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Research article

Asymptotic behavior of solutions of the third-order nonlinear advanced differential equations

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Abstract: The aim of this work is to study some asymptotic properties of a class of third-order advanced differential equations. We present new oscillation criteria that complete, simplify and improve some previous results. We also provide many different examples to clarify the significance of our results.

Keywords: third-order differential equations; advanced differential equation; oscillation criteria; nonoscillatory solution
Mathematics Subject Classification: 34C10, 34K11

1. Introduction

Differential equations (DEs) are mathematical models used to study phenomena that occur in nature, where each dependent variable represents a quantity in the modeled phenomenon. Differential equations made it possible to understand many complex phenomena in our daily lives and play a pivotal role in many applications in engineering [1–6]. They have become important tools in applied sciences and technology, used for studying telephone signals, media, conversations and the statistics of online purchasing. More traditionally, they were used in astronomy to describe the orbits of planets and the motion of stars [7]. They are also have many applications in biology and the medical sciences. Recently, differential equations were used to describe the evolution of COVID-19 pandemic [8–10]. By describing those phenomena with variables that symbolize time and place, differential equations can provide insights about the phenomena's future.

Differential equations with delays, known as delay differential equations (DDEs), are used to model systems where time delays play a significant role in the dynamics. They are used to model phenomenon where the current state of the system depends not only on its current inputs and initial conditions, but also on its past inputs or states over a certain time interval. These equations have been used in ecological models of population dynamics, chemical kinetics of reactions, neurobiology and neuroscience. By contrast, advanced differential equations (ADEs) are used to describe phenomenon in which the evolution of the system depends on both present and future time. The possibility of introducing an advance into the equation to take into account future influence that may actually affect the present, makes such equations a useful tool in various economic problems, population dynamics and in mechanical control [11].

The problem of establishing the oscillation criteria for differential equations with deviating arguments remained a stumbling block for scientists until the appearance of Fite's paper [12] in 1921. Since then, the study of oscillation criteria for equations of different orders has become a very active field [13–17]. For a differential equation, the presence of oscillating solutions typically indicates the presence of periodic terms or sinusoidal functions in the solution. This can be seen through trigonometric functions such as sine or cosine. In contrast, non-oscillating solutions generally do not involve periodic terms or sinusoidal functions. They can take various forms such as exponential decay, polynomial functions or constant values.

It should be noted that the vast majority of the published paper are concerned with differential equations with delay, while the equations with advanced arguments did not receive the attention they deserve. Furthermore, those studies that considered advanced arguments were restricted to second order differential equations [18–20]. In [21, 22] the authors established new oscillation criteria for the linear second-order advanced differential equation

$$\rho''(\iota) + h(\iota)\rho(\Omega(\iota)) = 0.$$

Dzurina in [23] investigated the advanced canonical equations of the form

$$(\nu\rho')'(\iota) + h(\iota)\rho(\Omega(\iota)) = 0$$

and presented new properties of nonoscillatory solutions. Several papers also studied similar equations [24–27].

For the third-order delay differential equations (DDEs), the authors in [28–33] studied the following third-order nonlinear delay differential equation

$$\left(\nu(\iota)(\rho''(\iota))^{\alpha}\right)' + h(\iota)\rho^{\alpha}(\Omega(\iota))) = 0$$

and established some results of oscillation in both the canonical and noncanonical cases. In [20,34,35], different oscillation results for the third-order quasilinear delay differential equation

$$\left(\nu(\iota)(\rho''(\iota))^{\alpha}\right)' + h(\iota)\rho^{\beta}(\Omega(\iota))) = 0$$

were achieved. Li et al. [36] obtained sufficient conditions for the solution ρ for the equation

$$\left(\nu_1(\iota)(\nu_2(\iota)\rho'(\iota))'\right)' + h(\iota)\rho^{\alpha}(\Omega(\iota)) = 0$$

to be oscillatory or satisfy $\lim_{\iota\to\infty}\rho(\iota) = 0$.

The oscillation of the following advanced differential equation

$$\left(\nu_2(\iota)\left(\left(\nu_1(\iota)\left(\rho'(\iota)\right)^{\alpha}\right)'\right)^{\beta}\right)' + h(\iota)\rho(\Omega(\iota)) = 0, \quad \iota \geq \iota_0 > 0 \tag{1.1}$$

was discussed in [37]. The author in [37] obtained some conditions that guarantee that the solutions of Eq (1.1) are either oscillatory or tend to zero under conditions

$$\int^{\iota} \nu_i^{-1/\alpha}(s)\,\mathrm{d}s < \infty, \quad i = 1, 2$$

and

$$\Omega'(\iota) \geq 0. \tag{1.2}$$

The oscillation criteria for the equation

$$\left(\nu(\iota)\left(\rho'(\iota)\right)^{\alpha}\right)'' + h(\iota)\rho(\Omega(\iota)) = 0, \quad \iota \geq \iota_0 > 0$$

were studied by Dzurina and Baculikova [38, 39] under conditions $\int^{\iota} \nu^{-1/\alpha}(s)\,\mathrm{d}s = \infty$ and (1.2).

