Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(1): 2296–2307.
DOI: 10.3934/math.2023119
Received: 07 June 2022
Revised: 01 October 2022
Accepted: 06 October 2022
Published: 31 October 2022

Research article

Reducibility for a class of almost periodic Hamiltonian systems which are degenerate

Jia Li[1,*], Xia Li[2] and Chunpeng Zhu[1]

[1] School of Mathematics and Statistics, Xuzhou Institute of Technology, Xuzhou 221111, China
[2] Shandong Information Technology Industry Development Research Institute, Jinan 250014, China

* **Correspondence:** Email: lijia831112@163.com; Tel: +8615852179834.

Abstract: This paper studies the reducibility for a class of Hamiltonian almost periodic systems that are degenerate in a small perturbation parameter. We prove for most of the sufficiently small parameter, the Hamiltonian system is reducible by a symplectic almost periodic mapping.

Keywords: KAM; reducibility; Hamiltonian; almost periodic; degenerate
Mathematics Subject Classification: 37J40, 34C27

1. Introduction and main results

Definition 1.1. If $f(t) = L(\omega_1 t, \omega_2 t, \cdots, \omega_r t)$ with $\theta_j = \omega_j t$, $j = 1, 2\cdots, r$, and $L(\theta_1, \theta_2, \cdots, \theta_r)$ is 2π periodic with respect to all θ_j, we say a function f is quasi-periodic with frequencies $\omega = (\omega_1, \omega_2, \cdots, \omega_r)$. Further, if $L(\theta)$ $(\theta = (\theta_1, \theta_2, \cdots, \theta_r))$ is analytic on $D_\rho = \{\theta \in C^r |\ |Im\theta_j| \le \rho, j = 1, 2, \cdots, r\}$, we say $f(t)$ is analytic quasi-periodic on D_ρ. The norm of f on D_ρ is defined as $\|f\|_\rho = \sup_{\theta \in D_\rho} |L(\theta)|$.

Definition 1.2. If $p_{ij}(t)(i, j = 1, 2\cdots n)$ are all analytic quasi-periodic on D_ρ, we say a matrix function $P(t) = (p_{ij}(t))_{1\le i,j\le n}$ is analytic quasi-periodic on D_ρ.

Define the norm of the matrix P by $\| P \|_\rho = \max_{1\le i\le n} \sum_{j=1}^{n} \| p_{ij} \|_\rho$. Obviously, $\| P_1 P_2 \|_\rho \le \| P_1 \|_\rho \| P_2 \|_\rho$. For simplicity, if the matrix P is constant, denote $\| P \| = \| P \|_\rho$.

For almost periodic Hamiltonian systems, we use notations and definitions of finite spatial structure [1].

Definition 1.3. Assume τ is a family of subsets of N and N is a natural number set. If τ fulfills (i) $\cup_{\Lambda\in\tau}\Lambda = N$; (ii) if $\Lambda_1, \Lambda_2 \in \tau$, then $\Lambda_1 \cup \Lambda_2 \in \tau$; (iii) $\phi \in \tau$, where ϕ is an empty set, we say $(\tau,[\cdot])$ is a finite spatial structure. Moreover, $[\cdot]$ is called a weight function on τ if $[\phi]$=0 and $[\Lambda_1 \cup \Lambda_2] \le [\Lambda_1] + [\Lambda_2]$.

Definition 1.4. Assume $k=(k_1,k_2,\cdots)\in Z^{\infty}$. Define the support of k by supp$k=\{i\mid k_i\neq 0\}$. Denote $|k|=\sum\limits_{i=1}^{\infty}|k_i|$. The weight of its support is defined as $[k]=\inf_{suppk\subseteq\Lambda\in\tau}[\Lambda]$.

Definition 1.5. If $P(t)=\sum\limits_{\Lambda\in\tau}P_{\Lambda}(t)$, where $P_{\Lambda}(t)$ is a quasi-periodic matrix with frequencies $\omega_{\Lambda}=\{\omega_i|i\in\Lambda\}$, we say $P(t)$ is an almost periodic matrix with weighted spatial structure (τ,[$\cdot$]). In the context of integer modulus, frequencies ω of $Q(t)$ is the the biggest subset of $\bigcup\omega_{\Lambda}$.

Definition 1.6. Denote $P(t)=\sum\limits_{\Lambda\in\tau}P_{\Lambda}(t)$. When $m>0,\rho>0$, $|||P|||_{m,\rho}=\sum\limits_{\Lambda\in\tau}e^{m[\Lambda]}\|P_{\Lambda}(t)\|_{\rho}$ (see [1]) is defined as a weighted norm of $P(t)$. Clearly, for $m>0,\rho>0$, $\|P(t)\|_{\rho}\leq|||P(t)|||_{0,\rho}\leq|||P(t)|||_{m,\rho}$.

If the quasi-periodic equation

$$\dot{x}=B(t)x, x\in R^n, \tag{1.1}$$

by a non-sigular mapping $x=\psi(t)y$, where $\psi(t)^{-1}$ and $\psi(t)$ are bounded and quasi-periodic, (1.1) can become

$$\dot{y}=Cy$$

with the matrix C is constant, we call (1.1) is reducible. When the matrix $B(t)$ is periodic, the famous Floquent theorem tells us by a (double-)periodic transformation, $\dot{x}=B(t)x$ is reducible. However,for the quasi-periodic situation it is not true. Under some "full spectrum" conditions, the authors [2] obtained the quasi-periodic system (1.1) is reducible. For linear systems, the authors in [3] studied the quasi-periodic system

$$\dot{x}=(A+\varepsilon Q(t))x, x\in R^n, \tag{1.2}$$

where A is an $n\times n$ constant matrix with different eigenvalues $\lambda_1,\lambda_2,\cdots,\lambda_n$. If non-degeneracy conditions

$$\frac{d}{d\varepsilon}(\bar{\lambda}_i(\varepsilon)-\bar{\lambda}_j(\varepsilon))|_{\varepsilon=0}\neq 0,\ i\neq j, \tag{1.3}$$

and non-resonance conditions $|\ \langle k,\omega\rangle\sqrt{-1}+\lambda_i-\lambda_j\ |\geq\frac{\alpha_0}{|k|^{\tau}}$ are satisfied, where $\forall k\in Z^r\setminus\{0\}$, $\forall i,j=1,2,\cdots,n$, $\alpha_0>0$ is a small constant, $\tau>r-1$, $\bar{\lambda}_i(\varepsilon)(i=1,2,\cdots,n)$ are eigenvalues of $A+\varepsilon\bar{Q}$ and $\bar{Q}$ is the average of $Q(t)$, for $\varepsilon\in E$ with the nonempty Cantor subset E, (1.2) is reducible.

