alpha(t^\alpha \sigma_{ap}(A)) \subseteq \sigma_{ap}(S_\alpha(t))$.
(3) $E_\alpha(t^\alpha \sigma_p(A)) \subseteq \sigma_p(S_\alpha(t))$.
(4) $E_\alpha(t^\alpha \sigma_r(A)) \subseteq \sigma_r(S_\alpha(t))$.

Since the spectral mapping theorem is closely connected with the stability of the resolvent family and the decay rate can be given by spectral bound by using spectral mapping theorem (for example, [8, Proposition 1.7, Lemma 1.9]), it is very important to prove the spectral mapping theorem or construct a counter-example. Our first result is that we constrcut a normal operator A, which generates a fractional resolvent family $\{S_\alpha(t)\}$, such that $\lambda \in \sigma_{ap}(A)/\sigma_p(A)$ but $E_\alpha(\lambda) \in \sigma_p(S_\alpha(1))$.

Another topic we discussed here is decay estimate, which is also an important subject. There are numerous articles and books discussing this subject and giving very detailed results, but we only mentioned one of these results here,

Theorem 1.2. [2, Theorem 5.1.9] *Let T be a C_0-semigroup on Banach space X with generator A. Then*

$$s(A) = hol(\hat{T}) \leq \omega_1(T) = abs(T) \leq s_0(A) \leq \omega(T). \tag{1.3}$$

An important question is whether we can prove a similar theorem for fractional resolvent families. The decay estimate of the fractional resolvent family has been given in many pieces of literature ([20, Proposition 3.3] and [21, Proposition 3.1]) and they show this theorem does not hold for the fractional resolvent family in general. In this paper, we proved that if fractional resolvent family $\{S_\alpha(t)\}$ generated by a normal operator A is stable, then it is polynomial stable and the constants are determined by spectral bound $s(A)$, which can be seen as the Datko-Pazy theorem for fractional resolvent family. Additionally, we give some applications of this decay estimate.

The third one is the partial answer to the question: under which condition can $S_\alpha(t)x$ have a continuous derivative of order $\beta > 0$? It was proved by [17] that $t \to S_1(t)x$ has a continuous fractional derivative of order $\alpha > 0$ if and only if x belongs to $D(bI - A)^\alpha$. In [9], the author proved that for $\{S_2(t)\}$, the strongly continuous cosine functions, if $S_2(t)x$ has a continuous Riemann-Liouville fractional derivative of order $\alpha \neq n = \frac{1}{2}, n = 0.1.2.....$, then $x \in D(bI - A)^\alpha$.

Theorem 1.3. [9, Theorem 3.1] *Let* $\alpha > 0$*,* $\alpha \neq n + \frac{1}{2}$*,* $n = 0, 1, ...$ *Then*

$$E^-_{2\alpha,\beta} \subseteq F_\alpha, E^+_{2\alpha,\beta} \subseteq F_\alpha.$$

If we consider the Caputo fractional derivatives and let $\alpha \in (0, 2)$, suppose $\{S_\alpha(t)\}$ is the fractional resolvent family generated by normal operator A. Then, by using spectral measure presentation, we can prove that $S_\alpha(t)x$ has a continuous derivative of order $\alpha\beta > 0$ if and only if $x \in D(I - A)^\beta$.

This paper is organized as follows. In Section 2, we give some necessary definitions and properties of Mittag-Leffler functions and fractional resolvent families. Section 3 focuses on the conditions for the generation of resolvent families by normal operators and the representation of resolvent families. Proofs of the above main results are given in Section 4, together with some examples.

2. Preliminaries

2.1. Basic notations

Throughout this paper, H is a separable Hilbert space, and $L(H)$ is the Banach algebra of all bounded linear operators on H. We always assume that A is a closed unbounded operator, densely defined on H. We will denote by $N(A)$, $D(A)$, and $R(A)$ the kernel, domain, and range of A respectively. Additionally, by $\rho(A)$, $\sigma(A)$, $\sigma_p(A)$, $\sigma_r(A)$ we denote the resolvent set, spectrum, point spectrum and residual spectrum of A, respectively. $R(\lambda, A) := (\lambda - A)^{-1}$ means the resolvent of A at λ if $\lambda \in \rho(A)$, and notation $s(A)$ means the spectral bound of $\sigma(A)$, $s(A) := \sup\{\Re(\lambda)|\lambda \in \sigma(A)\}$; by the Hahn-Banach theorem $\sigma_r(A) = \sigma_p(A^*)$, where A^* is the adjoint of A. By $W(A)$ we denote the numerical range of A if A is defined on Hilbert space H,

$$W(A) = \{\langle Ax, x\rangle \in \mathbb{C} | x \in D(A), \|x\| = 1\}.$$

And sector Σ_θ is defined as

$$\Sigma_\theta = \{\lambda \in \mathbb{C} | \lambda \neq 0 \quad and \quad |arg\lambda| < \theta\}$$

for $\theta \in (0, \pi)$ and $\Sigma_0 = (0, \infty)$.

2.2. Special functions

We recall two important functions in the theory of fractional calculus. For details of these special functions and the general theory of fractional calculus, we refer to [1, 24] and the references therein.

