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Research article

Numerical analysis of fractional-order Whitham-Broer-Kaup equations with non-singular kernel operators

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Abstract: This paper solves a fractional system of non-linear Whitham-Broer-Kaup equations using a natural decomposition technique with two fractional derivatives. Caputo-Fabrizio and Atangana-Baleanu fractional derivatives were applied in a Caputo-manner. In addition, the results of the suggested method are compared to those of well-known analytical techniques such as the Adomian decomposition technique, the Variation iteration method, and the optimal homotopy asymptotic method. Two non-linear problems are utilized to demonstrate the validity and accuracy of the proposed methods. The analytical solution is then utilized to test the accuracy and precision of the proposed methodologies. The acquired findings suggest that the method used is very precise, easy to implement, and effective for analyzing the nature of complex non-linear applied sciences.

Keywords: system of Whitham-Broer-Kaup equations; Caputo-Fabrizio and Atangana-Baleanu operators; natural transform Adomian decomposition method
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1. Introduction

In recent years, fractional calculus (FC) has been successfully applied to describe the mathematical issues in real materials [1, 2]. For instance, [3] provided the representation of the constitutive relationship (RCR) of the fractional mechanical element with fractional derivative (FD) of the Riemann-Liouville type, and [4] presented its generalised form. In [5], the RCR with FC of the Liouville-Caputo type was taken into consideration. It has been demonstrated that the local fractional derivative is a wonderful mathematical tool for addressing fractal issues and some challenging natural phenomena [6, 7]. The Caputo-Fabrizio type's RCR through fractional derivative was introduced in [8]. The study of fractional differential equations has expanded owing to the Caputo-Fabrizio fractional derivative. The nonsingular kernel of the new derivative [9], is what makes it so beautiful. The Caputo-Fabrizio derivative has the same additional motivating characteristics of heterogeneity and configuration [10, 11] with different scales as it does in the Caputo and Riemann-Liouville fractional derivatives despite being created through the convolution of an ordinary derivative and an exponential function. In the past two years, numerous studies related to the the new Caputo-Fabrizio fractional derivative have been published. It has been shown that modelling using the fractional derivative of the Atangana-Baleanu has a brief random walk. Additionally, it has been found that the Mittag-Leffler function is a more significant and practical filter tool than the power and exponential law functions, making the Atangana-Baleanu fractional derivative in the sense of Caputo an effective mathematical tool for simulating more difficult real-world problems [12, 13].

Using partial differential equations, it is possible to express a specific relationship between its partial derivatives (PDEs) and an unknown function. PDEs may be found in almost every field of engineering and research. In recent years, the application of PDEs in fields such as finance, biology, image graphics and processing, and social sciences has risen. As a result, when certain independent variables interact with one another in each of the above-mentioned fields, appropriate functions in these variables may be established, allowing for the modelling of a variety of processes via the use of equations for the related functions [14–16]. The study of PDEs has various elements. The traditional method, which dominated the nineteenth century, was to develop procedures for identifying explicit answers [17–19]. Theoretical PDE analysis offers a wide range of applications. It's worth noting that there are certain really difficult equations that even supercomputers can't solve. All one can do in these situations is attempt to get qualitative data. Moreover, the formulation of the equation, as well as its associated side conditions, is significant. A model of a physical or technical issue is frequently used to generate the equation. It is not immediately clear that the model is continuous in the view that it results to a solved PDE [20–22]. Furthermore, in most circumstances, it is preferable that the answer be one of a kind and stable under minor data disruptions. Theoretical comprehension of the equation aids in determining whether or not these requirements are satisfied. Several methods for solving classical PDEs have already been suggested, and several solutions have been discovered [23–29].

The propagation of shallow water is represented by numerous well-known integral models, such as the Boussinesq equation, KdV equation, Whitham-Broer-Kaup equation, and others. Whitham, Broer, and Kaup [30–34] created non-linear Whitham-Broer-Kaup equations employing the Boussinesq approximation

$$\begin{cases} \psi_\tau + \psi\psi_\upsilon + \varphi_\upsilon + q\psi_{\upsilon\upsilon} = 0 \\ \varphi_\tau + \varphi\psi_\upsilon + \psi\varphi_\upsilon - q\varphi_{\upsilon\upsilon} + p\psi_{\upsilon\upsilon\upsilon} = 0, \end{cases} \tag{1.1}$$

where $\psi = \psi(\upsilon, \tau), \varphi = \varphi(\upsilon, \tau)$ represents the velocity of horizontal and fluid height, which fluctuates substantially from equilibriums, and q, p are constants made up of many diffusion power. Wang and Zheng [32] implemented riccati sub-equation method to obtain the result of fractional order Whitham-Broer-Kaup equations. The analytical and numerical methods have been applied to find the solutions of

Whitham-Broer-Kaup equations, such as reduced differential transform method, residual power series method (1.1), finite element method [35], power-series method [36], the finite difference approach [37], the exponential-function technique [38], the variation iteration technique, the homotopy perturbation technique, the homotopy analysis technique and others [39–41].

Adomian introduced the Adomian decomposition approach in 1980, which is an effective approach for locating numeric and explicit solutions to a wide variety of differential equations that describe physical conditions. This method is applicable to initial value problems, boundary value problems, partial and ordinary differential equations, including linear and nonlinear equations, and stochastic systems. The Natural transform decomposition method is constructed by combining the Adomian decomposition method with the Natural transform method (NTDM). NTDM has also been used to analyze fractional-order non-linear partial differential equations numerically in a number of articles [42–44].

In this research, we use NTDM in combination with two different derivatives to investigate the general as well as numerical solution of the coupled system of Whitham-Broer-Kaup equations of fractional order, as inspired by the above papers. NTDM is a simple and effective technique that does not require any perturbation. We compare our suggested method's results to those of other well-known approaches like the variational iteration method, Adomain decomposition method, and Optimal homoptoy analysis method. We can observe that the proposed method is superior to the previously discussed methods for finding nonlinear fractional order partial differential equation solutions. We use Maple to perform the calculations. The suggested techniques' convergence is also ensured by extending the concept mentioned in [45, 46].

2. Basic definitions

Fractional derivatives and integrals have a great number of properties and definitions. We propose changes to some basic fractional calculus definitions and preliminaries used in this study.

Definition 2.1. The Riemann-Liouville integral of order fraction for a function $j \in C_{\wp}, \wp \geq -1$, is given as [47–49]

$$\begin{aligned} I^{\varrho} j(\eta) &= \frac{1}{\Gamma(\varrho)} \int_0^{\eta} (\eta - \wp)^{\varrho - 1} j(\wp) d\wp, \quad \varrho > 0, \quad \eta > 0. \\ \textit{and} \quad I^0 j(\eta) &= j(\eta). \end{aligned} \tag{2.1}$$

Definition 2.2. The derivative with order fraction for $j(\eta)$ in Caputo manner is stated as [47–49]

$$D^{\varrho}_{\eta} j(\eta) = I^{m-\varrho} D^m j(\eta) = \frac{1}{m - \varrho} \int_0^{\eta} (\eta - \wp)^{m - \varrho - 1} j^{(m)}(\wp) d\wp, \tag{2.2}$$

for $m - 1 < \varrho \leq m, \ \ m \in N, \ \ \eta > 0, j \in C^m_{\wp}, \wp \geq -1$.

Definition 2.3. The derivative with order fraction for $j(\eta)$ in CF sense is stated as [47]

$$D^{\varrho}_{\eta} j(\eta) = \frac{F(\varrho)}{1 - \varrho} \int_0^{\eta} \exp\left(\frac{-\varrho(\eta - \wp)}{1 - \varrho} \right) D(j(\wp)) d\wp, \tag{2.3}$$

having $0 < \varrho < 1$ and $F(\varrho)$ is a normalization function having $F(0) = F(1) = 1$.

Definition 2.4. The derivative with order fraction for $j(\eta)$ in ABC sense is stated as [47]

$$D^{\varrho}_{\eta} j(\eta) = \frac{B(\varrho)}{1-\varrho} \int_0^{\eta} E_{\varrho}\left(\frac{-\varrho(\eta-\wp)}{1-\varrho}\right) D(j(\wp))d\wp, \tag{2.4}$$

having $0 < \varrho < 1$, $B(\varrho)$ is normalization function and $E_{\varrho}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(m\varrho+1)}$ is the Mittag-Leffler function.

Definition 2.5. For a function $\psi(\tau)$, the natural transform (NT) is stated as

$$\mathbf{N}(\psi(\tau)) = \mathcal{U}(\kappa,\ell) = \int_{-\infty}^{\infty} e^{-\kappa\tau}\psi(\ell\tau)d\tau, \quad \kappa,\ell \in (-\infty,\infty), \tag{2.5}$$

and for $\tau \in (0,\infty)$, the NT of $\psi(\tau)$ is stated as

$$\mathbf{N}(\psi(\tau)H(\tau)) = \mathbf{N}^{+} = \mathcal{U}^{+}(\kappa,\ell) = \int_{0}^{\infty} e^{-\kappa\tau}\psi(\ell\tau)d\tau, \quad \kappa,\ell \in (0,\infty), \tag{2.6}$$

where $H(\tau)$ is the Heaviside function.

Definition 2.6. For the function $\psi(\kappa,\ell)$, the inverse NT is stated as

$$\mathbf{N}^{-1}[\mathcal{U}(\kappa,\ell)] = \psi(\tau), \quad \forall\tau \geq 0. \tag{2.7}$$

Lemma 2.7. *If the NT of $\psi_1(\tau)$ and $\psi_2(\tau)$ are $\mathcal{U}_1(\kappa,\ell)$ and $\mathcal{U}_2(\kappa,\ell)$ respectively, so*

$$\mathbf{N}[c_1\psi_1(\tau) + c_2\psi_2(\tau)] = c_1\mathbf{N}[\psi_1(\tau)] + c_2\mathbf{N}[\psi_2(\tau)] = c_1\mathcal{U}_1(\kappa,\ell) + c_2\mathcal{U}_2(\kappa,\ell), \tag{2.8}$$

having c_1 and c_2 constants.

Lemma 2.8. *If $\psi_1(\tau)$ and $\psi_2(\tau)$ are the inverse NT of $\psi_1(\kappa,\ell)$ and $\psi_2(\kappa,\ell)$ respectively, so*

$$\{N\}^{-1}[c_1\mathcal{U}_1(\kappa,\ell) + c_2\mathcal{U}_2(\kappa,\ell)] = c_1\mathbf{N}^{-1}[\mathcal{U}_1(\kappa,\ell)] + c_2\mathbf{N}^{-1}[\mathcal{U}_2(\kappa,\ell)] = c_1\psi_1(\tau) + c_2\psi_2(\tau), \tag{2.9}$$

having c_1 and c_2 constants.

Definition 2.9. The NT of $D^{\varrho}_{\tau}\psi(\tau)$ in Caputo sense is stated as [47]

$$\mathbf{N}[D^{\varrho}_{\tau}\psi(\tau)] = \left(\frac{\kappa}{\ell}\right)^{\varrho}\left(\mathbf{N}[\psi(\tau)] - \left(\frac{1}{\kappa}\right)\psi(0)\right). \tag{2.10}$$

Definition 2.10. The NT of $D^{\varrho}_{\tau}\psi(\tau)$ in CF sense is stated as [47]

$$\mathbf{N}[D^{\varrho}_{\tau}\psi(\tau)] = \frac{1}{1-\varrho+\varrho(\frac{\ell}{\kappa})}\left(\mathbf{N}[\psi(\tau)] - \left(\frac{1}{\kappa}\right)\psi(0)\right). \tag{2.11}$$

Definition 2.11. The NT of $D^{\varrho}_{\tau}\psi(\tau)$ in ABC sense is stated as [47]

$$\mathbf{N}[D^{\varrho}_{\tau}\psi(\tau)] = \frac{M[\varrho]}{1-\varrho+\varrho(\frac{\ell}{\kappa})^{\varrho}}\left(\mathbf{N}[\psi(\tau)] - \left(\frac{1}{\kappa}\right)\psi(0)\right), \tag{2.12}$$

with $M[\varrho]$ denoting a normalization function.

3. General implementation of method

Consider the fractional partial differential equation

$$D_{\tau}^{\varrho}\psi(\upsilon,\tau) = \mathcal{L}(\psi(\upsilon,\tau)) + N(\psi(\upsilon,\tau)) + h(\upsilon,\tau) = M(\upsilon,\tau), \tag{3.1}$$

with the initial condition

$$\psi(\upsilon,0) = \phi(\upsilon), \tag{3.2}$$

which $\mathcal{L}$ is linear, N non-linear functions and sources function $h(\upsilon,\tau)$.

