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Research article

Numerical analysis of some partial differential equations with fractal-fractional derivative

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Abstract: In this study, we expanded the partial differential equation framework to which fractal-fractional differentiation can be applied. For this, we employed the generalized Mittag-Leffler function, and the fractal-fractional derivatives based on the power-law kernel. A general partial differential equation with the fractal-fractional derivative, the power law kernel and the generalized Mittag-Leffler function was thoroughly examined. There is almost no numerical scheme for solving partial differential equations with fractal-fractional derivatives, as less investigation has been done in this direction in the last decades. In this work, therefore, we shall attempt to provide a numerical method that might be used to solve these equations in each circumstance. The heat equation was taken into consideration for the application and numerically solved using a few simulations for various values of fractional and fractal orders. It is observed that, when the fractal order is 1, one obtains fractional partial differential equations which have been known to replicate nonlocal behaviors. Meanwhile, if the fractional order is 1, one obtains fractal-partial differential equations. Thus, when the fractional order and fractal dimension are different from zero, nonlocal processes with similar features are developed.

Keywords: fractal-fractional; power-law kernel; Mittag-Leffler kernel; partial differential equation; numerical scheme, heat equation
Mathematics Subject Classification: 35-xx

1. Introduction

The Riemann-Stieltjes integral, which bears the names of Thomas Joannes Stieltjes and Bernhard Riemann in mathematics, is an extension of the Riemann integral. Stieltjes published the definition of this integral for the first time in 1894 [1–4]. It functions as a helpful and educational predecessor to the Lebesgue integral and as a crucial tool for combining comparable versions of statistical theorems that apply to discrete and continuous probability. Numerous areas of mathematics and statistics have found a use for this generalization. The dual space of the Banach space $C[a,b]$ of continuous functions in the interval $[a,b]$ is, for instance, represented as Riemann-Stieltjes integrals against functions with limited variation in the original formulation of F. Riesz's theorem [2–4]. Later, a measure-based version of that theorem was developed. Additionally, it is mentioned in the definition of the spectral theorem for (non-compact) self-adjoint operators in a Hilbert space [2–4]. This theorem examines the integral in relation to a family of spectral projections. However, there is no recognised differential operator related to this integral. When the idea of fractal derivatives was first proposed, an attempt was made, but it did not receive the attention it deserved, possibly because such a derivative did not satisfy some fundamental characteristics of the classical derivative, such as the index law, and because its geometrical interpretation was not well understood [5]. However, a fractal integral was produced and discovered to be a specific class of the Riemann-Stieltjes integral utilizing the basic theorem of calculus and assuming the differentiability of the function. Keep in mind that one of the essential requirements is that the function must be classically differentiable [2–4]. The idea of fractal-fractional differentiation and integration was developed, and it has been applied to numerous problems with some success, using the fact that an integral operator is differential [6]. The lack of analytical and numerical methods that may be utilized to solve these problems may be the fundamental reason why their application to partial differential equations has received little attention. In this study, an attempt will be made to apply this notion to the framework of partial differential equations and present some numerical techniques.

2. General partial differential equation with fractal-fractional derivative with power- law kernel

Due to their ability to recreate occurrences as a function of time and place, partial differential equations have found use in every branch of science, technology and engineering. Here are a few noteworthy instances of partial differential equation applications. In scientific disciplines that are heavily reliant on mathematics, including physics and engineering, partial differential equations are commonplace. In the contemporary scientific knowledge of, for example, sound, heat, diffusion, electrostatics, electrodynamics, thermodynamics, fluid dynamics, elasticity, general relativity and quantum mechanics, they serve as the cornerstone. They also result from other purely mathematical issues, such as differential geometry and the calculus of variations. Among other famous uses, they serve as the main means of proving the geometric topological Poincaré conjecture. In this work, we will consider two general types of fractal-fractional partial differential equations, the first class with the fractal-fractional derivative with the power-law kernel, and the second with the generalized Mittag-Leffler kernel. First, we present some important definitions. Let a function $y(t)$ be continuous in the entire positive real numbers or even in some closed interval $[a,b]$. A fractal-fractional derivative of the function $y(t)$ with the power-law kernel is given as [6]

$$\frac{df(t)}{dt^{\beta}} = \lim_{l\to t}\frac{f(l)-f(t)}{l^{\beta}-t^{\beta}},$$

$${}^{FFP}_{\ \ 0}D_t^{\alpha,\beta}y(t) = \frac{d}{dt^{\beta}}\int_0^t y(\tau)\frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}d\tau.$$

