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Research article

Involvement of the topological degree theory for solving a tripled system of multi-point boundary value problems

Hasanen A. Hammad[1,2], **Hassen Aydi**[3,4,5,*] **and Mohra Zayed**[6,*]

[1] Department of Mathematics, Unaizah College of Sciences and Arts, Qassim University, Buraydah 52571, Saudi Arabia

[2] Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt

[3] Université de Sousse, Institut Supérieur d'Informatique et des Techniques de Communication, H. Sousse 4000, Tunisia

[4] China Medical University Hospital, China Medical University, Taichung 40402, Taiwan

[5] Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa

[6] Mathematics Department, College of Science, King Khalid University, Abha, Saudi Arabia

* **Correspondence:** Email: hassen.aydi@isima.rnu.tn, mzayed@kku.edu.sa.

Abstract: This article investigates the existence and uniqueness (EU) of positive solutions to the tripled system of multi-point boundary value problems (M-PBVPs) for fractional order differential equations (FODEs). The topological degree theory technique is employed to derive sufficient requirements for the (EU) of positive solutions to the proposed system. To justify the efficiency and validity of our study, an illustrative example is considered.

Keywords: degree theory; boundary value problem; fractional-order differential equation; positive solution
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1. Introduction

Fractional calculus is a powerful tool concerned with investigating integrals and derivatives of arbitrary order and their applications in physics, engineering, fluid mechanics, optics, signals processing, biology, etc. (see [1–5]). Several remarkable mathematicians made significant contributions to the field of fractional calculus. The first mention of the fractional calculus was provoked by Lacroix [6] in 1819 by introducing the common n-th derivative of the power function

$y = x^p$ using the Gamma function. In 1822, Fourier [7] introduced the derivative of arbitrary order via the Fourier transform of a function. Two years later, Fourier presented the fractional derivative of a function in terms of its Fourier transform. In 1832, Liouville [8] published two definitions of fractional derivatives of a fairly restrictive class of functions. In 1847, Riemann [9] developed a theory of fractional integration. Later in 1869, one of the first treatments of the Riemann-Liouville definition of the fractional integral was considered by Ya Sonin [10]. It is worth mentioning the fractional Riemann-Liouville version is one of the most frequently used in the literature. An alternative definition of a fractional derivative was also initiated by Caputo. For more related details, see [11–14].

The theory of fractional differential equations is a fruitful branch of mathematics by which various real-life phenomena in several fields of engineering and science can be expressed. In this context, enormous contributions to exploring useful applications have been attained in recent times. For instance, boundary value problems of nonlinear fractional differential equations with various boundary conditions have been investigated. Fractional differential equations represent an essential point of research. Nonlinear boundary value problems (BVPs for short) arise in several fields of physics, biology, chemistry, and applied mathematics. They are related to the theory of nonlinear diffusion generated by nonlinear sources, in chemical or biological problems, in the theory of elastics stability, and in thermal ignition of gases (for details, see [15–19]). Many phenomena in viscoelasticity, electrochemistry, electromagnetism, control theory, etc., can be expressed as fractional differential equations. Consequently, nonlinear BVPs are of great importance. The methods of nonlinear analysis, such as the Leray-Schauder continuation theorem, the coincidence degree theory of Mawhin, the fixed point theorems of Krasnoselskii and Schauder, fixed point theorems for mixed monotone operators, and others, are frequently used to solve fractional boundary value problems. For instance, see [20–23].

Only a few studies have used the degree theory arguments to prove the (EU) to boundary value problems (BVPs) [24–30]. However, to the best our knowledge, no previous research has discussed at the (EU) of solutions to tripled systems of (M-PBVPs) for (FODEs) using the topological degree technique. By this technique, Wang et al. [27] investigated the (EU) of solutions to a class of nonlocal Cauchy problems below:

$$\begin{cases} D^{\ell}\varpi(z) = \Lambda\left(z, \varpi(z)\right), \ z \in [0, T], \\ \varpi(0) + \vartheta(\varpi) = \varpi_0, \end{cases}$$

where D^{ℓ} is the Caputo fractional derivative (CFD) of order $\ell \in (0, 1)$, $\varpi_0 \in \mathbb{R}$, $\Lambda : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Chen et al. [25] explored necessary criteria for existence results for the following two-point boundary value problem (BVP) and extended the result above to the case of the (BVP):

$$\begin{cases} D^{u}_{0^+}\varphi_p\left(D^{v}_{0^+}\varpi(z)\right) = \Lambda\left(z, \varpi(z), D^{v}_{0^+}\varpi(z)\right), \\ D^{v}_{0^+}\varpi(0) = D^{v}_{0^+}\varpi(1) = 0, \end{cases}$$

where $D^{u}_{0^+}$ and $D^{v}_{0^+}$ are (CFDs), $0 < u, v \leq 1$, and $1 < u + v \leq 2$. The following two-point (BVP) for (FDEs) with various boundary conditions was investigated by Wang et al. [26]:

$$\begin{cases} D^{u}_{0^+}\varphi_p\left(D^{v}_{0^+}\varpi(z)\right) = \Lambda\left(z, \varpi(z), D^{v}_{0^+}\varpi(z)\right), \\ \varpi(0) = 0, \ D^{v}_{0^+}\varpi(0) = D^{v}_{0^+}\varpi(1), \end{cases}$$

where $D^{u}_{0^+}$, and $D^{v}_{0^+}$ are (CFDs), $0 < u, v \leq 1$ and $1 < u + v \leq 2$.

Motivated by the previous discussion, in this paper, we use a coincidence degree theory approach for condensing maps to derive appropriate criteria for the (EU) of solutions to more broad tripled systems of nonlinear (M-PBVPs). There are also nonlinear boundary conditions. The system's structure can be described as follows:

$$\begin{cases} D^{\ell}\varpi(z) = \Lambda_1\left(z, \varpi(z), \rho\left(z\right), \varrho\left(z\right)\right), \ z \in [0,1], \\ D^{\gamma}\rho\left(z\right) = \Lambda_2\left(z, \varpi(z), \rho\left(z\right), \varrho\left(z\right)\right), \ z \in [0,1], \\ D^{\varkappa}\varrho\left(z\right) = \Lambda_3\left(z, \varpi(z), \rho\left(z\right), \varrho\left(z\right)\right), \ z \in [0,1], \\ \quad \varpi(0) = \vartheta_1(\varpi), \ \varpi(1) = \eta_1\varpi(\xi_1), \ \xi_1 \in (0,1), \\ \quad \rho(0) = \vartheta_2(\rho), \ \rho(1) = \eta_2\rho(\xi_2), \ \xi_2 \in (0,1), \\ \quad \varrho(0) = \vartheta_3(\varrho), \ \varrho(1) = \eta_3\varrho(\xi_3), \ \xi_3 \in (0,1), \end{cases} \tag{1.1}$$

where $\ell, \gamma, \varkappa \in (1,2]$, D refers to the standard (CFD), $\eta_1, \eta_2, \eta_3 \in (0,1)$ are parameters so that $\eta_1\xi_1^{\ell} < 1$, $\eta_2\xi_2^{\gamma} < 1$, $\eta_3\xi_3^{\varkappa} < 1$, $\vartheta_1, \vartheta_2, \vartheta_3 \in C(I,\mathbb{R})$, and $\Lambda_1, \Lambda_2, \Lambda_3 : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ are boundary continuous functions.