3.1. Case I ($NTDM_{CF}$)

The fractional CF derivative and natural transformation, Eq (3.1) can be stated as,

$$\frac{1}{p(\varrho,\ell,\kappa)}\left(\mathbf{N}[\psi(\upsilon,\tau)] - \frac{\phi(\upsilon)}{\kappa}\right) = \mathbf{N}[M(\upsilon,\tau)], \tag{3.3}$$

with

$$p(\varrho,\ell,\kappa) = 1 - \varrho + \varrho(\frac{\ell}{\kappa}). \tag{3.4}$$

On employing natural inverse transformation, we get

$$\psi(\upsilon,\tau) = \mathbf{N}^{-1}\left(\frac{\phi(\upsilon)}{\kappa} + p(\varrho,\ell,\kappa)\mathbf{N}[M(\upsilon,\tau)]\right). \tag{3.5}$$

Thus, for $\psi(\upsilon,\tau)$, the series type solution is given as

$$\psi(\upsilon,\tau) = \sum_{i=0}^{\infty}\psi_i(\upsilon,\tau), \tag{3.6}$$

and $N(\psi(\upsilon,\tau))$ can be decomposed as

$$N(\psi(\upsilon,\tau)) = \sum_{i=0}^{\infty}A_i(\psi_0,...,\psi_i), \tag{3.7}$$

the Adomian polynomials A_i is given as

$$A_n = \frac{1}{n!}\frac{d^n}{d\varepsilon^n}N(t,\Sigma_{k=0}^{n}\varepsilon^k\psi_k)|_{\varepsilon=0.}$$

Putting Eqs (3.7) and (3.6) into (3.5), we obtain

$$\begin{aligned}\sum_{i=0}^{\infty}\psi_i(\upsilon,\tau) =& \mathbf{N}^{-1}\left(\frac{\phi(\upsilon)}{\kappa} + p(\varrho,\ell,\kappa)\mathbf{N}[h(\upsilon,\tau)]\right)\\ &+ \mathbf{N}^{-1}\left(p(\varrho,\ell,\kappa)\mathbf{N}\left[\sum_{i=0}^{\infty}\mathcal{L}(\psi_i(\upsilon,\tau)) + A_\tau\right]\right).\end{aligned} \tag{3.8}$$

From (3.8), we get,

$$\begin{aligned}
\psi_0^{CF}(\upsilon,\tau) =& \mathbf{N}^{-1}\left(\frac{\phi(\upsilon)}{\kappa} + p(\varrho,\ell,\kappa)\mathbf{N}[h(\upsilon,\tau)]\right),\\
\psi_1^{CF}(\upsilon,\tau) =& \mathbf{N}^{-1}\left(p(\varrho,\ell,\kappa)\mathbf{N}\left[\mathcal{L}(\psi_0(\upsilon,\tau)) + A_0\right]\right),\\
&\vdots\\
\psi_{l+1}^{CF}(\upsilon,\tau) =& \mathbf{N}^{-1}\left(p(\varrho,\ell,\kappa)\mathbf{N}\left[\mathcal{L}(\psi_l(\upsilon,\tau)) + A_l\right]\right), \quad l = 1,2,3,\cdots.
\end{aligned} \tag{3.9}$$

Thus, we obtain the result of (3.1) by putting (3.9) into (3.6) applying $NTDM_{CF}$,

$$\psi^{CF}(\upsilon,\tau) = \psi_0^{CF}(\upsilon,\tau) + \psi_1^{CF}(\upsilon,\tau) + \psi_2^{CF}(\upsilon,\tau) + \cdots. \tag{3.10}$$

3.2. Case II ($NTDM_{ABC}$)

The fractional ABC derivative and natural transformation, Eq (3.1) is given as,

$$\frac{1}{q(\varrho,\ell,\kappa)}\left(\mathbf{N}[\psi(\upsilon,\tau)] - \frac{\phi(\upsilon)}{\kappa}\right) = \mathbf{N}[M(\upsilon,\tau)], \tag{3.11}$$

with

$$q(\varrho,\ell,\kappa) = \frac{1-\varrho+\varrho(\frac{\ell}{\kappa})^{\varrho}}{B(\varrho)}. \tag{3.12}$$

On employing natural inverse transformation, we get

$$\psi(\upsilon,\tau) = \mathbf{N}^{-1}\left(\frac{\phi(\upsilon)}{\kappa} + q(\varrho,\ell,\kappa)\mathbf{N}[M(\upsilon,\tau)]\right). \tag{3.13}$$

The Adomain decomposition, we obtain as

$$\begin{aligned}
\sum_{i=0}^{\infty}\psi_i(\upsilon,\tau) =& \mathbf{N}^{-1}\left(\frac{\phi(\upsilon)}{\kappa} + q(\varrho,\ell,\kappa)\mathbf{N}[h(\upsilon,\tau)]\right)\\
&+ \mathbf{N}^{-1}\left(q(\varrho,\ell,\kappa)\mathbf{N}\left[\sum_{i=0}^{\infty}\mathcal{L}(\psi_i(\upsilon,\tau)) + A_\tau\right]\right).
\end{aligned} \tag{3.14}$$

From (3.8), we get,

$$\begin{aligned}
\psi_0^{ABC}(\upsilon,\tau) =& \mathbf{N}^{-1}\left(\frac{\phi(\upsilon)}{\kappa} + q(\varrho,\ell,\kappa)\mathbf{N}[h(\upsilon,\tau)]\right),\\
\psi_1^{ABC}(\upsilon,\tau) =& \mathbf{N}^{-1}\left(q(\varrho,\ell,\kappa)\mathbf{N}\left[\mathcal{L}(\psi_0(\upsilon,\tau)) + A_0\right]\right),\\
&\vdots\\
\psi_{l+1}^{ABC}(\upsilon,\tau) =& \mathbf{N}^{-1}\left(q(\varrho,\ell,\kappa)\mathbf{N}\left[\mathcal{L}(\psi_l(\upsilon,\tau)) + A_l\right]\right), \quad l = 1,2,3,\cdots.
\end{aligned} \tag{3.15}$$

Thus, we obtain the result of (3.1), by applying $NTDM_{ABC}$

$$\psi^{ABC}(\upsilon,\tau) = \psi_0^{ABC}(\upsilon,\tau) + \psi_1^{ABC}(\upsilon,\tau) + \psi_2^{ABC}(\upsilon,\tau) + \cdots. \tag{3.16}$$

4. Convergence analysis

In this part, we give the uniqueness and convergence of the $NTDM_{ABC}$ and $NTDM_{CF}$.

Theorem 4.1. *Let* $|\mathcal{L}(\psi) - \mathcal{L}(\psi^*)| < \gamma_1|\psi - \psi^*|$ *and* $|N(\psi) - N(\psi^*)| < \gamma_2|\psi - \psi^*|$, *where* $\psi := \psi(\wp, \tau)$ *and* $\psi^* := \psi^*(\wp, \tau)$ *are values of two different functions and* γ_1, γ_2 *are Lipschitz constants.*

$\mathcal{L}$ and N are the operators stated in (3.1). Then for $NTDM_{CF}$ the solution of (3.1) is unique when $0 < (\gamma_1 + \gamma_2)(1 - \varrho + \varrho\tau) < 1$ *for all* τ.

Proof. Let $H = (C[J], \|.\|)$ having norm $\|\phi(\tau)\| = max_{\tau\in J}|\phi(\tau)|$ is the Banach space, ∀ continuous function over the interval $J = [0, T]$. Let $I : H \to H$ a non-linear mapping, with

$$\psi^C_{l+1} = \psi^C_0 + \mathbf{N}^{-1}[p(\varrho, \ell, \kappa)\mathbf{N}[\mathcal{L}(\psi_l(\wp, \tau)) + N(\psi_l(\wp, \tau))]], \quad l \geq 0.$$

$$\begin{aligned} \|I(\psi) - I(\psi^*)\| &\leq max_{\tau\in J}|\mathbf{N}^{-1}\Big[p(\varrho, \ell, \kappa)\mathbf{N}[\mathcal{L}(\psi) - \mathcal{L}(\psi^*)] \\ &+ p(\varrho, \ell, \kappa)\mathbf{N}[N(\psi) - N(\psi^*)]|\Big] \\ &\leq max_{\tau\in J}\Big[\gamma_1\mathbf{N}^{-1}[p(\varrho, \ell, \kappa)\mathbf{N}[|\psi - \psi^*|]] \\ &+ \gamma_2\mathbf{N}^{-1}[p(\varrho, \ell, \kappa)\mathbf{N}[|\psi - \psi^*|]]\Big] \\ &\leq max_{\tau\in J}(\gamma_1 + \gamma_2)\left[\mathbf{N}^{-1}[p(\varrho, \ell, \kappa)\mathbf{N}|\psi - \psi^*|]\right] \\ &\leq (\gamma_1 + \gamma_2)\left[\mathbf{N}^{-1}[p(\varrho, \ell, \kappa)\mathbf{N}\|\psi - \psi^*\|]\right] \\ &= (\gamma_1 + \gamma_2)(1 - \varrho + \varrho\tau)\|\psi - \psi^*\|. \end{aligned} \tag{4.1}$$

So, I is contraction as $0 < (\gamma_1 + \gamma_2)(1 - \varrho + \varrho\tau) < 1$. The solution of (3.1) is unique as according to Banach fixed point theorem. □

Theorem 4.2. *As according to the above theorem, the solution of (3.1) is unique for $NTDM_{ABC}$ when* $0 < (\gamma_1 + \gamma_2)(1 - \varrho + \varrho\frac{\tau^\varrho}{\Gamma(\varrho+1)}) < 1$ *for all* τ.

Proof. As from above theorem let $H = (C[J], \|.\|)$ be the Banach space, ∀ continuous function over the interval J. Let $I : H \to H$ is non-linear mapping, with

$$\psi^C_{l+1} = \psi^C_0 + \mathbf{N}^{-1}[p(\varrho, \ell, \kappa)\mathbf{N}[\mathcal{L}(\psi_l(\wp, \tau)) + N(\psi_l(\wp, \tau))]], \quad l \geq 0.$$

$$\begin{aligned} \|I(\psi) - I(\psi^*)\| &\leq max_{\tau\in J}|\mathbf{N}^{-1}\Big[q(\varrho, \ell, \kappa)\mathbf{N}[\mathcal{L}(\psi) - \mathcal{L}(\psi^*)] \\ &+ q(\varrho, \ell, \kappa)\mathbf{N}[N(\psi) - N(\psi^*)]|\Big] \\ &\leq max_{\tau\in J}\Big[\gamma_1\mathbf{N}^{-1}[q(\varrho, \ell, \kappa)\mathbf{N}[|\psi - \psi^*|]] \\ &+ \gamma_2\mathbf{N}^{-1}[q(\varrho, \ell, \kappa)\mathbf{N}[|\psi - \psi^*|]]\Big] \\ &\leq max_{\tau\in J}(\gamma_1 + \gamma_2)\left[\mathbf{N}^{-1}[q(\varrho, \ell, \kappa)\mathbf{N}|\psi - \psi^*|]\right] \\ &\leq (\gamma_1 + \gamma_2)\left[\mathbf{N}^{-1}[q(\varrho, \ell, \kappa)\mathbf{N}\|\psi - \psi^*\|]\right] \\ &= (\gamma_1 + \gamma_2)(1 - \varrho + \varrho\frac{\tau^\varrho}{\Gamma(\varrho + 1)})\|\psi - \psi^*\|. \end{aligned} \tag{4.2}$$

So, I is contraction as $0 < (\gamma_1 + \gamma_2)(1 - \varrho + \varrho\frac{\tau^\varrho}{\Gamma(\varrho+1)}) < 1$. The solution of (3.1) is unique as according to Banach fixed point theorem. □

Theorem 4.3. *Let $\mathcal{L}$ and N are a Lipschitz functions as stated in the above theorems, then the $NTDM_{CF}$ solution of (3.1) is convergent.*

