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Research article

A note on positive partial transpose blocks

Moh. Alakhrass *

Department of Mathematics, University of Sharjah, Sharjah, UAE

* **Correspondence:** Email: malakhrass@sharjah.ac.ae.

Abstract: In this article, we study the class of positive partial transpose blocks. We introduce several inequalities related to this class with an emphasis on comparing the main diagonal and off-diagonal components of a 2×2 positive partial transpose block.

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1. Introduction

Let $\mathbb{M}_n$ be the algebra of all $n \times n$ complex matrices. For $X \in \mathbb{M}_n$, the notation $X \geq 0$ (resp. $X > 0$) will be used to mean that X is positive semidefinite (resp. positive definite). If $X, Y \in \mathbb{M}_n$ are two Hermitian matrices in $\mathbb{M}_n$, we write $X \leq Y$ to mean $Y - X \geq 0$. The unitarily invariant norm of $X \in \mathbb{M}_n$ is denoted by $\|X\|$. Recall that a norm $\| \cdot \|$ on $\mathbb{M}_n$ is said to be unitarily invariant if it satisfies the property $\|UXV\| = \|X\|$ for all $X \in \mathbb{M}_n$ and all unitaries $U, V \in \mathbb{M}_n$.

Let $A, B, X \in \mathbb{M}_n$. Throughout this note, we consider the 2×2 block matrix H in the form

$$H = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}.$$

It is well known that H is positive if and only if the Schur complement of A in H is positive semidefinite provided that A is strictly positive. That is, $H \geq 0$ if and only if

$$H/A = B - X^* A^{-1} X \geq 0. \tag{1.1}$$

The 2×2 blocks play an important role in studying matrices and positive matrices in particular. Bhatia book [5] provides a comprehensive survey about block matrices. Furthermore, a positive 2×2 block can be a very useful tool in studying sectorial matrices, see for example [1–3].

The partial transpose of the block H is defined by

$$H^{\tau} = \begin{pmatrix} A & X^* \\ X & B \end{pmatrix}.$$

It is quite clear that the positivity of H does not generally imply the positivity of H^{τ}. The block H is said to be positive partial transpose (PPT) if both H and H^{τ} are positive semidefinite. The Schure criterion for positivity implies that H is PPT if and only if

$$B - X^*A^{-1}X \geq 0 \quad \text{and} \quad B - XA^{-1}X^* \geq 0,$$

provided that A is strictly positive.

The Peres–Horodecki criterion (PPT criterion) plays an important roll in the quantum information theory. For example, PPT condition is a necessary condition for a mixed quantum state to be separable. Moreover, in low dimensional composite spaces (two and three) this condition (PPT) is also sufficient. See [8, 11]. It is noteworthy that 2×2 blocks, positive blocks and the PPT blocks serve as key tools in fast inversion and fast multiplications of triangular matrices. For further details, refer to [12] and the cited references.

The class of PPT matrices possess many interesting properties. Therefore, it has attracted a huge interest. See [7, 9, 10]. Given a PPT block $H = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$.

Recently, many interesting inequalities connecting the main and the off-diagonal of the PPT Block H have been established. For example, in [10] Lin proved that if H is PPT then

$$tr\,(X^*X) \leq tr\,(AB)\,. \tag{1.2}$$

In the sense of Loewner, it has been proved in [9] that for some unitary $U \in \mathbb{M}_n$

$$|X| \leq \frac{A\#B + U^*(A\#B)U}{2}. \tag{1.3}$$

An improvement of the inequality (1.3) was given in [7]. The authors proved that if H is PPT then

$$|X| \leq (A\#B)\,\#U^*(A\#B)U \tag{1.4}$$

for some unitary $U \in \mathbb{M}_n$.