2. Basic concepts

This part provides some basic definitions and findings from fractional calculus and topological degree theory. We suggest [31–35] for a more in-depth investigation.

Definition 2.1. For the function $\varpi \in L^1([a,b],\mathbb{R})$, the fractional integral of order ℓ is described as

$$I_{0^+}^{\ell}\varpi(z) = \frac{1}{\Gamma(\ell)}\int_a^z (z-r)^{\ell-1} z(r)dr.$$

The CFD is given as

$$D_{0^+}^{\ell}\varpi(z) = \frac{1}{\Gamma(m-\ell)}\int_a^z (z-r)^{m-\ell-1} z^{(m)}(r)dr,$$

where $m = [\ell] + 1$ and $[\ell]$ refers to the integer part of ℓ.

Lemma 2.1. *For (FDEs), the following result holds:*

$$I^{\ell}D^{\ell}\varpi(z) = \varpi(z) + a_0 + a_1 z + a_2 z^2 + \cdots + a_m d^{m-1},$$

for arbitrary $a_j \in \mathbb{R}$, $j = 1, 2, ..., m-1$.

Let $E = C\left([0,1],\mathbb{R}\right), \widetilde{E} = C\left([0,1],\mathbb{R}\right)$ and $\widehat{E} = C\left([0,1],\mathbb{R}\right)$ be the space of all continuous functions on $[0,1]$. Clearly, theses spaces are Banach spaces under the norms $\|\varpi\| = \sup_{z\in[0,1]} |\varpi(z)|$, $\|\rho\| = \sup_{z\in[0,1]} |\rho(z)|$ and $\|\varrho\| = \sup_{z\in[0,1]} |\varrho(z)|$, respectively. Moreover, the product $E \times \widetilde{E} \times \widehat{E}$ is a Banach space equipped with the norm $\|(\varpi,\rho,\varrho)\| = \|\varpi\| + \|\rho\| + \|\varrho\|$ or $|(\varpi,\rho,\varrho)| = \max\{\|\varpi\|, \|\rho\|, \|\varrho\|\}$.

Assume that Δ is the class of bounded sets of $\mho(E \times \widetilde{E} \times \widehat{E})$, where $E \times \widetilde{E} \times \widehat{E}$ is a Banach space. In what follows, we present basis notions and results which are very essential in the sequel (see [36]).

Definition 2.2. The Kuratowski measure of non-compactness $\mu : \Delta \to \mathbb{R}^+$ is described as

$$\mu(\upsilon) = \inf\{a > 0 : \ \upsilon \text{ admits a finite cover by sets of diameter } \leq a\},$$

where $\upsilon \in \Delta$.

Proposition 2.1. *The Kuratowski measure μ fulfills the hypotheses below:*

(1) υ *is relatively compact if and only if* $\mu(\upsilon) = 0$,
(2) $\mu(\beta\upsilon) = |\beta|\mu(\upsilon), \beta \in \mathbb{R}$, *and* $\mu(\upsilon_1 + \upsilon_2) \leq \mu(\upsilon_1) + \mu(\upsilon_2)$, *that is,* υ *is a seminorm,*
(3) $\upsilon_1 \subset \upsilon_2$ *implies* $\mu(\upsilon_1) \leq \mu(\upsilon_2)$, *and* $\mu(\upsilon_1 \cup \upsilon_2) = \max\{\mu(\upsilon_1), \mu(\upsilon_2)\}$,
(4) $\mu(conv\ \upsilon) = \mu(\upsilon)$,
(5) $\mu(\overline{\upsilon}) = \mu(\upsilon)$.

Definition 2.3. Assume that the mapping $\phi : \nabla \to E$ is continuous and bounded, where $\nabla \subset E$. Then ϕ is μ-Lipschitzian if there is $U \geq 0$ so that for all $\upsilon \subset \nabla$ bounded, $\mu(\phi(\upsilon)) \leq U\mu(\upsilon)$.
In addition, ϕ is called a strict υ-contraction if $U < 1$.

Definition 2.4. A function ϕ is called υ-condensing if $\mu(\phi(\upsilon)) < \mu(\upsilon)$ for all $\upsilon \subset \nabla$ bounded with $\mu(\upsilon) > 0$. Or, equivalently, $\mu(\phi(\upsilon)) \geq \mu(\upsilon)$ implies $\mu(\upsilon) = 0$.

For the bounded continuous mapping $\phi : \nabla \to E$, $D_\upsilon(\nabla)$ represents the class of all strict υ-contractions, and $\widetilde{D}_\upsilon(\nabla)$ refers to the class of all υ-condensing maps.

Remark 2.1. Note that $D_\upsilon(\nabla) \subset \widetilde{D}_\upsilon(\nabla)$, and every $\phi \in \widetilde{D}_\upsilon(\nabla)$ is υ-Lipschitz with constant $U = 1$. In addition, we recall that $\phi : \nabla \to E$ is Lipschitz if there is $U > 0$ such that

$$\text{for all } \varpi, \rho \in \nabla, \ \ \|\phi(\varpi) - \phi(\rho)\| \leq U\|\varpi - \rho\|.$$

Moreover, ϕ is called a strict contraction if $U < 1$.

Seeking clarification for the reader, we present the following results, quoted from [34], which we rely on through this study.

Proposition 2.2. *(i) If* $\phi, \supsetneq : \nabla \to E$ *are* υ*-Lipschitz with* U *and* U^*, *then* $\phi + \supsetneq : \nabla \to E$ *is* υ*-Lipschitz with* $U + U^*$.
(ii) If $\phi : \nabla \to E$ *is* υ*-Lipschitz with* U, *then* ϕ *is* υ*-Lipschitz with the same constant* U.
(iii) If $\phi : \nabla \to E$ *is compact, then* ϕ *is* υ*-Lipschitz with constant* $U = 0$.

The following result deduced by Isaia [34] is crucial to our main finding.

Theorem 2.1. *Assume that* $\Xi : \Lambda \to \Lambda$ *is* μ*-condensing, and*

$$\varphi = \{\varpi \in \Lambda : \ \text{there is } \varsigma \in [0,1] \text{ so that } \varpi = \varsigma\Xi\varpi\}.$$

If the set φ *is a bounded in* Λ, *there is* $s > 0$ *so that* $\varphi \subset U_s(0)$, *and the degree*

$$Q(I - \varsigma\Xi, U_s(0), 0) = 1, \text{ for all } \varsigma \in [0,1].$$

As a result, Ξ *owns at least one (FP), and the set of (FPs) is contained in* $U_s(0)$.