In this paper, we establish some properties of third-order advanced differential equations of the form

$$\left(\nu(\iota)\left(\rho''(\iota)\right)^{\alpha}\right)' + \int_c^d h(\iota, s) f(\rho(\Omega(\iota, s)))\,\mathrm{d}s = 0, \quad \iota \geq \iota_0 > 0. \tag{1.3}$$

As to our knowledge, the above equation and the advanced differential equations of the third order in general did not receive the attention of researchers due to the difficulty of obtaining relationships to reach conditions that guarantee the oscillation of all their solutions.

The obtained results also apply to the following third-order advanced differential equation

$$\left(\nu(\iota)\left|\rho''(\iota)\right|^{\alpha-1}\rho''(\iota)\right)' + \int_c^d h(\iota, s)\left|\rho(\Omega(\iota, s))\right|^{\alpha-1}\rho(\Omega(\iota, s))\,\mathrm{d}s = 0, \text{ where } \alpha > 0. \tag{1.4}$$

The purpose of this research is to contribute to the less-developed oscillation theory of third-order equations with advanced argument. Using the new approach taken in this paper, we present new and more general results than the previous studies mentioned above. The paper is organized as follows. The second section presents background results that are necessary to obtain the main results. In Section 3, Theorems 3.1, 3.3 and 3.4 and Corollaries 3.1 and 3.2 present some conditions that guarantee the exclusion of positive increasing solutions. Theorem 3.2 guarantees that any nonoscillatory solution to Eq (1.3), under certain conditions, tends to zero. Examples given in this paper illustrate the significance of our results and improvements to known oscillation criteria are provided in Section 4.

2. Preliminaries

In this section, we present background definitions and results needed for later sections. Throughout this paper, we assume the following:

(H_1) $h(\iota, s) \in C([\iota_0, \infty) \times [c, d], (0, \infty))$, $\Omega(\iota, s) \in C([\iota_0, \infty) \times [c, d], (0, \infty))$, $\nu \in C([\iota_0, \infty), (0, \infty))$, $\Omega(\iota, s) \geq \iota \geq \iota_0 > 0$, $\Omega'(\iota, s) \geq \Omega_0 > 0$, $h(\iota, s)$ does not vanish identically and

$$\int_{\iota_0}^{\iota} \frac{1}{\nu^{1/\alpha}(s)}\mathrm{d}s = \infty. \tag{2.1}$$

(H_2) $f \in C(\mathbb{R}, \mathbb{R})$ such that $\rho f(\rho) > 0$ for $\rho \neq 0$ and satisfies the following condition:

$$f(\rho)/\delta > \rho^{\alpha} \text{ for all } \rho \neq 0,$$

where $\delta > 0$ and α is a quotient of odd positive integers.

Note that the conditions in the first line of (H_1) ensure that Eq (1.3) has a solution.

Definition 2.1. *We say ρ is a solution of Eq (1.3) if $\rho \in C^2\left([\iota_\rho, \infty), [0, \infty)\right)$, $\iota_\rho > \iota_0$, with $\nu(\rho'')^{\alpha} \in C^1\left([\iota_\rho, \infty), [0, \infty)\right)$ and satisfies Eq (1.3) on $[\iota_\rho, \infty)$.*

We consider those solutions of Eq (1.3) defined on some half-line $[\iota_\rho, \infty)$ and satisfying

$$\sup\{|\rho(\iota)| : T \leq \iota < \infty\} > 0 \text{ for any } T \geq \iota_\rho.$$

Definition 2.2. *A solution ρ of Eq (1.3) is said to be oscillatory if it has arbitrary large zeros on $[\iota_\rho, \infty)$, otherwise, it is called nonoscillatory. If all solutions of Eq (1.3) are oscillatory, then Eq (1.3) is said to be oscillatory.*

The next lemma classifies the sign of nonoscillatory solutions.

Lemma 2.1. *If $\rho > 0$ be a solution of Eq (1.3), then*

$$\rho(\iota)\left(\nu(\iota)(\rho''(\iota))^{\alpha}\right)' < 0, \qquad \rho(\iota)\rho''(\iota) > 0$$

and only one of the following cases holds:

$$\rho(\iota)\rho'(\iota) < 0, \tag{2.2}$$

$$\rho(\iota)\rho'(\iota) > 0, \text{ eventually.} \tag{2.3}$$

Proof. Let $\rho > 0$ be a solution of Eq (1.3), for some $\iota \geq \iota_0$. By Eq (1.3), we have

$$\left(\nu(\iota)(\rho''(\iota))^{\alpha}\right)' < 0, \quad \text{eventually.}$$