Proof. Suppose H be the Banach space determined above and let $\psi_m = \sum_{r=0}^{m}\psi_r(\wp,\tau)$. To prove that ψ_m is a Cauchy sequence in H. Let,

$$\begin{aligned}
\|\psi_m-\psi_n\| &= max_{\tau\in J}|\sum_{r=n+1}^{m}\psi_r|, \quad n=1,2,3,\cdots\\
&\le max_{\tau\in J}\left|\mathbf{N}^{-1}\left[p(\varrho,\ell,\kappa)\mathbf{N}\left[\sum_{r=n+1}^{m}(\mathcal{L}(\psi_{r-1})+N(\psi_{r-1}))\right]\right]\right|\\
&= max_{\tau\in J}\left|\mathbf{N}^{-1}\left[p(\varrho,\ell,\kappa)\mathbf{N}\left[\sum_{r=n+1}^{m-1}(\mathcal{L}(\psi_{r})+N(\psi_{r}))\right]\right]\right|\\
&\le max_{\tau\in J}|\mathbf{N}^{-1}[p(\varrho,\ell,\kappa)\mathbf{N}[(\mathcal{L}(\psi_{m-1})-\mathcal{L}(\psi_{n-1})+N(\psi_{m-1})-N(\psi_{n-1}))]]|\\
&\le \gamma_1 max_{\tau\in J}|\mathbf{N}^{-1}[p(\varrho,\ell,\kappa)\mathbf{N}[(\mathcal{L}(\psi_{m-1})-\mathcal{L}(\psi_{n-1}))]]|\\
&+\gamma_2 max_{\tau\in J}|\mathbf{N}^{-1}[p(\varrho,\ell,\kappa)\mathbf{N}[(N(\psi_{m-1})-N(\psi_{n-1}))]]|\\
&= (\gamma_1+\gamma_2)(1-\varrho+\varrho\tau)\|\psi_{m-1}-\psi_{n-1}\|.
\end{aligned} \tag{4.3}$$

Let $m = n+1$, then

$$\|\psi_{n+1}-\psi_n\| \le \gamma\|\psi_n-\psi_{n-1}\| \le \gamma^2\|\psi_{n-1}\psi_{n-2}\| \le \cdots \le \gamma^n\|\psi_1-\psi_0\|, \tag{4.4}$$

where $\gamma = (\gamma_1+\gamma_2)(1-\varrho+\varrho\tau)$. Thus, we get

$$\begin{aligned}
\|\psi_m-\psi_n\| &\le \|\psi_{n+1}-\psi_n\| + \|\psi_{n+2}\psi_{n+1}\| + \cdots + \|\psi_m-\psi_{m-1}\|,\\
&(\gamma^n+\gamma^{n+1}+\cdots+\gamma^{m-1})\|\psi_1-\psi_0\|\\
&\le \gamma^n\left(\frac{1-\gamma^{m-n}}{1-\gamma}\right)\|\psi_1\|.
\end{aligned} \tag{4.5}$$

As $0<\gamma<1$, we have $1-\gamma^{m-n}<1$. Thus,

$$\|\psi_m-\psi_n\| \le \frac{\gamma^n}{1-\gamma} max_{\tau\in J}\|\psi_1\|. \tag{4.6}$$

Since $\|\psi_1\|<\infty$, $\|\psi_m-\psi_n\|\to 0$ when $n\to\infty$. As a result, ψ_m is a Cauchy sequence in H, stated that the series ψ_m is convergent. □

Theorem 4.4. *Let $\mathcal{L}$ and N are a Lipschitz functions as stated in the above theorems, then the $NTDM_{ABC}$ solution of (3.1) is convergent.*

Proof. Suppose $\psi_m = \sum_{r=0}^{m}\psi_r(\wp,\tau)$. To prove that ψ_m is a Cauchy sequence in H. Let,

$$\begin{aligned}
\|\psi_m-\psi_n\| &= max_{\tau\in J}|\sum_{r=n+1}^{m}\psi_r|, \quad n=1,2,3,\cdots\\
&\le max_{\tau\in J}\left|\mathbf{N}^{-1}\left[q(\varrho,\ell,\kappa)\mathbf{N}\left[\sum_{r=n+1}^{m}(\mathcal{L}(\psi_{r-1})+N(\psi_{r-1}))\right]\right]\right|\\
&= max_{\tau\in J}\left|\mathbf{N}^{-1}\left[q(\varrho,\ell,\kappa)\mathbf{N}\left[\sum_{r=n+1}^{m-1}(\mathcal{L}(\psi_{r})+N(u_{r}))\right]\right]\right|\\
&\le max_{\tau\in J}|\mathbf{N}^{-1}[q(\varrho,\ell,\kappa)\mathbf{N}[(\mathcal{L}(\psi_{m-1})-\mathcal{L}(\psi_{n-1})+N(\psi_{m-1})-N(\psi_{n-1}))]]|\\
&\le \gamma_1 max_{\tau\in J}|\mathbf{N}^{-1}[q(\varrho,\ell,\kappa)\mathbf{N}[(\mathcal{L}(\psi_{m-1})-\mathcal{L}(\psi_{n-1}))]]|\\
&+\gamma_2 max_{\tau\in J}|\mathbf{N}^{-1}[p(\varrho,\ell,\kappa)\mathbf{N}[(N(\psi_{m-1})-N(\psi_{n-1}))]]|\\
&=(\gamma_1+\gamma_2)(1-\varrho+\varrho\frac{\tau^{\varrho}}{\Gamma(\varrho+1)})\|\psi_{m-1}-\psi_{n-1}\|.
\end{aligned}$$

Suppose $m=n+1$, thus

$$\|\psi_{n+1}-\psi_n\|\le\gamma\|\psi_n-\psi_{n-1}\|\le\gamma^2\|\psi_{n-1}-\psi_{n-2}\|\le\cdots\le\gamma^n\|\psi_1-\psi_0\|, \tag{4.7}$$

with $\gamma=(\gamma_1+\gamma_2)(1-\varrho+\varrho\frac{\tau^{\varrho}}{\Gamma(\varrho+1)})$. Thus, we get

$$\begin{aligned}
\|\psi_m-\psi_n\| &\le \|\psi_{n+1}-\psi_n\|+\|\psi_{n+2}-\psi_{n+1}\|+\cdots+\|\psi_m-\psi_{m-1}\|,\\
&\le(\gamma^n+\gamma^{n+1}+\cdots+\gamma^{m-1})\|\psi_1-\psi_0\|\\
&\le\gamma^n\left(\frac{1-\gamma^{m-n}}{1-\gamma}\right)\|\psi_1\|.
\end{aligned} \tag{4.8}$$

As $0<\gamma<1$, we have $1-\gamma^{m-n}<1$. Thus,

$$\|\psi_m-\psi_n\|\le\frac{\gamma^n}{1-\gamma}max_{\tau\in J}\|\psi_1\|. \tag{4.9}$$

Since $\|\psi_1\|<\infty$, $\|\psi_m-\psi_n\|\to 0$ when $n\to\infty$. As a result, ψ_m is a Cauchy sequence in H, stated that the series ψ_m is convergent. □

5. Applications

Example 1. Consider the fractional-order system of Whitham-Broer-Kaup equations is given as

$$\begin{aligned}
&D_\tau^{\varrho}\psi(\upsilon,\tau)+\psi(\upsilon,\tau)\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}+\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}+\frac{\partial\varphi(\upsilon,\tau)}{\partial\upsilon}=0,\ \ 0<\varrho\le 1,\quad -1<\tau\le 1,\quad -10\le\upsilon\le 10,\\
&D_\tau^{\varrho}\varphi(\upsilon,\tau)+\psi(\upsilon,\tau)\frac{\partial\varphi(\upsilon,\tau)}{\partial\upsilon}+\varphi(\upsilon,\tau)\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}+3\frac{\partial^3\psi(\upsilon,\tau)}{\partial\upsilon^3}-\frac{\partial^2\varphi(\upsilon,\tau)}{\partial\upsilon^2}=0,
\end{aligned} \tag{5.1}$$

the initial condition

$$\begin{aligned}
\psi(\upsilon,0)&=\frac{1}{2}-8\tanh(-2\upsilon),\\
\varphi(\upsilon,0)&=16-16\tanh^2(-2\upsilon).
\end{aligned} \tag{5.2}$$

On employing natural transform, we have

$$\begin{aligned}\mathbf{N}[D_\tau^\varrho\psi(\upsilon,\tau)] &= -\mathbf{N}\left[\psi(\upsilon,\tau)\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}+\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}+\frac{\partial\varphi(\upsilon,\tau)}{\partial\upsilon}\right],\\ \mathbf{N}[D_\tau^\varrho\varphi(\upsilon,\tau)] &= -\mathbf{N}\left[\psi(\upsilon,\tau)\frac{\partial\varphi(\upsilon,\tau)}{\partial\upsilon}+\varphi(\upsilon,\tau)\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}+3\frac{\partial^3\psi(\upsilon,\tau)}{\partial\upsilon^3}-\frac{\partial^2\varphi(\upsilon,\tau)}{\partial\upsilon^2}\right],\end{aligned}\tag{5.3}$$

Thus, we have

$$\begin{aligned}\frac{1}{\kappa^\varrho}\mathbf{N}[\psi(\upsilon,\tau)]-\kappa^{2-\varrho}\psi(\upsilon,0) &= -\mathbf{N}\left[\psi(\upsilon,\tau)\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}+\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}+\frac{\partial\varphi(\upsilon,\tau)}{\partial\upsilon}\right],\\ \frac{1}{\kappa^\varrho}\mathbf{N}[\varphi(\upsilon,\tau)]-\kappa^{2-\varrho}\psi(\upsilon,0) &= -\mathbf{N}\left[\psi(\upsilon,\tau)\frac{\partial\varphi(\upsilon,\tau)}{\partial\upsilon}+\varphi(\upsilon,\tau)\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}\right.\\ &\qquad\left.+3\frac{\partial^3\psi(\upsilon,\tau)}{\partial\upsilon^3}-\frac{\partial^2\varphi(\upsilon,\tau)}{\partial\upsilon^2}\right].\end{aligned}\tag{5.4}$$

On simplification we have

$$\begin{aligned}\mathbf{N}[\psi(\upsilon,\tau)] &= \kappa^2\left[\frac{1}{2}-8\tanh(-2\upsilon)\right]-\frac{\varrho(\kappa-\varrho(\kappa-\varrho))}{\kappa^2}\mathbf{N}\left[\psi(\upsilon,\tau)\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}\right.\\ &\qquad\left.+\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}+\frac{\partial\varphi(\upsilon,\tau)}{\partial\upsilon}\right],\\ \mathbf{N}[\varphi(\upsilon,\tau)] &= \kappa^2\left[16-16\tanh^2(-2\upsilon)\right]-\frac{\varrho(\kappa-\varrho(\kappa-\varrho))}{\kappa^2}\mathbf{N}\left[\psi(\upsilon,\tau)\frac{\partial\varphi(\upsilon,\tau)}{\partial\upsilon}+\varphi(\upsilon,\tau)\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}\right.\\ &\qquad\left.+3\frac{\partial^3\psi(\upsilon,\tau)}{\partial\upsilon^3}-\frac{\partial^2\varphi(\upsilon,\tau)}{\partial\upsilon^2}\right].\end{aligned}\tag{5.5}$$

Using the inverse natural transform, we get

$$\begin{aligned}\psi(\upsilon,\tau) &= \left[\frac{1}{2}-8\tanh(-2\upsilon)\right]\\ &\quad-\mathbf{N}^{-1}\left[\frac{\varrho(\kappa-\varrho(\kappa-\varrho))}{\kappa^2}\mathbf{N}\left\{\psi(\upsilon,\tau)\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}+\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}+\frac{\partial\varphi(\upsilon,\tau)}{\partial\upsilon}\right\}\right],\\ \varphi(\upsilon,\tau) &= \left[16-16\tanh^2(-2\upsilon)\right]\\ &\quad-\mathbf{N}^{-1}\left[\frac{\varrho(\kappa-\varrho(\kappa-\varrho))}{\kappa^2}\mathbf{N}\left\{\psi(\upsilon,\tau)\frac{\partial\varphi(\upsilon,\tau)}{\partial\upsilon}+\varphi(\upsilon,\tau)\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}\right.\right.\\ &\quad\left.\left.+3\frac{\partial^3\psi(\upsilon,\tau)}{\partial\upsilon^3}-\frac{\partial^2\varphi(\upsilon,\tau)}{\partial\upsilon^2}\right\}\right].\end{aligned}\tag{5.6}$$

5.1. $NTDM_{CF}$ solution

The series form solution for the unknown term $\psi(\upsilon,\tau)$ and $\varphi(\upsilon,\tau)$ is given as

$$\psi(\upsilon,\tau)=\sum_{l=0}^{\infty}\psi_l(\upsilon,\tau)\quad and\quad \varphi(\upsilon,\tau)=\sum_{l=0}^{\infty}\psi_l(\upsilon,\tau). \tag{5.7}$$