The Mittag-leffler function $E_{\alpha,\beta}(z)$ is defined by

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^\beta}{\Gamma(\alpha n + \beta)} = \frac{1}{2\pi i} \int_{H_a} \frac{\mu^{\alpha-\beta} e^\mu}{\mu^\alpha - z} d\mu, \quad z \in \mathbb{C},$$

where $\alpha, \beta > 0$, H_a is the Hankel contour which starts and ends at $-\infty$, and encircles the disc $|t| \leq |z|^{\frac{1}{\alpha}}$ counter clockwise. We use $E_\alpha(t) := E_{\alpha,1}(t)$ for short. The Mittag-Leffler function $E_\alpha(t)$ satisfies the fractional differential equation

$$D_t^\alpha E_\alpha(\omega t^\alpha) = \omega E_\alpha(\omega t^\alpha),$$

where D_t^α is the Caputo derivative of α-order (see [4]). The most important properties of this function are associated with their Laplace integral

$$\int_0^\infty e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(st^\alpha)dt = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - s}, \quad \Re(\lambda) > |s|^{\frac{1}{\alpha}},$$

and their asymptotic expansion as $z \to \infty$. For $0 < \alpha < 2$ and $\beta = 1$,

Proposition 2.1. [24, Proposition 3.5] *Let* $\alpha \in (0, 2)$ *and*

$$\frac{\alpha\pi}{2} < \theta < \min\{\pi, \alpha\pi\}.$$

Then we have the following asymptotics for formulas in which N is an arbitrary positive integer

$$E_\alpha(z) = \frac{1}{\alpha} \exp(z^{\frac{1}{\alpha}}) + \epsilon_\alpha(z), \quad |\arg z| \leq \theta, \quad |z| \to \infty, \tag{2.1}$$

$$E_\alpha(z) = \epsilon_\alpha(z), \qquad \theta \leq |\arg(z)| \leq \pi, \qquad |z| \to \infty, \tag{2.2}$$

where

$$\epsilon_\alpha(z) = -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(1 - \alpha n)} + O(|z|^{-N}).$$

From the asymptotic expansion one knows that $E_\alpha(-\omega t^\alpha) = O(t^{-\alpha})$ as $t \to \infty$ when $\omega > 0$.

2.3. Resolvent family

Here we define fractional resolvent families and list some basic properties [4, 18].

Definition 2.2. Let $0 < \alpha \leq 2$, a family $\{S_\alpha(t)\}_{t\geq 0} \subset L(X)$ is called an α-times resolvent family generated by A if the following conditions are satisfied:

(1) $S_\alpha(t)$ is strongly continuous for $t \geq 0$ and $S_\alpha(0) = I$;

(2) $S_\alpha(t)A \subset AS_\alpha(t)$ for $t \geq 0$;

(3) for $x \in D(A)$, the resolvent equation

$$S_\alpha(t)x = x + A\int_0^t g_\alpha(t-s)S_\alpha(t)xds$$

holds for all $t \geq 0$, where $g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$.

By this definition, we know that a 1-times resolvent family is exactly a C_0-semigroup, and a 2-times resolvent family is a cosine operator.

Definition 2.3. An α-times resolvent family $S_\alpha(t)$ is said to be exponentially bounded if there exists a constant $M \geq 1$ and $\omega \geq 0$ such that $\|S_\alpha(t)\| \leq Me^{\omega t}$ for every $t \geq 0$. $S_\alpha(t)$ is called bounded if ω can be taken as 0, i.e., $\|S_\alpha(t)\| \leq M$ for all $t \geq 0$.

Let $\theta_0 \in (0, \pi]$, an α-times resolvent family $S_\alpha(t)$ is called analytic of angle θ_0 if $S_\alpha(t)$ admits an analytic extension to the sectorial sector $\Sigma_{\theta_0} := \{z \in \mathbb{C} : z \neq 0 \text{ and } |\arg z| < \theta_0\}$. An analytic α-times resolvent family $S_\alpha(t)$ is called to be bounded if $\|S_\alpha(z)\|$ is uniformly bounded for $z \in \Sigma_\theta$ for any $0 < \theta < \theta_0$.

Lemma 2.4. [4, Theorems 2.8 and 2.9] *Let* $0 < \alpha \leq 2$, $S_\alpha(t)$ *be an* α*-times resolvent family generated by A. Then* $\|S_\alpha(t)\| \leq Me^{\omega t}$ *for every* $t \geq 0$ *if and only if* $(\omega^\alpha, \infty) \in \rho(A)$ *and*

$$\left\|\frac{d^n}{d\lambda^n}(\lambda^{\alpha-1}R(\lambda^\alpha, A))\right\| \leq \frac{Mn!}{(\lambda - \omega)^{n+1}}, \quad \lambda > \omega, n \in \mathbb{N}_0.$$

In this case $\{\lambda^\alpha : \operatorname{Re}\lambda > \omega\} \subset \rho(A)$ *and*

$$\lambda^{\alpha-1}R(\lambda^\alpha, A)x = \int_0^\infty e^{-\lambda t}S_\alpha(t)x\,dt, \quad \Re(\lambda) > \omega$$

for every $x \in X$*. In particular, if* $S_\alpha(t)$ *is bounded, then* $\sup_{\lambda>0}\|\lambda R(\lambda, A)\| < \infty$.

The relationship between the generator of the analytic bounded resolvent operator and the sectorial operator can be narrated as follows.

Lemma 2.5. [5, Lemma 2.7] *Let* $\alpha \in (0, 2)$ *and* $\theta_0 \in (0, min\left\{\frac{\pi}{2}, \frac{\pi}{\alpha} - \frac{\pi}{2}\right\})$*. The following assertions are equivalent.*

(1) *A generates a bounded analytic* α*-times resolvent operator of angle* θ_0.

(2) $\Sigma_{\alpha(\frac{\pi}{2}+\theta)} \in \rho(A)$ *and for every* $\theta \in (0, \theta_0)$ *there is a constant* M_θ *such that*

$$\|\lambda R(\lambda, A)\| \leq M_\theta, \quad \lambda \in \Sigma_{\alpha(\frac{\pi}{2}+\theta)}.$$

(3) $-A \in sect(\pi - \alpha(\frac{\pi}{2} + \theta_0))$.