In this paper, we show that if H is PPT and $t \in [0, 1]$ then

$$\begin{aligned} |X| &\leq (A\#_t B)\,\#U^*(A\#_{1-t}B)U \\ &\leq \frac{A\#_t B + U^*(A\#_{1-t}B)U}{2} \end{aligned} \tag{1.5}$$

for some unitary $U \in \mathbb{M}_n$. Then we present several consequences of (1.5) including inequalities such as (1.2)–(1.4). Finally, we present some inequalities that connect the diagonal components to the real part of the off-diagonal components of H. Note that in this context, $A\#B$ and $A\#_t B$ denote the geometric mean and the weighted geometric mean of the two positive matrices A and B, respectively. The detailed definition will be provided at the final paragraph of the preliminary section.

2. Preliminaries

In this section we present some basic properties of positive and PPT blocks. These properties are summarized in Proportions 2.1–2.3. To make this note self-contained, we outline the proofs of these propositions. We also recall some important facts about weighted geometric mean of two positive matrices.

Proposition 2.1. *If* $H = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \geq 0$ *then*

(1). $\begin{pmatrix} A & -X \\ -X^* & B \end{pmatrix} \geq 0$ *and* $\begin{pmatrix} B & X^* \\ X & A \end{pmatrix} \geq 0.$

(2). $\begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} \leq \frac{1}{2}H.$

Proof. To see the first part, observe that

$$\begin{pmatrix} A & -X \\ -X^* & B \end{pmatrix} = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \geq 0$$

and

$$\begin{pmatrix} B & X^* \\ X & A \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} A & X^* \\ X & B \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \geq 0.$$

For the second part, notice that

$$\frac{1}{2}H - \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} A & -X \\ -X^* & B \end{pmatrix} \geq 0.$$

□

Proposition 2.2. *If* $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ *is PPT then the following blocks are positive semidefinite.*

$\begin{pmatrix} A & \mp X \\ \mp X^* & B \end{pmatrix}, \begin{pmatrix} A & \mp X^* \\ \mp X & B \end{pmatrix}, \begin{pmatrix} B & \mp X^* \\ \mp X & A \end{pmatrix}, \begin{pmatrix} B & \mp X \\ \mp X^* & A \end{pmatrix}.$

Proof. The semi positivity of the first two blocks follows from the definition of PPT and the first part of Proposition 2.1. The semi positivity of the second two blocks results from conjugating the first two blocks by the unitary $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. □

Proposition 2.3. *Let* $H = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ *be PPT. Then,*

$$\begin{pmatrix} A & e^{i\theta}X \\ e^{-i\theta}X^* & B \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} \frac{A+B}{2} & X \\ X^* & \frac{A+B}{2} \end{pmatrix} \quad \textit{are PPT.}$$

Proof. Let $W = \begin{pmatrix} e^{i\theta}I & 0 \\ 0 & I \end{pmatrix}$. Notice that

$$\begin{pmatrix} A & e^{i\theta}X \\ e^{-i\theta}X^* & B \end{pmatrix} = W\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}W^* \geq 0$$

and

$$\begin{pmatrix} A & e^{-i\theta}X^* \\ e^{i\theta}X & B \end{pmatrix} = W^* \begin{pmatrix} A & X^* \\ X & B \end{pmatrix} W \geq 0.$$

This implies that the first block is PPT.

Since $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ is PPT, Proposition 2.2 implies that $\begin{pmatrix} B & -X \\ -X^* & A \end{pmatrix} \geq 0$. Therefore,

$$H = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \leq \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} + \begin{pmatrix} B & -X \\ -X^* & A \end{pmatrix} = \begin{pmatrix} A+B & 0 \\ 0 & A+B \end{pmatrix}.$$

So,

$$\frac{1}{2}H \leq \begin{pmatrix} \frac{A+B}{2} & 0 \\ 0 & \frac{A+B}{2} \end{pmatrix}. \tag{2.1}$$

The second part of Proposition 2.1 implies that

$$\begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} \leq \frac{1}{2}H. \tag{2.2}$$

Hence, combining (2.1) and (2.2) gives

$$\begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} \leq \begin{pmatrix} \frac{A+B}{2} & 0 \\ 0 & \frac{A+B}{2} \end{pmatrix}.$$