The non-linear functions define by Adomian polynomials are given as $\psi\psi_\upsilon = \sum_{m=0}^{\infty}\mathcal{A}_m$, $\psi\varphi_\upsilon=\sum_{m=0}^{\infty}\mathcal{B}_m$ and $\varphi\psi_\upsilon=\sum_{m=0}^{\infty}C_m$, thus by means of these functions Eq (5.6) can be calculated as

$$\begin{aligned}\sum_{l=0}^{\infty}\psi_{l+1}(\upsilon,\tau)&=\frac{1}{2}-8\tanh(-2\upsilon)\\&\quad-\mathbf{N}^{-1}\left[\frac{\varrho(\kappa-\varrho(\kappa-\varrho))}{\kappa^2}\mathbf{N}\left\{\sum_{l=0}^{\infty}\mathcal{A}_l\right.\right.\\&\quad\left.\left.+\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}+\frac{\partial\varphi(\upsilon,\tau)}{\partial\upsilon}\right\}\right],\\\sum_{l=0}^{\infty}\varphi_{l+1}(\upsilon,\tau)&=16-16\tanh^2(-2\upsilon)\\&\quad-\mathbf{N}^{-1}\left[\frac{\varrho(\kappa-\varrho(\kappa-\varrho))}{\kappa^2}\mathbf{N}\left\{\sum_{l=0}^{\infty}\mathcal{B}_l+\sum_{l=0}^{\infty}C_l\right.\right.\\&\quad\left.\left.+3\frac{\partial^3\psi(\upsilon,\tau)}{\partial\upsilon^3}-\frac{\partial^2\varphi(\upsilon,\tau)}{\partial\upsilon^2}\right\}\right].\end{aligned}\tag{5.8}$$

Both side comparison of Eq (5.8), we achieve

$$\begin{aligned}\psi_0(\upsilon,\tau)&=\frac{1}{2}-8\tanh(-2\upsilon),\\\varphi_0(\upsilon,\tau)&=16-16\tanh^2(-2\upsilon),\end{aligned}$$

$$\begin{aligned}\psi_1(\upsilon,\tau)&=-8\sec h^2(-2\upsilon)\left(\varrho(\tau-1)+1\right),\\\varphi_1(\upsilon,\tau)&=-32\,\mathrm{sech}^2(-2\upsilon)\tanh(-2\upsilon)\left(\varrho(\tau-1)+1\right),\end{aligned}\tag{5.9}$$

$$\begin{aligned}\psi_2(\upsilon,\tau)&=-16\,\mathrm{sech}^2(-2\upsilon)\Big(4\,\mathrm{sech}^2(-2\upsilon)\\&\quad-8\tanh^2(-2\upsilon)+3\tanh(-2\upsilon)\Big)\Big((1-\varrho)^2\\&\quad+2\varrho(1-\varrho)\tau+\frac{\varrho^2\tau^2}{2}\Big),\\\varphi_2(\upsilon,\tau)&=-32\sec h^2(-2\upsilon)\{40\sec h^2(-2\upsilon)\tanh(-2\upsilon)\\&\quad+96\tanh(-2\upsilon)-2\tanh^2(-2\upsilon)-32\tanh^3(-2\upsilon)\\&\quad-25\sec h^2(-2\upsilon)\}\Big((1-\varrho)^2+2\varrho(1-\varrho)\tau+\frac{\varrho^2\tau^2}{2}\Big).\end{aligned}\tag{5.10}$$

Thus, for ψ_l and φ_l with ($l \geq 3$) the remaining terms are simply calculate. So, the result in series type is as

$$
\begin{aligned}
\psi(\upsilon,\tau) &= \sum_{l=0}^{\infty}\psi_l(\upsilon,\tau) = \psi_0(\upsilon,\tau) + \psi_1(\upsilon,\tau) + \psi_2(\upsilon,\tau) + \cdots, \\
\psi(\upsilon,\tau) &= \frac{1}{2} - 8\tanh(-2\upsilon) - 8\,\mathrm{sech}^2(-2\upsilon)\,(\varrho(\tau-1)+1) \\
&\quad - 16\sec h^2(-2\upsilon)\Big(4\sec h^2(-2\upsilon) - 8\tanh^2(-2\upsilon) \\
&\quad + 3\tanh(-2\upsilon)\Big)\Big((1-\varrho)^2 + 2\varrho(1-\varrho)\tau + \frac{\varrho^2\tau^2}{2}\Big) \\
&\quad + \cdots. \\
\varphi(\upsilon,\tau) &= \sum_{l=0}^{\infty}\varphi_l(\upsilon,\tau) = \varphi_0(\upsilon,\tau) + \varphi_1(\upsilon,\tau) + \varphi_2(\upsilon,\tau) + \cdots, \\
\varphi(\upsilon,\tau) &= 16 - 16\tanh^2(-2\upsilon) - 32\,\mathrm{sech}^2(-2\upsilon)\tanh(-2\upsilon)\,(\varrho(\tau-1)+1) \\
&\quad - 32\sec h^2(-2\upsilon)\{40\sec h^2(-2\upsilon)\tanh(-2\upsilon) \\
&\quad + 96\tanh(-2\upsilon) - 2\tanh^2(-2\upsilon) - 32\tanh^3(-2\upsilon) \\
&\quad - 25\,\mathrm{sech}^2(-2\upsilon)\}\Big((1-\varrho)^2 + 2\varrho(1-\varrho)\tau + \frac{\varrho^2\tau^2}{2}\Big) + \cdots.
\end{aligned}
\tag{5.11}
$$

5.2. NDM_{ABC} solution

The series form solution for the unknown term $\psi(\upsilon,\tau)$ and $\varphi(\upsilon,\tau)$ is given as

$$
\begin{aligned}
\psi(\upsilon,\tau) &= \sum_{l=0}^{\infty}\psi_l(\upsilon,\tau), \\
\varphi(\upsilon,\tau) &= \sum_{l=0}^{\infty}\psi_l(\upsilon,\tau).
\end{aligned}
\tag{5.12}
$$

The non-linear functions by mean of Adomian polynomials is given as $\psi\psi_\upsilon = \sum_{l=0}^{\infty}\mathcal{A}_l$ and $\psi^2\psi_\upsilon = \sum_{l=0}^{\infty}\mathcal{B}_l$, thus by mean of these functions Eq (5.6) can be calculated as

$$
\begin{aligned}
\sum_{l=0}^{\infty}\psi_{l+1}(\upsilon,\tau) &= \frac{1}{2} - 8\tanh(-2\upsilon) \\
&\quad - \mathbf{N}^{-1}\left[\frac{\ell^{\varrho}(\kappa^{\varrho} + \varrho(\ell^{\varrho} - \kappa^{\varrho}))}{\kappa^{2\varrho}}\mathbf{N}\left\{\sum_{l=0}^{\infty}\mathcal{A}_l + \frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon} + \frac{\partial\varphi(\upsilon,\tau)}{\partial\upsilon}\right\}\right], \\
\sum_{l=0}^{\infty}\varphi_{l+1}(\upsilon,\tau) &= 16 - 16\tanh^2(-2\upsilon) \\
&\quad - \mathbf{N}^{-1}\left[\frac{\ell^{\varrho}(\kappa^{\varrho} + \varrho(\ell^{\varrho} - \kappa^{\varrho}))}{\kappa^{2\varrho}}\mathbf{N}\left\{\sum_{l=0}^{\infty}\mathcal{B}_l + \sum_{l=0}^{\infty}C_l + 3\frac{\partial^3\psi(\upsilon,\tau)}{\partial\upsilon^3} - \frac{\partial^2\varphi(\upsilon,\tau)}{\partial\upsilon^2}\right\}\right].
\end{aligned}
\tag{5.13}
$$

Both side comparison of Eq (5.13), we get

$$\begin{aligned}
\psi_0(\upsilon,\tau) &= \frac{1}{2} - 8\tanh(-2\upsilon),\\
\varphi_0(\upsilon,\tau) &= 16 - 16\tanh^2(-2\upsilon),\\
\psi_1(\upsilon,\tau) &= -8\,\mathrm{sech}^2(-2\upsilon)\left(1-\varrho+\frac{\varrho\tau^{\varrho}}{\Gamma(\varrho+1)}\right),\\
\varphi_1(\upsilon,\tau) &= -32\,\mathrm{sech}^2(-2\upsilon)\tanh(-2\upsilon)\left(1-\varrho+\frac{\varrho\tau^{\varrho}}{\Gamma(\varrho+1)}\right),
\end{aligned} \tag{5.14}$$

$$\begin{aligned}
\psi_2(\upsilon,\tau) &= -16\sec h^2(-2\upsilon)\left(4\sec h^2(-2\upsilon) - 8\tanh^2(-2\upsilon) + 3\tanh(-2\upsilon)\right)\left[\frac{\varrho^2\tau^{2\varrho}}{\Gamma(2\varrho+1)}\right.\\
&\quad \left. + 2\varrho(1-\varrho)\frac{\tau^{\varrho}}{\Gamma(\varrho+1)} + (1-\varrho)^2\right],\\
\varphi_2(\upsilon,\tau) &= -32\sec h^2(-2\upsilon)\{40\sec h^2(-2\upsilon)\tanh(-2\upsilon) + 96\tanh(-2\upsilon) - 2\tanh^2(-2\upsilon)\\
&\quad - 32\tanh^3(-2\upsilon) - 25\sec h^2(-2\upsilon)\}\left[\frac{\varrho^2\tau^{2\varrho}}{\Gamma(2\varrho+1)} + 2\varrho(1-\varrho)\frac{\tau^{\varrho}}{\Gamma(\varrho+1)} + (1-\varrho)^2\right].
\end{aligned} \tag{5.15}$$

Thus, for ψ_l with ($l \geq 3$) the remaining component are simply calculated. So, the series form solution is given as

$$\begin{aligned}
\psi(\upsilon,\tau) &= \sum_{l=0}^{\infty}\psi_l(\upsilon,\tau) = \psi_0(\upsilon,\tau)+\psi_1(\upsilon,\tau)+\psi_2(\upsilon,\tau)+\cdots,\\
\psi(\upsilon,\tau) &= \frac{1}{2} - 8\tanh(-2\upsilon) - 8\,\mathrm{sech}^2(-2\upsilon)\left(1-\varrho+\frac{\varrho\tau^{\varrho}}{\Gamma(\varrho+1)}\right)\\
&\quad - 16\sec h^2(-2\upsilon)\Big(4\sec h^2(-2\upsilon) - 8\tanh^2(-2\upsilon)\\
&\quad + 3\tanh(-2\upsilon)\Big)\left[\frac{\varrho^2\tau^{2\varrho}}{\Gamma(2\varrho+1)} + 2\varrho(1-\varrho)\frac{\tau^{\varrho}}{\Gamma(\varrho+1)} + (1-\varrho)^2\right]+\cdots.\\
\varphi(\upsilon,\tau) &= \sum_{l=0}^{\infty}\varphi_l(\upsilon,\tau) = \varphi_0(\upsilon,\tau)+\varphi_1(\upsilon,\tau)+\varphi_2(\upsilon,\tau)+\cdots,\\
\varphi(\upsilon,\tau) &= 16 - 16\tanh^2(-2\upsilon) - 32\,\mathrm{sech}^2(-2\upsilon)\tanh(-2\upsilon)\left(1-\varrho+\frac{\varrho\tau^{\varrho}}{\Gamma(\varrho+1)}\right)\\
&\quad - 32\sec h^2(-2\upsilon)\{40\sec h^2(-2\upsilon)\tanh(-2\upsilon) + 96\tanh(-2\upsilon) - 2\tanh^2(-2\upsilon)\\
&\quad - 32\tanh^3(-2\upsilon) - 25\sec h^2(-2\upsilon)\}\left[\frac{\varrho^2\tau^{2\varrho}}{\Gamma(2\varrho+1)} + 2\varrho(1-\varrho)\frac{\tau^{\varrho}}{\Gamma(\varrho+1)} + (1-\varrho)^2\right]+\cdots.
\end{aligned} \tag{5.16}$$

The exact result is

$$\begin{aligned}
\psi(\upsilon,\tau) &= \frac{1}{2} - 8\tanh\left\{-2\left(\upsilon-\frac{\tau}{2}\right)\right\},\\
\varphi(\upsilon,\tau) &= 16 - 16\tanh^2\left\{-2\left(\upsilon-\frac{\tau}{2}\right)\right\}.
\end{aligned} \tag{5.17}$$

Example 2. Consider the fractional-order system of Whitham-Broer-Kaup equations is given as

$$
\begin{aligned}
&D_\tau^\varrho \psi(\upsilon,\tau)+\psi(\upsilon,\tau)\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}+\frac{1}{2}\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}+\frac{\partial\varphi(\upsilon,\tau)}{\partial\upsilon}=0,\\
&D_\tau^\varrho \varphi(\upsilon,\tau)+\psi(\upsilon,\tau)\frac{\partial\varphi(\upsilon,\tau)}{\partial\upsilon}+\varphi(\upsilon,\tau)\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}-\frac{1}{2}\frac{\partial^2\varphi(\upsilon,\tau)}{\partial\upsilon^2}=0,\\
&0<\varrho\le 1,\quad -1<\tau\le 1,\quad -10\le\upsilon\le 10,
\end{aligned}
\tag{5.18}
$$

with the initial condition

$$
\begin{aligned}
\psi(\upsilon,0)&=\lambda-\kappa\coth[\kappa(\upsilon+\theta)],\\
\varphi(\upsilon,0)&=-\kappa^2csch^2[\kappa(\upsilon+\theta)].
\end{aligned}
\tag{5.19}
$$