The following subordination principle is a very powerful tool. For a more general subordination principle for the fractional powers of the generators see [18], and for regularized resolvent families see [1].

Lemma 2.6. *(Subordination principle) Let* $0 < \beta < \alpha \leq 2$*. If A generates an exponentially bounded* α*-times resolvent family* $S_\alpha(t)$*, then A generates exponentially bounded analytic* β*-times resolvent family* $S_\beta(t)$ *which is subordinated to* $S_\alpha(t)$ *by*

$$S_\beta(t) = \int_0^\infty t^{-\frac{\beta}{\alpha}}W_{-\frac{\beta}{\alpha},1-\frac{\beta}{\alpha}}(st^{-\frac{\beta}{\alpha}})S_\alpha(s)ds, \quad t > 0. \tag{2.3}$$

Moreover, if $S_\alpha(t)$ *is bounded, then* $S_\beta(t)$ *is analytic and bounded in a sector with an angle smaller than* $(\alpha/\beta - 1)\pi/2$.

Where $W_{-\frac{\beta}{\alpha},1-\frac{\beta}{\alpha}}(st^{-\frac{\beta}{\alpha}})$ is the Wright-type function, for details of this function, we refer to [1,24,28].

3. Fractional resolvent family generated by normal operators

Let (Ω, Σ, μ) be a σ-finite measure space. Let $1 \le p < \infty$, define Banach space $X := L^p(\Omega, \mu)$, and suppose q is a measurable function on X, define set $q_{ess}(\Omega)$:

$$q_{ess}(\Omega) := \{\lambda \in \mathbb{C} : \mu(\{s \in \Omega : |q(s) - \lambda| < \epsilon\}) \neq 0, \forall \epsilon > 0\}$$

be the essential range of function q. Using function q we can define a multiplication operator M_q on X.

$$M_q f := q \cdot f \quad f \in D(M_q) := \{f \in X : q \cdot f \in X\}. \tag{3.1}$$

Some properties of the multiplication operator are summarized as follows.

Lemma 3.1. [8, Proposition 4.10] *Let M_q be the multiplication operator on $X = L^p(\Omega, \mu)$ defined by measurable function q and (3.1), the following conclusion is valid:*

(1) *M_q is a closed operator with a dense domain.*

(2) *M_q is a bounded operator if and only if q is an essential bounded function, that is, essential range $q_{ess}(\Omega)$ is a bounded set, and*

$$\|M_q\| = \|q\|_\infty := \sup\{|\lambda| : \lambda \in q_{ess}(\Omega)\}.$$

(3) *The spectral of M_q is equal to the essential range of q.*

Next, we give a conclusion about the generation of resolvent families by multiplication operators.

Theorem 3.2. *Let $0 < \alpha < 2$ and q is a measurable function, $q : \Omega \mapsto \mathbb{C}$, if*

$$\widetilde{q} := \sup\{\Re(q(x)^{\frac{1}{\alpha}}) : q(x) \in q_{ess}(\Omega) \cap \overline{\Sigma_{\frac{\alpha\pi}{2}}}, x \in \Omega\} < \infty.$$

Then the operator family $S^q_\alpha(t)$ defined by

$$S^q_\alpha(t) g := E_\alpha(t^\alpha q) g, \quad g \in L^p(\Omega, \mu)$$

is a α-times resolvent family generated by M_q. And $S^q_\alpha(t)$ is uniformly bounded if and only if $q_{ess} \subseteq \mathbb{C} - \Sigma_{\frac{\alpha\pi}{2}}$.

Proof. By asymptotic estimate of Mittag-Leffler function (2.2),

$$\sup\{|E_\alpha(z)| : |\arg(z)| \ge \frac{\alpha\pi}{2}\} < \infty,$$

and when $z \to \infty$,

$$E_\alpha(z) = O(e^{\Re(z^{\frac{1}{\alpha}})}), \quad |\arg(z)| \le \frac{\alpha\pi}{2}.$$

Because $\widetilde{q} < \infty$, by Lemma 3.1, operator family $S^q_\alpha(t)$ is exponentially bounded,

$$\|S^q_\alpha(t)\| = \|E_\alpha(t^\alpha q)\|_\infty \le M e^{t\widetilde{q}}.$$

Since

$$\|S^q_\alpha(t) g - g\|^p = \int_\Omega |E_\alpha(t^\alpha q(x)) - 1|^p \cdot |g(x)|^p dx,$$

so the strong convergence of $S^q_\alpha(t)$ can be proved directly by the dominant convergence theorem. Then it is easy to see that operator family $S^q_\alpha(t)$ is an α-times resolvent family generated by M_q. $\square$

The following unitary isomorphism theorem is a classical theorem describing normal operators.

Theorem 3.3. [15, Appendix D, Spectral Theorem] *Suppose operator A is a normal operator on H, then there exist a σ-finite measure space (Ω, Σ, μ) and measurable function $q : \Omega \mapsto \mathbb{C}$, such that operator A is unitary isomorphic to a multiplication operator M_q defined on $L^2(\Omega, \mu)$. This means there exists a unitary operator $U \in L(H, L^2(\Omega, \mu))$, such that*

$$A = U^* M_q U = U^{-1} M_q U,$$

and $\sigma(A) = \sigma(M_q) = q_{ess}(\Omega)$.

From the above discussion, it can be seen that if the normal operator A unitary isomorphic to the multiplication operator M_q which is defined on $L^2(\Omega, \mu)$, and $q_{ess}(\Omega) \subseteq \mathbb{C} - \Sigma_{\frac{\alpha}{2}\pi}$, then A generates a bounded α-times resolvent family. Additionally, the converse can be proved directly by Theorems 2.4 and 3.3.