Consequently, $\begin{pmatrix} \frac{A+B}{2} & -X \\ -X^* & \frac{A+B}{2} \end{pmatrix} \geq 0$ and then by Proposition 2.1, we have $\begin{pmatrix} \frac{A+B}{2} & X \\ X^* & \frac{A+B}{2} \end{pmatrix} \geq 0$. A similar argument implies that $\begin{pmatrix} \frac{A+B}{2} & X^* \\ X & \frac{A+B}{2} \end{pmatrix} \geq 0$. This proves that $\begin{pmatrix} \frac{A+B}{2} & X \\ X^* & \frac{A+B}{2} \end{pmatrix}$ is PPT. □

In the following paragraph, we present the definition of the weighted geometric mean of two positive matrices and we state some of its properties.

For positive definite $X, Y \in \mathbb{M}_n$ and $t \in [0, 1]$, the weighted geometric mean of X and Y is defined as follows

$$X\#_t Y = X^{1/2}(X^{-1/2}YX^{-1/2})^t X^{1/2}.$$

When $t = \frac{1}{2}$, we drop t from the above definition and we simply write $X\#Y$ and call it the geometric mean of X and Y. It is well-known that

$$X\#_t Y \leq (1-t)X + tY. \tag{2.3}$$

See [5, Chapter 4].

When $t = \frac{1}{2}$, an extremal property of the geometric mean of positive definite $X, Y \in \mathbb{M}_n$ is given as follows

$$X\#Y = \max\left\{Z : Z = Z^*, \begin{pmatrix} X & Z \\ Z & Y \end{pmatrix} \geq 0\right\}. \tag{2.4}$$

See [5, Theorem 4.1.3].

For every unitarily invariant norm, we have

$$\begin{aligned}\|X\#_t Y\| &\le \|X^{1-t}Y^t\| \\ &\le \|(1-t)X + tY\|. \end{aligned} \tag{2.5}$$

See [4, Theorem 3].

3. Main results

We start this section by the following two lemmas.

Lemma 3.1. *If* $\begin{pmatrix} A_j & X \\ X^* & B_j \end{pmatrix} \ge 0 \quad (j = 1, 2)$ *then*

$$\begin{pmatrix} A_1\#_t A_2 & X \\ X^* & B_1\#_t B_2 \end{pmatrix} \ge 0, \forall t \in [0, 1].$$

Proof. Without loss of generality we may assume that for $j = 1, 2$ the block $\begin{pmatrix} A_j & X \\ X^* & B_j \end{pmatrix}$ is positive definite, otherwise we use the well know continuous argument. Therefore, by Schure criterion (1.1), we have

$$X^*A_1^{-1}X \le B_1 \quad \text{and} \quad X^*A_2^{-1}X \le B_2.$$

Observe,

$$\begin{aligned} X^*(A_1\#_t A_2)^{-1})X &= X^*(A_1^{-1}\#_t A_2^{-1})X \\ &= (X^*A_1^{-1}X)\#_t(X^*A_2^{-1}X) \\ &\le B_1\#_t B_2 \quad \text{(by the increasing property of means).} \end{aligned}$$

And so, $B_1\#_t B_2 \ge X^*(A_1\#_t A_2)^{-1})X$. This implies the result. □

Lemma 3.2. *If* $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ *is PPT then for every* $t \in [0, 1]$ *the block* $\begin{pmatrix} A\#_t B & X \\ X^* & A\#_{1-t} B \end{pmatrix}$ *is PPT.*

Proof. The result follows directly from Lemma 3.1, Proposition 2.2 and the fact that $B\#_t A = A\#_{1-t} B$. □

Recall that the absolute value of $X \in \mathbb{M}_n$ is defined as $|X| = (X^*X)^{1/2}$.

The main result can be stated as follows.