In [4], $\bar{\lambda}_i(\varepsilon)-\bar{\lambda}_j(\varepsilon)$ are called degenerate if non-degeneracy conditions (1.3) do not hold. The authors [4] considered this degenerate case. They proved a similar result under weaker non-degeneracy conditions .

Previously, the reducibility for analytic quasi-periodic systems were mainly considered. The finitely smooth case was considered in [5].

In KAM theorems, non-degeneracy conditions are always necessary. But when the hamiltonian system is two degrees of freedom, a special result [6] is obtained. Without any non-degeneracy condition, the authors [7] obtained the reducible result for the linear two-dimensional quasi-periodic system depending on a small parameter analytically. For the case that depends on the small parameter smoothly, there is a similar result [8]. Without any non-degeneracy condition, the authors [9] obtained the reducible result for the nonlinear two-dimensional quasi-periodic system. Recently, for the two dimensional almost periodic system, we also obtain a similar result in [10].

For nonlinear quasi-periodic systems, the authors [11] studied the following system

$$\dot{x}=(A+\varepsilon Q(t,\varepsilon))x+\varepsilon g(t,\varepsilon)+h(x,t,\varepsilon), \tag{1.4}$$

where the matrix A is constant and h is $O(x^2)$. If non-degeneracy conditions and non-resonance conditions are satisfies, using an analogous way as [3], the system (1.4) is reducible. When the system (1.4) becomes the hamiltonian system with multiple eigenvalues, we obtain an analogous result in [12].

In [13], under non-resonance and non-degeneracy conditions, Xu further considered the reducibility for the almost periodic system.

Motivated by [1, 4, 13], here we consider the reducibility for the higher dimensional Hamiltonian almost periodic system under weaker non-degeneracy conditions, which is called degenerate in [4].

Here non-resonance conditions are presented by so called approximation function. If $\Delta : [1, +\infty) \to [1, +\infty)$, $\Delta(1) = 1$,

$$\frac{\log \Delta(t)}{t} \searrow 0, \ 1 \leq t \to \infty,$$

and

$$\int_1^{\infty} \frac{\log \Delta(t)}{t^2} dt < +\infty,$$

we say an increasing function $\Delta(t)$ is an approximation function [1]. obviously, when $\Delta(t)$ is an approximation function, so is $\Delta^4(t)$.

Let

$$\Gamma(\varrho) = \sup_{t\geq 0}(\Delta^3(t)e^{-\varrho t}), \ \psi(\varrho) = \frac{1}{2} \inf_{\varrho_0+\varrho_1+\cdots\varrho_n+\cdots\leq\varrho} \prod_{\nu=1}^{\infty} (\Gamma(\varrho_\nu))^{(\frac{3}{2})^{-(\nu+1)}}.$$

There exists a sequence $\bar{\varrho}_1 \geq \bar{\varrho}_2 \geq \cdots \geq 0$, such that $\sum_{\nu=0}^{\infty} \bar{\varrho}_\nu = \varrho$ and $\psi(\varrho) = \frac{1}{2} \prod_{\nu=0}^{\infty} (\Gamma(\bar{\varrho}_\nu))^{(\frac{3}{2})^{-(\nu+1)}}$. For the details, see [1].

Suppose $\omega = (\omega_1, \omega_2, \cdots)$ is frequencies of $Q(t)$, $\lambda_1, \lambda_2, \cdots, \lambda_{2n}$ are the different eigenvalues of A, $\Delta(t)$ is an approximation function that fulfills

$$\sum_{k\in Z^\infty} \frac{1}{\Delta(|k|)\Delta([k])} < +\infty. \tag{1.5}$$

For Theorem 1.1 of this paper, non-resonance conditions are

$$|\langle k, \omega\rangle \sqrt{-1} - \lambda_i + \lambda_j| \geq \frac{\alpha}{\Delta(|k|)\Delta([k])}, \ \forall k \in Z^\infty \backslash \{0\}, \ i, j = 1, 2, \cdots, 2n. \tag{1.6}$$

Since [1], when we choose $\Delta(t)$ which satisfies (1.5) and $[\Lambda] = 1 + \sum_{i\in\Lambda} \log^r(1 + |i|)$ $(r > 2)$, there exists $\omega = (\omega_1, \omega_2, \cdots)$ [1] which fulfills non-resonance conditions (1.6). The following theorem is the main result of this paper.

Theorem 1.1. *Consider the linear Hamiltonian system*

$$\dot{x} = (A + \varepsilon Q(t, \varepsilon))x, \ x \in R^{2n}, \tag{1.7}$$

where $A = diag(\lambda_1, \lambda_2, \cdots, \lambda_{2n})$ *is a* $2n \times 2n$ *constant Hamiltonian matrix with* $\lambda_i \neq \lambda_j$, $i \neq j$, $1 \leq i, j \leq 2n$, *and* $\lambda_{p+n} = -\lambda_p$, $p = 1, 2, \cdots, n$. *Suppose a small parameter* $\varepsilon \in (0, \varepsilon_0)$, $Q(t, \varepsilon) = \sum_{\Lambda\in\tau} Q_\Lambda(t, \varepsilon)$ *is Hamiltonian analytic almost periodic in t with frequencies* $\omega = (\omega_1, \omega_2, \cdots)$ *on* D_ρ *and analytic in* ε.

Assume

(A_1) *(non-resonance conditions) The frequencies* $\omega = (\omega_1, \omega_2, \cdots)$ *satisfies*

$$| \langle k, \omega \rangle \sqrt{-1} - \lambda_i + \lambda_j | \geq \frac{\alpha}{\Delta(|k|)\Delta([k])} \tag{1.8}$$

for $\forall k \in Z^\infty \backslash \{0\}$, $1 \leq i, j \leq 2n$, *where* $\alpha > 0$ *is a small constant.*

(A_2) *Let* $\bar{q}_{ii}$ *be the average of* $q_{ii}(t)$ *and* $\bar{R}_0 = diag(\bar{q}_{11}, \bar{q}_{22}, \cdots, \bar{q}_{2n,2n})$. *Assume when* $j \neq i$, $\varepsilon(\bar{q}_{jj} - \bar{q}_{ii})$ *satisfies one of the following forms:*

$$\mu_1 \varepsilon^{l_1} + o(\varepsilon^{l_1}),\ \mu_2 \varepsilon^{l_2} + o(\varepsilon^{l_2}), \cdots, \mu_p \varepsilon^{l_p} + o(\varepsilon^{l_p}),$$

where $\mu_i \neq 0$, $i = 1, 2, \cdots, p$, $1 \leq l_1 < l_2 < \cdots < l_p$, *and* $o(\varepsilon^l)$ *is of order smaller than* ε^l *as* $\varepsilon \to 0$.

