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Research article

On the number of zeros of Abelian integrals for a kind of quadratic reversible centers*

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Abstract: Hilbert′s 16th problem is extensively studied in mathematics and its applications. Arnold proposed a weakened version focusing on differential equations. While significant progress has been made for Hamiltonian systems, less attention has been given to integrable non-Hamiltonian systems. In recent years, investigating quadratic reversible systems in integrable non-Hamiltonian systems has gained widespread attention and shown promising advancements. In this academic context, our study is based on qualitative analysis theory. It explores the upper bound of the number of zeros of Abelian integrals for a specific class of quadratic reversible systems under perturbations with polynomial degrees of n. The Picard-Fuchs equation method and the Riccati equation method are employed in our investigation. The research findings indicate that when the degree of the perturbing polynomial is n ($n \geq 5$), the upper bound for the number of zeros of Abelian integrals is determined to be $7n-12$. To achieve this, we first numerically transform the Hamiltonian function of the quadratic reversible system into a standard form. By applying a combination of the Picard-Fuchs equation method and the Riccati equation method, we derive the representation of the Abelian integrals. Using relevant theorems, we estimate the upper bound for the number of zeros of the Abelian integrals, which consequently provides an upper bound for the number of limit cycles in the system. The research results demonstrate that when the perturbation polynomial degree is high or equal to n, the Picard-Fuchs equation method and the Riccati equation method can be applied to estimate the upper bound of the number of zeros of the Abelian integrals.

Keywords: quadratic reversible systems; polynomial perturbations; Picard-Fuchs equation; Riccati equation; upper bound; Abelian integrals
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1. Introduction

Differential dynamical systems find extensive practical applications in various fields, including modeling ecological system equilibrium, spreading infectious diseases, radar technology, radio wave propagation and the orbital trajectories of celestial bodies. Since its first formulation in 1977, Hilbert′s 16th problem has remained a prominent and heavily researched topic within the international mathematical community. Scholars worldwide have researched this problem extensively, leading to significant progress. However, several unresolved questions persist regarding the maximum number of limit cycles and their positional relationships.

In light of this context, we aim to investigate the number of zeros of Abelian integrals for a specific class of quadratic reversible centers, thereby contributing to a further understanding of Hilbert′s 16th problem. Before delving into the particular details of this research, it is vital to provide an overview of the historical background, the significance of the study and the significant accomplishments in existing research.

The perturbed Hamiltonian system is given by

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\partial H(x,y)}{\partial y} + \mu q(x,y), \\ \dfrac{dy}{dt} = -\dfrac{\partial H(x,y)}{\partial x} + \mu p(x,y), \end{cases} \tag{1.1}$$

here, $0 <| \mu |\ll 1$, $H(x,y)$ is a real polynomial of degree $m+1$ in x and y, and $f(x,y)$ and $g(x,y)$ are real polynomials of degree not exceeding n in x and y. When $\mu = 0$, let $\Gamma(h)$ be a closed family of trajectories of (1.1). Then, $\Gamma_h(D)$ is the largest domain of existence of $\Gamma(h)$, given by

$$\Gamma_h(D) = \{(x,y) \in R^2 | H(x,y) = h, h \in \Gamma_h(D)\}. \tag{1.2}$$

The integral given by

$$I(h) = \oint_{\Gamma_h} f(x,y)dy - g(x,y)dx, h \in \Gamma_h(D) \tag{1.3}$$

is commonly known as the Abelian integral. One of the main objectives in this study is to determine the smallest upper bound, denoted as $Z(m,n)$, for the number of zeros of the Abelian integrals. This problem is also referred to as the weak Hilbert′s 16th problem or the Hilbert-Arnold problem (see [1]). This problem remains an active area of research, with numerous related studies conducted. Notably, Khovansky and Varchenko independently established the finiteness of $Z(m,n)$; however, they did not provide an explicit expression (see [2,3]). In a separate study, Li Chengzhi and Zhang Zhifen were able to determine that $Z(2,2) = 4$ (see [4]), among other results. Nevertheless, the majority of current research on determining the upper bound for the number of isolated zeros of Abelian integrals primarily focuses on perturbing Hamiltonian systems. Exploring perturbed integrable yet non-Hamiltonian systems poses additional challenges in the investigation process.

The paper discussed the classification of the complex form of quadratic systems with at least one center, as presented in reference [5]. It presented five types of systems, namely A, B, C, D and E, and noted that the quadratic reversible system B had received significant attention due to its unique properties (see [5]). However, research on this system had been challenging due to a lack of research methods.

References [6,7] have made significant contributions to the study of Abelian integrals for cubic vector fields, with [6] establishing minimum and upper bounds for the number of zeros of Abelian integrals and [7] providing linear estimates of the zeros of Abelian integrals of the cubic Hamiltonian function (see [6,7]).

It is crucial to emphasize the significant phenomena of soliton solutions and limit cycles in nonlinear dynamical systems, as they can undergo mutual transformation. The understanding of soliton solutions and their dynamics has been dramatically enriched by the valuable insights presented in the literature [8–10], which significantly complement our investigation into the existence and characteristics of limit cycles (see [8–10]).

When exploring the upper bound of Abelian integrals in Hamiltonian systems, researchers often rely on the effectiveness of the Picard-Fuchs equation method and Riccati equation method to estimate the number of zeros for the upper bound of Abelian integrals. These methodologies have proven particularly effective in analyzing higher-order polynomial perturbations ($n \geq 3$). Previous studies have successfully applied these techniques to explore various types of Hamiltonian systems, including those exhibiting polynomial potentials or terms of the form $x^i y^j$ (see [11–17]).