A fractal-fractional derivative of the function $y(t)$ with the generalized-Mittag-Leffler kernel is given as [6]:

$${}^{FFM}_{\ \ 0}D_t^{\alpha,\beta}y(t) = \frac{1}{1-\alpha}\frac{d}{dt^{\beta}}\int_0^t y(\tau)E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\right]d\tau.$$

Let a function $f(t)$ be continuous in the entire positive real number or even in some closed interval $[a,b]$: A fractal-fractional integral of the function $f(t)$ with the power-law kernel is given as:

$${}^{FFP}_{\ \ 0}J_t^{\alpha,\beta}f(t) = \frac{\beta}{\Gamma(\alpha)}\int_0^t f(\tau)(t-\tau)^{\alpha-1}\tau^{\beta-1}d\tau.$$

A fractal-fractional integral of the function $f(t)$ with the power-law kernel is given as [6]:

$${}^{FFM}_{\ \ 0}J_t^{\alpha,\beta}f(t) = \beta(1-\alpha)\tau^{\beta-1}f(t) + \frac{\alpha\beta}{\Gamma(\alpha)}\int_0^t f(\tau)(t-\tau)^{\alpha-1}\tau^{\beta-1}d\tau.$$

2.1. Fractal-fractional partial differential equation with power-law kernel

The ease with which some general classes of systems create power-law interactions is a contributing factor in the scientific interest in them. A few notable examples of power laws are Pareto's law of income distribution, the structural self-similarity of fractals and scaling laws in biological systems [7–11]. The demonstration of a power-law relation in some data can point to specific kinds of mechanisms that might underlie the natural phenomenon in question and can indicate a deep connection with other, seemingly unrelated systems [13,15]. Numerous scientific disciplines, including physics, computer science, linguistics, geophysics, neurology, systematics, sociology and economics, are actively researching the origins of power-law interactions as well as ways to detect and validate them in the actual world [7–11]. In order to incorporate long-tailed behaviors caused by the power-law into the mathematical model, a differential operator with a power-law kernel is essential [14,15]. We consider first the following general fractal-fractional partial differential equations:

$$\begin{cases} {}^{FFP}_{\ \ 0}D_t^{\alpha,\beta}u(x,t) = f\big(x,t,u(x,t)\big), t>0, \\ u(0,t) = g(t), \\ u(x,0) = l(x)\,, \end{cases} \tag{1}$$

where the function f is a generic function that can be linear or nonlinear, containing partial derivatives with respect to the parameter x, as for example $\frac{\partial}{\partial x}$, $\frac{\partial^2}{\partial x^2}$, … … $\frac{\partial^n}{\partial x^n}$ under the condition that the function $u(x,t)$ is n-times differentiable with respect to the variable x.

Assumptions

- $g(t)$ and $l(x)$ are continuous bounded functions.
- The function f is twice differentiable with respect to time and bounded $\forall(x,t)\in X\times\pi$ where X is the space domain and π is the time domain.

- $u(x,t)$ is bounded $\forall(x,t)\in X\times\pi$ and differentiable with respect to time and $n\geq 2$ differentiable with respect to space.
- $0<\alpha\leq 1$, $\beta>0$, but preferably less than 1. $x_i-x_{i-1}=l$, $t_j-t_{j-1}=h$.