(A_3) *There exists* $m > 0$ *satisfying* $|||Q(t, \varepsilon)|||_{m,\rho} < +\infty$.

Then for $\varepsilon \in \tilde{E}$, *there exists an analytic symplectic almost periodic mapping* $x = \phi(t, \varepsilon)y$, *where* $\phi(t, \varepsilon)$ *and* $Q(t, \varepsilon)$ *have the same spatial structure and frequencies, such that (1.7) becomes the Hamiltonian system*

$$\dot{y} = A_\infty(\varepsilon)y,\ y \in R^{2n}, \tag{1.9}$$

where $\tilde{E} \subset (0, \varepsilon_0)$ *is a non-empty Cantor subset of positive Lebesgue measure satisfying meas*$((0, \varepsilon_0) \setminus \tilde{E}) = o(\varepsilon_0)$ *when* $\varepsilon_0 \to 0$, *and a constant matrix* A_∞ *is Hamiltonian.*

Remark 1: We understand the smoothness of the function in ε for Cantor set $\tilde{E}$ in the sense of Whitney [14].

Remark 2: Generally, Q depends on ε. Sometimes this dependence is not shown explicitly for simplicity.

Remark 3: If α is small enough and $\forall \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{2n})$ is given, by [1], there exists $\omega \in R^\infty$ satisfying (1.8).

Remark 4: Now the Hamiltonian system is

$$\dot{x} = JS(t, \varepsilon)x = (A + \varepsilon Q(t, \varepsilon))x,\ x \in R^{2n},$$

where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Since it is the Hamiltonian system, there exists a symmetric matrix $S(t, \varepsilon)$ such that $JS(t, \varepsilon) = A + \varepsilon Q(t, \varepsilon)$.

Remark 5: In [4], the degenerate case is also the condition (A_2). However, it is the quasi-periodic case for [4] and it is the almost periodic case for this paper.

2. The lemmas

To prove Theorem 1.1, in this section we formulate some lemmas which will be used in the next section. Below $c > 0$ indacate a constant.

Lemma 2.1. *Suppose* $D(t)$ *and* $G(t)$ *are almost periodic matrices with the same spatial structure and frequencies. If* $|||D(t)|||_{m,\rho}, |||G(t)|||_{m,\rho} < +\infty$, *then DG is also an almost periodic matrix with the same spatial structure and frequencies as D and G. Furthermore,* $|||DG|||_{m,\rho} \leq |||D|||_{m,\rho} |||G|||_{m,\rho}$.

The proof can be seen in [13]. □

To solve the transformation equation, we give the following lemma.

Lemma 2.2. *Consider the equation*

$$\dot{P} = AP - PA + Q(t), \tag{2.1}$$

where $A = diag(\lambda_1, \lambda_2, \cdots, \lambda_{2n})$, $|\lambda_l - \lambda_m| \geq \mu$ *with a constant* $\mu > 0$, $l \neq m$, $1 \leq l, m \leq 2n$, *and* $\lambda_{i+n} = -\lambda_i$, $1 \leq i \leq n$. *Suppose* $Q(t) = (q_{ij}(t))_{1\leq i,j\leq 2n} = \sum_{\Lambda\in\tau} Q_\Lambda(t)$ *is analytic Hamiltonian almost periodic in t with frequencies* $\omega = (\omega_1, \omega_2, \cdots)$ *on* D_ρ *with finite spatial structure* $(\tau,[\cdot])$. *Suppose* $\bar{q}_{ii} = 0$, $i = 1, 2, \cdots, 2n$, *where* $\bar{q}_{ii}$ *is the average of* $q_{ii}(t)$ *in t. Assume*

$$|\langle k, \omega\rangle \sqrt{-1} - \lambda_i + \lambda_j| \geq \frac{\alpha}{\Delta^3(|k|)\Delta^3([k])}, \tag{2.2}$$

for $\forall k \in Z^\infty\backslash\{0\}$, $1 \leq i, j \leq 2n$. *Then there exists a unique analytic Hamiltonian almost periodic solution* $P(t)$ *with the same frequencies and spatial structure as* $Q(t)$, *and* $|||P|||_{m-\bar{m},\rho-\bar{\rho}} \leq \frac{c\Gamma(\bar{m})\Gamma(\bar{\rho})}{\alpha}|||Q|||_{m,\rho}$, $|||\varepsilon\frac{\partial P}{\partial\varepsilon}|||_{m-\bar{m},\rho-\bar{\rho}} \leq \frac{c\Gamma^2(\frac{\bar{m}}{2})\Gamma^2(\frac{\bar{\rho}}{2})}{\alpha^2}(|||Q|||_{m,\rho} + |||\varepsilon\frac{\partial Q}{\partial\varepsilon}|||_{m,\rho})$, *where* $\Gamma(\varrho) = \sup_{t\geq 0}(\Delta^3(t)e^{-\varrho t})$, $0 < \bar{m} < m$, $0 < \bar{\rho} < \rho$.

Proof. Now we need solve the equation

$$\dot{P}_\Lambda = AP_\Lambda - P_\Lambda A + Q_\Lambda, \tag{2.3}$$

Let

$$Q_\Lambda = (q^{ij}_\Lambda),\ q^{ij}_\Lambda = \sum_{\text{supp}k\subseteq\Lambda} q^{ij}_{\Lambda k} e^{\langle k,w\rangle\sqrt{-1}t},$$

$$P_\Lambda = (p^{ij}_\Lambda),\ p^{ij}_\Lambda = \sum_{\text{supp}k\subseteq\Lambda} p^{ij}_{\Lambda k} e^{\langle k,w\rangle\sqrt{-1}t},$$

Comparing the coefficients of (2.3), it follows $p^{ii}_{\Lambda 0} = 0$; or else,

$$p^{ij}_{\Lambda k} = \frac{q^{ij}_{\Lambda k}}{\langle k, \omega\rangle\sqrt{-1} - \lambda_i + \lambda_j}.$$