This means that the function $\nu(\rho'')^{\alpha}$ of fixed sign eventually. If $\nu(\iota)(\rho''(\iota))^{\alpha} < 0$, then both $\rho(\iota) < 0$ and $\rho'(\iota) < 0$ which leads to a contradiction. That is,

$$\nu(\iota)(\rho''(\iota))^{\alpha} > 0, \quad \text{eventually.}$$

Thus, $\rho(\iota)$ is of fixed sign for all ι large enough, i.e., either Cases (2.2) or (2.3) holds. □

Definition 2.3. *We say that Eq (1.3) has property (A) if every positive solutions of Eq (1.3) satisfies*

$$\rho(\iota)\rho'(\iota) < 0.$$

Lemma 2.2. *If $\rho(\iota) > 0$ and $\rho'(\iota)$ is positive increasing, eventually, then*

$$\iota\,\rho(\Omega(\iota, s)) - K_0\,\Omega(\iota, s)\rho(\iota) \geq 0, \; K_0 \in (0, 1), \quad \text{eventually.} \tag{2.4}$$

Proof. Since $\rho'(\iota)$ is positive increasing, we have

$$\begin{aligned}\rho(\Omega(\iota,s)) - \rho(\iota) &= \int_{\iota}^{\Omega(\iota,s)} \rho'(s)\,\mathrm{d}s \\ &\geq \rho'(\iota)(\Omega(\iota,s) - \iota).\end{aligned}$$

Equivalently,

$$\frac{\rho(\Omega(\iota,s))}{\rho(\iota)} - 1 \geq \frac{\rho'(\iota)}{\rho(\iota)}(\Omega(\iota,s) - \iota). \tag{2.5}$$

Using the fact $\lim_{\iota\to\infty}\rho(\iota) = \infty$, there exists a ι_1 large enough, such that

$$\begin{aligned}K_0\,\rho(\iota) &\leq \rho(\iota) - \rho(\iota_1) = \int_{\iota_1}^{\iota} \rho'(s)\,\mathrm{d}s \\ &\leq \rho'(\iota)(\iota - \iota_1) \leq \rho'(\iota)\iota, \text{ for any } K_0 \in (0,1),\end{aligned}$$

i.e.,

$$\iota\,\rho'(\iota) \geq K_0\,\rho(\iota). \tag{2.6}$$

Substituting in (2.5), we get

$$\frac{\rho(\Omega(\iota,s))}{\rho(\iota)} \geq \frac{K_0(\Omega(\iota,s) - \iota)}{\iota} + 1 \geq \frac{K_0\,\Omega(\iota,s)}{\iota},$$

which implies the result. □

3. Main results

The current section contains the main results of this work. To ease notations, we set

$$\Psi(\iota) = \delta\int_{\iota}^{\infty} h(\iota,s)\left(\frac{\Omega(\iota,s)}{s}\right)^{\alpha}\mathrm{d}s \quad\text{and}\quad \Gamma(\iota) = \int_{\iota_1}^{\iota} \nu^{-1/\alpha}(s)\,\mathrm{d}s. \tag{3.1}$$

Theorem 3.1. *If*

$$\liminf_{\iota\to\infty}\frac{1}{\Psi(\iota)}\int_{\iota}^{\infty}\Gamma(s)\,\Psi^{1+1/\alpha}(s)\,\mathrm{d}s > \frac{1}{(\alpha+1)^{1+1/\alpha}}, \tag{3.2}$$

then Eq (1.3) has property (A).

Proof. Let $\rho > 0$ be a solution of Eq (1.3) and satisfying Case (2.3). From Eq (1.3), we obtain

$$\begin{aligned}(\nu(\iota)(\rho''(\iota))^{\alpha})' &\leq -\int_{c}^{d}\delta\,h(\iota,\vartheta)\,\rho^{\alpha}(\Omega(\iota,\vartheta))\,\mathrm{d}\vartheta \\ &\leq -\delta\,\rho^{\alpha}(\Omega(\iota,c))\int_{c}^{d}h(\iota,\vartheta)\,\mathrm{d}\vartheta.\end{aligned} \tag{3.3}$$

Using (2.4), we have

$$(\nu(\iota)(\rho''(\iota))^{\alpha})' \leq -\delta K\left(\frac{\Omega(\iota,c)}{\iota}\right)^{\alpha}\rho^{\alpha}(\iota)\int_{c}^{d}h(\iota,\vartheta)\,\mathrm{d}\vartheta, \tag{3.4}$$

where $K = (K_0)^{\alpha}$. Define the positive function

$$w(\iota) = \frac{\nu(\iota)(\rho''(\iota))^{\alpha}}{\rho^{\alpha}(\iota)}. \tag{3.5}$$

That is

$$w'(\iota) = \frac{1}{\rho^{\alpha}(\iota)}\left(\nu(\iota)(\rho''(\iota))^{\alpha}\right)' - \alpha\nu(\iota)(\rho''(\iota))^{\alpha}\frac{\rho'(\iota)}{\rho^{\alpha}(\iota)\rho(\iota)}. \tag{3.6}$$