In an alternate scenario where the degree of the perturbing polynomial in the system is relatively low, a combination of detection functions and numerical exploration methods can be employed to explore the number and location of limit cycles. Using these techniques, researchers can ascertain the existence and properties of limit cycles in such systems. For further in-depth exploration and specific references on this topic, please refer to the cited works (see [18–23]).

Motivated by these earlier works, our research is primarily focused on investigating the upper bound of the number of zeros of Abelian integrals in perturbed quadratic reversible systems, where cubics form the majority of orbits. To tackle these challenges, we employ two powerful analytical methods: the Picard-Fuchs equation and the Riccati equation. These methods allow us to rigorously analyze the properties of Abelian integrals and derive compelling results concerning their distribution of zeros.

One of the key innovations of our research lies in determining the upper bound for the number of zeros of Abelian integrals in perturbed quadratic reversible systems under n ($n \geq 5$) degree polynomial perturbations, which is given by $7n - 12$. A thorough review of related literature revealed that the upper bound for the number of zeros of Abelian integrals in this specific class of quadratic reversible systems has yet to be previously explored by other researchers. Therefore, our research results are novel and original, further enriching the studies on Hilbert′s 16th problem.

Furthermore, our research explicitly applies the Picard-Fuchs and Riccati equation methods to quadratic reversible systems under n-degree polynomial perturbations. While these methods have been widely used in Hamiltonian systems to explore the upper bounds of the number of zeros of Abelian integrals, their combined application in the context of non-Hamiltonian systems, especially for quadratic reversible systems, has received relatively less attention. Therefore, our research methodology exhibits a certain degree of innovation and applicability in exploring the upper bounds of limit cycles in quadratic reversible systems under n-degree polynomial perturbations, contributing to a comprehensive analysis of these systems.

2. Methods and analysis

2.1. Theoretical methods: Picard-Fuchs and Riccati equation methods

Consider the following system:

$$\begin{cases} \dfrac{dx}{dt} = -\dfrac{\partial H(x,y)}{\partial y} + \varepsilon f(x,y), \\ \dfrac{dy}{dt} = \dfrac{\partial H(x,y)}{\partial x} + \varepsilon g(x,y), \end{cases} \tag{2.1}$$

here, Γ^h is a family of closed curves for the Hamiltonian system corresponding to $\varepsilon = 0$, $\Gamma^h(D)$ is the enclosed area by Γ^h and $I(h)$ is the Abelian integral corresponding to the system (2.1).

Suppose H is a polynomial of degree $n+1$, and its degree $n+1$ terms at infinity can be expressed as products of $n+1$ linear terms. In that case, we call the corresponding differential equation $X^{'}(h) = A(h)X(h)$ a linear Picard-Fuchs equation, where $A(h)$ is an $n \times n$ matrix with rational function elements of h, and its singularities correspond to the critical values of H. In general, $I(h)$ can be written as a linear combination of $\{I_1(h), I_2(h)...I_k(h)\}$ or a subset of them, where $I(h)$ coefficients are polynomials of h. By applying the Picard-Fuchs equation, we can investigate the number of zeros of $I(h)$ (see [24]).

2.2. Normal form

Next, we will explore the standard form of quadratic reversible systems. The quadratic reversible system of type Q_3^R is given by the equation

$$B : \dot{z} = -iz + az^2 + 2|z|^2 + b\bar{z}^2, \tag{2.2}$$

where $z \in \mathbb{C}$, $a, b \in \mathbb{R}$, and i is the imaginary unit.

Let $z = x + yi$. Then the B-reversible type system can be transformed into the following form

$$\begin{cases} \dot{x} = (a+b+2)x^2 - (a+b-2)y^2 + y, \\ \dot{y} = -x[1 - 2(a-b)y], \end{cases} \tag{2.3}$$

let $Y = x$, $X = 1 - 2(a-b)y$, $d\tau = -2(a-b)dt$, then (2.3) can be transformed into

$$\begin{cases} \dot{X} = -XY, \\ \dot{Y} = -\dfrac{a+b+2}{2(a-b)}Y^2 + \dfrac{a+b-2}{8(a-b)^3}X^2 - \dfrac{b-1}{2(a-b)^3}X - \dfrac{a-3b+2}{8(a-b)^3}. \end{cases} \tag{2.4}$$

The first integral of the system (2.4) is presented in the literature [25] as follows (see [25])

$$H(X,Y) = X^{\lambda}\left[\frac{1}{2}Y^2 + \frac{1}{8(a-b)^2}\left(\frac{a+b-2}{a-3b-2}X^2 + 2\frac{b-1}{b+1}X + \frac{a-3b+2}{a+b+2}\right)\right], \tag{2.5}$$

where $\lambda = -\dfrac{a+b+2}{a-b}$ and $M(X,Y) = X^{\lambda-1}$ is its integrating factor. We investigate the nth perturbed system of system (2.4), which is given by

$$\begin{cases} \dot{X} = -XY + \mu f(X,Y), \\ \dot{Y} = -\dfrac{a+b+2}{2(a-b)}Y^2 + \dfrac{a+b-2}{8(a-b)^3}X^2 - \dfrac{b-1}{2(a-b)^3}X - \dfrac{a-3b+2}{8(a-b)^3} + \mu g(X,Y), \end{cases} \tag{2.6}$$

where $f(X, Y) = \sum_{0\le i+j\le n} a_{ij}X^iY^j$ and $g(X, Y) = \sum_{0\le i+j\le n} b_{ij}X^iY^j$ are n-th order polynomials in x and y, and $0 < \mu \ll 1$.