Now, we will state the hypotheses that will help us to achieve our objectives in this paper:

(H_1) There are constants $A_{\vartheta_1}, A_{\vartheta_2}, A_{\vartheta_3}$ so that, for $\varpi_1, \varpi_2, \rho_1, \rho_2, \varrho_1, \varrho_2 \in \mathbb{R}$,

$$\begin{aligned} |\vartheta_1(\varpi_2) - \vartheta_1(\varpi_1)| &\leq A_{\vartheta_1} |\varpi_2 - \varpi_1|, \\ |\vartheta_2(\rho_2) - \vartheta_2(\rho_1)| &\leq A_{\vartheta_2} |\rho_2 - \rho_1|, \\ |\vartheta_3(\varrho_2) - \vartheta_3(\varrho_1)| &\leq A_{\vartheta_3} |\varrho_2 - \varrho_1|. \end{aligned}$$

(H_2) There are constants $D_{\vartheta_1}, D_{\vartheta_2}, D_{\vartheta_3}, O_{\vartheta_1}, O_{\vartheta_2}, O_{\vartheta_3}$ so that, for $\varpi, \rho, \varrho \in \mathbb{R}$,

$$|\vartheta_1(\varpi)| \leq D_{\vartheta_1} |\varpi| + O_{\vartheta_1}, \ \ |\vartheta_2(\rho)| \leq D_{\vartheta_2} |\rho| + O_{\vartheta_2} \text{ and } |\vartheta_3(\varrho)| \leq D_{\vartheta_3} |\varrho| + O_{\vartheta_3}.$$

(H_3) There are constants p_i, q_i, t_i $(i = 1, 2, 3)$ and $O_{\Lambda_1}, O_{\Lambda_2}, O_{\Lambda_3}$ so that, for $\varpi, \rho, \varrho \in \mathbb{R}$,

$$\begin{aligned} |\Lambda_1(z, \varpi, \rho, \varrho)| &\leq p_1 |\varpi| + p_2 |\rho| + p_3 |\varrho| + O_{\Lambda_1}, \\ |\Lambda_2(z, \varpi, \rho, \varrho)| &\leq q_1 |\varpi| + q_2 |\rho| + q_3 |\varrho| + O_{\Lambda_2}, \\ |\Lambda_3(z, \varpi, \rho, \varrho)| &\leq t_1 |\varpi| + t_2 |\rho| + t_3 |\varrho| + O_{\Lambda_3}. \end{aligned}$$

(H_4) There are constants $\hbar_{\Lambda_1}, \hbar_{\Lambda_2}, \hbar_{\Lambda_3}$ so that, for $\varpi_1, \varpi_2, \rho_1, \rho_2, \varrho_1, \varrho_2 \in \mathbb{R}$,

$$\begin{aligned} |\Lambda_1(z, \varpi_2, \rho_2, \varrho_2) - \Lambda_1(z, \varpi_1, \rho_1, \varrho_1)| &\leq \hbar_{\Lambda_1} \left[|\varpi_2 - \varpi_1| + |\rho_2 - \rho_1| + |\varrho_2 - \varrho_1| \right], \\ |\Lambda_2(z, \varpi_2, \rho_2, \varrho_2) - \Lambda_2(z, \varpi_1, \rho_1, \varrho_1)| &\leq \hbar_{\Lambda_2} \left[|\varpi_2 - \varpi_1| + |\rho_2 - \rho_1| + |\varrho_2 - \varrho_1| \right], \\ |\Lambda_3(z, \varpi_2, \rho_2, \varrho_2) - \Lambda_3(z, \varpi_1, \rho_1, \varrho_1)| &\leq \hbar_{\Lambda_3} \left[|\varpi_2 - \varpi_1| + |\rho_2 - \rho_1| + |\varrho_2 - \varrho_1| \right]. \end{aligned}$$

3. Main results

In this section, the (EU) of solutions to the BVP (1.1) are discussed. We start stating and proving the lemma below.

Lemma 3.1. *The solutions of the BVP*

$$\begin{cases} D^{\ell}\varpi(z) = \Lambda_1(z), & z \in [0, 1], \\ \varpi(0) = \vartheta_1(\varpi), \quad \varpi(1) = \eta_1 \varpi(\xi_1), & \xi_1 \in (0, 1), \end{cases} \tag{3.1}$$

are equivalent to the solutions of the following Fredholm integral equation:

$$\varpi(z) = \left(1 - \frac{z(1 - \eta_1)}{1 - \eta_1 \xi_1}\right) \vartheta_1(\varpi) + \int_0^1 \eth_{\ell}(z, r) \Lambda_1(r)\, dr, \ z \in [0, 1],$$

where $\Lambda_1 : I \to \mathbb{R}$ is an ℓ_1 times integrable function, and $\eth_{\ell_1}(z, r)$ is defined by

$$\eth_{\ell}(z, r) = \frac{1}{\Gamma(\ell)} \begin{cases} (z - r)^{\ell-1} + \frac{z\eta_1(\xi_1 - r)^{\ell-1}}{1 - \eta_1\xi_1} - \frac{z(1-r)^{\ell-1}}{1-\eta_1\xi_1}, & 0 \leq r \leq z \leq \xi_1 \leq 1, \\ (z - r)^{\ell-1} - \frac{z(1-r)^{\ell-1}}{1-\eta_1\xi_1}, & 0 \leq \xi_1 \leq r \leq z \leq 1, \\ \frac{z\eta_1(\xi_1 - r)^{\ell-1}}{1-\eta_1\xi_1} - \frac{z(1-r)^{\ell-1}}{1-\eta_1\xi_1}, & 0 \leq z \leq r \leq \xi_1 \leq 1, \\ -\frac{z(1-r)^{\ell-1}}{1-\eta_1\xi_1}, & 0 \leq \xi_1 \leq z \leq r \leq 1. \end{cases} \tag{3.2}$$

Proof. Reflecting I^{ℓ} on (3.1) and applying Lemma 2.1, we have

$$\varpi(z) = I^{\ell}\Lambda_1(z) + a_0 + a_1 z,$$

for some $a_0, a_1 \in \mathbb{R}$. It follows from the conditions $\varpi(0) = \vartheta_1(\varpi)$ and $\varpi(1) = \eta_1\varpi(\xi_1)$ that $a_0 = \vartheta_1(\varpi)$ and

$$a_1 = \frac{\eta_1}{1-\xi_1\eta_1} I^{\ell}\Lambda_1(\xi_1) - \frac{1-\eta_1}{1-\xi_1\eta_1}\vartheta_1(\varpi) - \frac{1}{1-\xi_1\eta_1} I^{\ell}\Lambda_1(1).$$

Hence, we have

$$\varpi(z) = I^{\ell}\Lambda_1(z) + \vartheta_1(\varpi) + z\left[\frac{\eta_1}{1-\xi_1\eta_1} I^{\ell}\Lambda_1(\xi_1) - \frac{1-\eta_1}{1-\xi_1\eta_1}\vartheta_1(\varpi) - \frac{1}{1-\xi_1\eta_1} I^{\ell}\Lambda_1(1)\right].$$

After a simple rearrangement, one can write

$$\varpi(z) = \left(1 - \frac{z(1-\eta_1)}{1-\xi_1\eta_1}\right)\vartheta_1(\varpi) + \int_0^1 \eth_{\ell}(z,r)\Lambda(r)dr.$$