Then we have

$$\|p^{ij}_\Lambda\|_{\rho-\bar{\rho}} \leq \sum_{\text{supp}k\subseteq\Lambda} \frac{\Delta^3(|k|)e^{-\bar{\rho}|k|}}{\alpha}\Delta^3([k])\|q^{ij}_\Lambda\|_\rho.$$

So

$$\|P_\Lambda\|_{\rho-\bar{\rho}} \leq \frac{c\Gamma(\bar{\rho})\Delta^3([\Lambda])}{\alpha} \parallel Q_\Lambda \parallel_\rho . \tag{2.4}$$

Denote $P = \sum_{\Lambda\in\tau} P_\Lambda$. Since (2.4), it follows

$$\begin{aligned} |||P|||_{m-\bar{m},\rho-\bar{\rho}} &= \sum_{\Lambda\in\tau} \|P_\Lambda\|_{\rho-\bar{\rho}} e^{(m-\bar{m})[\Lambda]} \\ &\leq \sum_{\Lambda\in\tau} \frac{c\Gamma(\bar{\rho})\Delta^3([\Lambda])}{\alpha} \parallel Q_\Lambda \parallel_\rho e^{(m-\bar{m})[\Lambda]} \end{aligned}$$

$$\leq \frac{c\Gamma(\bar{\rho})\Gamma(\bar{m})}{\alpha} \sum_{\Lambda \in \tau} \| Q_\Lambda \|_\rho e^{m[\Lambda]}$$
$$= \frac{c\Gamma(\bar{m})\Gamma(\bar{\rho})}{\alpha} |||Q|||_{m,\rho}.$$

Let us estimate $\| \varepsilon \frac{\partial P}{\partial \varepsilon} \|_{m-\bar{m},\rho-\bar{\rho}}$. Moreover, $\frac{dp^{ii}_{\Lambda 0}(\varepsilon)}{d\varepsilon} = 0$, and

$$\frac{dp^{ij}_{\Lambda k}}{d\varepsilon} = \frac{-(\frac{d\lambda_j(\varepsilon)}{d\varepsilon} - \frac{d\lambda_i(\varepsilon)}{d\varepsilon})q^{ij}_{\Lambda k} + (\langle k, \omega\rangle \sqrt{-1} - \lambda_i + \lambda_j)\frac{dq^{ij}_{\Lambda k}(\varepsilon)}{d\varepsilon}}{(\langle k, \omega\rangle \sqrt{-1} - \lambda_i + \lambda_j)^2} \text{ for } |i-j| + |k| \neq 0.$$

Then it follows

$$\|\varepsilon \frac{\partial p^{ij}_\Lambda}{\partial \varepsilon}\|_{\rho-\bar{\rho}} \leq \sum_{\mathrm{supp} k \subseteq \Lambda} (\frac{c\Delta^6(|k|)e^{-\bar{\rho}|k|}}{\alpha^2}\Delta^6([k])\|q^{ij}_\Lambda\|_\rho + \frac{\Delta^3(|k|)e^{-\bar{\rho}|k|}}{\alpha}\Delta^3([k])\|\varepsilon\frac{\partial q^{ij}_\Lambda}{\partial\varepsilon}\|_\rho)$$
$$\leq \frac{c\Gamma^2(\frac{\bar{\rho}}{2})}{\alpha^2}\Delta^6([\Lambda])(\|q^{ij}_\Lambda\|_\rho + \|\varepsilon\frac{\partial q^{ij}_\Lambda}{\partial \varepsilon}\|_\rho).$$

So

$$\|\varepsilon \frac{\partial P_\Lambda}{\partial \varepsilon}\|_{\rho-\bar{\rho}} \leq \frac{c\Gamma^2(\frac{\bar{\rho}}{2})}{\alpha^2}\Delta^6([\Lambda])(\|Q_\Lambda\|_\rho + \|\varepsilon\frac{\partial Q_\Lambda}{\partial\varepsilon}\|_\rho).$$

Then

$$|||\varepsilon\frac{\partial P}{\partial\varepsilon}|||_{m-\bar{m},\rho-\bar{\rho}} = \sum_{\Lambda\in\tau}\|\varepsilon\frac{\partial P_\Lambda}{\partial\varepsilon}\|_{\rho-\bar{\rho}}e^{(m-\bar{m})[\Lambda]}$$
$$\leq \sum_{\Lambda\in\tau}\frac{c\Gamma^2(\frac{\bar{\rho}}{2})\Delta^6([\Lambda])}{\alpha^2}(\|Q_\Lambda\|_\rho + \|\varepsilon\frac{\partial Q_\Lambda}{\partial\varepsilon}\|_\rho)e^{(m-\bar{m})[\Lambda]}$$
$$\leq \frac{c\Gamma^2(\frac{\bar{\rho}}{2})\Gamma^2(\frac{\bar{m}}{2})}{\alpha^2}\sum_{\Lambda\in\tau}(\|Q_\Lambda\|_\rho + \|\varepsilon\frac{\partial Q_\Lambda}{\partial\varepsilon}\|_\rho)e^{m[\Lambda]}$$
$$= \frac{c\Gamma^2(\frac{\bar{m}}{2})\Gamma^2(\frac{\bar{\rho}}{2})}{\alpha^2}(|||Q|||_{m,\rho} + |||\varepsilon\frac{\partial Q}{\partial\varepsilon}|||_{m,\rho}).$$

Moreover, by A and Q are Hamiltonian, it follows $Q = JQ_J$ and $A = JA_J$, where Q_J and A_J are symmetric. Denote $P_J = J^{-1}P$. Below we prove P_J is symmetric. (2.1) becomes

$$\dot{P}_J = A_J J P_J - P_J J A_J + Q_J. \tag{2.5}$$

(2.5) becomes

$$(\dot{P}_J)^T = A_J J (P_J)^T - (P_J)^T J A_J + Q_J.$$

By the solution of (2.5) is unique, it follows $(P_J) = (P_J)^T$. So P is Hamiltonian. □

The following lemma is used for the estimate of the measure.