Recall that for a normal operator A, there is a unique resolution of identity $P(\lambda)$, which satisfies

$$\langle Ax, y\rangle = \int_{\sigma(A)} \lambda d\langle P(\lambda)x, y\rangle, x \in D(A), y \in H. \tag{3.2}$$

Then for every measureble function $f : \sigma(A) \to \mathbb{C}$, we can define operator $f(A)$ as follows:

$$f(A) := \int_{\sigma(A)} f(\lambda) dP(\lambda),$$

with domain

$$D(f(A)) := \{x \in H : \int_{\sigma(A)} |f(\lambda)| d\langle P(\lambda)x, x\rangle < \infty\}.$$

For more information about the resolution of identity and proofs please refer to [25, Section 13], especially [25, Lemma 13.22, Theorems 13.23 and 13.24].

Now suppose that A is a normal operator with $\sigma(A) \subseteq \mathbb{C} - \Sigma_{\frac{\alpha}{2}\pi}$, since $E_\alpha(t^\alpha \lambda)$ is bounded in $\sigma(A)$, therefore we can define an operator family $\{E_\alpha(t^\alpha A)\}_{t \geq 0}$ with domain $D(E_\alpha(t^\alpha A)) = H$,

$$E_\alpha(t^\alpha A) := \int_{\sigma(A)} E_\alpha(t^\alpha \lambda) dP(\lambda).$$

The operator family $\{E_\alpha(t^\alpha A)\}_{t \geq 0}$ is uniformly bounded and for every $\mu > 0$,

$$\int_0^\infty E_\alpha(t^\alpha A) e^{-\mu t} dt = \int_{\sigma(A)} \frac{\mu^{\alpha-1}}{\mu - \lambda} dP(\lambda) = \mu^{\alpha-1} R(\mu, A).$$

Since the strong continuity of $E_\alpha(t^\alpha A)$ can be easily deduced by the dominant convergence theorem, which means that $\{E_\alpha(t^\alpha A)\}_{t \geq 0}$ is the bounded α-times resolvent family generated by A.

Combining the above discussion, we deduce the following proposition.

Proposition 3.4. *Suppose A is a densely defined, closed normal operator on H, $\alpha \in (0, 2)$. Then operator A generates an bounded α-times resolvent family $\{S_\alpha(t)\}_{t \geq 0}$ if and only if $\sigma(A) \subseteq \mathbb{C} - \Sigma_{\frac{\alpha}{2}\pi}$. Moreover, if A generates a bounded fractional resolvent family $\{S_\alpha(t)\}_{t \geq 0}$, then it can be represented as:*

$$S_\alpha(t) = \int_{\sigma(A)} E_\alpha(t^\alpha \lambda) dP(\lambda), \tag{3.3}$$

where $P(\lambda)$ is the resolution of identity corresponding to A which satisfies the Eq (3.2).

4. Applications

In this section, we give some applications of Proposition 3.4. We shall use the properties of zeros of the Mittag-Leffler function several times, then distributions of zeros of the Mittag-Leffler function can be found in [24, Sections 3.5 and 4.6].

4.1. A counter-example for point-spectral mapping theorem

It has been proved that if A generates a strong-continuous semigroup $\{T(t)\}$, then we have

$$\sigma_p(A)\backslash\{0\} = e^{t\sigma_p(A)}. \tag{4.1}$$

This equation is called the spectral mapping theorem for point spectral or point-spectral mapping theorem, the proof of this equation can be found in [8, Chapter 4, Section 3.7]. In paper [19], authors prove the point-spectral inclusion theorem for a fractional resolvent family

$$E_\alpha(\sigma_p(A)t^\alpha) \subseteq \sigma_p(S_\alpha(t)) \tag{4.2}$$

by using the following lemma.

Lemma 4.1. [19, Lemma 3.1] *Denote $m_a(t) = t^{\alpha-1}E_{\alpha,\alpha}(at^\alpha)$, $a \in \mathbb{C}$. Suppose $\{S_\alpha(t)\}$ is a fractional resolvent family generated by A with $\alpha \in (0, 2]$, then*

$$(a - A)\int_0^t m_a(s)S_\alpha(t - s)xds = E_\alpha(at^\alpha)x - S_\alpha(t)x, \quad x \in X. \tag{4.3}$$

$$\int_0^t m_a(s)S_\alpha(t - s)(a - A)xds = E_\alpha(at^\alpha)x - S_\alpha(t)x, \quad x \in D(A). \tag{4.4}$$

Here, we will use Proposition 3.4, Lemma 4.1, and properties of resolution of identity to construct an operator A such that

$$0 \neq \lambda \in \sigma_{ap}(A)\backslash\sigma_p(A), \quad E_\alpha(\lambda t^\alpha) \in \sigma_p(S_\alpha(t)), \quad 1 \neq \alpha \in (0, 2]. \tag{4.5}$$

Let A be a normal operator which satisfies Proposition 3.4, then the fractional resolvent family $\{S_\alpha(t)\}$ generated by A is given by

$$S_\alpha(t) = \int_{\sigma(A)} E_\alpha(t^\alpha\lambda)dP(\lambda), \tag{4.6}$$

where $P(\lambda)$ is the resolution of identity corresponding to A. Moreover, suppose operator A satisfies the following condition.

Condition 1: Let $\lambda_0 \in \sigma(A)\backslash\sigma_p(A)$ and $\lambda_1 \in \sigma(A)\backslash\{\lambda_0\}$ satisfies $P(\lambda_1) > 0$ and $E_{\alpha,\alpha+1}(\lambda_0) = E_{\alpha,\alpha+1}(\lambda_1) = 0$.