In the light of Lemma 3.1, solutions of (M-PBVPs) (1.1) are solutions of the Fredholm integral equations below:

$$\begin{cases} \varpi(z) = \left(1 - \frac{z(1-\eta_1)}{1-\xi_1\eta_1}\right)\vartheta_1(\varpi) + \int_0^1 \eth_{\ell}(z,r)\Lambda_1(r,\varpi(r),\rho(r),\varrho(r))\,dr, \\ \rho(z) = \left(1 - \frac{z(1-\eta_2)}{1-\xi_2\eta_2}\right)\vartheta_2(\varpi) + \int_0^1 \eth_{\gamma}(z,r)\Lambda_2(r,\varpi(r),\rho(r),\varrho(r))\,dr, \\ \varrho(z) = \left(1 - \frac{z(1-\eta_3)}{1-\xi_3\eta_3}\right)\vartheta_3(\varpi) + \int_0^1 \eth_{\varkappa}(z,r)\Lambda_3(r,\varpi(r),\rho(r),\varrho(r))\,dr, \end{cases} \tag{3.3}$$

where $\eth_{\gamma}(z,r)$ and $\eth_{\varkappa}(z,r)$ are defined by

$$\eth_{\gamma}(z,r) = \frac{1}{\Gamma(\gamma)}\begin{cases} (z-r)^{\gamma-1} + \frac{z\eta_2(\xi_2-r)^{\gamma-1}}{1-\eta_2\xi_2} - \frac{z(1-r)^{\gamma-1}}{1-\eta_2\xi_2}, & 0 \le r \le z \le \xi_2 \le 1, \\ (z-r)^{\gamma-1} - \frac{z(1-r)^{\gamma-1}}{1-\eta_2\xi_2}, & 0 \le \xi_2 \le r \le z \le 1, \\ \frac{z\eta_2(\xi_2-r)^{\gamma-1}}{1-\eta_2\xi_2} - \frac{z(1-r)^{\gamma-1}}{1-\eta_2\xi_2}, & 0 \le z \le r \le \xi_2 \le 1, \\ -\frac{z(1-r)^{\gamma-1}}{1-\eta_2\xi_2}, & 0 \le \xi_2 \le z \le r \le 1. \end{cases} \tag{3.4}$$

and

$$\eth_{\varkappa}(z,r) = \frac{1}{\Gamma(\varkappa)}\begin{cases} (z-r)^{\varkappa-1} + \frac{z\eta_3(\xi_3-r)^{\varkappa-1}}{1-\eta_3\xi_3} - \frac{z(1-r)^{\varkappa-1}}{1-\eta_3\xi_3}, & 0 \le r \le z \le \xi_3 \le 1, \\ (z-r)^{\varkappa-1} - \frac{z(1-r)^{\varkappa-1}}{1-\eta_3\xi_3}, & 0 \le \xi_3 \le r \le z \le 1, \\ \frac{z(1-r)^{\varkappa-1}}{1-\eta_3\xi_3} - \frac{z(1-r)^{\varkappa-1}}{1-\eta_3\xi_3}, & 0 \le z \le r \le \xi_3 \le 1, \\ -\frac{z(1-r)^{\varkappa-1}}{1-\eta_3\xi_3}, & 0 \le \xi_3 \le z \le r \le 1. \end{cases} \tag{3.5}$$

It is clear that

$$\max_{z\in[0,1]} |\eth_{\ell}(z,r)| = \frac{(1-r)^{\ell-1}}{(1-\eta_1\xi_1)\Gamma(\ell)}, \quad \max_{z\in[0,1]} \left|\eth_{\gamma}(z,r)\right| = \frac{(1-r)^{\gamma-1}}{(1-\eta_2\xi_2)\Gamma(\gamma)}$$

$$\text{and } \max_{z\in[0,1]} |\eth_{\varkappa}(z,r)| = \frac{(1-r)^{\varkappa-1}}{(1-\eta_3\xi_3)\,\Gamma(\varkappa)}. \tag{3.6}$$

Define the operators $\phi_1 : E \to E$, $\phi_2 : \widetilde{E} \to \widetilde{E}$, $\phi_3 : \widehat{E} \to \widehat{E}$ by

$$\begin{aligned}
\phi_1(\varpi)(z) &= \left(1 - \frac{z\,(1-\eta_1)}{1-\xi_1\eta_1}\right)\vartheta_1(\varpi),\ \ \phi_2(\rho)(z) = \left(1 - \frac{z\,(1-\eta_2)}{1-\xi_2\eta_2}\right)\vartheta_2(\rho),\\
\text{and } \phi_3(\varrho)(z) &= \left(1 - \frac{z\,(1-\eta_3)}{1-\xi_3\eta_3}\right)\vartheta_3(\varrho),
\end{aligned}$$

and the operators $\eth_1, \eth_2, \eth_3 : E \times \widetilde{E} \times \widehat{E} \to E \times \widetilde{E} \times \widehat{E}$ by

$$\begin{aligned}
\eth_1\left(\varpi,\rho,\varrho\right)(z) &= \int_0^1 \eth_{\ell}(z,r)\Lambda_1\left(r,\varpi(r),\rho\left(r\right),\varrho\left(r\right)\right)dr,\\
\eth_2\left(\varpi,\rho,\varrho\right)(z) &= \int_0^1 \eth_{\gamma}(z,r)\Lambda_2\left(r,\varpi(r),\rho\left(r\right),\varrho\left(r\right)\right)dr,\\
\eth_3\left(\varpi,\rho,\varrho\right)(z) &= \int_0^1 \eth_{\varkappa}(z,r)\Lambda_3\left(r,\varpi(r),\rho\left(r\right),\varrho\left(r\right)\right)dr.
\end{aligned}$$

Now, consider $\phi = (\phi_1, \phi_2, \phi_3)$, $\eth = (\eth_1, \eth_2, \eth_3)$ and $\psi = \phi + \eth$. Then, the suggested problem (3.3) can be written as an operator equation as follows:

$$(\varpi,\rho,\varrho) = \psi\left(\varpi,\rho,\varrho\right) = \phi\left(\varpi,\rho,\varrho\right) + \eth\left(\varpi,\rho,\varrho\right).$$

Hence, the solutions of the proposed problem (3.3) are FPs of ψ.

Lemma 3.2. *Under the assumptions* (H_1) *and* (H_2), *the operator* ϕ *satisfies the Lipschitz condition and the following condition is true:*

$$\text{for each } (\varpi,\rho,\varrho) \in E \times \widetilde{E} \times \widehat{E},\ \ \|\phi\left(\varpi,\rho,\varrho\right)\| \le D\,\|(\varpi,\rho,\varrho)\| + O,$$

where $D = \max\{D_{\vartheta_1}, D_{\vartheta_2}, D_{\vartheta_3}\}$, $O = \max\{O_{\vartheta_1}, O_{\vartheta_2}, O_{\vartheta_3}\}$.