Eqations (3.4) and (3.6) imply

$$\begin{aligned} w'(\iota) &\le \frac{\left(\nu(\iota)(\rho''(\iota))^{\alpha}\right)'}{\rho^{\alpha}(\iota)} - \alpha\frac{\nu(\iota)(\rho''(\iota))^{\alpha}}{\rho^{\alpha}(\iota)}\frac{\rho'(\iota)}{\rho(\iota)} \\ &\le -\delta K\left(\frac{\Omega(\iota,c)}{\iota}\right)^{\alpha}\int_c^d h(\iota,\vartheta)\,\mathrm{d}\vartheta - \alpha w(\iota)\frac{\rho'(\iota)}{\rho(\iota)}. \end{aligned} \tag{3.7}$$

Using that $\left(\nu(\iota)(\rho''(\iota))^{\alpha}\right)' \le 0$, we obtain

$$\begin{aligned} \rho'(\iota) &\ge \int_{\iota_1}^{\iota}\left(\nu(s)(\rho''(s))^{\alpha}\right)^{1/\alpha}\frac{1}{\nu^{1/\alpha}(s)}\mathrm{d}s \ge \left(\nu(\iota)(\rho''(\iota))^{\alpha}\right)^{1/\alpha}\int_{\iota_1}^{\iota}\nu^{-1/\alpha}(s)\,\mathrm{d}s \\ &\ge K\left(\nu(\iota)(\rho''(\iota))^{\alpha}\right)^{1/\alpha}\Gamma(\iota). \end{aligned} \tag{3.8}$$

Equation (3.7) yields

$$w'(\iota) \le -\delta K\left(\frac{\Omega(\iota,c)}{\iota}\right)^{\alpha}\int_c^d h(\iota,s)\,\mathrm{d}s - \alpha w^{1+1/\alpha}(\iota)\Gamma(\iota).$$

Integrating from ι to ∞, we get

$$w(\iota) \ge K\,\Psi(\iota) + K\int_{\iota}^{\infty}\alpha w^{1+1/\alpha}(s)\Gamma(s)\,\mathrm{d}s. \tag{3.9}$$

Equivalently,

$$\frac{w(\iota)}{K\,\Psi(\iota)} \ge \alpha K^{1+1/\alpha}\frac{1}{\Psi(\iota)}\int_{\iota}^{\infty}\Gamma(s)\Psi^{1+1/\alpha}(s)\left(\frac{w(s)}{K\,\Psi(s)}\right)^{1+1/\alpha}\mathrm{d}s + 1.$$

Since $w(\iota) - K\ \Psi(\iota) > 0$, $\inf_{\iota\ge\iota_1} w(\iota)/K\,\Psi(\iota) = \lambda$, $\lambda \in [0,\infty)$. i.e.,

$$\frac{w(\iota)}{K\,\Psi(\iota)} \ge \alpha(K\lambda)^{1+1/\alpha}\frac{1}{\Psi(\iota)}\int_{\iota}^{\infty}\Gamma(s)\Psi^{1+1/\alpha}(s)\,\mathrm{d}s + 1. \tag{3.10}$$

Using Eq (3.2), we obtain

$$\liminf_{\iota\to\infty} K^{1+1/\alpha}\frac{1}{\Psi(\iota)}\int_{\iota}^{\infty}\Gamma(s)\Psi^{1+1/\alpha}(s)\,\mathrm{d}s > \frac{1}{(\alpha+1)^{1+1/\alpha}},$$

for $0 < K < 1$. Thus, there exists a positive η such that

$$K^{1+1/\alpha}\frac{1}{\Psi(\iota)}\int_{\iota}^{\infty}\Gamma(s)\Psi^{1+1/\alpha}(s)\,\mathrm{d}s > \eta > \frac{1}{(\alpha+1)^{1+1/\alpha}}. \tag{3.11}$$

Substituting (3.10) in (3.11) yields

$$\frac{w(\iota)}{K\,\Psi(\iota)} \geq \alpha\,\eta\left(\lambda^{1+1/\alpha}\right)+1.$$

i.e.,

$$\lambda \geq \alpha\,\eta\,\lambda^{1+1/\alpha}+1 > \frac{\alpha\left(\lambda^{1+1/\alpha}\right)}{(\alpha+1)^{1+1/\alpha}}+1.$$

Hence,

$$\frac{1}{\alpha+1}+\frac{1}{\alpha+1}\frac{\alpha}{(\alpha+1)^{1+1/\alpha}}\lambda^{1+1/\alpha}-\frac{1}{\alpha+1}\lambda<0.$$

Set

$$g(x)=\frac{1}{\alpha+1}+\frac{1}{\alpha+1}x^{1+1/\alpha}-x.$$

This contradicts the fact that $g(x)>0$ for all $x>0$, which completes the proof. □

Corollary 3.1. *If either*

$$\int_{\iota_0}^{\infty}\frac{\Omega^{\alpha}(\iota,c)}{s^{\alpha}}\left(\int_c^d h(\iota,\vartheta)\,\mathrm{d}\vartheta\right)\mathrm{d}s=\infty \tag{3.12}$$

or

$$\int_{\iota_0}^{\infty}\Psi(s)^{1+1/\alpha}\,\Gamma(s)\,\mathrm{d}s=\infty \tag{3.13}$$

is satisfied, then Eq (1.3) has property (A).