There indeed exists an operator A satisfies Condition 1, since there are infinitely zeros of $E_{\alpha,\alpha+1}(\lambda)$ lies in the $\mathbb{C} - \Sigma_{\frac{\alpha}{2}\pi}$.

Example 4.2. By using Lemma 4.1, [24, (4.4.10)] and let $t = 1$, we have

$$\begin{aligned}
& (\lambda_0 - A)\int_0^1 m_{\lambda_0}(s)S_\alpha(1-s)xds \\
= & (\lambda_0 - A)\int_0^1 m_{\lambda_0}(s)\int_{\sigma(A)} E_\alpha((1-s)^\alpha\lambda)dP(\lambda)xds \\
= & (\lambda_0 - A)\int_{\sigma(A)}\int_0^1 m_{\lambda_0}(s)E_\alpha((1-s)^\alpha\lambda)dsdP(\lambda)x \\
= & (\lambda_0 - A)\int_{\sigma(A)} f_{\lambda_0}(\lambda)dP(\lambda)x \\
= & E_\alpha(\lambda_0)x - S_\alpha(1)x,
\end{aligned}$$

where

$$f_{\lambda_0}(\lambda) = -\frac{\lambda E_{\alpha,\alpha+1}(\lambda)}{\lambda_0 - \lambda}. \tag{4.7}$$

Then we have

$$f_{\lambda_0}(\lambda_1) = 0,$$

Since $P(\lambda_1) > 0$ we know that $P(\omega) > 0$, where $\omega = \{\lambda \in \sigma(A) : f_{\lambda_0}(\lambda) = 0\}$.

Then choose $0 \neq x_0 \in R(P(\omega))$, by using the proof of [25, Theorem 13.27(a)] we have

$$\int_{\sigma(A)} f_{\lambda_0}(\lambda)dP(\lambda)x_0 = 0. \tag{4.8}$$

That is,

$$E_\alpha(\lambda_0)x_0 - S_\alpha(1)x_0 = 0, \tag{4.9}$$

this means $E_\alpha(\lambda_0) \in \sigma_p(S_\alpha(1))$ with $\lambda_0 \notin \sigma_p(A)$.

Since we construct this example in Hilbert space and A is a normal operator, then we can prove the following claim for A satisfies Condition 1 directly,

$$E_\alpha(\sigma_r t^\alpha) \subsetneq \sigma_r(S_\alpha(t)). \tag{4.10}$$

One question is how to add some more conditions on normal generator A to ensure the correctness of the point-spectral mapping theorem. Notice that we prove $E_\alpha(\lambda_0)x_0 - S_\alpha(1)x_0 = 0$ only for $t = 1$, so if we want to prove the point-spectral mapping theorem for all fractional resolvent family, we at least should prove that for every $t > 0$,

$$\int_{\sigma(A)} \frac{\lambda_0 E_{\alpha,\alpha+1}(\lambda_0 t^\alpha) - \lambda E_{\alpha,\alpha+1}(\lambda t^\alpha)}{\lambda_0 - \lambda} t^\alpha dP(\lambda) \tag{4.11}$$

is an injective operator then the point-spectral mapping theorem is valid for the fractional resolvent family with this normal generator. However, this is difficult since instead of explicit representation of zeros, we only have the asymptotic behavior of zeros of Mittag-Leffler function $E_{\alpha,\alpha}(\lambda)$ except $\alpha = 1$ [24, Sections 3.5 and 4.6], in this case, $E_{1,1}(\lambda) = e^\lambda$ has no zeros and the fractional resolvent operator is a strongly continuous semigroup and satisfies the point-spectral mapping theorem.

It should be noticed that in strong continuous semigroup sense ($\alpha = 1$), the spectral mapping theorem holds if this semigroup is eventually norm-continuous [8, Chapter 4, 3.10], but we can find an operator satisfies Condition 1 which generates an analytic fractional resolvent family ($\alpha \in (0,2), \alpha \neq 1$), so we conclude that the point-spectral mapping theorem does not hold for the fractional resolvent family in general, even if for analytic fractional resolvent family or vector-valued cosine function.

The method we used in the above example can be used to construct another example that shows that there exists a fractional resolvent family $\{S_\alpha(t)\}$ and a positive constant t_0 such that

$$S_\alpha(t_0) = 0.$$

It is well known that if there is a t_0 and a semigroup $\{T(t)\}$ such that

$$T(t_0) = 0,$$

then

$$T(t) = 0, \quad t \geq t_0,$$

then we conclude that for any postive constant ω, we can find another constant M such that

$$\|T(t)\| \leq Me^{-\omega t},$$

then we conclude that the generator of $\{T(t)\}$ has an empty spectrum set, which is impossible if $\{T(t)\}$ is a semigroup of normal operator,

Example 4.3. Denote set $B = \{\lambda : E_\alpha(\lambda) = 0, \quad \lambda \in \mathbb{C} - \Sigma_{\frac{\alpha\pi}{2}}\}$ and let A be the normal operator with $\sigma(A) = B$. Then we know that A generates a fractional resolvent family $\{S_\alpha(t)\}$ with representation

$$S_\alpha(t) = \int_{\sigma(A)} E_\alpha(\lambda t^\alpha) dP(\lambda). \tag{4.12}$$

Then we have

$$S_\alpha(1) = \int_B E_\alpha(\lambda) dP(\lambda) = 0. \tag{4.13}$$

This construction can not be applied on semigroup because $E_1(\lambda) = e^\lambda$ has no zeros.

4.2. The constant of decay estimate

It has been proved that if A generates a stable semigroup, then this semigroup is exponentially stable and there exists a constant $\omega < 0$ such that

$$\sigma(A) \subseteq \{\lambda : \Re(\lambda) < \omega\}.$$

But a similar property has not been proved for the general fractional resolvent family. By using Proposition 3.4 we can prove the following theorem.