Let $\{\Gamma_h\} = \{(X, Y) \mid H(X, Y) = h, h \in \prod\}$ denote the family of closed trajectories of the system (2.6), where $\prod$ is the maximal existence interval of the family of closed orbits Γ_h. The problem of how many limit cycles can bifurcate from the closed orbits of the system (2.4) is equivalent to determining the number of isolated zeros of the successor function defined on the untouchable segment passing through Γ_h, given by

$$d(h, \mu) = \mu I(h) + \mu^2 M_2(h) + ..., \tag{2.7}$$

where $I(h)$ is called the Abelian integral and is given by

$$I(h) = \oint_{\Gamma_h} M(X, Y)g(X, Y)dX - M(X, Y)f(X, Y)dY. \tag{2.8}$$

In the upcoming discussion, our focus will be on the case of Abelian integration related to the system (2.3) with the specific parameter values $a = 3$ and $b = 1$. The integrating factor for this system is given by $M(X, Y) = X^{-3}$, and the first integration of the system corresponds to the loss-grid case labeled as 1.

When $a = 3$ and $b = 1$, system (2.4) becomes

$$\begin{cases} \dot{X} = -XY, \\ \dot{Y} = -\dfrac{3}{2}Y^2 + \dfrac{1}{32}X^2 - \dfrac{1}{32}. \end{cases} \tag{2.9}$$

It is known that system (2.9) is an integrable non-Hamiltonian system, and one of its integral curves is given by $X = 0$. The system has a center-type singularity at $(1, 0)$. In the interval $h \in (-\frac{1}{48}, 0)$, there exists a family of closed trajectories $\{\Gamma_h\}$ (see Figure 1). The first integral of system (2.9) is given by

$$H(X, Y) = X^{-3}(\frac{1}{2}Y^2 - \frac{1}{32}X^2 + \frac{1}{96}) = h + 1, \tag{2.10}$$

where X^{-4} is its integrating factor.

We study the number of zeros of the Abelian integrals of the following system:

$$\begin{cases} \dot{X} = -YX^{-3} + \mu f(X, Y), \\ \dot{Y} = -\dfrac{3}{2}Y^2X^{-4} + \dfrac{1}{32}X^{-2} - \dfrac{1}{32}X^{-4} + \mu g(X, Y), \end{cases} \tag{2.11}$$

which is equivalent to studying the number of zeros of the Abelian integrals of system the following system, given by

$$\begin{cases} \dot{X} = -YX^{-3} + \mu X^{-4} f(X, Y), \\ \dot{Y} = -\dfrac{3}{2}Y^2X^{-4} + \dfrac{1}{32}X^{-2} - \dfrac{1}{32}X^{-4} + \mu X^{-4} g(X, Y). \end{cases} \tag{2.12}$$

Thus, solving for the number of zeros of the Abelian integrals of the system (2.11) is equivalent to solving for the number of zeros of the Abelian integrals of the system (2.12) under the condition of (2.10). This can be further simplified to finding the number of zeros of the Abelian integrals given by

$$I(h) = \oint_{\Gamma_h} X^{-4} \sum_{0\le i+j\le n} b_{ij}X^iY^jdX - \oint_{\Gamma_h} X^{-4} \sum_{0\le i+j\le n} a_{ij}X^iY^jdY, \tag{2.13}$$

where X^{-4} is the integrating factor, and $h \in (-\frac{1}{48}, 0)$. This is equivalent to solving for the number of zeros of the Abelian integrals given by

$$I(h) = \oint_{\Gamma_h} \sum_{0 \le i+j \le n} b_{ij} X^{i-4} Y^j dX - \oint_{\Gamma_h} \sum_{0 \le i+j \le n} a_{ij} X^{i-4} Y^j dY, \tag{2.14}$$

where $h \in (-\frac{1}{48}, 0)$, $i = 0, 1, 2, 3...n$; $j = 0, 1, 2, 3...n$.

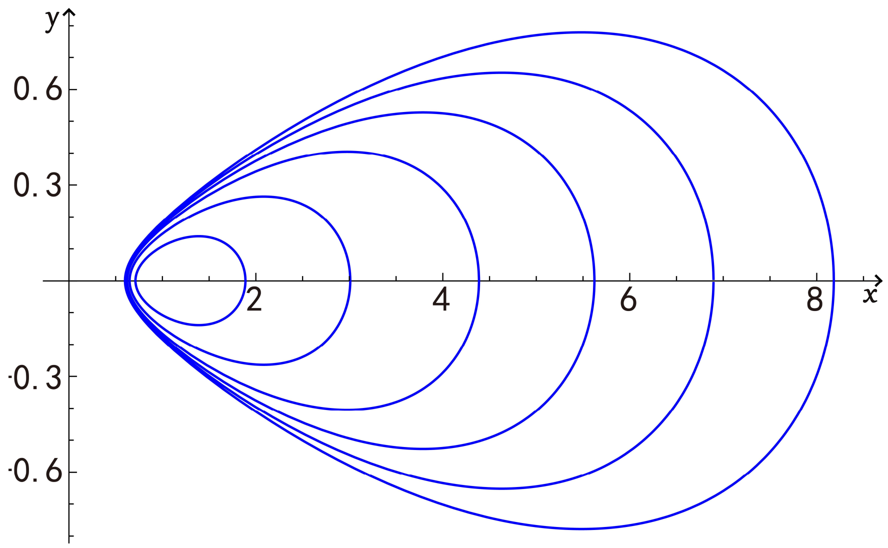

Figure 1. Phase portrait of system (2.9).