Proof. Assume that Eq (1.3) satisfies Case (2.3). Similar to the proof of Theorem 3.1, we obtain (3.9), which contradicts (3.12). Using (3.9) and $w(\iota)-K\,\Psi(\iota)>0$, we obtain

$$w(\iota_1)\geq K\left(\Psi(\iota_1)+K^{1+1/\alpha}\int_{\iota_1}^{\infty}\alpha\Psi^{1+1/\alpha}(s)\,\Gamma(s)\,\mathrm{d}s\right),$$

which contradicts (3.13). □

Theorem 3.2. *Assume that Eq (1.3) has property (A). If*

$$\int_{\iota_0}^{\infty}\int_{v}^{\infty}\nu^{-1/\alpha}(u)\left(\int_u^{\infty}\left(\int_c^d h(\iota,\vartheta)\,\mathrm{d}\vartheta\right)\mathrm{d}s\right)^{1/\alpha}\mathrm{d}u\,\mathrm{d}v=\infty, \tag{3.14}$$

then every nonoscillatory solution $\rho(\iota)$ of Eq (1.3) tends to zero as $\iota\to\infty$.

Proof. Let ρ be a solution of Eq (1.3) such $\rho(\iota)$ satisfies Case (2.2). Therefore, $\lim_{\iota\to\infty}\rho(\iota)=l\geq 0$. If $l\neq 0$, then l is positive, and $\rho(\Omega(\iota,s))>l$. Integrating (3.3) yields

$$\begin{aligned}\nu(\iota)\left(\rho''(\iota)\right)^{\alpha} &\geq \delta\int_{\iota}^{\infty}\left(\int_c^d h(\iota,\vartheta)\,\mathrm{d}\vartheta\right)\rho^{\alpha}(\Omega(\iota,s))\mathrm{d}s\\ &\geq \delta\,l^{\alpha}\int_{\iota}^{\infty}\left(\int_c^d h(\iota,\vartheta)\,\mathrm{d}\vartheta\right)\mathrm{d}s, \end{aligned}\tag{3.15}$$

which implies that (3.15) becomes

$$\rho''(\iota) \geq \frac{\delta l}{\nu^{\frac{1}{\alpha}}(\iota)} \left(\int_{\iota}^{\infty} \left(\int_{c}^{d} h(\iota, \vartheta)\, \mathrm{d}\vartheta \right) \mathrm{d}s \right)^{\frac{1}{\alpha}}. \tag{3.16}$$

By integrating (3.16), we obtain

$$-\rho'(\iota) \geq \delta\, l \int_{\iota}^{\infty} \frac{1}{\nu^{1/\alpha}(u)} \left(\int_{u}^{\infty} \left(\int_{c}^{d} h(\iota, \vartheta)\, \mathrm{d}\vartheta \right) \mathrm{d}s \right)^{1/\alpha} \mathrm{d}u.$$

Integrating again from ι_1 to ∞ implies

$$\rho(\iota_1) \geq \delta\, l \int_{\iota_1}^{\infty} \int_{v}^{\infty} \frac{1}{\nu^{1/\alpha}(u)} \left(\int_{u}^{\infty} \left(\int_{c}^{d} h(\iota, \vartheta)\, \mathrm{d}\vartheta \right) \mathrm{d}s \right)^{1/\alpha} \mathrm{d}u\mathrm{d}v,$$

which contradicts (3.14). Thus, $\lim_{\iota\to\infty} \rho(\iota) = 0$. □

Definition 3.1. *Let* $A_0(\iota) = K\,\Psi(\iota)$, $K \in (0,1)$ *and for each* $\gamma = 0, 1, 2, ...$

$$A_{\gamma+1}(\iota) = A_0(\iota) + \alpha K \int_{\iota}^{\infty} A_{\gamma}^{1+1/\alpha}(s)\, \Gamma(s)\, \mathrm{d}s. \tag{3.17}$$

Theorem 3.3. *If there exists some* $A_\gamma(\iota)$ *such that*

$$\int_{\iota_0}^{\infty} \left(\int_{c}^{d} h(\iota, \vartheta)\, \mathrm{d}\vartheta \right) \frac{\Omega^{\alpha}(\iota, c)}{\iota^{\alpha}} \left(e^{\alpha K \int_{\iota_0}^{\iota} \Gamma(s) A_{\gamma}^{1/\alpha}(s) \mathrm{d}s} \right) \mathrm{d}\iota = \infty \text{ for some } K \in (0,1), \tag{3.18}$$

then Eq (1.3) has property (A).