Existence and uniqueness will not be subjects of this paper. In this paper, we shall present a numerical method that could be used to derive a numerical solution to the above type of partial derivative. To proceed, we first convert the above equation into an integral equation by applying the corresponding integral to obtain

$$\begin{cases} u(x,t)=\dfrac{\beta}{\Gamma(\alpha)}\displaystyle\int_0^t \tau^{\beta-1}f\big(x,\tau,u(x,\tau)\big)(t-\tau)^{\alpha-1}d\tau,\\ u(x,0)=l(x),\\ u(0,t)=g(t).\end{cases} \tag{2}$$

In order to create new data points within the range of a discrete set of known data points, a curve fitting technique known as linear interpolation uses linear polynomials. Using two known values of the function f at different positions, linear interpolation is frequently used to approximate the value of a function. Since the beginning of time, tables have been filled in by linear interpolation. In this section, we derive a numerical method for partial differential equations with fractal fractional derivatives, where the space is approximated using the forward, backward, central, or Crank-Nicolson approximations, and the temporal discretization is approximated using linear interpolation. At (x_i,t_{n+1}) we have

$$\begin{cases} u(x_i,t_{n+1})=\dfrac{\beta}{\Gamma(\alpha)}\displaystyle\int_0^{t_{n+1}} \tau^{\beta-1}f\big(x_i,\tau,u(x_i,\tau)\big)(t_{n+1}-\tau)^{\alpha-1}d\tau,\\ u(x_i,0)=l(x_i),\\ u(0,t_{n+1})=g(t_{n+1}),\end{cases} \tag{3}$$

$$\begin{cases} u(x_i,t_{n+1})=\dfrac{\beta}{\Gamma(\alpha)}\displaystyle\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} \tau^{\beta-1}f\big(x_i,\tau,u(x_i,\tau)\big)(t_{n+1}-\tau)^{\alpha-1}d\tau,\\ u(x_i,0)=l(x_i),\\ u(0,t_{n+1})=g(t_{n+1}).\end{cases} \tag{4}$$

Within $[t_j,t_{j+1}]$, we approximate $f\big(x_i,\tau,u(x_i,\tau)\big)\approx P_j(\tau)$
where $P_j(\tau)$ is a linear interpolation.

$$P_j(\tau)=f\Big(x_i,t_j,u\big(x_i,t_j\big)\Big)+\frac{f\Big(x_i,t_{j+1},u\big(x_i,t_{j+1}\big)\Big)-f\Big(x_i,t_j,u\big(x_i,t_j\big)\Big)}{h}\big(\tau-t_j\big). \tag{5}$$

Substituting into the original equation, we obtain

$$u(x_i,t_{n+1})\approx\frac{\beta}{\Gamma(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\tau^{\beta-1}P_j(\tau)(t_{n+1}-\tau)^{\alpha-1}d\tau, \tag{6}$$

$$u_i^{n+1} = \frac{\beta}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \tau^{\beta-1} \Big\{ f\big(x_i, t_j, u_i^j\big) + \big(\tau - t_j\big) \frac{f\big(x_i, t_{j+1}, u_i^{j+1}\big) - f\big(x_i, t_j, u_i^j\big)}{h} (t_{n+1} - \tau)^{\alpha-1} d\tau \Big\}, \tag{7}$$

$$u_i^{n+1} = \frac{\beta}{\Gamma(\alpha)} \sum_{j=0}^{n} f\big(x_i, t_j, u_i^j\big) \int_{t_j}^{t_{j+1}} \tau^{\beta-1} (t_{n+1} - \tau)^{\alpha-1} d\tau + \frac{\beta}{\Gamma(\alpha)} \sum_{j=0}^{n} \frac{\big(f_i^{j+1} - f_i^j\big)}{h} \int_{t_j}^{t_{j+1}} \tau^{\beta-1} (t_{n+1} - \tau)^{\alpha-1} \big(\tau - t_j\big) d\tau. \tag{8}$$

We have to evaluate the following integral:

$$\int_{t_j}^{t_{j+1}} \tau^{\beta-1} (t_{n+1} - \tau)^{\alpha-1} \text{ and } \int_{t_j}^{t_{j+1}} \tau^{\beta-1} (t_{n+1} - \tau)^{\alpha-1} \big(\tau - t_j\big) d\tau. \tag{9}$$

We should recall that the generalization of the complete beta function is defined as

$$B(x, a, b) = \int_0^x t^{a-1} (1 - t)^{b-1} d\tau. \tag{10}$$

Thus,

$$\int_{t_j}^{t_{j+1}} \tau^{\beta-1} (t_{n+1} - \tau)^{\alpha-1} \, d\tau = \int_0^{t_{j+1}} \tau^{\beta-1} (t_{n+1} - \tau)^{\alpha-1} \, d\tau - \int_0^{t_j} \tau^{\beta-1} (t_{n+1} - \tau)^{\alpha-1} \, d\tau. \tag{11}$$