Lemma 2.3. *Assume*

$$|\langle k,\omega\rangle-\chi|\geq\frac{\alpha}{\Delta(|k|)\Delta([k])},\ \forall\, k\in Z^{\infty}\backslash\{0\},$$

where $\chi\in R$. *Let* $\tilde{\alpha}\leq\frac{\alpha}{2}$, $\sigma\neq 0$, *and*

$$O=\{\varepsilon\in(0,\varepsilon_0)\ \big|\ |\langle k,\omega\rangle-(\chi+\sigma\varepsilon^q+\varepsilon^q g(\varepsilon))|\geq\frac{\tilde{\alpha}}{\Delta^3(|k|)\Delta^3([k])},\ \forall k\neq 0\},$$

where $q\in Z^+$ *and* $\Delta(t)$ *is an approximation function that fulfills (1.5). Suppose* $g(\varepsilon)$ *fulfills* $|g'(\varepsilon)|\leq c$ *for* $\varepsilon\in(0,\varepsilon_0)$, *and* $g(\varepsilon)\to 0$ *when* $\varepsilon\to 0$. *If* ε_0 *is small enough, it follows*

$$meas((0,\varepsilon_0)\setminus O)\leq\frac{c\tilde{\alpha}}{\alpha}\varepsilon_0^{q+1}.$$

Proof. Denote $\varphi(\varepsilon)=\langle k,\omega\rangle-(\chi+\sigma\varepsilon^q+\varepsilon^q g(\varepsilon))$ and fix $k\neq 0$. Let

$$I_k=\{\varepsilon\in(0,\varepsilon_*)\ \big|\ |\varphi(\varepsilon)|<\frac{\tilde{\alpha}}{\Delta^3(|k|)\Delta^3([k])}\}.$$

We first consider the case $\varepsilon^q\leq\frac{\alpha}{4|\sigma|\Delta(|k|)\Delta([k])}$. If $\varepsilon^q\leq\frac{\alpha}{4|\sigma|\Delta(|k|)\Delta([k])}$, it follows $|\sigma\varepsilon^q+\varepsilon^q g(\varepsilon)|\leq\frac{\alpha}{2\Delta(|k|)\Delta([k])}$. So

$$|\varphi(\varepsilon)|\geq\frac{\alpha}{\Delta(|k|)\Delta([k])}-\frac{\alpha}{2\Delta(|k|)\Delta([k])}\geq\frac{\tilde{\alpha}}{\Delta^3(|k|)\Delta^3([k])}.$$

Thus, we only consider the case $\varepsilon_0^q\geq\varepsilon^q\geq\frac{\alpha}{4|\sigma|\Delta(|k|)\Delta([k])}$. So

$$\frac{1}{\Delta(|k|)\Delta([k])}\leq\frac{4|\sigma|\varepsilon_0^q}{\alpha}. \tag{2.6}$$

For ε_0 sufficiently small, we get

$$|\frac{d\varphi}{d\varepsilon}(\varepsilon)|\geq\frac{|\sigma|}{2}\varepsilon^{q-1}\geq\frac{\alpha}{8\Delta(|k|)\Delta([k])\varepsilon_0}. \tag{2.7}$$

By (2.6) and (2.7), we have

$$\begin{aligned}\text{meas}(I_k)&\leq\frac{2\tilde{\alpha}}{\Delta^3(|k|)\Delta^3([k])}\frac{8\Delta(|k|)\Delta([k])\varepsilon_0}{\alpha}\\&=\frac{16\tilde{\alpha}\varepsilon_0}{\alpha}\frac{4|\sigma|\varepsilon_0^q}{\alpha}\frac{1}{\Delta(|k|)\Delta([k])}.\end{aligned}$$

Then since (1.5), it follows

$$\begin{aligned}\text{meas}((0,\varepsilon_0)\backslash O)&\leq\sum_{k\in Z^{\infty}}\text{meas}(I_k)\\&\leq\frac{c\tilde{\alpha}}{\alpha}\varepsilon_0\varepsilon_0^q\sum_{k\in Z^{\infty}}\frac{1}{\Delta(|k|)\Delta([k])}\\&\leq\frac{c\tilde{\alpha}}{\alpha}\varepsilon_0^{q+1}.\end{aligned}$$

□

The following lemma is used for the convergance of KAM iteration.

Lemma 2.4. *([11]) A sequence $\{\eta_\nu\}$ satisfies*

$$\eta_{\nu+1} \leq (\bar{\gamma} z^\nu)^{\bar{\gamma} z^\nu} \eta_\nu^2, \ \forall \nu \geq 0,$$

where η_ν are all positive real numbers, $1 < z < 2$ and $\bar{\gamma} > 0$. It follows that

$$\eta_\nu \leq [(\bar{\gamma} z^{\frac{z}{2-z}})^{\frac{\bar{\gamma}}{2-z}} \eta_0]^{2^\nu}.$$

This Lemma is used for the convergence of KAM iteration. □

3. Proof of Theorem 1.1

KAM-step. At ν-th step, consider the Hamiltonian system

$$\dot{x}_\nu = (A_\nu + \varepsilon^{2^\nu} Q_\nu(t, \varepsilon)) x_\nu, \ \nu \geq 0, \tag{3.1}$$

where $A_0 = A$, $Q_0 = Q$, $A_\nu = \mathrm{diag}(\lambda_1^\nu, \lambda_2^\nu, \cdots, \lambda_{2n}^\nu)$, $|\lambda_i^\nu - \lambda_j^\nu| \geq \mu > 0$, $i \neq j$, $1 \leq i, j \leq 2n$, $\lambda_{d+n} = -\lambda_d$, $1 \leq d \leq n$, and Q_ν is almost periodic on D_{ρ_ν}.

Let $Q_\nu = (q_{ij}^\nu)_{1 \leq i,j \leq 2n}$ $R_\nu = \mathrm{diag}(q_{11}^\nu, q_{22}^\nu, \cdots, q_{2n,2n}^\nu)$, $R_0 = \mathrm{diag}(q_{11}, q_{22}, \cdots, q_{2n,2n})$. Denote the average of R_ν by $\bar{R}_\nu = \mathrm{diag}(\bar{q}_{11}^\nu, \bar{q}_{22}^\nu, \cdots, \bar{q}_{2n,2n}^\nu)$. Hamiltonian system (3.1) becomes

$$\dot{x}_\nu = (A_{\nu+1} + \varepsilon^{2^\nu} \widetilde{Q}_\nu(t, \varepsilon)) x_\nu, \tag{3.2}$$

where $A_{\nu+1} = A_\nu + \varepsilon^{2^\nu} \bar{R}_\nu = \mathrm{diag}(\lambda_1^{\nu+1}, \lambda_2^{\nu+1}, \cdots, \lambda_{2n}^{\nu+1})$ and $\widetilde{Q}_\nu = Q_\nu - \bar{R}_\nu$.