Proof. Let $\rho > 0$ a solution of Eq (1.3) and satisfy Case (2.3). Similar to the proof of Theorem 3.1, we obtain (3.9). By using (3.9) and definition of $A_0(\iota)$, we have $w(\iota) \geq A_0(\iota)$. Thus,

$$\begin{aligned} A_1(\iota) &= A_0(\iota) + \alpha K \int_{\iota}^{\infty} A_0^{1+1/\alpha}(s)\, \Gamma(s)\, \mathrm{d}s \\ &\leq A_0(\iota) + \alpha K \int_{\iota}^{\infty} w^{1+1/\alpha}(s)\, \Gamma(s)\, \mathrm{d}s \leq w(\iota). \end{aligned}$$

By induction, the sequence $\left\{A_\gamma(\iota)\right\}_{\gamma=0}^{\infty}$ is nondecreasing and $w(\iota) - A_\gamma(\iota) \geq 0$. So, $\{A_\gamma(\iota)\}_{\gamma=0}^{\infty}$ tends to $A(\iota)$. Let $\gamma \to \infty$. By Lebesgue monotone theorem, the equation in (3.17) implies

$$A(\iota) = A_0(\iota) + \alpha K \int_{\iota}^{\infty} \Gamma(s)\, A^{1+1/\alpha}(s)\, \mathrm{d}s.$$

Taking into account $A(\iota) - A_\gamma(\iota) \geq 0$, we obtain

$$A'(\iota) \leq -\delta K \left(\frac{\Omega(\iota, c)}{\iota} \right)^{\alpha} \left(\int_{c}^{d} h(\iota, \vartheta)\, \mathrm{d}\vartheta \right) - \alpha K A(\iota)\, A_{\gamma}^{1/\alpha}(\iota)\, \Gamma(\iota), \text{ for } \iota \geq \iota_1.$$

i.e.,

$$\left(A(\iota)\left(e^{\alpha K\int_{\iota_1}^{\iota}\Gamma(s)A_{\gamma}^{1/\alpha}(s)\mathrm{d}s}\right)\right)' \leq -\frac{\delta K\Omega^{\alpha}(\iota,c)}{\iota^{\alpha}}\left(\int_c^d h(\iota,\vartheta)\,\mathrm{d}\vartheta\right)\left(e^{\alpha K\int_{\iota_1}^{\iota}\Gamma(s)A_{\gamma}^{1/\alpha}(s)\mathrm{d}s}\right).$$

Integrating from ι_1 to ι, we get

$$\begin{aligned} 0 &\leq A(\iota)\left(e^{\alpha K\int_{\iota_1}^{\iota}A_{\gamma}^{1/\alpha}(s)\Gamma(s)\mathrm{d}s}\right) \\ &\leq A(\iota_1)-\delta K\int_{\iota_1}^{\iota}\frac{\Omega^{\alpha}(\iota,c)}{u^{\alpha}}\left(\int_c^d h(\iota,\vartheta)\,\mathrm{d}\vartheta\right)\left(e^{\alpha K\int_{\iota_1}^{u}\Gamma(s)A_{\gamma}^{1/\alpha}(s)\mathrm{d}s}\right)\mathrm{d}u, \end{aligned}$$

which implies that

$$K\int_{\iota_1}^{\iota}\frac{\Omega^{\alpha}(\iota,c)}{u^{\alpha}}\left(\int_c^d h(\iota,\vartheta)\,\mathrm{d}\vartheta\right)\left(e^{\alpha K\int_{\iota_1}^{u}\Gamma(s)A_{\gamma}^{1/\alpha}(s)\mathrm{d}s}\right)\mathrm{d}u \leq \frac{A(\iota_1)}{\delta},$$

which contradict the assumption. □

Theorem 3.4. *If there exists some $A_{\gamma}(\iota)$ such that*

$$\limsup_{\iota\to\infty}\left(\int_{\iota_1}^{\iota}(\Gamma(s)-\Gamma(\iota_1))\,\mathrm{d}s\right)^{\alpha}A_{\gamma}(\iota)>1, \tag{3.19}$$

then Eq (1.3) has property (A).