Proof. From Hypothesis (H_1), one can write

$$\begin{aligned}
& \left|\phi\left(\varpi,\rho,\varrho\right)(z) - \phi\left(\varpi^*,\rho^*,\varrho^*\right)(z)\right| \\
= & \left|\left(1 - \frac{z\,(1-\eta_1)}{1-\xi_1\eta_1}\right)\left(\vartheta_1(\varpi) - \vartheta_1^*(\varpi)\right) + \left(1 - \frac{z\,(1-\eta_2)}{1-\xi_2\eta_2}\right)\left(\vartheta_2(\rho) - \vartheta_2^*(\rho)\right)\right.\\
& \left. + \left(1 - \frac{z\,(1-\eta_3)}{1-\xi_3\eta_3}\right)\left(\vartheta_3(\varrho) - \vartheta_3^*(\varrho)\right)\right| \\
\le & \ A_{\vartheta_1}\left|\vartheta_1(\varpi) - \vartheta_1^*(\varpi)\right| + A_{\vartheta_2}\left|\vartheta_2(\varpi) - \vartheta_2^*(\varpi)\right| + A_{\vartheta_3}\left|\vartheta_3(\varpi) - \vartheta_3^*(\varpi)\right| \\
\le & \ A\left|(\varpi,\rho,\varrho) - (\varpi^*,\rho^*,\varrho^*)\right|,\ A = \max\{A_{\vartheta_1}, A_{\vartheta_2}, A_{\vartheta_3}\}.
\end{aligned} \tag{3.7}$$

Applying Proposition 2.2 (ii), we have ϕ is υ-Lipschitz with constant A.

Now, from (H_2), we have

$$\|\phi(\varpi,\rho,\varrho)\| = \|(\phi_1(\varpi),\phi_2(\rho),\phi_3(\varrho))\| \leq D\|(\varpi,\rho,\varrho)\| + O,$$

where $D = \max\{D_{\vartheta_1}, D_{\vartheta_2}, D_{\vartheta_3}\}$, $O = \max\{O_{\vartheta_1}, O_{\vartheta_2}, O_{\vartheta_3}\}$. This completes the proof.

Lemma 3.3. *The operator $\supset$ is continuous, and it satisfies the following growth condition under the postulate (H_3):*

$$\|\supset(\varpi,\rho,\varrho)\| \leq \Theta\|(\varpi,\rho,\varrho)\| + \Upsilon, \tag{3.8}$$

where $\Theta = \theta(p+q+t)$, $\theta = \max\left\{\frac{1}{(1-\xi_1\eta_1)\Gamma(\ell)}, \frac{1}{(1-\xi_2\eta_2)\Gamma(\gamma)}, \frac{1}{(1-\xi_3\eta_3)\Gamma(\varkappa)}\right\}$, $p = \max\{p_1,p_2,p_3\}$, $q = \max\{q_1,q_2,q_3\}$, $t = \max\{t_1,t_2,t_3\}$, *and* $\Upsilon = \theta\left(O_{\Lambda_1} + O_{\Lambda_2} + O_{\Lambda_3}\right)$.

Proof. Assume that $\{(\varpi_k,\rho_k,\varrho_k)\}$ is a sequence of the bounded set $V_s = \left\{\|(\varpi,\rho,\varrho)\| \leq s : (\varpi,\rho,\varrho) \in E \times \widetilde{E} \times \widehat{E}\right\}$ so that $\{(\varpi_k,\rho_k,\varrho_k)\} \to (\varpi,\rho,\varrho)$ in V_s. We want to show that $\|(\varpi_k,\rho_k,\varrho_k) - (\varpi,\rho,\varrho)\| \to 0$. Consider

$$\begin{aligned}
&|\supset_1(\varpi_k,\rho_k,\varrho_k)(z) - \supset_1(\varpi,\rho,\varrho)(z)| \\
\leq\ & \frac{1}{\Gamma(\ell)}\left[\int_0^z (z-r)^{\ell-1}\left|\Lambda_1(r,\varpi_k(r),\rho_k(r),\varrho_k(r)) - \Lambda_1(r,\varpi(r),\rho(r),\varrho(r))\right| dr\right. \\
& + \frac{\eta_1}{1-\eta_1\xi_1}\int_0^{\xi_1} (\xi_1-r)^{\ell-1}\left|\Lambda_1(r,\varpi_k(r),\rho_k(r),\varrho_k(r)) - \Lambda_1(r,\varpi(r),\rho(r),\varrho(r))\right| dr \\
& \left. - \frac{1}{1-\eta_1\xi_1}\int_0^{1} (1-r)^{\ell-1}\left|\Lambda_1(r,\varpi_k(r),\rho_k(r),\varrho_k(r)) - \Lambda_1(r,\varpi(r),\rho(r),\varrho(r))\right| dr\right].
\end{aligned}$$

The continuity of Λ_1 leads to $\Lambda_1(r,\varpi_k(r),\rho_k(r),\varrho_k(r)) \to \Lambda_1(r,\varpi(r),\rho(r),\varrho(r))$ as $k \to \infty$. For all $z \in [0,1]$, from (H_3), we get

$$(z-r)^{\ell-1}\left|\Lambda_1(r,\varpi_k(r),\rho_k(r),\varrho_k(r)) - \Lambda_1(r,\varpi(r),\rho(r),\varrho(r))\right| \leq 3(z-r)^{\ell-1}\left[(p_1+p_2+p_3)s + O_{\Lambda_1}\right],$$

which leads to the integrability for $z, r \in [0,1]$. Applying the Lebesgue dominated convergence theorem, we have

$$\int_0^z (z-r)^{\ell-1}\left|\Lambda_1(r,\varpi_k(r),\rho_k(r),\varrho_k(r)) - \Lambda_1(r,\varpi(r),\rho(r),\varrho(r))\right| \to 0 \text{ as } k \to \infty.$$

Analogously, the rest terms tend to 0 as $k \to \infty$. This implies that

$$|\supset_1(\varpi_k,\rho_k,\varrho_k) - \supset_1(\varpi,\rho,\varrho)| \to 0 \text{ as } k \to \infty.$$

Similarly, one can obtain that

$$|\supset_2(\varpi_k,\rho_k,\varrho_k) - \supset_2(\varpi,\rho,\varrho)| \to 0 \text{ as } k \to \infty,$$

and

$$\left|\eth_3(\varpi_k,\rho_k,\varrho_k)-\eth_3(\varpi,\rho,\varrho)\right|\to 0 \text{ as } k\to\infty.$$