We now make the symplectic mapping $x_\nu = e^{\varepsilon^{2^\nu} P_\nu(t)} x_{\nu+1}$ to (3.2) to obtain

$$\begin{aligned} \dot{x}_{\nu+1} = &(e^{-\varepsilon^{2^\nu} P_\nu} (A_{\nu+1} + \varepsilon^{2^\nu} \widetilde{Q}_\nu - \varepsilon^{2^\nu} \dot{P}_\nu) e^{\varepsilon^{2^\nu} P_\nu} \\ &+ e^{-\varepsilon^{2^\nu} P_\nu} (\varepsilon^{2^\nu} \dot{P}_\nu e^{\varepsilon^{2^\nu} P_\nu(t)} - \frac{d}{dt}(e^{\varepsilon^{2^\nu} P_\nu(t)}))) x_{\nu+1}, \end{aligned} \tag{3.3}$$

Expand $e^{\varepsilon^{2^\nu} P_\nu}$ and $e^{-\varepsilon^{2^\nu} P_\nu}$ into $e^{\varepsilon^{2^\nu} P_\nu} = I + \varepsilon^{2^\nu} P_\nu + B_\nu$ and $e^{-\varepsilon^{2^\nu} P_\nu} = I - \varepsilon^{2^\nu} P_\nu + \widetilde{B}_\nu$, where $B_\nu = \frac{(\varepsilon^{2^\nu} P_\nu)^2}{2!} + \frac{(\varepsilon^{2^\nu} P_\nu)^3}{3!} + \cdots$ and $\widetilde{B}_\nu = \frac{(\varepsilon^{2^\nu} P_\nu)^2}{2!} - \frac{(\varepsilon^{2^\nu} P_\nu)^3}{3!} + \cdots$. (3.3) becomes

$$\begin{aligned} \dot{x}_{\nu+1} &= ((I - \varepsilon^{2^\nu} P_\nu + \widetilde{B}_\nu)(A_{\nu+1} + \varepsilon^{2^\nu} \widetilde{Q}_\nu - \varepsilon^{2^\nu} \dot{P}_\nu)(I + \varepsilon^{2^\nu} P_\nu + B_\nu) \\ &\quad + e^{-\varepsilon^{2^\nu} P_\nu} (\varepsilon^{2^\nu} \dot{P}_\nu e^{\varepsilon^{2^\nu} P_\nu} - \frac{d}{dt}(e^{\varepsilon^{2^\nu} P_\nu(t)}))) x_{\nu+1} \\ &= (A_{\nu+1} + \varepsilon^{2^\nu} \widetilde{Q}_\nu - \varepsilon^{2^\nu} \dot{P}_\nu + \varepsilon^{2^\nu} A_{\nu+1} P_\nu - \varepsilon^{2^\nu} P_\nu A_{\nu+1} + Q_\nu^{(1)}) x_{\nu+1}, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} Q_\nu^{(1)} = &-\varepsilon^{2^{\nu+1}} P_\nu (\widetilde{Q}_\nu - \dot{P}_\nu) + \varepsilon^{2^{\nu+1}} (\widetilde{Q}_\nu - \dot{P}_\nu) P_\nu - \varepsilon^{2^{\nu+1}} P_\nu (A_{\nu+1} + \varepsilon^{2^\nu} \widetilde{Q}_\nu - \varepsilon^{2^\nu} \dot{P}_\nu) \\ &P_\nu - \varepsilon^{2^\nu} P_\nu (A_{\nu+1} + \varepsilon^{2^\nu} \widetilde{Q}_\nu - \varepsilon^{2^\nu} \dot{P}_\nu) B_\nu + (A_{\nu+1} + \varepsilon^{2^\nu} \widetilde{Q}_\nu - \varepsilon^{2^\nu} \dot{P}_\nu) B_\nu \\ &+ \widetilde{B}_\nu (A_{\nu+1} + \varepsilon^{2^\nu} \widetilde{Q}_\nu - \varepsilon^{2^\nu} \dot{P}_\nu) e^{\varepsilon^{2^\nu} P_\nu} + e^{-\varepsilon^{2^\nu} P_\nu} (\varepsilon^{2^\nu} \dot{P}_\nu e^{\varepsilon^{2^\nu} P_\nu} - \frac{d}{dt} e^{\varepsilon^{2^\nu} P_\nu}). \end{aligned}$$

We want to have that

$$\widetilde{Q}_v - \dot{P}_v + A_{v+1}P_v - P_vA_{v+1} = 0;$$

that is,

$$\dot{P}_v = A_{v+1}P_v - P_vA_{v+1} + \widetilde{Q}_v. \tag{3.5}$$

Clearly, A_{v+1} and $\widetilde{Q}_v$ are Hamiltonian. Since Lemma 2.2, if

$$|< k, \omega > \sqrt{-1} - \lambda_i^{v+1} + \lambda_j^{v+1} | \geq \frac{\alpha_{v+1}}{\Delta^3(|k|)\Delta^3([k])}, \tag{3.6}$$

for $\forall k \in Z^\infty \backslash \{0\}$, where $1 \leq i, j \leq 2n$, $\alpha_{v+1} = \frac{\alpha}{(v+1)^2}$, for the Eq (3.5), we find an unique analytic almost periodic Hamiltonian solution $P_v(t)$ with frequencies ω, and

$$|||P_v|||_{m_v-\bar{m}_v,\rho_v-\bar{\rho}_v} \leq \frac{c\Gamma(\bar{m}_v)\Gamma(\bar{\rho}_v)}{\alpha_{v+1}}|||Q_v|||_{m_v,\rho_v},$$

$$|||\varepsilon\frac{\partial P_v}{\partial \varepsilon}|||_{m_v-\bar{m}_v,\rho_v-\bar{\rho}_v} \leq \frac{c\Gamma^2(\frac{\bar{m}_v}{2})\Gamma^2(\frac{\bar{\rho}_v}{2})}{\alpha_{v+1}^2}(|||Q_v|||_{m_v,\rho_v} + |||\varepsilon\frac{\partial Q_v}{\partial \varepsilon}|||_{m_v,\rho_v}), \tag{3.7}$$

where $\Gamma(\rho) = \sup_{t\geq 0}(\Delta^3(t)e^{-\rho t})$, $0 < \bar{m}_v < m_v$, $0 < \bar{\rho}_v < \rho_v$.

Now (3.4) is changed to the Hamiltonian system

$$\dot{x}_{v+1} = (A_{v+1} + \varepsilon^{2^{v+1}}Q_{v+1}(t, \varepsilon))x_{v+1}, \tag{3.8}$$

where $\varepsilon^{2^{v+1}}Q_{v+1} = Q_v^{(1)}$.