Proof. Let $\rho(\iota)$ be a solution of Eq (1.3) and $\rho(\iota)>0$ satisfies Case (2.3). By (3.8), since $\iota<\Omega(\iota,s)$, we have

$$\rho(\iota)\geq \nu^{\frac{1}{\alpha}}(\iota)\rho''(\iota)\int_{\iota_1}^{\iota}\int_{\iota_1}^{u}\nu^{-1/\alpha}(s)\,\mathrm{d}s\mathrm{d}u. \tag{3.20}$$

Combining (3.5) with (3.20) yields

$$\frac{1}{w(\iota)}=\frac{\rho^{\alpha}(\iota)}{\nu(\iota)(\rho''(\iota))^{\alpha}}\geq\left(\int_{\iota_1}^{\iota}\Gamma(s)-\Gamma(\iota_1)\,\mathrm{d}s\right)^{\alpha}.$$

Therefore,

$$\left(\int_{\iota_1}^{\iota}(\Gamma(s)-\Gamma(\iota_1))\,\mathrm{d}s\right)^{\alpha}A_{\gamma}(\iota)\leq\left(\int_{\iota_1}^{\iota}(\Gamma(s)-\Gamma(\iota_1))\,\mathrm{d}s\right)^{\alpha}w(\iota)\leq 1,$$

which contradicts the assumption (3.19). □

Remark 3.1. *Note that since the sequence $\{A_{\gamma}(\iota)\}_{\gamma=0}^{\infty}$ is increasing, the greater value of γ in (3.18) and (3.19), the better criteria is obtained.*

The following result is obtained by letting $\gamma=0$ and $\gamma=1$ in Theorem 3.4.

Corollary 3.2. *If either*

$$\limsup_{\iota\to\infty}\left(\int_{t1}^{\iota}\Gamma(s)-\Gamma(\iota_1)\,\mathrm{d}s\right)^{\alpha}\int_{\iota}^{\infty}\frac{\Omega^{\alpha}(\iota,s)}{s^{\alpha}}\left(\int_{c}^{d}h(\iota,\vartheta)\,\mathrm{d}\vartheta\right)\mathrm{d}s>1 \tag{3.21}$$

or

$$\limsup_{\iota\to\infty}\left(\int_{\iota_1}^{\iota}\Gamma(s)-\Gamma(\iota_1)\,\mathrm{d}s\right)^{\alpha}\left(\Psi(\iota)+\alpha\left(\frac{\Omega_0\varsigma_0}{\varsigma_0+p_0^{\alpha}}\right)^{1/\alpha}\int_{\iota}^{\infty}\Psi^{1+1/\alpha}(s)\,\Gamma(s)\,\mathrm{d}s\right)>1, \tag{3.22}$$

then Eq (1.3) has property (A).

By summarizing the results of this section, we obtain criteria that ensure that every solution of Eq (1.3) is either oscillatory or tends to zero.

Theorem 3.5. *Assume that Eq (3.14) holds. If one the Eqs (3.2), (3.12) or (3.13) is satisfied, then every solution of Eq (1.3) oscillates or converges to zero.*

Theorem 3.6. *Assume that Eq (3.14) holds. If there exists some* $A_{\gamma}(\iota)$ *such that one of the Eqs (3.18), (3.19), (3.21) or (3.22) is satisfied, then every solution of Eq (1.3) oscillates or converges to zero.*

4. Examples and comments

Example 4.1. *Consider the following advanced differential equation*

$$\left(\iota\left(\rho''(\iota)\right)^{3}\right)'+\int_{0}^{1}\frac{\beta}{s^{6}}\rho^{3}(\lambda s)\,\mathrm{d}s=0,\ \ \beta>0,\ \ \lambda\in[1,\infty),\quad \iota\geq 1. \tag{4.1}$$

It is in the form of Eq (1.3) with $\nu(\iota)=\iota, f(\rho)=\rho^{3}, h(\iota,s)=\frac{\beta}{s^{6}}, \Omega(\iota,s)=\lambda s, \alpha=3, c=0, d=1$. Using Eq (3.1) to compute $\Psi(\iota)$ and $\Gamma(\iota)$ with $\delta=1, \iota_1=0$, we obtain

$$\Psi(\iota)=\frac{\lambda^{3}\beta}{5\iota^{5}}\quad\text{and}\quad\Gamma(\iota)=\frac{3\,\iota^{2/3}}{2}.$$

By Theorem 3.1, Eq (4.1) has property (A) if

$$\beta>\left(\frac{2}{3}\right)^{3}\left(\frac{5}{4}\right)^{4}\frac{1}{\lambda^{3}}.$$

By Theorem 3.2, the equation in (3.14) holds. Therefore, every nonoscillatory solution $\rho(\iota)$ of Eq (4.1) tends to zero as $\iota\to\infty$.