Thus,

$$\int_0^{t_{j+1}} \tau^{\beta-1} (t_{n+1} - \tau)^{\alpha-1} \, d\tau = t_{n+1}^{\alpha+\beta-1} \int_0^{\frac{t_{j+1}}{t_{n+1}}} y^{\beta-1} (1 - y)^{\alpha-1} \, d\tau = t_{n+1}^{\alpha+\beta-1} B\left(\frac{t_{j+1}}{t_{n+1}}, \beta, \alpha\right). \tag{12}$$

$$\int_0^{t_{j+1}} \tau^{\beta-1} (t_{n+1} - \tau)^{\alpha-1} \, d\tau = t_{n+1}^{\alpha+\beta-1} B\left(\frac{t_j}{t_{n+1}}, \beta, \alpha\right). \tag{13}$$

On the other hand, we have that

$$\int_{t_j}^{t_{j+1}} \tau^{\beta-1}(t_{n+1}-\tau)^{\alpha-1}\left(\tau-t_j\right) d\tau \tag{14}$$

$$= \int_{0}^{t_{j+1}} \tau^{\beta-1}(t_{n+1}-\tau)^{\alpha-1}\left(\tau-t_j\right) d\tau$$

$$- \int_{0}^{t_{j}} \tau^{\beta-1}(t_{n+1}-\tau)^{\alpha-1}\left(\tau-t_j\right) d\tau,$$

$$= \int_{0}^{t_{j+1}} \tau^{\beta}(t_{n+1}-\tau)^{\alpha-1} d\tau - t_j \int_{0}^{t_{j+1}} \tau^{\beta-1}(t_{n+1}-\tau)^{\alpha-1} d\tau \tag{15}$$

$$- \int_{0}^{t_{j}} \tau^{\beta}(t_{n+1}-\tau)^{\alpha-1} d\tau + t_j \int_{0}^{t_{j}} \tau^{\beta-1}(t_{n+1}-\tau)^{\alpha-1} d\tau,$$

$$= t_{n+1}^{\beta+\alpha} B\left(\frac{t_{j+1}}{t_{n+1}}, \beta+1+\alpha\right) - t_j\, t_{n+1}^{\alpha+\beta-1} B\left(\frac{t_{j+1}}{t_{n+1}}, \beta, \alpha\right) - t_{n+1}^{\alpha+\beta} B\left(\frac{t_{j}}{t_{n+1}}, \beta+1, \alpha\right) \tag{16}$$

$$+ t_j\, t_{n+1}^{\alpha+\beta-1} B\left(\frac{t_{j}}{t_{n+1}}, \beta, \alpha\right),$$

$$= t_{n+1}^{\beta+\alpha}\left(B\left(\frac{t_{j+1}}{t_{n+1}}, \beta+1, \alpha\right) - B\left(\frac{t_{j}}{t_{n+1}}, \beta+1, \alpha\right)\right) \tag{17}$$

$$+ t_{n+1}^{\beta+\alpha-1}\left(B\left(\frac{t_{j+1}}{t_{n+1}}, \beta, \alpha\right) - B\left(\frac{t_{j}}{t_{n+1}}, \beta, \alpha\right)\right),$$

Substituting into the original formula, we obtain the following:

$$u_i^{n+1} = \frac{\beta}{\Gamma(\alpha)} \sum_{j=0}^{n} \frac{\left(f_i^{j+1}-f_i^j\right)}{h} \left\{ t_{n+1}^{\alpha+\beta}\left(B\left(\frac{t_{j+1}}{t_{n+1}}, \beta+1, \alpha\right) - B\left(\frac{t_{j}}{t_{n+1}}, \beta+1, \alpha\right)\right)\right. \tag{18}$$

$$\left. + t_{n+1}^{\beta+\alpha-1}\left(B\left(\frac{t_{j+1}}{t_{n+1}}, \beta, \alpha\right) - B\left(\frac{t_{j}}{t_{n+1}}, \beta, \alpha\right)\right)\right\}$$

$$+ \frac{\beta}{\Gamma(\alpha)} \sum_{j=0}^{n} f_i^j\, t_{n+1}^{\alpha+\beta-1}\left(B\left(\frac{t_{j+1}}{t_{n+1}}, \beta, \alpha\right) - B\left(\frac{t_{j}}{t_{n+1}}, \beta, \alpha\right)\right).$$