Theorem 4.4. *Suppose A is a normal operator which generates a stable fractional resolvent family $\{S_\alpha(t)\}$, then there exists a constant $\omega > 0$ such that*

$$\omega + \sigma(A) \subseteq \mathbb{C} - \Sigma_{\frac{\alpha\pi}{2}}, \tag{4.14}$$

and

$$\|S_\alpha(t)\| \leq \frac{1}{\omega\Gamma(1-\alpha)} t^{-\alpha} + o(t^{-2\alpha}), \quad t \to \infty. \tag{4.15}$$

Proof. We prove the first claim by a contradiction. If there is no such a constant ω satisfies the Eq (4.14), then there must be a sequence $\{z_n\} \subseteq \sigma(A)$ such that

$$\Re(z_n^{\frac{1}{\alpha}}) \to 0, \quad as \quad n \to \infty. \tag{4.16}$$

Since $\{S_\alpha(t)\}$ is stable, then $0 \in \rho(A)$ and we can choose t_0 big enough and $\epsilon < \frac{1}{3\alpha}$ such that $\|S_\alpha(t)\| \leq \epsilon$, and $t_0^\alpha z_n$ satisfies Proposition 2.1,

$$E_\alpha(t_0^\alpha z_n) = \frac{1}{\alpha}\exp((t_0^\alpha z_n)^{\frac{1}{\alpha}}) + \epsilon_\alpha((t_0^\alpha z_n)).$$

Since $\Re(z_n^{\frac{1}{\alpha}}) \to 0$, we can choose n_0 big enough and $t_1 > t_0$ such that $\|S_\alpha(t_1)\| < \epsilon$ and

$$|E_\alpha(t_1^\alpha z_{n_0})| > \frac{1}{2\alpha}. \tag{4.17}$$

By spectral inclusion theorem [19, Theorem 3.2] we have

$$E_\alpha(t_1^\alpha z_{n_0}) \in \sigma(S_\alpha(t_1)). \tag{4.18}$$

Thus

$$\frac{1}{3\alpha} > \epsilon > \|S_\alpha(t_1)\| \geq |E_\alpha(t_1^\alpha z_{n_0})| > \frac{1}{2\alpha}. \tag{4.19}$$

This is a contradiction. Then there exists a constant $\omega > 0$ such that

$$\omega + \sigma(A) \subseteq \mathbb{C} - \Sigma_{\frac{\alpha\pi}{2}}.$$

The second claim can be proved by Propositions 2.1 and 3.4 directly. Since

$$\|S_\alpha(t)\| = \|\int_{\sigma(A)} E_\alpha(\lambda t^\alpha) dP(\lambda)\|, \tag{4.20}$$

then for t big enough, we have

$$\begin{aligned}
\|S_\alpha(t)\| &\leq \max\{\|E_\alpha(\lambda t^\alpha), \lambda \in \sigma(A)\} \\
&\leq \frac{1}{\alpha}\exp((-\omega t^\alpha)^{\frac{1}{\alpha}}) + \epsilon_\alpha(-\omega t^\alpha) \\
&\leq \frac{1}{\omega\Gamma(1-\alpha)} t^{-\alpha} + o(t^{-2\alpha}).
\end{aligned}$$

We shall give some examples to show how to use Theorem 4.4, suppose Δ is the n-dimension Laplace operator. More details about the following operators can be found in [26, 29].

Example 4.5. Let $H = L^2(R^n), n > 3$ and operator $A = -\frac{1}{2}\Delta + \lambda V, \lambda > 1$. Then, by [26, Theorem B5.2] we know that if $0 \geq V \in L^p \cap L^q, p < \frac{n}{2} < q$, and

$$\alpha_2(1) = -\ln \|e^{-A}\| = 0,$$

then

$$\lim_{t\to\infty} \|e^{-tA}\|$$

exists, then for every $\omega > 0$, $A + \omega = -\frac{1}{2}\Delta + \lambda V + \omega$ generates a exponentially stable semigroup. Since operator $A + \omega$ is a normal operator, by using Theorem 4.4 we know that the solution of fractional differential equation with $\alpha < 1$,

$$\begin{aligned} i^\alpha D_t^\alpha u(t,x) &= (-\frac{1}{2}\Delta + \lambda V(x))u(t,x) + \omega u(t,x),\ t > 0 \\ u(0,x) &= f(x) \in L^2(R^n) \end{aligned}$$

satisfies the Eq (4.15), that is

$$\|u(t,x)\|_{L^2(R^n)} \leq \frac{\cos(\frac{\alpha\pi}{2})}{\omega} \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + o(t^{-2\alpha}), \quad t \to \infty. \tag{4.21}$$

Example 4.6. Let Δ_S be the Laplace operator on $S^n, n > 1$, the n-dimension sphere, and define operator A on $L^2(S^n)$ as

$$A = (-\Delta_S + \frac{(n-1)^2}{4})^{\frac{1}{2}}.$$

Then by [29, Proposition 4.1] we know that A is self-adjoint and

$$\sigma(A) \subseteq \{\frac{1}{2}(n-1) + k : k = 0, 1, 2, ...\}$$

Thus, for every $\alpha \in (0, 2)$, the equation

$$\begin{aligned} i^\alpha D_t^\alpha u(t,x) &= -(-\Delta_S + \frac{(n-1)^2}{4})^{\frac{1}{2}} u(t,x),\ t > 0 \\ u(0,x) &= f(x) \in L^2(S^n) \end{aligned}$$

has a solution $u(t, x)$ satisfies

$$\|u(t,x)\|_{L^2(S^n)} \leq \frac{2\cos(\frac{\alpha\pi}{2})}{n-1} \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + o(t^{-2\alpha}), \quad t \to \infty. \tag{4.22}$$