Now, for growth condition on $\eth$, using (H_3) and (3.6), we have

$$\begin{aligned}
&\left|\eth_1(\varpi,\rho,\varrho)\right| \\
= &\left|\int_0^1 \eth_\ell(z,r)\Lambda_1(r,\varpi(r),\rho(r),\varrho(r))\,dr\right| \le \frac{1}{(1-\eta_1\xi_1)\Gamma(\ell)}\left(p_1\|\varpi\|+p_2\|\rho\|+p_3\|\varrho\|+O_{\Lambda_1}\right), \\
&\left|\eth_2(\varpi,\rho,\varrho)\right| \\
= &\left|\int_0^1 \eth_\gamma(z,r)\Lambda_2(r,\varpi(r),\rho(r),\varrho(r))\,dr\right| \le \frac{1}{(1-\eta_2\xi_2)\Gamma(\gamma)}\left(q_1\|\varpi\|+q_2\|\rho\|+q_3\|\varrho\|+O_{\Lambda_2}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\left|\eth_3(\varpi,\rho,\varrho)\right| \\
= &\left|\int_0^1 \eth_\varkappa(z,r)\Lambda_3(r,\varpi(r),\rho(r),\varrho(r))\,dr\right| \le \frac{1}{(1-\eta_3\xi_3)\Gamma(\varkappa)}\left(t_1\|\varpi\|+t_2\|\rho\|+t_3\|\varrho\|+O_{\Lambda_3}\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|\eth(\varpi,\rho,\varrho)\| &= \|\eth_1(\varpi,\rho,\varrho)\|+\|\eth_2(\varpi,\rho,\varrho)\|+\|\eth_3(\varpi,\rho,\varrho)\| \\
&\le \theta\left(p_1\|\varpi\|+p_2\|\rho\|+p_3\|\varrho\|+O_{\Lambda_1}\right)+\theta\left(q_1\|\varpi\|+q_2\|\rho\|+q_3\|\varrho\|+O_{\Lambda_2}\right) \\
&\quad +\theta\left(t_1\|\varpi\|+t_2\|\rho\|+t_3\|\varrho\|+O_{\Lambda_3}\right) \\
&\le \theta(p+q+t)(\|\varpi\|+\|\rho\|+\|\varrho\|)+\theta\left(O_{\Lambda_1}+O_{\Lambda_2}+O_{\Lambda_3}\right) \\
&= \Theta\|(\varpi,\rho,\varrho)\|+\Upsilon.
\end{aligned}$$

This finishes the desired result.

Lemma 3.4. *The mapping $\eth: E\times\widetilde{E}\times\widehat{E}\to E\times\widetilde{E}\times\widehat{E}$ is compact. As a result, $\eth$ is υ-Lipschitz with constant zero.*

Proof. Consider a bounded set $\mho\subset V_s\subseteq E\times\widetilde{E}\times\widehat{E}$ and a sequence $\{(\varpi_k,\rho_k,\varrho_k)\}$ in $\mho$. Then from (3.8), we obtain that

$$\|\eth(\varpi_k,\rho_k,\varrho_k)\|\le\Theta s+\Upsilon, \text{ for every } (\varpi,\rho,\varrho)\in E\times\widetilde{E}\times\widehat{E},$$

which implies that $\eth(\mho)$ is bounded. Now, for equi-continuity and for given $\varepsilon>0$, put

$$\begin{aligned}
\delta = \min&\left\{\delta_1=\frac{1}{3}\left(\frac{\varepsilon\Gamma(1+\ell)}{6\left([p_1+p_2+p_3]s+O_{\Lambda_1}\right)}\right)^{\frac{1}{\ell}},\ \delta_2=\frac{1}{3}\left(\frac{\varepsilon\Gamma(1+\gamma)}{6\left([q_1+q_2+q_3]s+O_{\Lambda_2}\right)}\right)^{\frac{1}{\gamma}},\right. \\
&\left.\delta_3=\frac{1}{3}\left(\frac{\varepsilon\Gamma(1+\varkappa)}{6\left([t_1+t_2+t_3]s+O_{\Lambda_3}\right)}\right)^{\frac{1}{\varkappa}}\right\}.
\end{aligned}$$

If $z, \lambda \in [0,1]$, and $\lambda - z \in (0, \delta_1)$, then for each $(\varpi_k, \rho_k, \varrho_k) \in \mho$, we claim that

$$|\eth_1(\varpi_k,\rho_k,\varrho_k)(z) - \eth_1(\varpi_k,\rho_k,\varrho_k)(\lambda)| < \frac{\varepsilon}{3}.$$

Consider

$$\begin{aligned}
& |\eth_1(\varpi_k,\rho_k,\varrho_k)(z) - \eth_1(\varpi_k,\rho_k,\varrho_k)(\lambda)| \\
= & \left| \frac{1}{\Gamma(\ell)} \int_0^z \left[(z-r)^{\ell-1} - (\lambda - r)^{\ell-1} \right] \Lambda_1(r, \varpi_k(r), \rho_k(r), \varrho_k(r))\, dr \right. \\
& + \frac{1}{\Gamma(\ell)} \int_z^\lambda (\lambda - r)^{\ell-1} \Lambda_1(r, \varpi_k(r), \rho_k(r), \varrho_k(r))\, dr \\
& \left. + \frac{\eta_1(z-\lambda)}{(1-\eta_1\xi_1)\Gamma(\ell)} \int_0^{\xi_1} (\xi_1 - r)^{\ell-1} \Lambda_1(r, \varpi_k(r), \rho_k(r), \varrho_k(r))\, dr \right| \\
\leq & \frac{(p_1|\varpi| + p_2|\rho| + p_3|\varrho| + O_{\Lambda_1})}{\Gamma(\ell+1)} \left[\left(z^\ell - \lambda^\ell \right) + 3(\lambda - z)^\ell \right] \\
\leq & \frac{(p_1 + p_2 + p_3)s + O_{\Lambda_1}}{\Gamma(\ell+1)} \left[\left(z^\ell - \lambda^\ell \right) + 3(\lambda - z)^\ell \right].
\end{aligned}$$

Now, we realize the following cases:

($\bullet$) If $\delta_1 \leq z < \lambda < 1$, we have

$$\begin{aligned}
|\eth_1(\varpi_k,\rho_k,\varrho_k)(z) - \eth_1(\varpi_k,\rho_k,\varrho_k)(\lambda)| & < \frac{(p_1 + p_2 + p_3)s + O_{\Lambda_1}}{\Gamma(\ell+1)} (3+\ell)\,\delta_1^{\ell-1}(\lambda - z) \\
& < \frac{(p_1 + p_2 + p_3)s + O_{\Lambda_1}}{\Gamma(\ell+1)} (3+\ell)\,\delta_1^{\ell} < \frac{\varepsilon}{3}.
\end{aligned}$$

($\bullet\bullet$) If $0 \leq z < \delta_1$, $\lambda < 2\delta_1$, we get

$$|\eth_2(\varpi_k,\rho_k,\varrho_k)(z) - \eth_2(\varpi_k,\rho_k,\varrho_k)(\lambda)| < \frac{\varepsilon}{3},$$

and

$$|\eth_3(\varpi_k,\rho_k,\varrho_k)(z) - \eth_3(\varpi_k,\rho_k,\varrho_k)(\lambda)| < \frac{\varepsilon}{3}.$$

This proves that $\eth(\mho)$ is equi-continuous. In light of the Arzelà-Ascoli theorem, $\eth(\mho)$ is compact. Based on Proposition 2.2 (iii), $\eth$ is a υ-Lipschitz with a constant zero.