2.3. The algebraic structure of the Abelian integrals $I(h)$

To investigate the number of zeros of the Abelian integrals for system (2.11), we need to study the algebraic structure of the Abelian integrals $I(h)$ related to (2.14). To this end, we introduce the integral function as follows:

$$I_{i,j}(h) = \oint_{\Gamma_h} X^{i-4} Y^j dX, \quad h \in (-\frac{1}{48}, 0), \tag{2.15}$$

where $i = 0, 1, 2...n$ and $j = 0, 1, 2...n$. When $j = 1$, we denote it as Using Green's formula, we obtain

$$\oint_{\Gamma_h} X^{i-4} Y^j dY = \iint_{\Gamma_h(D)} (i-4) X^{i-5} Y^j dX dY = -\frac{i-4}{j+1} \oint_{\Gamma_h} X^{i-5} Y^{j+1} dX, \tag{2.16}$$

which is equivalent to

$$\oint_{\Gamma_h} X^{i-4} Y^j dY = -\frac{i-4}{j+1} \oint_{\Gamma_h} X^{i-5} Y^{j+1} dX. \tag{2.17}$$

Equation (2.17) shows that $\oint_{\Gamma_h} X^{i-4} Y^j dY$ can be expressed in terms of $I_{ij}(h)$. We use $\deg(\alpha(h))$ to denote the degree of polynomial $\alpha(h)$, $[n]$ to denote the largest integer not exceeding n, and let $\hbar = \frac{1}{h}$.

Proposition 2.1. The algebraic structure of the Abelian integrals (2.14) can be expressed as

$$\begin{cases} I(h) = \hbar J(h), \\ J(h) = \alpha(\hbar)J_{-1}(h) + \beta(\hbar)J_0(h) + \gamma(\hbar)J_1(h), \end{cases} \tag{2.18}$$

where for $n \geq 5$, we have $\deg(\alpha(\hbar)) \leq n-4$, $\deg(\beta(\hbar)) \leq n-3$ and $\deg(\gamma(\hbar)) \leq n-4$.
Proof. From Eq (2.10), we have

$$H(X,Y) = \frac{1}{2}Y^2X^{-3} - \frac{1}{32}X^{-1} + \frac{1}{96}X^{-3} = h. \tag{2.19}$$

Differentiating both sides concerning X yields:

$$-\frac{3}{2}X^{-4}Y^2 + X^{-3}Y\frac{\partial Y}{\partial X} + \frac{1}{32}X^{-2} - \frac{1}{32}X^{-4} = 0. \tag{2.20}$$

Multiplying both sides of Eq (2.20) by $X^iY^{j-2}dX$, we obtain

$$-\frac{3}{2}X^{i-4}Y^j + X^{i-3}Y^{j-2}\frac{\partial Y}{\partial X} + \frac{1}{32}X^{i-2}Y^{j-2} - \frac{1}{32}X^{i-4}Y^{j-2} = 0. \tag{2.21}$$

Integrating both sides of Eq (2.21) along Γ_h, we have

$$\oint_{\Gamma_h}(-\frac{3}{2}X^{i-4}Y^j + X^{i-3}Y^{j-2}\frac{\partial Y}{\partial X} + \frac{1}{32}X^{i-2}Y^{j-2} - \frac{1}{32}X^{i-4}Y^{j-2})dX = 0. \tag{2.22}$$

Since $\frac{\partial Y}{\partial X}dX = dY$, the following equation holds:

$$\oint_{\Gamma_h}(-\frac{3}{2}X^{i-4}Y^j)dX + \oint_{\Gamma_h}X^{i-3}Y^{j-1}dY + \oint_{\Gamma_h}\frac{1}{32}Y^{j-2}(X^{i-2} - X^{i-4})dX = 0. \tag{2.23}$$

Equation (2.23) is equivalent to the following equation:

$$-\frac{3}{2}I_{i,j} - \frac{i-3}{j}I_{i,j} + \frac{1}{32}I_{i+2,j-2} - \frac{1}{32}I_{i,j-2} = 0, \tag{2.24}$$

after simplification, Eq (2.24) can be expressed as:

$$I_{i,j}(h) = \frac{1}{16}\frac{\left(I_{i+2,j-2} - I_{i,j-2}\right)j}{3j+2i-6}. \tag{2.25}$$

When j is even, we can deduce that $I_{ij}(h) = 0$ based on the properties of curve integrals. When j is odd, we can use Eq (2.25) to reduce the value of j to 1, which implies that $I_{ij}(h)$ can be expressed as a linear combination of $J_i(h)$ for $i = 0, 1, 2, 3 \ldots$. Therefore, the following equation holds true

$$I_{i,j}(h) = \sum_{k=0}^{\frac{j-1}{2}} c_{ik}J_{i+2k}. \tag{2.26}$$

By utilizing expression (2.26), we can derive the following equation, which provides a relationship between $\hbar I_{i,j}(h)$ and $\hbar J_{i+2k}$:

$$\hbar I_{i,j}(h) = \sum_{k=0}^{\frac{j-1}{2}} c_{ik}\hbar J_{i+2k}. \tag{2.27}$$

This equation shows that $I_{i,j}(h)$ can be expressed as a linear combination of $\hbar J_{i+2k}$ with coefficients c_{ik}, where k ranges from 0 to $\frac{j-1}{2}$.