Example 4.7. Let Δ_S be the Laplace operator on $\mathcal{H}^n, n > 1$, the n-dimension hyperbolic space, defined as

$$\mathcal{H}^n = \{v \in R^{n+1} : \langle v, v\rangle = 1, v_{n+1} > 0\},$$

Since $\mathcal{H}^n$ is a compact Riemannian manifold, then by [29, Proposition 2.1] and the proof of [29, Proposition 5.1] we know that Δ is self-adjoint and

$$\sigma(\Delta) \subseteq (-\infty, \frac{1}{4}(n-1)^2].$$

Then we know that for every $\alpha \in (0, 2)$, the equation

$$\begin{aligned} i^{\alpha} D_t^{\alpha} u(t, x) &= \Delta u(t, x), \ t > 0 \\ u(0, x) &= f(x) \in L^2(\mathcal{H}^n) \end{aligned}$$

has a solution $u(t, x)$ satisfies

$$\|u(t, x)\|_{L^2(\mathcal{H}^n)} \leq \frac{4\cos(\frac{\alpha\pi}{2})}{(n-1)^2} \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + o(t^{-2\alpha}), \quad t \to \infty. \tag{4.23}$$

4.3. Fractional domain and fractional derivative

It has been proved in [17] that for every exponentially bounded semigroup $\{T(t)\}$ with generator A and $x \in A$, that $T(t)x$ has a continuous fractional derivative of order $\alpha > 0$ if and only if x belongs to $D((bI - A)^{\alpha})$ for some $b \in \rho(A)$. More precisely, in [9], the following theorem has been proved.

Theorem 4.8. [9, Theorem 2.1] *Let $\alpha > 0$ and $\{T(t)\}$ be the exponentially bounded semigroup generated by A which satisfies*

$$\|T(t)\| \leq Me^{\omega t}.$$

Then

$$E^+_{\alpha,\beta} = F_{\alpha}. \tag{4.24}$$

Where $F_{\alpha} := D((bI - A)^{\alpha})$ with $b \in \rho(A)$ and $\beta > \omega$, $x \in E^+_{\alpha,\beta}$ means there exists a continuous function f such that

$$e^{-\beta t} T(t)x = \frac{e^{i\pi\alpha}}{\Gamma(\alpha)} \int_t^{\infty} (s-t)^{\alpha-1} f(s) ds.$$

And similar results for vector-valued cosine function family are also proved in [9]. It should be noticed that the fractional derivative used in these papers are Riemann-Liouville fractional derivative, which is different from the Caputo fractional derivative we used here, by using Proposition 3.4 we can prove a similar result for fractional resolvent family generated by the normal operator.

Theorem 4.9. *Let $\{S_{\alpha}(t)\}$ be the bounded fractional resolvent family generated by normal operator A. $F_{\beta} = D((I - A)^{\beta}), \beta > 0$ and $x \in E_{\alpha,\beta}$ means there exists a continuous operator family $\{f(t)\}$ such that*

$$D_t^{\alpha\beta} S_{\alpha}(t)x = f(t)x, \tag{4.25}$$

then the following two assertions hold:

(1) if $\alpha\beta < 1$, then

$$F_{\beta} = E_{\alpha,\beta}.$$

(2) If $\alpha\beta \geq 1$, then

$$F_{\beta} \subseteq E_{\alpha,\beta}.$$

Proof. We only need to prove this theorem for $\beta < 1$. If $\beta \geq 1$, then $x \in D(A^{\beta}) \subseteq D(A)$ means $Ax \in D(A^{\beta-1})$, by definition of fractional resolvent family we know that

$$x \in D(A) \quad \text{iff} \quad D_t^{\alpha} S_{\alpha}(t)x = AS_{\alpha}(t)x = S_{\alpha}(t)Ax. \tag{4.26}$$

Now we assume $\beta < 1$ and $x \in F_\beta$, then $S_\alpha(x) = (I - A)^{-\beta} S_\alpha(t)(I - A)^\beta x$, since $\frac{1}{(1-\lambda)^\beta}$ is bounded in $\sigma(A)$ and $(I - A)^{-\beta}$ is a bound operator, then we have

$$\begin{aligned} S_\alpha(t)x &= (I - A)^{-\beta} S_\alpha(t)(I - A)^\beta x \\ &= \int_{\sigma(A)} \frac{1}{(1-\lambda)^\beta} E_\alpha(\lambda t^\alpha) dP(\lambda)(I - A)^\beta x. \end{aligned}$$

Since we have ([24, Equation 4.4.5])

$$\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} E_\alpha(\lambda s^\alpha) ds = t^\beta E_{\alpha,\beta+1}(\lambda t^\alpha). \tag{4.27}$$

Then we can define the operator $h(t)$ as

$$h(t) := t^{\alpha(1-\beta)} \int_{\sigma(A)} \frac{\lambda}{(1-\lambda)^\beta} E_{\alpha,\alpha(1-\beta)+1}(\lambda t^\alpha) dP(\lambda). \tag{4.28}$$