Example 4.2. *Consider the advanced differential equation*

$$\left(\iota\left(\rho''(\iota)\right)^{3}\right)'+\int_{0}^{1}\frac{\beta}{s^{9}}\rho^{3}\left(s^{2}\right)\mathrm{d}s=0,\ \beta>0,\quad \iota\geq 1. \tag{4.2}$$

It is in the form of Eq (1.3) with $\nu(\iota)=\iota, f(\rho)=\rho^{3}, h(\iota,s)=\frac{\beta}{s^{9}}, \Omega(\iota,s)=s^{2}, \alpha=3, c=0, d=1$. Similar to the previous example, we compute $\Psi(\iota)$ and $\Gamma(\iota)$ with $\delta=1, \iota_1=0$, to obtain

$$\Psi(\iota)=\frac{\beta}{5\iota^{5}}\quad\text{and}\quad\Gamma(\iota)=\frac{3\,\iota^{2/3}}{2}.$$

By Theorem 3.1, Eq (4.2) has property (A) if

$$\beta > \left(\frac{2}{3}\right)^3 \left(\frac{5}{4}\right)^4 .$$

Example 4.3. *Consider the equation*

$$\left(\iota^2 \left(\rho''(\iota)\right)^3\right)' + \int_0^1 \frac{\beta}{s^5} \rho^3 (\lambda s)\,\mathrm{d}s = 0, \quad \beta > 0, \lambda \in [1, \infty),\ \iota \geq 1. \tag{4.3}$$

It is an advanced differential equation in the form of Eq (1.3) with $\nu(\iota) = \iota^2, f(\rho) = \rho^3, h(\iota, s) = \frac{\beta}{s^5}, \Omega(\iota, s) = \lambda s, \alpha = 3, c = 0, d = 1$. Computing $\Psi(\iota)$ and $\Gamma(\iota)$ with $\delta = 1, \iota_1 = 0$, we obtain

$$\Psi(\iota) = \frac{\lambda^3 \beta}{4\iota^4} \text{ and } \Gamma(\iota) = 3\,\iota^{1/3}.$$

By Corollary 3.2, Eq (4.3) has property (A) if

$$\beta > \frac{4^4}{9^3} \frac{1}{\lambda^3}, \tag{4.4}$$

since this implies

$$\frac{9^3}{4^4} \lambda^3 \beta > 1 - \frac{9^4}{4^{16/3}} \lambda^4 \beta^{4/3}, \tag{4.5}$$

then Eq (3.14) holds. Therefore, by Theorem 3.2, every nonoscillatory solution $\rho(\iota)$ of Eq (4.3) tends to zero as $\iota \to \infty$.

Example 4.4. *Consider the advanced differential equation of the form*

$$\left(\iota^a \left|\rho''(\iota)\right|^{\alpha-1} \rho''(\iota)\right)' + \int_0^1 \frac{\beta}{s^b} \left|\rho(s^t)\right|^{\alpha-1} \rho(s^t) = 0, \quad \iota \geq 1, \tag{4.6}$$

where $0 < a < \alpha$, $b, \beta > 0$, $t \geq 1$. It is in the form of Eq (1.4) with $\nu(\iota) = \iota^a, h(\iota, s) = \frac{\beta}{s^b}, \Omega(\iota, s) = s^t, c = 0, d = 1$.

Computing $\Psi(\iota)$ and $\Gamma(\iota)$ with $\delta = 1, \iota_1 = 0$, we obtain

$$\Psi(\iota) = \beta s^{\alpha t - \alpha - b + 1} \left(\alpha t - \alpha - b + 1\right)^{-1} \Big|_\iota^\infty \text{ and } \Gamma(\iota) = \frac{\iota^{1-\frac{a}{\alpha}}}{1 - \frac{a}{\alpha}}.$$

Therefore, Eq (4.6) has property (A) if one of the following conditions holds

- $1 - \alpha \geq b - \alpha t$ (by Corollary 3.1) or
- $1 - \alpha < b - \alpha t$ and $\frac{1}{\alpha} - \alpha + 1 \geq \frac{a}{\alpha+2} + \left(\frac{1}{\alpha} - 1\right) b - \alpha t$ (by Corollary 3.2) or
- $1 - \alpha < b - \alpha t$, $\frac{1}{\alpha} - \alpha + 1 < \frac{a}{\alpha+2} + \left(\frac{1}{\alpha} - 1\right) b - \alpha t\ \frac{1}{\alpha} + 1 = \frac{a}{\alpha} + \frac{b}{\alpha} - t$ and $\frac{1}{(\alpha - a)(b + \alpha - \alpha t - 1)^{1+1/\alpha}} > \frac{1}{\alpha \beta^{1/\alpha} (\alpha+1)^{1+1/\alpha}}$ (by Theorem 3.1).

5. Conclusions

In this work, we classified the positive solutions of the equation in 1.3 according to the sign of its derivatives and studied some properties of these solutions. Using these properties, we found different conditions that ensure that Eq (1.3) satisfies the property (A). We also established condition 3.14 to guarantee every nonoscillatory solutions tends to zero as $\iota \to \infty$. Finally, we obtained new criteria that guarantee the solutions of (1.3) are either oscillatory or converge to zero. We hope this work inspire other researchers to extend the results to the following advanced differential equation:

$$\left(\nu_1(\iota)\left(\nu_2(\iota)\left(\rho''(\iota)\right)^{\alpha}\right)\right)' + \int_c^d h(\iota, s) f(\rho(\Omega(\iota, s)))\,\mathrm{d}s = 0, \quad \iota \geq \iota_0 > 0.$$

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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