Since $\widetilde{Q}_v - \dot{P}_v = P_vA_{v+1} - A_{v+1}P_v$, it follows

$$\begin{aligned}
\varepsilon^{2^{v+1}}Q_{v+1} = Q_v^{(1)} &= -\varepsilon^{2^{v+1}}P_v(P_vA_{v+1} - A_{v+1}P_v) + \varepsilon^{2^{v+1}}(P_vA_{v+1} - A_{v+1}P_v)P_v \\
&- \varepsilon^{2^{v+1}}P_v(A_{v+1} + \varepsilon^{2^v}P_vA_{v+1} - \varepsilon^{2^v}A_{v+1}P_v)P_v \\
&- \varepsilon^{2^v}P_v(A_{v+1} + \varepsilon^{2^v}P_vA_{v+1} - \varepsilon^{2^v}A_{v+1}P_v)B_v \\
&+ (A_{v+1} + \varepsilon^{2^v}P_vA_{v+1} - \varepsilon^{2^v}A_{v+1}P_v)B_v \\
&+ \widetilde{B}_v(A_{v+1} + \varepsilon^{2^v}P_vA_{v+1} - \varepsilon^{2^v}A_{v+1}P_v)e^{\varepsilon^{2^v}P_v} \\
&+ e^{-\varepsilon^{2^v}P_v}(\varepsilon^{2^v}\dot{P}_ve^{\varepsilon^{2^v}P_v} - \frac{d}{dt}e^{\varepsilon^{2^v}P_v}).
\end{aligned} \tag{3.9}$$

Under the symplectic mapping $x_v = e^{\varepsilon^{2^v}P_v}x_{v+1}$, Hamiltonian system (3.1) becomes Hamiltonian system (3.8).

KAM iteration. Let us prove the convergence of KAM iteration when $v \to \infty$. At v-th step, define $\alpha_{v+1} = \frac{\alpha}{(v+1)^2}, \rho_0 = \tilde{\rho}, m_0 = \tilde{m}, \bar{m}_v \searrow 0, \bar{\rho}_v \searrow 0, \sum_{v=0}^{\infty}\bar{m}_v = \frac{1}{2}\tilde{m}, \sum_{v=0}^{\infty}\bar{\rho}_v = \frac{1}{2}\tilde{\rho}, m_{v+1} = m_v - \bar{m}_v, \rho_{v+1} = \rho_v - \bar{\rho}_v$. Let $|||\cdot|||_v = |||\cdot|||_{m_v,\rho_v}$. Since Lemma A.1 of [1], there exists a constant $H > 0$, which satisfies

$$\Gamma(\bar{m}_v)\Gamma(\bar{\rho}_v), \Gamma^2(\frac{\bar{m}_v}{2})\Gamma^2(\frac{\bar{\rho}_v}{2}) \leq H^{(\frac{3}{2})^v}. \tag{3.10}$$

By (3.7) and (3.9), if $|||\varepsilon^{2^v}P_v|||_v \leq \frac{1}{2}$, it follows

$$|||Q_{v+1}|||_{v+1} \leq \frac{c\Gamma^2(\bar{m}_v)\Gamma^2(\bar{\rho}_v)}{\alpha_{v+1}^2}|||Q_v|||_v^2,$$

$$|||\varepsilon\frac{\partial Q_{\nu+1}}{\partial\varepsilon}|||_{\nu+1}\leq\frac{c\Gamma^4(\frac{\bar{m}_\nu}{2})\Gamma^4(\frac{\bar{\rho}_\nu}{2})}{\alpha_{\nu+1}^4}(|||Q_\nu|||_\nu+|||\varepsilon\frac{\partial Q_\nu}{\partial\varepsilon}|||_\nu)^2. \tag{3.11}$$

Denote $\eta_\nu = |||Q_\nu|||_\nu + |||\varepsilon\frac{\partial Q_\nu}{\partial\varepsilon}|||_\nu$. Since (3.10) and (3.11), there exists $1 < z < 2$ satisfying $\eta_{\nu+1} \leq (\bar{\gamma}z^\nu)^{\bar{\gamma}z^\nu}\eta_\nu^2$. Since Lemma 2.4, it follows $\eta_\nu \leq M^{2^\nu}$ with a constant $M > 0$. Thus,

$$|||Q_\nu|||_\nu, |||\varepsilon\frac{\partial Q_\nu}{\partial\varepsilon}|||_\nu \leq M^{2^\nu}. \tag{3.12}$$

If $0 < M\varepsilon < 1$, by (3.12) , it follows that

$$\lim_{\nu\to\infty}\varepsilon^{2^\nu}Q_\nu = 0.$$

By (3.2) , it follows that

$$\|A_{\nu+1}-A_\nu\| \leq c\varepsilon^{2^\nu}|||Q_\nu|||_\nu \leq (\varepsilon M)^{2^\nu}.$$

When $0 < M\varepsilon < 1$, A_ν is convergent as $\nu \to \infty$. Denote

$$\lim_{\nu\to\infty}A_\nu = A_\infty.$$

By (3.7), (3.10) and (3.12), we have

$$|||P_\nu|||_\nu \leq c^{2^\nu}. \tag{3.13}$$

Thus, there exists a symplectic mapping $x = \phi(t,\varepsilon)y$, such that Hamiltonian system (1.7) becomes Hamiltonian system (1.9).

Estimate of measure. Below let us prove if ε_0 is sufficiently small, for most $\varepsilon \in (0,\varepsilon_0)$, non-resonance conditions

$$|\langle k,\omega\rangle\sqrt{-1}-\lambda_i^{\nu+1}+\lambda_j^{\nu+1}|\geq\frac{\alpha_{\nu+1}}{\Delta^3(|k|)\Delta^3([k])} \tag{3.14}$$

hold, where $0 \neq k \in Z^\infty$, $\nu = 0,1,2\cdots$, and $i,j = 1,2,\cdots,2n$. Denote

$$E_{\nu+1} = \{\varepsilon\in(0,\varepsilon_0)\ \Big|\ |\langle k,\omega\rangle\sqrt{-1}-\lambda_i^{\nu+1}+\lambda_j^{\nu+1}|\geq\frac{\alpha_{\nu+1}}{\Delta^3(|k|)\Delta^3([k])},$$
$$\forall k\in Z^\infty\backslash\{0\},\ i,j=1,2,\cdots,2n\}.$$