Now using the asymptotic formula of Mittag-Leffler function [24, Theorem 4.3] and dominant convergence theorem, we deduce that $h(t)$ is a strongly continuous operator, and for every $x \in D((I - A)^\beta)$,

$$\begin{aligned} (g_\alpha * h)(t)(I - A)^\beta x &= \int_{\sigma(A)} g_\alpha(t) * (g_{\alpha(1-\beta)}(t) * \frac{\lambda}{(1-\lambda)^\beta} E_\alpha(\lambda t^\alpha)) dP(\lambda)(I - A)^\beta x \\ &= \int_{\sigma(A)} g_{\alpha(1-\beta)}(t) * \frac{1}{(1-\lambda)^\beta} (E_\alpha(\lambda t^\alpha) + k(t)) dP(\lambda)(I - A)^\beta x \\ &= (g_{\alpha(1-\beta)} * (I - A)^{-\beta} S_\alpha)(t)(I - A)^\beta x - (g_{\alpha(1-\beta)} * k)(t)(I - A)^\beta x \\ &= (g_{\alpha(1-\beta)} * S_\alpha)(t)x - (g_{\alpha(1-\beta)} * k)(t)(I - A)^\beta x, \end{aligned}$$

where $k(t)$ are defined as

$$k(t) = 1 \quad \text{if} \quad \alpha < 1, \text{or} \quad k(t) = \frac{t}{\Gamma(\alpha+1)}, \quad \text{if} \quad 1 < \alpha < 2. \tag{4.29}$$

Thus

$$S_\alpha(t)x = ((g_{\alpha\beta} * h)(t) + k(t))(I - A)^\beta x, \tag{4.30}$$

this shows that $S_\alpha(t)x$ is continuous differentiable of order $\alpha\beta$.

Next we suppose that $\alpha\beta < 1$ and $S_\alpha(t)x$ is continuous differentiable of order $\alpha\beta$, thus there exists a continuous operator $f(t)$ such that

$$S_\alpha(t)x = (g_{\alpha\beta} * f)(t)x. \tag{4.31}$$

Since

$$E_\alpha(\lambda t^\alpha) = g_{\alpha\beta}(t) * t^{-\alpha\beta} E_{\alpha,1-\alpha\beta}(\lambda t^\alpha),$$

thus by the uniqueness of Laplace transform we deduce that

$$f(t)x = t^{-\alpha\beta} \int_{\sigma(A)} E_{\alpha,1-\alpha\beta}(\lambda t^\alpha) dP(\lambda)x. \tag{4.32}$$

Then we can prove that
$$h(t)x = \int_{\sigma(A)} \frac{\lambda^{\beta+1}}{1-\lambda} E_\alpha(\lambda t^\alpha) dP(\lambda)x \tag{4.33}$$
is well defined too, by using the asymptotic formulas of $E_{\alpha,1-\alpha\beta}(\lambda t^\alpha)$ and $\lambda^\beta E_\alpha(\lambda t^\alpha)$. Then
$$(g_\alpha * h)(t)x = \int_{\sigma_(A)} E_\alpha(\lambda t^\alpha) \frac{\lambda^\beta}{1-\lambda} + k(t) dP(\lambda)x = S_\alpha A^\beta (I-A)^{-1} x + k(t)x, \tag{4.34}$$
this shows that $S_\alpha A^\beta (I-A)^{-1}x$ is continuous differentiable of order α, then $A^\beta(I-A)^{-1}x \in D(A)$ and $x \in D(A^\beta)$.

There is proof of assertion (2) without using Proposition 3.4, instead, we use the Theorem 3.3. By using this theorem and the uniqueness of the Laplace transform we deduce that
$$\begin{aligned} \int_0^\infty e^{-\lambda t} S_\alpha(t)xdt &= \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x \\ &= \lambda^{\alpha-1}(\lambda^\alpha - U^* q U)^{-1}x \\ &= U^* \lambda^{\alpha-1}(\lambda^\alpha - q)^{-1} Ux \\ &= U^* \int_0^\infty e^{-\lambda t} E_\alpha(qt^\alpha) dt Ux \\ &= \int_0^\infty e^{-\lambda t} U^* E_\alpha(qt^\alpha) Uxdt, \end{aligned}$$
that is
$$S_\alpha(t)x = U^* E_\alpha(qt^\alpha)Ux. \tag{4.35}$$
Since we know that $S_\alpha(t)x$ is continuous differentiable of order $\alpha\beta$, then
$$D_t^{\alpha\beta} S_\alpha(t)x = U^* D_t^{\alpha\beta} E_\alpha(qt^\alpha)Ux = U^* t^\alpha E_{\alpha.1-\alpha\beta}(qt^\alpha)Ux, \tag{4.36}$$
this means for every $t > 0$, $t^\alpha E_{\alpha.1-\alpha\beta}(qt^\alpha)Ux \in L^2(\Omega, \mu)$, then by using asymptotic beheavior of Mittag-Leffler function we have
$$\frac{q^{\beta+1}}{(1-q)} E_\alpha(qt^\alpha)Ux \in L^2(\Omega, \mu),$$
and
$$U^* \frac{q^{\beta+1}}{(1-q)} E_\alpha(qt^\alpha)Ux \in H,$$
thus
$$\begin{aligned} g_\alpha(t) * U^* \frac{q^{\beta+1}}{(1-q)} E_\alpha(qt^\alpha)Ux &= U^* g_\alpha(t) * \frac{q^{\beta+1}}{(1-q)} E_\alpha(qt^\alpha)Ux \\ &= U^* \frac{q^\beta}{(1-q)} E_\alpha(qt^\alpha)Ux + k(t)x \\ &= S_\alpha(t) A^\beta (I-A)^{-1}x + k(t)x. \end{aligned}$$
This shows $S_\alpha(t)A^\beta(I-A)^{-1}x$ is continuous diffenertiable of order α and $x \in D(A^\beta)$.

5. Conclusions

By using the resolution of identity of a normal operator A, we deduce an integral representation of the fractional resolvent family generated by A. And then by using this representation, some applications are given here, especially, we show that the spectral mapping theorem does not hold for the fractional resolvent family.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest.

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