Theorem 3.1. *The tripled system of nonlinear (M-PBVPs) (1.1) has at least one solution* $(\varpi, \rho, \varrho) \in E \times \widetilde{E} \times \widehat{E}$ *provided that the assumptions* (H_1)–(H_3) *hold and* $D + \Theta < 1$. *In addition, the set of solutions of problem (1.1) is bounded in* $E \times \widetilde{E} \times \widehat{E}$.

Proof. Based on Lemmas 3.2 and 3.4, ϕ is υ-Lipschitz with constant $A \in [0,1)$, and $\eth$ is υ-Lipschitz with constant zero, respectively. From Proposition 2.2 (i), we have, ψ is a strict υ-contraction with constant A. Define

$$G = \{(\varpi,\rho,\varrho) \in E \times \widetilde{E} \times \widehat{E} : \text{ there is } \varsigma \in [0,1] \text{ so that } (\varpi,\rho,\varrho) = \varsigma\psi(\varpi,\rho,\varrho)\}.$$

In order to prove that G is bounded, assume that $(\varpi,\rho,\varrho) \in G$. Then, in light of growth stipulations as in Lemmas 3.2 and 3.3, we get

$$\begin{aligned} \|(\varpi,\rho,\varrho)\| &= \|\varsigma\psi(\varpi,\rho,\varrho)\| = \varsigma\|\psi(\varpi,\rho,\varrho)\| = \varsigma\left[\|\phi(\varpi,\rho,\varrho)\| + \|\eth(\varpi,\rho,\varrho)\|\right] \\ &\leq \varsigma\left[D\|(\varpi,\rho,\varrho)\| + O + \Theta\|(\varpi,\rho,\varrho)\| + \Upsilon\right] \\ &= \varsigma(D+\Theta)\|(\varpi,\rho,\varrho)\| + \varsigma(O+\Upsilon) \\ &< \varsigma\|(\varpi,\rho,\varrho)\| + \varsigma(O+\Upsilon), \end{aligned}$$

which implies that G is bounded in $E \times \widetilde{E} \times \widehat{E}$. Hence, according to Theorem 2.1, we conclude that ψ has at least one (FP), and the set of (FPs) is bounded in $E \times \widetilde{E} \times \widehat{E}$.

Theorem 3.2. *The tripled system of nonlinear (M-PBVPs) (1.1) has a unique solution* $(\varpi,\rho,\varrho) \in E \times \widetilde{E} \times \widehat{E}$ *provided that the assumptions* (H_1)–(H_4) *are true and* $A + \theta(\hbar_{\Lambda_1} + \hbar_{\Lambda_2} + \hbar_{\Lambda_3}) < 1$.

Proof. For $(\varpi,\rho,\varrho), (\varpi^*,\rho^*,\varrho^*) \in \mathbb{R}^3$, it follows from the Banach FP theorem and (3.7) that

$$\|\phi(\varpi,\rho,\varrho) - \phi(\varpi^*,\rho^*,\varrho^*)\| \leq A\|(\varpi,\rho,\varrho) - (\varpi^*,\rho^*,\varrho^*)\|. \tag{3.9}$$

From (H_4) and (3.6), we have

$$\begin{aligned} |\eth_1(\varpi,\rho,\varrho) - \eth_1(\varpi^*,\rho^*,\varrho^*)| &= \int_0^1 |\eth_\ell(z,r)|\,|\Lambda_1(r,\varpi(r),\rho(r),\varrho(r)) - \Lambda_1(r,\varpi^*(r),\rho^*(r),\varrho^*(r))|\,dr \\ &\leq \theta\hbar_{\Lambda_1}\left[|\varpi - \varpi^*| + |\rho - \rho^*| + |\varrho - \varrho^*|\right], \end{aligned}$$

which yields that

$$\begin{aligned} \|\eth_1(\varpi,\rho,\varrho) - \eth_1(\varpi^*,\rho^*,\varrho^*)\| &\leq \theta\hbar_{\Lambda_1}\left[\|\varpi - \varpi^*\| + \|\rho - \rho^*\| + \|\varrho - \varrho^*\|\right] \\ &= \theta\hbar_{\Lambda_1}\|(\varpi - \varpi^*, \rho - \rho^*, \varrho - \varrho^*)\| \\ &= \theta\hbar_{\Lambda_1}\|(\varpi,\rho,\varrho) - (\varpi^*,\rho^*,\varrho^*)\|. \end{aligned} \tag{3.10}$$

Analogously, we can obtain

$$\|\eth_2(\varpi,\rho,\varrho) - \eth_2(\varpi^*,\rho^*,\varrho^*)\| \leq \theta\hbar_{\Lambda_2}\|(\varpi,\rho,\varrho) - (\varpi^*,\rho^*,\varrho^*)\|, \tag{3.11}$$

and

$$\|\eth_3(\varpi,\rho,\varrho) - \eth_3(\varpi^*,\rho^*,\varrho^*)\| \leq \theta\hbar_{\Lambda_3}\|(\varpi,\rho,\varrho) - (\varpi^*,\rho^*,\varrho^*)\|. \tag{3.12}$$

Combining (3.10)–(3.12), we get

$$\|\eth(\varpi,\rho,\varrho) - \eth(\varpi^*,\rho^*,\varrho^*)\| = \|\eth_1(\varpi,\rho,\varrho) - \eth_1(\varpi^*,\rho^*,\varrho^*)\| + \|\eth_2(\varpi,\rho,\varrho) - \eth_2(\varpi^*,\rho^*,\varrho^*)\|$$

$$
\begin{aligned}
& +\left\|\text{Ↄ}_3(\varpi,\rho,\varrho)-\text{Ↄ}_3\left(\varpi^*,\rho^*,\varrho^*\right)\right\| \\
\leq\ & \theta\left(\hbar_{\Lambda_1}+\hbar_{\Lambda_2}+\hbar_{\Lambda_3}\right)\left\|(\varpi,\rho,\varrho)-\left(\varpi^*,\rho^*,\varrho^*\right)\right\|. \qquad (3.13)
\end{aligned}
$$

Using (3.9) and (3.13), we have

$$
\begin{aligned}
\left\|\psi(\varpi,\rho,\varrho)-\psi\left(\varpi^*,\rho^*,\varrho^*\right)\right\| &= \left\|\phi(\varpi,\rho,\varrho)-\phi\left(\varpi^*,\rho^*,\varrho^*\right)\right\|+\left\|\text{Ↄ}(\varpi,\rho,\varrho)-\text{Ↄ}\left(\varpi^*,\rho^*,\varrho^*\right)\right\| \\
&\leq \left[A+\theta\left(\hbar_{\Lambda_1}+\hbar_{\Lambda_2}+\hbar_{\Lambda_3}\right)\right]\left\|(\varpi,\rho,\varrho)-\left(\varpi^*,\rho^*,\varrho^*\right)\right\|.
\end{aligned}
$$

This proves that ψ is a contraction mapping. From the Banach FP theorem, the suggested problem has a unique solution.