If $i = j$, since (1.8), (3.14) holds. So we merely need prove for most $\varepsilon \in (0,\varepsilon_0)$,

$$|\langle k,\omega\rangle\sqrt{-1}-\lambda_i^{\nu+1}+\lambda_j^{\nu+1}|\geq\frac{\alpha_{\nu+1}}{\Delta^3(|k|)\Delta^3([k])},\ i\neq j.$$

Without generality, since (A_2), we assume

$$\lambda_j^1-\lambda_i^1=\lambda_j-\lambda_i+\mu_1\varepsilon^{l_1}+o(\varepsilon^{l_1}),\ i\neq j. \tag{3.15}$$

There exists an integer $N \geq 0$ satisfy

$$2^N \leq l_1 \leq 2^{N+1}. \tag{3.16}$$

So after $N+2$ KAM steps, by (3.12), we have

$$\lambda_j^t-\lambda_i^t=\lambda_j^{N+1}-\lambda_i^{N+1}+\varepsilon^{2^{N+1}}f(\varepsilon),\ i\neq j, \tag{3.17}$$

where $|f'(\varepsilon)| \leq c$, $f(\varepsilon) \to 0$ when $\varepsilon \to 0$, and the integer $t \geq N+2$. Moreover, for previous $N+1$ KAM steps, by (3.15), we have

$$\lambda_j^s - \lambda_i^s = \lambda_j - \lambda_i + \sigma\varepsilon^q + o(\varepsilon^q), \tag{3.18}$$

where $s = 1, 2, \cdots, N+1$, $0 < q \leq l_1$ is an integer, and $\sigma \neq 0$ is a constant. Now substitute (3.18) ($s = N+1$) into (3.17), by (3.16), we have

$$\lambda_j^t - \lambda_i^t = \lambda_j - \lambda_i + \sigma\varepsilon^q + \varepsilon^q K(\varepsilon), \tag{3.19}$$

where $|K'(\varepsilon)| \leq c$, $K(\varepsilon) \to 0$ as $\varepsilon \to 0$. By (3.18), (3.19), (A_1), and Lemma 2.3, it follows

$$\text{meas}((0, \varepsilon_0) \setminus E_{\nu+1}) \leq c\frac{1}{(\nu+1)^2}\varepsilon_0^{q+1}. \tag{3.20}$$

Denote $\tilde{E} = \cap_{\nu=0}^{\infty} E_{\nu+1}$. By (3.20), it follows that

$$\begin{aligned} \text{meas}((0, \varepsilon_0) \setminus \tilde{E}) &\leq \sum_{\nu=0}^{\infty} \text{meas}((0, \varepsilon_0) \setminus E_{\nu+1}) \\ &\leq \sum_{\nu=0}^{\infty} c\frac{1}{(\nu+1)^2}\varepsilon_0^{q+1} = c\varepsilon_0^{q+1}. \end{aligned}$$

So when ε_0 is sufficiently small, non-resonance conditions (3.14) hold for most $\varepsilon \in (0, \varepsilon_0)$.

We prove Theorem 1.1 completely. □

Acknowledgments

The authors were supported by the Natural Science Foundations for Colleges and Universities in Jiangsu Province grant 18KJB110029.

Conflict of interest

The authors declare that they do not have any conflicts of interest regarding this paper.

References

1. J. Pöschel, Small divisors with spatial structure in infinite dimensional Hamiltonian systems, *Commun. Math. Phys.*, **127** (1990), 351–393. https://doi.org/10.1007/BF02096763
2. R. A. Johnson, G. R. Sell, Smoothness of spectral subbundles and reducibility of quasiperodic linear differential systems, *J. Differ. Equations*, **41** (1981), 262–288. https://doi.org/10.1016/0022-0396(81)90062-0
3. A. Jorba, C. Simó, On the reducibility of linear differential equations with quasiperiodic coefficients, *J. Differ. Equations*, **98** (1992), 111–124. https://doi.org/10.1016/0022-0396(92)90107-X

4. J. X. Xu, Q. Zheng, On the reducibility of linear differential equations with quasiperiodic coefficients which are degenerate, *PROC*, **126** (1998), 1445–1451.
5. J. Li, C. P. Zhu, On the reducibility of a class of finitely differentiable quasi-periodic linear systems, *J. Math. Anal. Appl.*, **413** (2014), 69–83. https://doi.org/10.1016/j.jmaa.2013.10.077
6. H. Rüssmann, Convergent transformations into a normal form in analytic Hamiltonian systems with two degrees of freedom on the zero energy surface near degenerate elliptic singularities, *Ergod. Theor. Dyn. Syst.*, **24** (2004), 1787–1832. https://doi.org/10.1017/S0143385703000774
7. J. X. Xu, X. Z. Lu, On the reducibility of two-dimensional linear quasi-periodic systems with small parameter, *Ergod. Theor. Dyn. Syst.*, **35** (2015), 2334–2352. https://doi.org/10.1017/etds.2014.31
8. J. X. Xu, K. Wang, M. Zhu, On the reducibility of 2-dimensional linear quasi-periodic systems with small parameters, *PROC*, **144** (2016), 4793–4805. http://doi.org/10.1090/proc/13088
9. X. C. Wang, J. X. Xu, On the reducibility of a class of nonlinear quasi-periodic system with small perturbation parameter near zero equilibrium point, *Nonlinear Anal.: Theory, Methods Appl.*, **69** (2008), 2318–2329. https://doi.org/10.1016/j.na.2007.08.016
10. J. Li, J. X. Xu, On the reducibility of a class of almost periodic Hamiltonian systems, *Discrete Cont. Dyn.-B*, **26** (2021), 3905–3919. https://doi.org/10.3934/dcdsb.2020268
11. A. Jorba, C. Simó, On quasi-periodic perturbations of elliptic equilibrium points, *Siam J. Math. Anal.*, **27** (1996), 1704–1737. https://doi.org/10.1137/S0036141094276913
12. J. Li, C. P. Zhu, S. T. Chen, On the reducibility of a class of quasi-periodic Hamiltonian systems with small perturbation parameter near the equilibrium, *Qual. Theory Dyn. Syst.*, **16** (2017), 127–147. https://doi.org/10.1007/s12346-015-0164-x
13. J. X. Xu, J. G. You, On reducibility of linear differential equations with almost-periodic coefficients, *Chinese Ann. Math. A*, **17** (1996), 607–616.
14. H. Whitney, Analytical extensions of differentiable functions defined in closed sets, In: *Hassler Whitney collected papers*, Boston: Birkhäuser, 1992. https://doi.org/10.1007/978-1-4612-2972-8_4

AIMS Press

© 2023 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)