On employing natural transform, we have

$$
\begin{aligned}
\mathbf{N}[D_\tau^\varrho\psi(\upsilon,\tau)]&=-\mathbf{N}\left[\psi(\upsilon,\tau)\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}+\frac{1}{2}\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}+\frac{\partial\varphi(\upsilon,\tau)}{\partial\upsilon}\right],\\
\mathbf{N}[D_\tau^\varrho\varphi(\upsilon,\tau)]&=-\mathbf{N}\left[\psi(\upsilon,\tau)\frac{\partial\varphi(\upsilon,\tau)}{\partial\upsilon}+\varphi(\upsilon,\tau)\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}-\frac{1}{2}\frac{\partial^2\varphi(\upsilon,\tau)}{\partial\upsilon^2}\right].
\end{aligned}
\tag{5.20}
$$

Thus, we have

$$
\begin{aligned}
&\frac{1}{\kappa^\varrho}\mathbf{N}[\psi(\upsilon,\tau)]-\kappa^{2-\varrho}\psi(\upsilon,0)=-\mathbf{N}\left[\psi(\upsilon,\tau)\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}+\frac{1}{2}\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}+\frac{\partial\varphi(\upsilon,\tau)}{\partial\upsilon}\right],\\
&\frac{1}{\kappa^\varrho}\mathbf{N}[\varphi(\upsilon,\tau)]-\kappa^{2-\varrho}\psi(\upsilon,0)=-\mathbf{N}\left[\psi(\upsilon,\tau)\frac{\partial\varphi(\upsilon,\tau)}{\partial\upsilon}+\varphi(\upsilon,\tau)\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}-\frac{1}{2}\frac{\partial^2\varphi(\upsilon,\tau)}{\partial\upsilon^2}\right].
\end{aligned}
\tag{5.21}
$$

On simplification we have

$$
\begin{aligned}
\mathbf{N}[\psi(\upsilon,\tau)]&=\kappa^2\Big[\lambda-\kappa\coth[\kappa(\upsilon+\theta)]\Big]-\frac{\varrho(\kappa-\varrho(\kappa-\varrho))}{\kappa^2}\mathbf{N}\Big[\psi(\upsilon,\tau)\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}\\
&\quad+\frac{1}{2}\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}+\frac{\partial\varphi(\upsilon,\tau)}{\partial\upsilon}\Big],\\
\mathbf{N}[\varphi(\upsilon,\tau)]&=\kappa^2\Big[-\kappa^2csch^2[\kappa(\upsilon+\theta)]\Big]-\frac{\varrho(\kappa-\varrho(\kappa-\varrho))}{\kappa^2}\mathbf{N}\Big[\psi(\upsilon,\tau)\frac{\partial\varphi(\upsilon,\tau)}{\partial\upsilon}\\
&\quad+\varphi(\upsilon,\tau)\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}-\frac{1}{2}\frac{\partial^2\varphi(\upsilon,\tau)}{\partial\upsilon^2}\Big].
\end{aligned}
\tag{5.22}
$$

Using the inverse natural transform, we have

$$
\begin{aligned}
\psi(\upsilon,\tau)&=\Big[\lambda-\kappa\coth[\kappa(\upsilon+\theta)]\Big]\\
&\quad-\mathbf{N}^{-1}\left[\frac{\varrho(\kappa-\varrho(\kappa-\varrho))}{\kappa^2}\mathbf{N}\left\{\psi(\upsilon,\tau)\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}+\frac{1}{2}\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}+\frac{\partial\varphi(\upsilon,\tau)}{\partial\upsilon}\right\}\right],\\
\varphi(\upsilon,\tau)&=\Big[-\kappa^2csch^2[\kappa(\upsilon+\theta)]\Big]\\
&\quad-\mathbf{N}^{-1}\left[\frac{\varrho(\kappa-\varrho(\kappa-\varrho))}{\kappa^2}\mathbf{N}\left\{\psi(\upsilon,\tau)\frac{\partial\varphi(\upsilon,\tau)}{\partial\upsilon}+\varphi(\upsilon,\tau)\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}-\frac{1}{2}\frac{\partial^2\varphi(\upsilon,\tau)}{\partial\upsilon^2}\right\}\right].
\end{aligned}
\tag{5.23}
$$

5.3. NDM_{CF} solution

The series form solution for the unknown term $\psi(\upsilon,\tau)$ and $\varphi(\upsilon,\tau)$ is given as

$$\psi(\upsilon,\tau)=\sum_{l=0}^{\infty}\psi_l(\upsilon,\tau)\quad and \quad \varphi(\upsilon,\tau)=\sum_{l=0}^{\infty}\psi_l(\upsilon,\tau). \tag{5.24}$$

The non-linear functions by Adomian polynomials is given as $\psi\psi_\upsilon=\sum_{m=0}^{\infty}\mathcal{A}_m$, $\psi\varphi_\upsilon=\sum_{m=0}^{\infty}\mathcal{B}_m$ and $\varphi\psi_\upsilon=\sum_{m=0}^{\infty}C_m$, thus by mean of these functions Eq (5.23) can be calculated as

$$\begin{aligned}\sum_{l=0}^{\infty}\psi_{l+1}(\upsilon,\tau)&=\lambda-\kappa\coth[\kappa(\upsilon+\theta)]\\&-\mathbf{N}^{-1}\left[\frac{\varrho(\kappa-\varrho(\kappa-\varrho))}{\kappa^2}\mathbf{N}\left\{\sum_{l=0}^{\infty}\mathcal{A}_l+\frac{1}{2}\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}+\frac{\partial\varphi(\upsilon,\tau)}{\partial\upsilon}\right\}\right],\\\sum_{l=0}^{\infty}\varphi_{l+1}(\upsilon,\tau)&=-\kappa^2csch^2[\kappa(\upsilon+\theta)]\\&-\mathbf{N}^{-1}\left[\frac{\varrho(\kappa-\varrho(\kappa-\varrho))}{\kappa^2}\mathbf{N}\left\{\sum_{l=0}^{\infty}\mathcal{B}_l+\sum_{l=0}^{\infty}C_l-\frac{1}{2}\frac{\partial^2\varphi(\upsilon,\tau)}{\partial\upsilon^2}\right\}\right].\end{aligned} \tag{5.25}$$

Both sides comparisons of Eq (5.25), we achieve

$$\begin{aligned}\psi_0(\upsilon,\tau)&=\lambda-\kappa\coth[\kappa(\upsilon+\theta)],\\\varphi_0(\upsilon,\tau)&=-\kappa^2csch^2[\kappa(\upsilon+\theta)],\end{aligned}$$

$$\begin{aligned}\psi_1(\upsilon,\tau)&=-\lambda\kappa^2csch^2[\kappa(\upsilon+\theta)]\left(\varrho(\tau-1)+1\right),\\\varphi_1(\upsilon,\tau)&=-\lambda\kappa^2csch^2[\kappa(\upsilon+\theta)]\coth[\kappa(\upsilon+\theta)]\left(\varrho(\tau-1)+1\right),\end{aligned} \tag{5.26}$$

$$\begin{aligned}\psi_2(\upsilon,\tau)&=\lambda\kappa^4csch^2[\kappa(\upsilon+\theta)]\Big\{2\lambda\kappa\Big\{(1-\varrho)^2 3\varrho\tau+(1-\varrho)^3+\frac{3\varrho^2(1-\varrho)\tau^2}{2}+\frac{\varrho^3\tau^3}{3!}\Big\}\\&-(3\coth^2([\kappa(\upsilon+\theta)]-1))\Big((1-\varrho)^2+2\varrho(1-\varrho)\tau+\frac{\varrho^2\tau^2}{2}\Big)\Big\},\\\varphi_2(\upsilon,\tau)&=[2\lambda\kappa^5csch^2[\kappa(\upsilon+\theta)]]\Big[\lambda\kappa csch^2(3\coth^2([\kappa(\upsilon+\theta)]-1))\Big\{(1-\varrho)^2 3\varrho\tau+\\&+(1-\varrho)^3\frac{3\varrho^2(1-\varrho)\tau^2}{2}+\frac{\varrho^3\tau^3}{3!}\Big\}+\frac{2\lambda\kappa csch^2\coth^2([\kappa(\upsilon+\theta)])\tau^{3\varrho}}{\Gamma(\varrho+1)\Gamma(3\varrho+1)}\\&-2\lambda\coth(3csch^2([\kappa(\upsilon+\theta)]-1))\Big((1-\varrho)^2+2\varrho(1-\varrho)\tau+\frac{\varrho^2\tau^2}{2}\Big)\Big].\end{aligned} \tag{5.27}$$

Thus, for ψ_l and φ_l with ($l \geq 3$) the remaining terms are simply calculate. So, the series form solution is as

$$\begin{aligned}
\psi(\upsilon,\tau) &= \sum_{l=0}^{\infty}\psi_l(\upsilon,\tau) = \psi_0(\upsilon,\tau)+\psi_1(\upsilon,\tau)+\psi_2(\upsilon,\tau)+\cdots,\\
\psi(\upsilon,\tau) &= \lambda-\kappa\coth[\kappa(\upsilon+\theta)]-\lambda\kappa^2 csch^2[\kappa(\upsilon+\theta)]\,(\varrho(\tau-1)+1)\\
&\quad+\lambda\kappa^4 csch^2[\kappa(\upsilon+\theta)]\Big\{2\lambda\kappa\Big\{(1-\varrho)^2 3\varrho\tau+(1-\varrho)^3+\frac{3\varrho^2(1-\varrho)\tau^2}{2}+\frac{\varrho^3\tau^3}{3!}\Big\}\\
&\quad-(3\coth^2([\kappa(\upsilon+\theta)]-1))\Big((1-\varrho)^2+2\varrho(1-\varrho)\tau+\frac{\varrho^2\tau^2}{2}\Big)\Big\}+\cdots.\\
\varphi(\upsilon,\tau) &= \sum_{l=0}^{\infty}\varphi_l(\upsilon,\tau) = \varphi_0(\upsilon,\tau)+\varphi_1(\upsilon,\tau)+\varphi_2(\upsilon,\tau)+\cdots, \qquad (5.28)\\
\varphi(\upsilon,\tau) &= -\kappa^2 csch^2[\kappa(\upsilon+\theta)]-\lambda\kappa^2 csch^2[\kappa(\upsilon+\theta)]\coth[\kappa(\upsilon+\theta)]\,(\varrho(\tau-1)+1)\\
&\quad+[2\lambda\kappa^5 csch^2[\kappa(\upsilon+\theta)]]\Big[\lambda\kappa csch^2(3\coth^2([\kappa(\upsilon+\theta)]-1))\Big\{(1-\varrho)^2 3\varrho\tau\\
&\quad+(1-\varrho)^3+\frac{3\varrho^2(1-\varrho)\tau^2}{2}+\frac{\varrho^3\tau^3}{3!}\Big\}+\frac{2\lambda\kappa csch^2\coth^2([\kappa(\upsilon+\theta)])\tau^{3\varrho}}{\Gamma(\varrho+1)\Gamma(3\varrho+1)}\\
&\quad-2\lambda\coth(3csch^2([\kappa(\upsilon+\theta)]-1))\Big((1-\varrho)^2+2\varrho(1-\varrho)\tau+\frac{\varrho^2\tau^2}{2}\Big)\Big]+\cdots.
\end{aligned}$$

5.4. NDM_{ABC} solution

The series form solution for the unknown term $\psi(\upsilon,\tau)$ and $\varphi(\upsilon,\tau)$ is given as

$$\begin{aligned}
\psi(\upsilon,\tau) &= \sum_{l=0}^{\infty}\psi_l(\upsilon,\tau),\\
\varphi(\upsilon,\tau) &= \sum_{l=0}^{\infty}\psi_l(\upsilon,\tau).
\end{aligned} \qquad (5.29)$$