4. Supportive example

Consider the tripled system of nonlinear (M-PBVPs) below

$$
\begin{cases}
D^{\frac{4}{3}}\varpi(z)=\frac{1}{75+z^2}\left(1+|\varpi(z)|+|\rho(z)|+|\varrho(z)|\right),\ z\in[0,1],\\
D^{\frac{4}{3}}\rho(z)=\frac{1+|\varpi(z)|+|\rho(z)|+|\varrho(z)|}{75+|\cos\varpi(z)|+|\sin\rho(z)|+|\cos\varrho(z)|},\ z\in[0,1],\\
D^{\frac{4}{3}}\varrho(z)=\frac{1+|\varpi(z)|+|\rho(z)|+|\varrho(z)|}{75+|\sin\varpi(z)|+|\cos\rho(z)|+|\sin\varrho(z)|},\ z\in[0,1],\\
\varpi(0)=\frac{\vartheta_1(\varpi)}{3},\ \varpi(1)=\frac{1}{3}\varpi(\frac{1}{3}),\\
\rho(0)=\frac{\vartheta_2(\rho)}{3},\ \rho(1)=\frac{1}{4}\rho(\frac{1}{4}),\\
\varrho(0)=\frac{\vartheta_3(\varrho)}{3},\ \varrho(1)=\frac{1}{5}\varrho(\frac{1}{5}).
\end{cases}
\tag{4.1}
$$

From the problem (4.1) we take $\ell=\gamma=\varkappa=\frac{4}{3}\in(1,2]$, $\eta_1=\xi_1=\frac{1}{3}$, $\eta_2=\xi_2=\frac{1}{4}$, $\eta_3=\xi_3=\frac{1}{5}$ with $\eta_1\xi_1^\ell=\frac{1}{3}\left(\frac{1}{3}\right)^{\frac{3}{4}}=0.1462<1$, $\eta_2\xi_2^\gamma<1$, $\eta_3\xi_3^\varkappa<1$ and $s=3>0$. The solution of the BVP (4.1) can be written as

$$
\begin{cases}
\varpi(z)=\frac{\vartheta_1(\varpi)}{3}\left(1-\frac{3z}{4}\right)+\int\limits_0^1\text{Ↄ}_\ell(z,r)\Lambda_1\left(r,\varpi(r),\rho(r),\varrho(r)\right)dr,\\
\rho(z)=\frac{\vartheta_2(\rho)}{3}\left(1-\frac{4z}{5}\right)+\int\limits_0^1\text{Ↄ}_\gamma(z,r)\Lambda_2\left(r,\varpi(r),\rho(r),\varrho(r)\right)dr,\\
\varrho(z)=\frac{\vartheta_3(\varrho)}{3}\left(1-\frac{5z}{6}\right)+\int\limits_0^1\text{Ↄ}_\varkappa(z,r)\Lambda_3\left(r,\varpi(r),\rho(r),\varrho(r)\right)dr.
\end{cases}
$$

where Ↄ_ℓ, Ↄ_γ and $\text{Ↄ}_\varkappa$ are the Green's functions, and they may be simply obtained as shown in (3.2), (3.4) and (3.5), respectively. Let us consider $\varsigma=\frac{1}{3}$, and then according to Theorem 3.2, we have $\hbar_{\Lambda_1}=\hbar_{\Lambda_2}=\hbar_{\Lambda_3}=\frac{1}{75}=p_i=q_i=t_i$ $(i=1,2,3)$, taking $A_{\vartheta_1}=A_{\vartheta_2}=A_{\vartheta_3}=\frac{1}{3}$. Then, the hypotheses (H_1)–(H_4) are fulfilled. We get

$$
\begin{aligned}
\phi_1(\varpi)(z) &= \frac{\vartheta_1(\varpi)}{3}\left(1-\frac{3z}{4}\right),\ \text{Ↄ}_1(\varpi)(z)=\int\limits_0^1\text{Ↄ}_\ell(z,r)\Lambda_1\left(r,\varpi(r),\rho(r),\varrho(r)\right)dr,\\
\phi_2(\rho)(z) &= \frac{\vartheta_2(\rho)}{3}\left(1-\frac{4z}{5}\right),\ \text{Ↄ}_2(\rho)(z)=\int\limits_0^1\text{Ↄ}_\gamma(z,r)\Lambda_2\left(r,\varpi(r),\rho(r),\varrho(r)\right)dr,
\end{aligned}
$$

$$\phi_3(\varrho)(z) = \frac{\vartheta_3(\varrho)}{3}\left(1-\frac{5z}{6}\right), \ \Game_3(\varrho)(z) = \int_0^1 \Game_{\varkappa}(z,r)\Lambda_3(r,\varpi(r),\rho(r),\varrho(r))\,dr.$$

The continuity and boundedness of $\phi_1, \phi_2, \phi_3, \Game_1, \Game_2$ and $\Game_3$ imply that $\phi = (\phi_1, \phi_2, \phi_3)$ and $\Game = (\Game_1, \Game_2, \Game_3)$ are also. Hence, $\psi = \phi + \Game$ is continuous and bounded. Moreover,

$$\|\Game(\varpi,\rho,\varrho) - \Game(\varpi^*,\rho^*,\varrho^*)\| \leq \frac{1}{3}\|(\varpi,\rho,\varrho) - (\varpi^*,\rho^*,\varrho^*)\|.$$

This illustrates that, if $\Game$ is υ-Lipschitz with constant $\frac{1}{3}$ and ϕ is υ-Lipschitz with constant 0, then ψ is a strict υ-contraction with constant $\frac{1}{3}$. Furthermore, it is easy to see that $\theta = 1.259845459$. Since

$$G = \{(\varpi,\rho,\varrho) \in C\left([0,1]\times\mathbb{R}^3,\mathbb{R}\right), \text{ there is } \varsigma \in [0,1] \text{ so that } (\varpi,\rho,\varrho) = \frac{1}{3}\psi(\varpi,\rho,\varrho)\},$$

the solution

$$\|(\varpi,\rho,\varrho)\| \leq \frac{1}{3}\|\psi(\varpi,\rho,\varrho)\| \leq 1,$$

implies that G is bounded. Using Theorem 3.1, the tripled system of nonlinear (M-PBVPs) (4.1) has a solution (ϖ,ρ,ϱ) in $C\left([0,1]\times\mathbb{R}^3,\mathbb{R}\right)$. In addition, $A + \theta(\hbar_{\Lambda_1} + \hbar_{\Lambda_2} + \hbar_{\Lambda_3}) = 0.38373 < 1$. Therefore, by Theorem 3.2, the suggested problem (4.1) has a unique solution.

5. Conclusions

The technique of a coincidence degree theory for condensing maps has been incorporated to obtain suitable conditions for the (EU) of positive solutions to tripled systems of nonlinear (M-PBVPs) under nonlinear boundary conditions. We provided an example to illustrate the obtained results. Our findings can be applied to further arbitrary fractional order differential equations, linear and nonlinear fractional integro-differential systems, Hadamard fractional derivatives, and other topics as future work.

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Conflict of interest

All authors declare that they have no conflicts of interest.

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