Moreover, by using formula (2.10), we can derive the following equation:

$$\frac{1}{2}Y^2 - \frac{1}{32}X^2 + \frac{1}{96} = hX^3. \tag{2.28}$$

Multiplying both sides of Eq (2.28) by $X^{i-4}Y^{j-2}dX$ yields the following equation:

$$\frac{1}{2}X^{i-4}Y^j - \frac{1}{32}X^{i-2}Y^{j-2} + \frac{1}{96}X^{i-4}Y^{j-2} = hX^{i-1}Y^{j-2}. \tag{2.29}$$

In Γ_h, integrating both sides of Eq (2.29) yields the following equation:

$$\oint_{\Gamma_h} \frac{1}{2}X^{i-4}Y^j dX - \oint_{\Gamma_h} \frac{1}{32}X^{i-2}Y^{j-2}dX + \oint_{\Gamma_h} \frac{1}{96}X^{i-4}Y^{j-2}dX = \oint_{\Gamma_h} hX^{i-1}Y^{j-2}dX. \tag{2.30}$$

Therefore, we can deduce the following equation:

$$hI_{i+3,j-2} = \frac{1}{2}I_{ij}(h) - \frac{1}{32}I_{i+2,j-2}(h) + \frac{1}{96}I_{i,j-2}(h). \tag{2.31}$$

Simplifying the previous Eq (2.31), we obtain:

$$I_{ij}(h) = 2hI_{i+3,j-2}(h) + \frac{1}{16}I_{i+2,j-2}(h) - \frac{1}{48}I_{i,j-2}(h). \tag{2.32}$$

Substituting Eq (2.25) into Eq (2.32), we obtain:

$$\frac{1}{16}\frac{\left(I_{i+2,j-2}(h) - I_{i,j-2}(h)\right)j}{3j+2i-6} = 2hI_{i+3,j-2} + \frac{1}{16}I_{i+2,j-2}(h) - \frac{1}{48}I_{i,j-2}(h). \tag{2.33}$$

Since $\oint_{\Gamma_h} X^{i-4}YdX = J_i(h)$, by substituting $j = 3$ into Eq (2.33), we obtain the transformed equation:

$$3J_{i+2} - 3J_i = 32h(2i+3)J_{i+3} + (2i+3)J_{i+2} - \frac{1}{3}(2i+3)J_i. \tag{2.34}$$

Simplifying Eq (2.34) yields the following equation:

$$96h(2i+3)J_{i+3}(h) = (2i-6)J_i(h) - 6iJ_{j+2}(h). \tag{2.35}$$

From Eq (2.35), we obtain:

$$J_i(h) = \frac{i-6}{48(2i-3)}\frac{1}{h}J_{j-3}(h) - \frac{i-3}{16(2i-3)}\frac{1}{h}J_{j-1}(h). \tag{2.36}$$

By substituting $\hbar = \frac{1}{h}$ into Eq (2.36), we can rewrite it as follows:

$$J_i(h) = \frac{i-6}{48(2i-3)}\hbar J_{j-3}(h) - \frac{i-3}{16(2i-3)}\hbar J_{j-1}(h). \tag{2.37}$$

According to Eq (2.37), the following results can be obtained:

(1) When $i \geq 2$, $J_i(h)$ can be expressed as a combination of $\hbar J_{i-3}(h)$ and $\hbar J_{i-1}(h)$. By iteratively applying Eq (2.37), $J_i(h)$ can be finally expressed as a linear combination of $J_{-1}(h)$, $J_0(h)$ and $J_1(h)$, with the following relation:

$$J_i(h) = \alpha_{i,-1}(\hbar)J_{-1}(h) + \beta_{i,0}(\hbar)J_0(h) + \gamma_{i,1}(\hbar)J_1(h), \tag{2.38}$$

when $i \geq 4$, $\deg(\alpha_{i,-1}(\hbar)) \leq i-3$, $\deg(\beta_{i,0}(\hbar)) \leq i-2$ and $\deg(\gamma_{i,1}(\hbar)) \leq i-3$;
when $i = 3$, $\deg(\alpha_{i,-1}(\hbar)) = 2$, $\deg(\beta_{i,0}(\hbar)) = 1$ and $\deg(\gamma_{i,1}(\hbar)) = 2$;
when $i = 2$, $\deg(\alpha_{i,-1}(\hbar)) = 1$, $\deg(\beta_{i,0}(\hbar)) = 0$ and $\deg(\gamma_{i,1}(\hbar)) = 1$.

(2) When $i = -1, 0, 1$, we can express $J_i(h)$ as a combination of $J_{-1}(h)$, $J_0(h)$ and $\hbar J_1(h)$, and the following equations hold:

$$\hbar J_{-1}(h) = \hbar J_{-1}(h), \hbar J_0(h) = \hbar J_0(h), \hbar J_1(h) = \hbar J_1(h), \tag{2.39}$$

when $i = -1$, $\deg(\alpha_{i,-1}(\hbar)) = 1$, $\deg(\beta_{i,0}(\hbar)) = 0$ and $\deg(\gamma_{i,1}(\hbar)) = 0$;
when $i = 0$, $\deg(\alpha_{i,-1}(\hbar)) = 0$, $\deg(\beta_{i,0}(\hbar)) = 1$ and $\deg(\gamma_{i,1}(\hbar)) = 0$;
when $i = 1$, $\deg(\alpha_{i,-1}(\hbar)) = 0$, $\deg(\beta_{i,0}(\hbar)) = 0$ and $\deg(\gamma_{i,1}(\hbar)) = 1$.