The non-linear function by mean of Adomian polynomials are define as $\psi\psi_\upsilon = \sum_{l=0}^{\infty}\mathcal{A}_l$ and $\psi^2\psi_\upsilon = \sum_{l=0}^{\infty}\mathcal{B}_l$, thus by mean of these function Eq (5.23) can be calculated as

$$\begin{aligned}
\sum_{l=0}^{\infty}\psi_{l+1}(\upsilon,\tau) &= \lambda-\kappa\coth[\kappa(\upsilon+\theta)]\\
&\quad+\mathbf{N}^{-1}\Big[\frac{\ell^{\varrho}(\kappa^{\varrho}+\varrho(\ell^{\varrho}-\kappa^{\varrho}))}{\kappa^{2\varrho}}\mathbf{N}\Big\{\sum_{l=0}^{\infty}\mathcal{A}_l+\frac{1}{2}\frac{\partial\psi(\upsilon,\tau)}{\partial\upsilon}+\frac{\partial\varphi(\upsilon,\tau)}{\partial\upsilon}\Big\}\Big],\\
\sum_{l=0}^{\infty}\varphi_{l+1}(\upsilon,\tau) &= -\kappa^2 csch^2[\kappa(\upsilon+\theta)]\\
&\quad+\mathbf{N}^{-1}\Big[\frac{\ell^{\varrho}(\kappa^{\varrho}+\varrho(\ell^{\varrho}-\kappa^{\varrho}))}{\kappa^{2\varrho}}\mathbf{N}\Big\{\sum_{l=0}^{\infty}\mathcal{B}_l+\sum_{l=0}^{\infty}C_l-\frac{1}{2}\frac{\partial^2\varphi(\upsilon,\tau)}{\partial\upsilon^2}\Big\}\Big].
\end{aligned} \qquad (5.30)$$

Both side comparison of Eq (5.30), we achieve

$$\psi_0(\upsilon,\tau) = \lambda - \kappa\coth[\kappa(\upsilon+\theta)],$$
$$\varphi_0(\upsilon,\tau) = -\kappa^2 csch^2[\kappa(\upsilon+\theta)],$$
$$\psi_1(\upsilon,\tau) = -\lambda\kappa^2 csch^2[\kappa(\upsilon+\theta)]\left(1-\varrho+\frac{\varrho\tau^{\varrho}}{\Gamma(\varrho+1)}\right),$$
$$\varphi_1(\upsilon,\tau) = -\lambda\kappa^2 csch^2[\kappa(\upsilon+\theta)]\coth[\kappa(\upsilon+\theta)]\left(1-\varrho+\frac{\varrho\tau^{\varrho}}{\Gamma(\varrho+1)}\right), \tag{5.31}$$

$$\begin{aligned}\psi_2(\upsilon,\tau) &= \lambda\kappa^4 csch^2[\kappa(\upsilon+\theta)]\Big\{2\lambda\kappa\Big\{(1-\varrho)^2 3\varrho\tau + (1-\varrho)^3 + \frac{3\varrho^2(1-\varrho)\tau^2}{2} + \frac{\varrho^3\tau^3}{3!}\Big\} \\ &\quad - (3\coth^2([\kappa(\upsilon+\theta)]-1))\left[\frac{\varrho^2\tau^{2\varrho}}{\Gamma(2\varrho+1)} + 2\varrho(1-\varrho)\frac{\tau^{\varrho}}{\Gamma(\varrho+1)} + (1-\varrho)^2\right]\Big\}, \\ \varphi_2(\upsilon,\tau) &= [2\lambda\kappa^5 csch^2[\kappa(\upsilon+\theta)]]\Big[\lambda\kappa csch^2(3\coth^2([\kappa(\upsilon+\theta)]-1))\Big\{(1-\varrho)^2 3\varrho\tau + (1-\varrho)^3 \\ &\quad + \frac{3\varrho^2(1-\varrho)\tau^2}{2} + \frac{\varrho^3\tau^3}{3!}\Big\} + \frac{2\lambda\kappa csch^2\coth^2([\kappa(\upsilon+\theta)])\tau^{3\varrho}}{\Gamma(\varrho+1)\Gamma(3\varrho+1)} \\ &\quad - 2\lambda\coth(3csch^2([\kappa(\upsilon+\theta)]-1))\left[\frac{\varrho^2\tau^{2\varrho}}{\Gamma(2\varrho+1)} + 2\varrho(1-\varrho)\frac{\tau^{\varrho}}{\Gamma(\varrho+1)} + (1-\varrho)^2\right]\Big].\end{aligned} \tag{5.32}$$

Thus, for ψ_l and φ_l with ($l \geq 3$), the remaining terms are simply calculated. So, the series form solution is as

$$\begin{aligned}\psi(\upsilon,\tau) &= \sum_{l=0}^{\infty}\psi_l(\upsilon,\tau) = \psi_0(\upsilon,\tau) + \psi_1(\upsilon,\tau) + \psi_2(\upsilon,\tau) + \cdots, \\ \psi(\upsilon,\tau) &= \lambda - \kappa\coth[\kappa(\upsilon+\theta)] - \lambda\kappa^2 csch^2[\kappa(\upsilon+\theta)]\left(1-\varrho+\frac{\varrho\tau^{\varrho}}{\Gamma(\varrho+1)}\right) \\ &\quad + \lambda\kappa^4 csch^2[\kappa(\upsilon+\theta)]\Big\{2\lambda\kappa\Big\{(1-\varrho)^2 3\varrho\tau + (1-\varrho)^3 + \frac{3\varrho^2(1-\varrho)\tau^2}{2} + \frac{\varrho^3\tau^3}{3!}\Big\} \\ &\quad - (3\coth^2([\kappa(\upsilon+\theta)]-1))\left[\frac{\varrho^2\tau^{2\varrho}}{\Gamma(2\varrho+1)} + 2\varrho(1-\varrho)\frac{\tau^{\varrho}}{\Gamma(\varrho+1)} + (1-\varrho)^2\right]\Big\} + \cdots. \\ \varphi(\upsilon,\tau) &= \sum_{l=0}^{\infty}\varphi_l(\upsilon,\tau) = \varphi_0(\upsilon,\tau) + \varphi_1(\upsilon,\tau) + \varphi_2(\upsilon,\tau) + \cdots, \\ \varphi(\upsilon,\tau) &= -\kappa^2 csch^2[\kappa(\upsilon+\theta)] - \lambda\kappa^2 csch^2[\kappa(\upsilon+\theta)]\coth[\kappa(\upsilon+\theta)]\left(1-\varrho+\frac{\varrho\tau^{\varrho}}{\Gamma(\varrho+1)}\right) \\ &\quad + [2\lambda\kappa^5 csch^2[\kappa(\upsilon+\theta)]]\Big[\lambda\kappa csch^2(3\coth^2([\kappa(\upsilon+\theta)]-1))\Big\{(1-\varrho)^2 3\varrho\tau + (1-\varrho)^3 \\ &\quad + \frac{3\varrho^2(1-\varrho)\tau^2}{2} + \frac{\varrho^3\tau^3}{3!}\Big\} + \frac{2\lambda\kappa csch^2\coth^2([\kappa(\upsilon+\theta)])\tau^{3\varrho}}{\Gamma(\varrho+1)\Gamma(3\varrho+1)} \\ &\quad - 2\lambda\coth(3csch^2([\kappa(\upsilon+\theta)]-1))\left[\frac{\varrho^2\tau^{2\varrho}}{\Gamma(2\varrho+1)} + 2\varrho(1-\varrho)\frac{\tau^{\varrho}}{\Gamma(\varrho+1)} + (1-\varrho)^2\right]\Big] + \cdots.\end{aligned} \tag{5.33}$$

We achieve the series type result at integer-order $\varrho = 1, \kappa = 0.1, \lambda = 0.005$, $\theta = 10$, as

$$\begin{aligned}\psi(\upsilon,\tau) =&0.005 - 0.1\coth(0.1\upsilon + 10) - 0.0005 csch^2(0.1\upsilon + 10)\tau\\ &+ 5\times 10^{-7} csch^2(0.1\upsilon + 10)0.003\tau^3\\ &-0.5\left(3\coth^2(0.1\upsilon + 10) - 1.\right)\tau^2,\\ \varphi(\upsilon,\tau) =& - 0.01 csch^2(0.1\upsilon + 10) - 0.000010 csch^2(0.1\upsilon + 10)\\ &\times\coth(0.1\upsilon + 10)\tau + 1.0\times 10^{-7} csch^2(0.1\upsilon + 10)\\ &\times\Big[8.3\times 10^{-5}\tau^3 csch^2(0.1\upsilon + 10)(3\coth(0.1\upsilon + 10) - 1)\\ &-\tau^2\coth(0.1\upsilon + 10)\left(3csch^2(0.1\upsilon + 10) - 1\right)\\ &+1.6\times 10^{-4}\tau^3 cosech^2(0.1\upsilon + 10)\coth(0.1\upsilon + 10)\Big].\end{aligned}$$

The exact result of Eq (5.18) at $\varrho = 1$, and taking $\kappa = 0.1$, $\lambda = 0.005$, $\theta = 10$,

$$\begin{aligned}\psi(\upsilon,\tau) &== \lambda - \kappa\coth[\kappa(\upsilon + \theta - \lambda\tau)],\\ \varphi(\upsilon,\tau) &= -\kappa^2 csch^2[\kappa(\upsilon + \theta - \lambda\tau)].\end{aligned} \tag{5.34}$$

6. Numerical results and discussion

In this study, we investigated the numerical solution of coupled fractional Whitham-Broer-Kaup equation systems using two novel approaches. For any order and different values of the space and time variables, numerical data for the coupled fractional Whitham-Broer-Kaup equations can be found using Maple. We create numerical simulations for system 1 at various υ and τ values in Tables 1 and 2. The Adomian decomposition approach, variational iteration method, optimal homotopy asymptotic method, and natural decomposition method are numerically compared in Tables 3 and 4 in terms of absolute error. Tables 5 and 6 display the results of calculations performed for the coupled system taken into account in Problem 2. We can conclude that the Natural decomposition approach yields more accurate results based on the data in the tables above. For $\psi(\upsilon,\tau)$ of Problem 1, Figure 1 shows the behavior of the exact and Natural decomposition approach result, while Figure 2 shows the behavior of the analytical result at various fractional-orders of ϱ. Figure 3 shows the solution to the equation $\psi(\upsilon,\tau)$ in various fractional orders, whereas Figure 4 shows the absolute error. The behavior of the exact and analytical solutions for $\varphi(\upsilon,\tau)$ is shown in Figure 5, while the behavior of the analytical result at various fractional orders of ϱ is shown in Figures 6 and 7. The behavior of the exact and analytical solution for $\psi(\upsilon,\tau)$ is depicted in Figure 8, while the absolute error for Problem 2 is shown in Figure 9. Similarly, the behavior of the exact and analytical solution for $\varphi(\upsilon,\tau)$ is depicted in Figure 10, while the absolute error is shown in Figure 11 for Problem 2.

Table 1. The suggested techniques result for $\psi(\upsilon,\tau)$ at various fractional-orders of Example 1.

(υ,τ)	$\psi(\upsilon,\tau)$ at $\varrho=0.6$	$\psi(\upsilon,\tau)$ at $\varrho=0.8$	$(NTDM_{ABC})$ at $\varrho=1$	$(NTDM_{CF})$ at $\varrho=1$	Exact solution
(0.2,0.01)	3.538802	3.538856	3.538907	3.538907	3.538907
(0.4,0.02)	5.811722	5.811755	5.811846	5.811846	5.811846
(0.6,0.03)	7.168821	7.168876	7.168992	7.168992	7.168992
(0.2,0.01)	3.525702	3.525764	3.525891	3.525891	3.525891
(0.4,0.02)	5.803176	5.803245	5.803337	5.803337	5.803337
(0.6,0.03)	7.164207	7.164278	7.164348	7.164348	7.164348
(0.2,0.01)	3.537413	3.537498	3.537537	3.537537	3.537537
(0.4,0.02)	5.810812	5.810856	5.810952	5.810952	5.810952
(0.6,0.03)	7.168423	7.168479	7.168504	7.168504	7.168504
(0.2,0.01)	3.536701	3.536746	3.536853	3.536853	3.536853
(0.4,0.02)	5.810411	5.810466	5.810504	5.810504	5.810504
(0.6,0.03)	7.168112	7.168160	7.168260	7.168260	7.168260
(0.2,0.01)	3.536076	3.536134	3.536168	3.536168	3.536168
(0.4,0.02)	5.810002	5.810010	5.810057	5.810057	5.810057
(0.6,0.03)	7.168000	7.168002	7.168016	7.168016	7.168016

Table 2. The suggested techniques solution for $\varphi(\upsilon,\tau)$ at various fractional-orders of Example 1.