Theorem 3.1. *Let* $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ *be PPT and let* $X = U|X|$ *be the polar decomposition of* X. *Then,*

$$|X| \le (A\#_t B)\,\#U^*(A\#_{1-t} B)U, \quad \forall t \in [0, 1].$$

Proof. Let $X = U|X|$ be the polar decomposition of X. Let W be the unitary defined as $W = \begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix}$. Since $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ is PPT, Lemma 3.2 implies that $\begin{pmatrix} A\#_t B & X \\ X^* & A\#_{1-t} B \end{pmatrix} \geq 0$ for every $t \in [0, 1]$. Therefore,

$$W^* \begin{pmatrix} A\#_t B & X \\ X^* & A\#_{1-t} B \end{pmatrix} W = \begin{pmatrix} U^*(A\#_t B)U & |X| \\ |X| & A\#_{1-t} B \end{pmatrix} \geq 0.$$

By the extremal property of the geometric mean (2.4) we get

$$|X| \leq (A\#_t B)\, \# U^*(A\#_{1-t} B)U.$$

This proves the result. □

Corollary 3.1. *Let* $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ *be PPT and let* $X = U|X|$ *be the polar decomposition of* X*. Then, for some unitary* $U \in \mathbb{M}_n$

$$|X| \leq \frac{(A\#_t B) + U^*(A\#_{1-t} B)U}{2}, \quad \forall t \in [0, 1].$$

In particular,

$$|X| \leq \frac{(A\#B) + U^*(A\#B)U}{2}.$$

We remark that the particular case $t = 1/2$ of Theorem 3.1 and Corollary 3.1 can be found in [7] and [9], respectively.

Corollary 3.2. *If* $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ *is PPT then for every unitarily invariant norm* $\|\cdot\|$ *and for* $t \in [0, 1]$

$$\begin{aligned} \|X\| &\leq \|(A\#_t B)\#U^*(A\#_{1-t} B)U\| \\ &\leq \left\| \frac{(A\#_t B) + U^*(A\#_{1-t} B)U}{2} \right\| \\ &\leq \frac{\|A\#_t B\| + \|A\#_{1-t} B\|}{2} \\ &\leq \frac{\|A^{1-t}B^t\| + \|A^t B^{1-t}\|}{2} \\ &\leq \frac{\|(1-t)A + tB\| + \|tA + (1-t)B\|}{2}, \end{aligned}$$

for some unitary $U \in \mathbb{M}_n$.

Proof. The first inequality follows directly from Theorem 3.1. The third is the triangle inequality. The other inequalities follow from (2.5). □

In particular, when $t = 1/2$ we have the following result.

Corollary 3.3. *If* $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ *is PPT then for every unitarily invariant norm* $\|\cdot\|$ *and for* $t \in [0, 1]$

$$\begin{aligned}\|X\| &\leq \|(A\#B)\#U^*(A\#B)U\| \\ &\leq \left\|\frac{(A\#B)+U^*(A\#B)U}{2}\right\| \\ &\leq \|A\#B\| \\ &\leq \|A^{1/2}B^{1/2}\| \\ &\leq \left\|\frac{A+B}{2}\right\|,\end{aligned}$$

for some unitary $U \in \mathbb{M}_n$.

If we square the inequalities in Corollary 3.3 and choose the Hilbert-Schmidt norm, $\|\cdot\|_2$, we get the following result, which is an improvement of the trace inequality (1.2). Recall that the Hilbert-Schmidt norm is defined as $\|X\|_2^2 = tr(X^*X)$.

Corollary 3.4. *If $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ is PPT then*

$$\begin{aligned}tr(X^*X) &\leq tr(A\#B)^2 \\ &\leq tr(AB). \\ &\leq tr\left(\frac{A+B}{2}\right)^2.\end{aligned}$$

For any $X \in \mathbb{M}_n$, let $s_j(X)$, $j = 1, 2, ..., n$ denote the singular values of X arranged in decreasing order. It is known that for any $X, Y \in \mathbb{M}_n$ and any indices i, j such that $i + j \leq n + 1$, we have $s_{i+j-1}(XY) \leq s_i(X)s_j(Y)$. (see [6, Page 75]). By utilizing this fact and Theorem 3.1 we can derive the following result.