Furthermore, using Eqs (2.14) and (2.26) along with Eq (2.17), we can deduce the following relationship:

$$I(h) = \sum_{0\leq i+j\leq n} b_{ij} \sum_{k=0}^{\frac{j-1}{2}} c_{ik} J_{i+2k}(h) + \frac{i-4}{j+1} \sum_{0\leq i+j\leq n} a_{ij} \sum_{k=0}^{\frac{j}{2}} c_{i-1,k} J_{i-1+2k}. \tag{2.40}$$

Given that $J(h) = \hbar I(h)$, we can conclude the validity of the following relationship:

$$J(h) = \hbar \sum_{0\leq i+j\leq n} b_{ij} \sum_{k=0}^{\frac{j-1}{2}} c_{ik} J_{i+2k}(h) + \hbar \frac{i-4}{j+1} \sum_{0\leq i+j\leq n} a_{ij} \sum_{k=0}^{\frac{j}{2}} c_{i-1,k} J_{i-1+2k}. \tag{2.41}$$

Considering Eqs (2.14), (2.40) and (2.41), it can be seen from Eq (2.38) that when $n \geq 5$, $\deg(\alpha(\hbar)) \leq n-4$, $\deg(\beta(\hbar)) \leq n-3$ and $\deg(\gamma(\hbar)) \leq n-4$. Therefore, Proposition 2.1 holds.

Corollary 2.1. When $n \geq 5$, the Abelian integrals (2.18) can be expressed as

$$J(h) = h^{n-3} I(h), \quad J(h) = \alpha_1(h)J_{-1}(h) + \beta_1(h)J_0(h) + \gamma_1(h)J_1(h), \tag{2.42}$$

where $\deg(\alpha_1(h)) \leq n-4$, $\deg(\beta_1(h)) \leq n-3$, $\gamma_1 \deg((h)) \leq n-4$ and $\alpha_1(h) = h\bar{\alpha}_1(h)$, $\gamma_1(h) = h\bar{\gamma}_1(h)$, where $\bar{\alpha}_1(h)$ and $\bar{\gamma}_1(h)$ are both polynomials.

Proof. When $n \geq 5$, it follows from Proposition 2.1 and formula (2.41) that the Eq (2.42) holds.

2.4. Picard-Fuchs equation and Riccati equation

Lemma 2.1. The integrals $J_i(h)$ $(i=-1,0,1)$ satisfy the following Picard-Fuchs equation

$$\begin{pmatrix} J_{-1}(h) \\ J_0(h) \\ J_1(h) \end{pmatrix} = \begin{pmatrix} \frac{3}{4}h & \frac{3}{128} & \frac{3}{8}h \\ \frac{1}{48} & h & 0 \\ 0 & \frac{1}{32} & \frac{3}{2}h \end{pmatrix} \begin{pmatrix} J'_{-1}(h) \\ J'_0(h) \\ J'_1(h) \end{pmatrix}. \tag{2.43}$$

Proof. Since $H(X,Y)=X^{-3}(\frac{1}{2}Y^2-\frac{1}{32}X^2+\frac{1}{96})=h$, we have:

$$Y^2 = 2hX^3 + \frac{1}{16}X^2 - \frac{1}{48}. \tag{2.44}$$

Taking partial derivatives of (2.44) with respect to h and X yields:

$$\begin{cases} \frac{\partial Y}{\partial h} = \frac{X^3}{Y}, \\ \frac{\partial Y}{\partial X} = \frac{3hX^2+\frac{1}{16}X}{Y}. \end{cases} \tag{2.45}$$

Moreover, since $\oint_{\Gamma_h} X^{i-4}YdX = J_i(h)$, taking derivative of (2.45) with respect to h yields:

$$J'_i(h) = \oint_{\Gamma_h} X^{i-4}\frac{\partial Y}{\partial h}dX = \oint_{\Gamma_h} X^{i-4}\frac{X^3}{Y}dX = \oint_{\Gamma_h} \frac{X^{i-1}}{Y}dX. \tag{2.46}$$

Furthermore, we have

$$J_i(h) = \oint_{\Gamma_h} \frac{X^{i-4}Y^2}{Y}dX = \oint_{\Gamma_h} \frac{X^{i-4}(2hX^3+\frac{1}{16}X^2-\frac{1}{48})}{Y}dX, \tag{2.47}$$

which implies

$$J_i(h) = \oint_{\Gamma_h} \frac{2hX^{i-1}+\frac{1}{16}X^{i-2}-\frac{1}{48}X^{i-4}}{Y}dX = 2hJ'_i(h) + \frac{1}{16}J'_{i-1} - \frac{1}{48}J'_{i-3}. \tag{2.48}$$

On the other hand, we have

$$J_i(h) = \oint_{\Gamma_h} X^{i-4}YdX = \iint_{\Gamma_h(D)} X^{i-4}dXdY = -\frac{1}{i-3}\oint_{\Gamma_h} X^{i-3}dY, \tag{2.49}$$

and $dY=\frac{\partial Y}{\partial X}dX$ holds. Therefore, we obtain

$$(i-3)J_i(h) = \oint_{\Gamma_h} YdX^{i-3} = -\oint_{\Gamma_h} \frac{X^{i-3}(\frac{1}{16}+3hX^2)}{Y}dX, \tag{2.50}$$

which simplifies to yield:

$$(i-3)J_i(h) = -3hJ'_i(h) - \frac{1}{16}J'_{i-1}(h). \tag{2.51}$$

Simplify Eqs (2.48) and (2.51) to obtain the following equation:

$$(i-2)J_i(h) = -hJ_i^{'}(h) - \frac{1}{48}J_{i-3}^{'}(h). \tag{2.52}$$

Setting $i = -1, 0, 1$ in Eqs (2.51) and (2.52), we obtain the following equations:

$$\begin{cases} -4J_{-1}(h) = -3hJ_{-1}^{'}(h) - \dfrac{1}{16}J_{-2}^{'}(h), \\ -3J_0(h) = -3hJ_0^{'}(h) - \dfrac{1}{16}J_{-1}^{'}(h), \\ -2J_1(h) = -3hJ_1^{'}(h) - \dfrac{1}{16}J_0^{'}(h), \\ -J_1(h) = -hJ_1^{'}(h) - \dfrac{1}{48}J_{-2}^{'}(h). \end{cases} \tag{2.53}$$