(υ,τ)	$\varphi(\upsilon,\tau)$ at $\varrho=0.6$	$\varphi(\upsilon,\tau)$ at $\varrho=0.8$	$(NTDM_{ABC})$ at $\varrho=1$	$(NTDM_{CF})$ at $\varrho=1$	Exact solution
(0.2,0.01)	13.691122	13.691178	13.691260	13.691260	13.691260
(0.4,0.02)	8.946002	8.946023	8.946070	8.946070	8.946070
(0.6,0.03)	4.881079	4.881103	4.881133	4.881133	4.881133
(0.2,0.01)	13.710837	13.710898	13.710995	13.710995	13.710995
(0.4,0.02)	8.968516	8.968579	8.968653	8.968653	8.968653
(0.6,0.03)	4.896526	4.896588	4.896615	4.896615	4.896615
(0.2,0.01)	13.693268	13.693302	13.693340	13.693340	13.693340
(0.4,0.02)	8.948378	8.948412	8.948446	8.948446	8.948446
(0.6,0.03)	4.882679	4.882705	4.882761	4.882761	4.882761
(0.2,0.01)	13.694302	13.694323	13.694380	13.694380	13.694380
(0.4,0.02)	8.949587	8.949612	8.949634	8.949634	8.949634
(0.6,0.03)	4.883501	4.883525	4.883575	4.883575	4.883575
(0.2,0.01)	13.695389	13.695402	13.695420	13.695420	13.695420
(0.4,0.02)	8.950736	8.950778	8.950823	8.950823	8.950823
(0.6,0.03)	4.884288	4.884326	4.884389	4.884389	4.884389

Table 3. Absolute Error (AE) comparison for $\psi(\upsilon,\tau)$ at $\varrho = 1$ obtained by different techniques.

(υ,τ)	AE Of ADM [50]	AE Of VIM [51]	AE Of OHAM [52]	AE of $(NTDM_{ABC})$	AE of $(NTDM_{CF})$
(0.1,0.2)	1.05983×10^{-5}	1.34144×10^{-6}	1.18169×10^{-7}	1.45621×10^{-10}	1.45621×10^{-10}
(0.1,0.4)	9.75585×10^{-6}	3.78688×10^{-6}	3.15656×10^{-7}	2.95673×10^{-09}	2.95673×10^{-09}
(0.1,0.6)	8.77423×10^{-6}	6.27984×10^{-6}	4.92412×10^{-7}	1.10432×10^{-8}	1.10432×10^{-8}
(0.2,0.2)	4.37319×10^{-5}	1.38978×10^{-6}	1.12395×10^{-7}	1.44301×10^{-10}	1.44301×10^{-10}
(0.2,0.4)	3.82189×10^{-5}	3.51189×10^{-6}	2.86457×10^{-6}	2.35213×10^{-09}	2.35213×10^{-09}
(0.2,0.6)	3.51272×10^{-5}	6.10117×10^{-5}	4.51245×10^{-6}	1.02145×10^{-08}	1.02145×10^{-08}
(0.3,0.2)	9.62833×10^{-5}	1.25698×10^{-5}	1.13664×10^{-6}	2.27511×10^{-10}	2.27511×10^{-10}
(0.3,0.4)	8.84418×10^{-5}	3.61977×10^{-5}	2.62353×10^{-6}	2.02314×10^{-09}	2.02314×10^{-09}
(0.3,0.6)	8.33563×10^{-5}	5.96721×10^{-5}	4.46642×10^{-6}	1.03541×10^{-08}	1.03541×10^{-08}
(0.4,0.2)	1.86687×10^{-4}	1.24938×10^{-5}	9.24537×10^{-5}	1.32601×10^{-10}	1.32601×10^{-10}
(0.4,0.4)	1.72542×10^{-4}	3.52859×10^{-5}	2.63564×10^{-5}	1.11427×10^{-09}	1.11427×10^{-09}
(0.4,0.6)	1.58687×10^{-4}	5.81821×10^{-5}	4.65446×10^{-5}	1.87965×10^{-08}	1.87965×10^{-08}
(0.5,0.2)	2.88628×10^{-4}	1.21847×10^{-5}	9.72736×10^{-5}	1.78145×10^{-10}	1.78145×10^{-10}
(0.5,0.4)	2.47825×10^{-4}	3.44373×10^{-5}	2.33457×10^{-5}	1.43901×10^{-09}	1.43901×10^{-09}
(0.5,0.6)	2.47295×10^{-4}	5.47346×10^{-5}	4.38895×10^{-5}	1.43421×10^{-08}	1.43421×10^{-08}

Table 4. Absolute Error (AE) comparison for $\varphi(\upsilon,\tau)$ at $\varrho = 1$ obtained by different techniques.

(υ,τ)	AE Of ADM [50]	AE Of VIM [51]	AE Of OHAM [52]	AE of $(NTDM_{ABC})$	AE of $(NTDM_{CF})$
(0.1,0.2)	6.52318×10^{-4}	1.23581×10^{-5}	5.72451×10^{-6}	1.73281×10^{-10}	1.73281×10^{-10}
(0.1,0.4)	5.87694×10^{-4}	3.53456×10^{-5}	3.24632×10^{-6}	2.00931×10^{-09}	2.00931×10^{-09}
(0.1,0.6)	5.72618×10^{-4}	5.63261×10^{-5}	3.38923×10^{-6}	1.08631×10^{-08}	1.08631×10^{-08}
(0.2,0.2)	1.44292×10^{-3}	1.18127×10^{-5}	5.45771×10^{-6}	1.11621×10^{-10}	1.11621×10^{-10}
(0.2,0.4)	1.33452×10^{-3}	3.34512×10^{-5}	2.86341×10^{-5}	2.07941×10^{-09}	2.07941×10^{-09}
(0.2,0.6)	1.25527×10^{-3}	5.47838×10^{-5}	2.82545×10^{-5}	1.03361×10^{-08}	1.03361×10^{-08}
(0.3,0.2)	2.14563×10^{-3}	1.14848×10^{-5}	5.36746×10^{-5}	1.16031×10^{-10}	1.16031×10^{-10}
(0.3,0.4)	1.84963×10^{-3}	3.22828×10^{-5}	2.74231×10^{-5}	2.006841×10^{-09}	2.006841×10^{-09}
(0.3,0.6)	1.72318×10^{-3}	5.32558×10^{-4}	2.66463×10^{-5}	2.86931×10^{-08}	2.86931×10^{-08}
(0.4,0.2)	2.98211×10^{-3}	1.11468×10^{-4}	5.23838×10^{-5}	4.09741×10^{-10}	4.09741×10^{-10}
(0.4,0.4)	2.59845×10^{-3}	3.13456×10^{-4}	2.72338×10^{-3}	3.16439×10^{-09}	3.16439×10^{-09}
(0.4,0.6)	2.61896×10^{-3}	5.15382×10^{-3}	2.54328×10^{-3}	1.00693×10^{-09}	1.00693×10^{-09}
(0.5,0.2)	3.84384×10^{-3}	9.86396×10^{-3}	4.83832×10^{-3}	1.00631×10^{-10}	1.00631×10^{-10}
(0.5,0.4)	3.58728×10^{-3}	2.84228×10^{-3}	2.84563×10^{-3}	2.12692×10^{-09}	2.12692×10^{-09}
(0.5,0.6)	3.35348×10^{-3}	4.72446×10^{-3}	2.52741×10^{-3}	1.00471×10^{-09}	1.00471×10^{-09}

Table 5. The suggested techniques result for $\psi(\upsilon, \tau)$ at various fractional-orders of Example 2.

(υ, τ)	$\psi(\upsilon, \tau)$ at ϱ =0.5	$\psi(\upsilon, \tau)$ at ϱ =0.75	$(NTDM_{ABC})$ at $\varrho = 1$	$(NTDM_{CF})$ at $\varrho = 1$	Exact result
(0.2,0.01)	-0.124902	-0.124896	-0.124892	-0.124892	-0.124892
(0.4,0.01)	-0.123585	-0.123568	-0.123553	-0.123553	-0.123553
(0.6,0.01)	-0.122297	-0.122292	-0.122280	-0.122280	-0.122280
(0.2,0.02)	-0.124908	-0.124898	-0.124892	-0.124892	-0.124892
(0.4,0.02)	-0.123597	-0.123578	-0.123553	-0.123553	-0.123553
(0.6,0.02)	-0.122299	-0.122293	-0.122280	-0.122280	-0.122280
(0.2,0.03)	-0.124909	-0.124899	-0.124892	-0.124892	-0.124892
(0.4,0.03)	-0.123586	-0.123568	-0.123553	-0.123553	-0.123553
(0.6,0.03)	-0.122299	-0.122289	-0.122280	-0.122280	-0.122280
(0.2,0.04)	-0.124908	-0.124896	-0.124892	-0.124892	-0.124892
(0.4,0.04)	-0.123589	-0.123576	-0.123553	-0.123553	-0.123553
(0.6,0.04)	-0.122296	-0.122288	-0.122280	-0.122280	-0.122280
(0.2,0.05)	-0.124905	-0.124897	-0.124892	-0.124892	-0.124892
(0.4,0.05)	-0.123578	-0.123564	-0.123553	-0.123553	-0.123553
(0.6,0.05)	-0.122298	-0.122290	-0.122280	-0.122280	-0.122280

Table 6. The suggested techniques result for $\varphi(\upsilon, \tau)$ at various fractional-orders of Example 2.

(υ, τ)	$\varphi(\upsilon, \tau)$ at ϱ =0.5	$\varphi(\upsilon, \tau)$ at ϱ =0.75	$(NTDM_{ABC})$ at $\varrho = 1$	$(NTDM_{CF})$ at $\varrho = 1$	Exact result
(0.2,0.01)	-0.006894	-0.006886	-0.006872	-0.006872	-0.006872
(0.4,0.01)	-0.006546	-0.006537	-0.006525	-0.006525	-0.006525
(0.6,0.01)	-0.006219	-0.006213	-0.006200	-0.006200	-0.006200
(0.2,0.02)	-0.006894	-0.006885	-0.006872	-0.006872	-0.006872
(0.4,0.02)	-0.006542	-0.006534	-0.006525	-0.006525	-0.006525
(0.6,0.02)	-0.006219	-0.006213	-0.006200	-0.006200	-0.006200
(0.2,0.03)	-0.006889	-0.006882	-0.006872	-0.006872	-0.006872
(0.4,0.03)	-0.006543	-0.006536	-0.006525	-0.006525	-0.006525
(0.6,0.03)	-0.006216	-0.006208	-0.006200	-0.006200	-0.006200
(0.2,0.04)	-0.006889	-0.006881	-0.006872	-0.006872	-0.006872
(0.4,0.04)	-0.006541	-0.006532	-0.006525	-0.006525	-0.006525
(0.6,0.04)	-0.006218	-0.006207	-0.006200	-0.006200	-0.006200
(0.2,0.05)	-0.006889	-0.006883	-0.006872	-0.006872	-0.006872
(0.4,0.05)	-0.006546	-0.006538	-0.006525	-0.006525	-0.006525
(0.6,0.05)	-0.006223	-0.006213	-0.006200	-0.006200	-0.006200

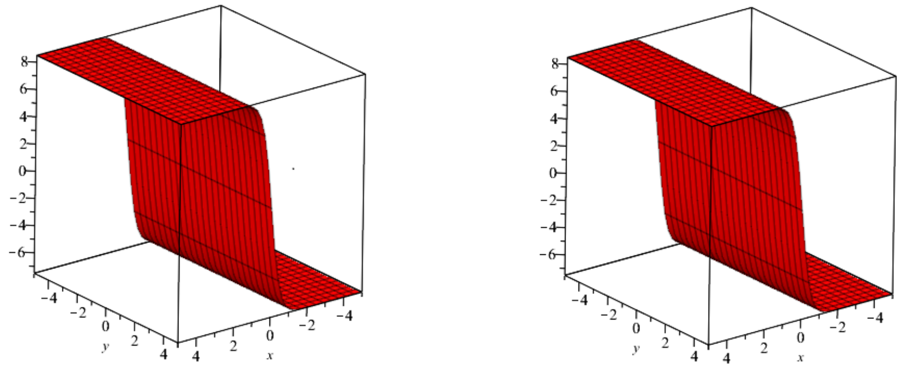

Figure 1. The actual and analytic solutions for $\psi(\upsilon, \tau)$ at $\varrho = 1$ of Example 1.