Corollary 3.5. *Let $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ be PPT. Then, for all $t \in [0, 1]$, we have*

$$s_{i+j-1}(X) \leq s_i(A\#_t B)\, s_j(A\#_{1-t}B).$$

Consequently,

$$s_{2j-1}(X) \leq s_j(A\#_t B)\, s_j(A\#_{1-t}B), \forall t \in [0, 1].$$

Proof. Let $t \in [0, 1]$. By Theorem 3.1, there exists a unitary matrix $U \in \mathbb{M}_n$ such that $|X| \leq (A\#_t B)\#U^*(A\#_{1-t}B)U$. Moreover, there exists a unitary $V \in \mathbb{M}_n$ such that $(A\#_t B)\#U^*(A\#_{1-t}B)U = (A\#_t B)\, VU^*(A\#_{1-t}B)U$. See [5, Page 108]. Now, observe that

$$\begin{aligned}s_{i+j-1}(X) &\leq s_{i+j-1}((A\#_t B)U^*(A\#_{1-t}B)U) \qquad (3.1)\\ &\leq s_i((A\#_t B))\, s_j((A\#_{1-t}B)U),\end{aligned}$$

which completes the proof. □

Finally, we study the connection between the diagonal components and the real part of the off-diagonal components of the PPT block H. Before doing so, we recall that every $X \in \mathbb{M}_n$ admits what is called the cartesian decomposition

$$X = Re(X) + iIm(X),$$

where $Re(X)$ and $Im(X)$ are the Hermitian matrices defined as $Re(X) = \frac{X+X^*}{2}, Im(X) = \frac{X-X^*}{2i}$ and are known, respectively, as the real and the imaginary parts of X.

Theorem 3.2. *Let* $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ *be PPT. Then,* $\forall t \in [0,1]$

$$Re(X) \leq (A\#_t B)\#(A\#_{1-t}B) \leq \frac{(A\#_t B) + (A\#_{1-t}B)}{2}$$

and

$$Im(X) \leq (A\#_t B)\#(A\#_{1-t}B) \leq \frac{(A\#_t B) + (A\#_{1-t}B)}{2}.$$

Proof. In first part, the second inequality follows from (2.5). For the second inequality, notice that by Lemma 3.2, we have

$$\begin{pmatrix} A\#_t B & X \\ X^* & A\#_{1-t}B \end{pmatrix} \geq 0 \quad \text{and} \quad \begin{pmatrix} A\#_t B & X^* \\ X & A\#_{1-t}B \end{pmatrix} \geq 0$$

for $t \in [0,1]$. Therefore,

$$\begin{pmatrix} A\#_t B & Re(X) \\ Re(X) & A\#_{1-t}B \end{pmatrix} = \frac{1}{2}\begin{pmatrix} A\#_t B & X \\ X^* & A\#_{1-t}B \end{pmatrix} + \frac{1}{2}\begin{pmatrix} A\#_t B & X^* \\ X & A\#_{1-t}B \end{pmatrix} \geq 0.$$

Therefore, by the extremal property of the geometric mean we have

$$Re(X) \leq (A\#_t B)\#(A\#_{1-t}B).$$

This implies the first inequality. To prove the second inequality just applying the first inequality to the block $G = \begin{pmatrix} A & -iX \\ iX^* & B \end{pmatrix}$. Note that G is PPT by Proposition 2.3. □

Corollary 3.6. *Let* $\begin{pmatrix} A & X \\ X & B \end{pmatrix} \geq 0$. *If X is Hermitian,*

$$X \leq (A\#_t B)\#(A\#_{1-t}B) \leq \frac{(A\#_t B) + (A\#_{1-t}B)}{2}, \forall t \in [0,1].$$

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that he has no conflict of interest.

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