Upon simplifying Eq (2.53), we arrive at the resulting expression, as shown below:

$$\begin{cases} J_{-1}(h) = \dfrac{3}{4}hJ_{-1}^{'}(h) + \dfrac{3}{128}J_0^{'}(h) + \dfrac{3}{8}hJ_1^{'}(h), \\ J_0(h) = \dfrac{1}{48}J_{-1}^{'}(h) + hJ_0^{'}(h), \\ J_1(h) = \dfrac{1}{32}J_0^{'}(h) + \dfrac{3}{2}hJ_1^{'}(h). \end{cases} \tag{2.54}$$

Equation (2.54) shows that Lemma 2.1 holds.
Proposition 2.2. The integrals $J_i(h)$ satisfy the following Picard-Fuchs equation:

$$\begin{pmatrix} J_{-1}^{''}(h) \\ J_0^{''}(h) \\ J_1^{''}(h) \end{pmatrix} = \frac{1}{A(h)} \begin{pmatrix} -9h^2 & 9h^2 \\ \dfrac{3}{16}h & -\dfrac{3}{16}h \\ -\dfrac{1}{256} & 9h^2 \end{pmatrix} \begin{pmatrix} J_{-1}^{'}(h) \\ J_1^{'}(h) \end{pmatrix}, \tag{2.55}$$

where $A(h) = 27h(\frac{1}{48^2} - h^2)$.
Proof. Differentiating both sides of Eq (2.54) yields

$$\begin{cases} J_{-1}^{'}(h) = 3hJ_{-1}^{''}(h) - 3hJ_1^{''}(h), \\ 0J_0^{'}(h) = 3hJ_0^{''}(h) + \dfrac{1}{16}J_{-1}^{''}(h), \\ J_1^{'}(h) = -\dfrac{1}{16}J_0^{''} - 3hJ_1^{''}(h). \end{cases} \tag{2.56}$$

From (2.56), we have

$$\begin{pmatrix} J_{-1}^{'}(h) \\ 0J_0^{'}(h) \\ J_1^{'}(h) \end{pmatrix} = \begin{pmatrix} 3h & 0 & -3h \\ \dfrac{1}{16}h & 3h & 0 \\ 0 & -\dfrac{1}{16} & -3h \end{pmatrix} \begin{pmatrix} J_{-1}^{''}(h) \\ J_0^{''}(h) \\ J_1^{''}(h) \end{pmatrix}. \tag{2.57}$$

Simplifying (2.57) gives the desired result of Proposition 2.2.

Corollary 2.2. When $h \in (-\frac{1}{48}, 0)$, we have $J_{-1}(-\frac{1}{48}) = J_0(-\frac{1}{48}) = J_1(-\frac{1}{48}) = 0$, and $J'_{-1}(h) > 0$, $J'_0(h) > 0$, and $J'_1(h) > 0$.

Proof. Substitute into Corollary 2.1 for computation.

Proposition 2.3. Let $W(h) = \dfrac{J'_{-1}(h)}{J'_1(h)}$. Then $W(h)$ satisfies the following Riccati equation

$$A(h)W'(h) = \frac{1}{256}W^2(h) - 18h^2W(h) + 9h^2. \tag{2.58}$$

Proof. By taking the derivative of $W(h)$ on both sides, we obtain the resulting equation, as indicated below:

$$W'(h) = \frac{J''_{-1}(h)J'_1(h) - J'_{-1}(h)J''_1(h)}{(J'_1(h))^2}. \tag{2.59}$$

Substituting (2.55) into (2.59) and simplifying yields (2.58).

2.5. The number of zeros of Abelian integrals

When $n \geq 5$, substituting Eq (2.43) into Eq (2.42) and manipulating yields the following equation:

$$J(h) = \alpha_2(h)J'_{-1}(h) + \beta_2(h)J'_0(h) + \gamma_2(h)J'_1(h), \tag{2.60}$$

where $\alpha_2(h) = \frac{3}{4}h\alpha_1(h) + \frac{1}{48}\beta_1(h)$, $\beta_2(h) = \frac{3}{128}\alpha_1(h) + h\beta_1(h) + \frac{1}{32}\gamma_1(h)$ and $\gamma_2(h) = \frac{3}{8}h\alpha_1(h) + \frac{3}{2}h\gamma_1(h)$, with polynomial degrees satisfying $\deg(\alpha_2(h)) \leq n-2$, $\deg(\beta_2(h)) \leq n-2$ and $\deg(\gamma_2(h)) \leq n-2$. Furthermore, $\beta_2(h) = h\bar{\beta}2(h)$ and $\gamma 2(h) = h^2\bar{\gamma}2(h)$, where $\bar{\beta}2(h)$ and $\bar{\gamma}_2(h)$ are both polynomials.