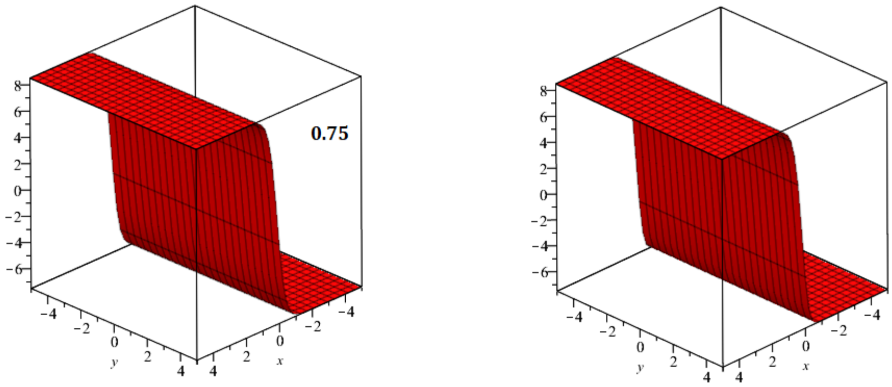

Figure 2. The analytic result for $\psi(\upsilon, \tau)$ at $\varrho = 0.8, 0.6$ of Example 1.

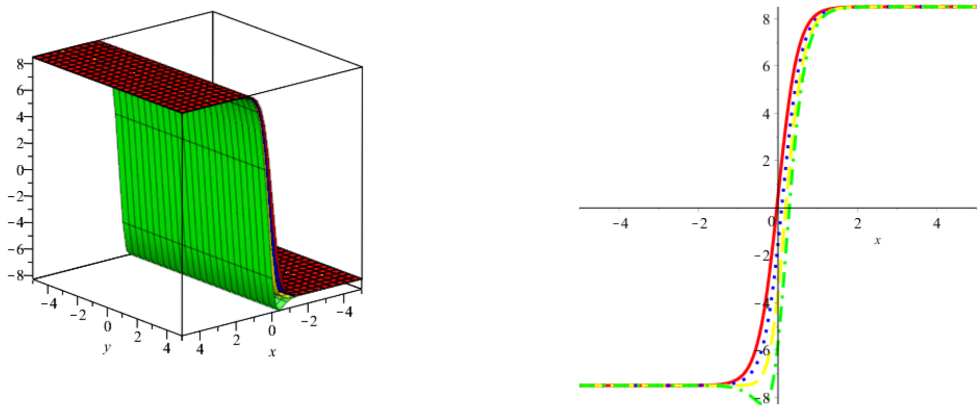

Figure 3. The analytic solution graph at different value of ϱ for $\psi(\upsilon, \tau)$ of Example 1.

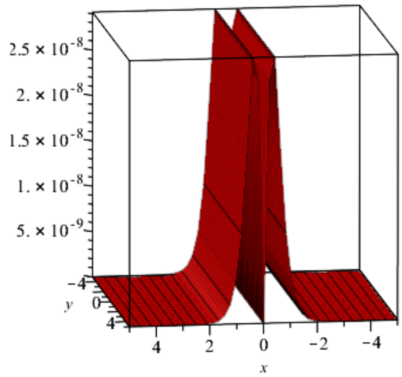

Figure 4. The absolute error for $\psi(\upsilon, \tau)$ of Example 1.

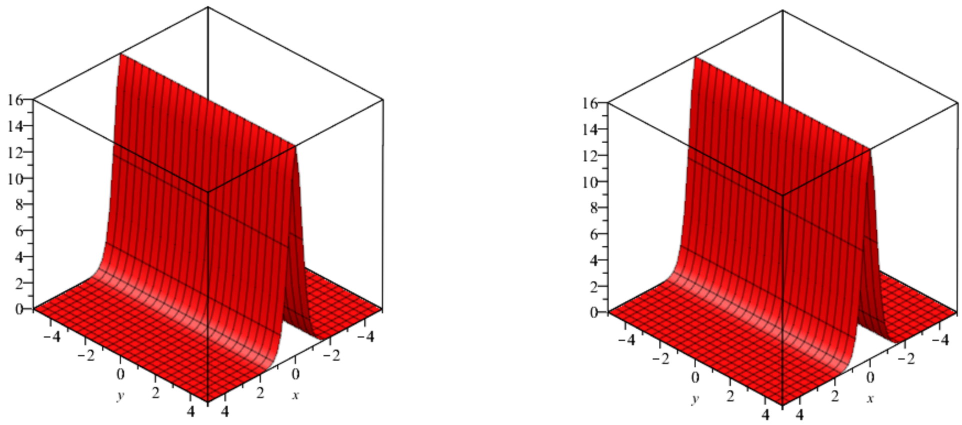

Figure 5. The actual and analytic result for $\varphi(\upsilon, \tau)$ at $\varrho = 1$ of Example 1.

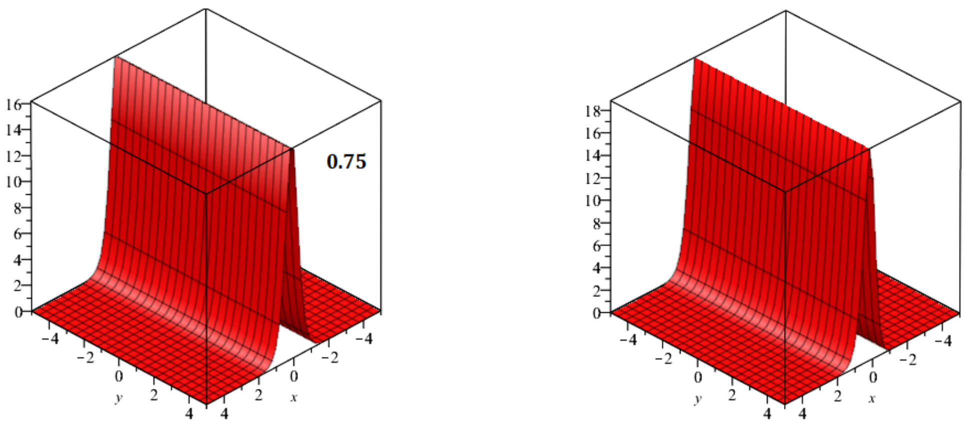

Figure 6. The analytic result graph for $\varphi(\upsilon, \tau)$ at $\varrho = 0.8, 0.6$ of Example 1.

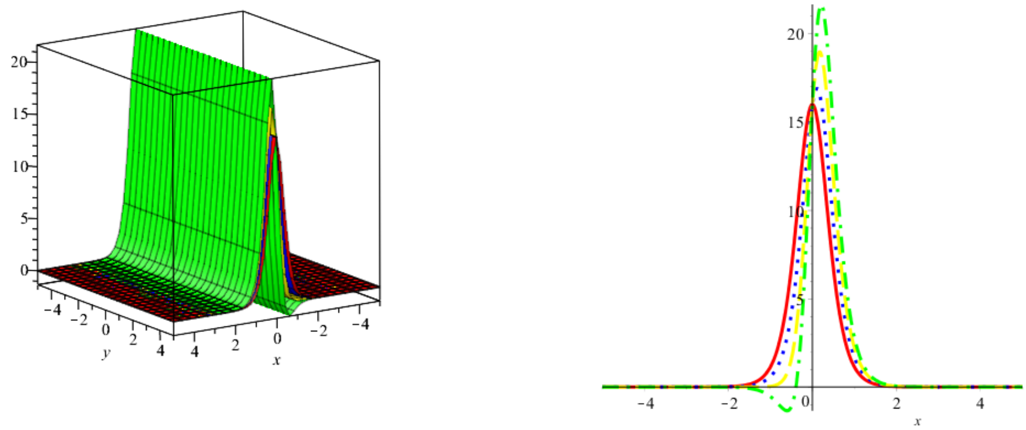

Figure 7. The analytic result graph at different value of ϱ for $\varphi(\upsilon, \tau)$ of Example 1.

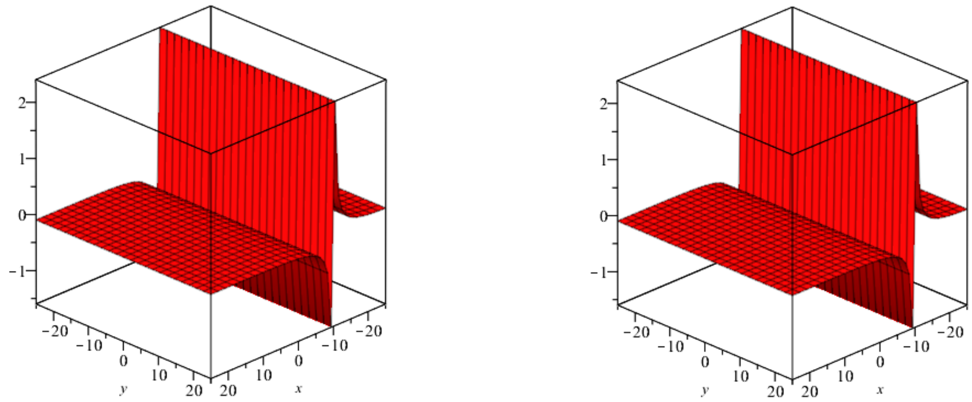

Figure 8. The exact and analytic solutions graph for $\psi(\upsilon, \tau)$ at $\varrho = 1$ of Example 2.

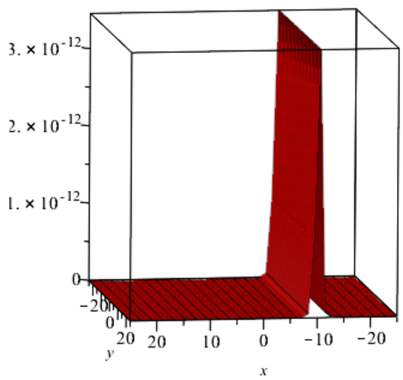

Figure 9. The absolute error for $\psi(\upsilon, \tau)$ of Example 2.

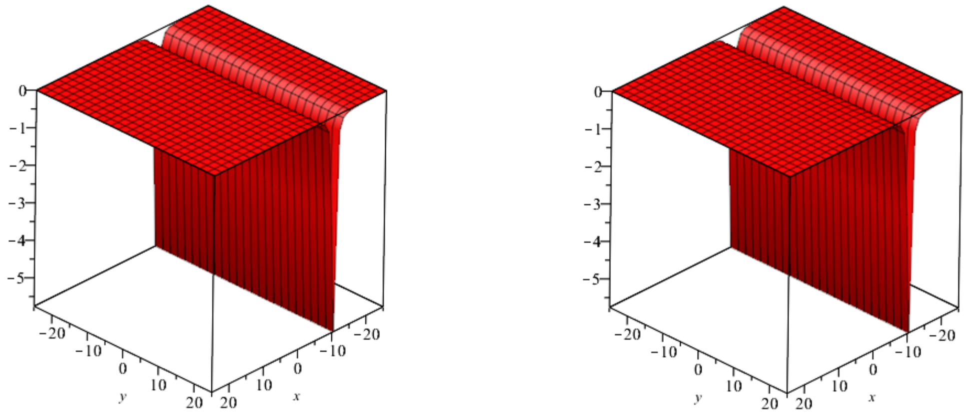

Figure 10. The exact and analytic solutions for $\varphi(\upsilon, \tau)$ at $\varrho = 1$ of Example 2.

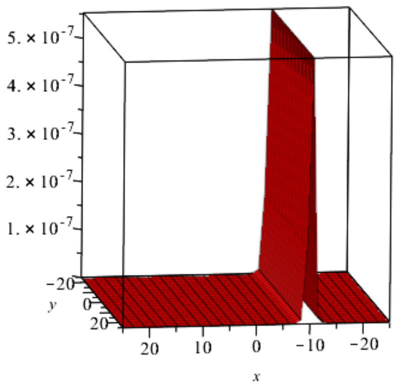

Figure 11. The absolute error for $\varphi(\upsilon, \tau)$ of Example 2.

7. Conclusions

The Atangana-Baleanu and Caputo-Fabrizio operators are used in this work to attempt a semi-analytic solution of fractional Whitham-Broer-Kaup equations. To show and confirm the efficiency of the recommended technique, two examples are solved. The numerical results show that the proposed method for solving time-fractional Whitham-Broer-Kaup equations is quite effective and accurate. According to numerical data, the method is very effective and reliable for getting close solutions for nonlinear fractional partial differential equations. Compared to other analytical methods, the proposed method is a quick and easy way to look into the numerical solution of nonlinear coupled systems of fractional partial differential equations. The proposed method gives solutions in the form of a series that is more accurate and take less time to figure out. The calculated results have been displayed graphically and in tables. For both considerably coupled systems, computations were done to determine the absolute error. Several computational solutions are contrasted with well-known analytical methods and the precise results at $\varrho = 1$. Fewer calculations and more precision are two advantages of the present approaches. Lastly, we can say that the proposed approaches are very effective and useful and that they can be used to study any nonlinear problems that come up in complex phenomena.

Conflict of interest

The authors declare that they have no competing interests.

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