When $n \geq 5$, differentiating both sides of Eq (2.60) with respect to h and substituting Eq (2.55) leads to the following equation:

$$A(h)J'(h) = \alpha_3(h)J'_{-1}(h) + A(h)\beta'_2(h)J'_0(h) + \gamma_3(h)J'_1(h), \tag{2.61}$$

where $\alpha_3(h) = A(h)\alpha'_2(h) - 9h^2\alpha_2(h) + \frac{3}{16}h\beta_2(h) - \frac{1}{256}\gamma_2(h)$ and $\gamma_3(h) = 9h^2\alpha_2(h) - \frac{3}{16}h\beta_2(h) + A(h)\gamma'_2(h) + 9h^2\gamma_2(h)$, with polynomial degrees satisfying $\deg(\alpha_3(h)) \leq n$ and $\deg(\gamma_3(h)) \leq n$, and where $\alpha_3(h) = h\bar{\alpha}3(h)$ and $\gamma 3(h) = h^2\bar{\gamma}3(h)$, and $\bar{\alpha}3(h)$ and $\bar{\gamma}_3(h)$ are both polynomials.

From Eqs (2.60) and (2.61), we obtain

$$A(h)\beta_2(h)J'(h) = A(h)\beta'_2(h)J(h) + S(h), \tag{2.62}$$

where $S(h)$ is given by

$$S(h) = \alpha_4(h)J'_{-1}(h) + \gamma_4(h)J'_1(h), \tag{2.63}$$

where $\alpha_4(h) = \alpha_3(h)\beta_2(h) - A(h)\alpha_2(h)\beta'_2(h)$ and $\gamma_4(h) = \beta_2(h)\gamma_3(h) - A(h)\beta'_2(h)\gamma_2(h)$. The degree of the polynomials satisfy $\deg(\alpha_4(h)) \leq 2n-2$ and $\deg(\gamma_4(h)) \leq 2n-2$, and $\alpha_4(h) = h^2\bar{\alpha}4(h)$ and $\gamma 4(h) = h^3\bar{\gamma}4(h)$, where $\bar{\alpha}4(h)$ and $\bar{\gamma}_4(h)$ are polynomials.

Lemma 2.2. Let $V(h) = \frac{S(h)}{J'_1(h)}$. Then $V(h)$ satisfies the following Riccati equation

$$A(h)\alpha_4(h)V'(h) = \frac{1}{256}V^2(h) + D(h)V(h) + E(h), \tag{2.64}$$

where $D(h)$ and $E(h)$ are given by $D(h) = A(h)\alpha_4'(h) - \frac{1}{128}\gamma_4(h) - 18h^2\alpha_4(h)$, $E(h) = 9h^2\alpha_4^2(h) + 18h^2\alpha_4(h)\gamma_4(h) + \frac{1}{256}\gamma_4^2(h) + A(h)\alpha_4(h)\gamma_4'(h) - A(h)\alpha_4'(h)\gamma_4(h)$, and $\deg(E(h)) \leq 4n-2$, and $E(h) = h^5\bar{E}(h)$, where $\bar{E}(h)$ is a polynomial.

Proof. Taking the derivative of $V(h)$ and using Eq (2.58), we obtain the desired result (2.64).

We use $\sharp I(h)$ to denote the zeros of $I(h)$. The following result holds.

Lemma 2.3. The number of zeros of the Abelian integrals $I(h)$ satisfies the following inequality:

$$\sharp I(h) = \sharp J(h) \leq \sharp A(h) + \sharp\beta_2(h) + \sharp S(h) + 1. \tag{2.65}$$

Proof. Using Eqs (2.42) and (2.62) and Lemma 5.3 from reference [26] , we can show that the above inequality (2.65) holds (see [26]).

Lemma 2.4. For $n \geq 5$, the number of zeros of the Abelian integrals $I(h)$ for system (2.11) is at most $7n-12$.

Proof. Since $\sharp S(h) = \sharp V(h)$, and according to (2.64) and Lemma 5.3 in reference [26], we have the following inequality

$$\sharp V(h) \leq \sharp A(h) + \sharp\alpha_4(h) + \sharp E(h) + 1. \tag{2.66}$$

By combining Eqs (2.65) and (2.66), we arrive at the following result:

$$\sharp I(h) \leq 2\sharp A(h) + \sharp\beta_2(h) + \sharp\alpha_4(h) + \sharp E(h) + 2. \tag{2.67}$$

When $n \geq 5$, since $A(h)$ has no zeros in $(-\frac{1}{48}, 0)$, and we have $\sharp(\beta_2(h)) = \sharp(\bar{\beta}2(h)) \leq n-3$, $\sharp(\alpha 4(h)) = \sharp(\bar{\alpha}_4(h)) \leq 2n-4$ and $\sharp(E(h)) = \sharp(\bar{E}(h)) \leq 4n-7$, the following equations holds:

$$\sharp I(h) \leq (n-3) + (2n-4) + (4n-7) + 2 = 7n-12. \tag{2.68}$$

Therefore, when $n \geq 5$, the number of zeros of the Abelian integrals $I(h)$ of the system (2.11) is no more than $7n-12$.

3. Conclusions

In conclusion, we investigate the upper bound of the number of zeros of Abelian integrals for quadratic reversible systems with trajectories almost forming cubic curves under n-degree polynomial perturbations. By applying the Picard-Fuchs equation method and the Riccati equation method, we have determined that the upper bound of the number of zeros of Abelian integrals in this case is $7n-12$ ($n \geq 5$). For the cases where $n < 5$, the number and locations of limit cycles can be effectively determined using the detection function and numerical exploration methods. However, due to limitations in our research approach, further investigations in this area are left for future analysis.

This result further confirms the linear dependency between the number of zeros of Abelian integrals and the order of perturbation polynomials. As for Hilbert′s 16th problem, despite making some progress, it has yet to be fundamentally resolved. The applicability of the Picard-Fuchs equation and the Riccati equation methods is limited, and they cannot radically address Hilbert′s 16th problem. Therefore, the key lies in finding new and more effective solving approaches to tackle Hilbert′s 16th problem.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors state no conflict of interest.

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