Note that,

$$f_i^{j+1} = f\left(x_i, t_{j+1}, u\left(x_i, t_{j+1}\right)\right), \tag{19}$$

$$f_i^{j} = f\left(x_i, t_{j}, u\left(x_i, t_{j}\right)\right). \tag{20}$$

Thus, the general method leads to

$$u_i^{n+1}=\frac{\beta}{\Gamma(\alpha)}\sum_{j=0}^{n}\Big(f\left(x_i,t_{j+1},u(x_i,t_{j+1})\right)$$
$$-f\left(x_i,t_j,u(x_i,t_j)\right)\Big)\Big\{t_{n+1}^{\alpha+\beta}\left[B\left(\frac{t_{j+1}}{t_{n+1}},\beta+1,\alpha\right)-B\left(\frac{t_j}{t_{n+1}},\beta+1,\alpha\right)\right]$$
$$+t_{n+1}^{\alpha+\beta-1}\left[B\left(\frac{t_{j+1}}{t_{n+1}},\beta,\alpha\right)-B\left(\frac{t_j}{t_{n+1}},\beta,\alpha\right)\right]\Big\}$$
$$+\frac{\beta}{\Gamma(\alpha)}\sum_{j=0}^{n}t_{n+1}^{\alpha+\beta-1}f\left(x_i,t_j,u(x_i,t_j)\right)\left[B\left(\frac{t_{j+1}}{t_{n+1}},\beta,\alpha\right)-B\left(\frac{t_j}{t_{n+1}},\beta,\alpha\right)\right]. \tag{21}$$

The equation can be reformulated as

$$u_i^{n+1}=\frac{\beta}{\Gamma(\alpha)}\sum_{j=0}^{n-1}\Big(f\left(x_i,t_{j+1},u(x_i,t_{j+1})\right)$$
$$-f\left(x_i,t_j,u(x_i,t_j)\right)\Big)\Big\{t_{n+1}^{\alpha+\beta}\left[B\left(\frac{t_{j+1}}{t_{n+1}},\beta+1,\alpha\right)-B\left(\frac{t_j}{t_{n+1}},\beta+1,\alpha\right)\right]$$
$$+t_{n+1}^{\alpha+\beta-1}\left[B\left(\frac{t_{j+1}}{t_{n+1}},\beta,\alpha\right)-B\left(\frac{t_j}{t_{n+1}},\beta,\alpha\right)\right]\Big\}$$
$$+\frac{\beta}{\Gamma(\alpha)}\sum_{j=0}^{n}f\left(x_i,t_j,u(x_i,t_j)\right)\Big[t_{n+1}^{\alpha+\beta-1}B\left(\frac{t_{j+1}}{t_{n+1}},\beta,\alpha\right)$$
$$-t_{n+1}^{\alpha+\beta-1}B\left(\frac{t_j}{t_{n+1}},\beta,\alpha\right)\Big]$$
$$+\frac{\beta}{\Gamma(\alpha)}\Big(f\left(x_i,t_{n+1},u^p(x_i,t_{n+1})\right)$$
$$-f\left(x_i,t_j,u(x_i,t_n)\right)\Big)\Big[\Big\{B(1,\beta+1,\alpha)-B\left(\frac{t_n}{t_{n+1}},\beta+1,\alpha\right)\Big\}t_{n+1}^{\alpha+\beta}$$
$$+t_{n+1}^{\alpha+\beta-1}\Big\{B(1,\beta,\alpha)-B\left(\frac{t_j}{t_{n+1}},\beta,\alpha\right)\Big\}\Big], \tag{22}$$

where the $u^p(x_i,t_{n+1})$ is the predictor term that can be obtained like in [12]

$$u^p(x_i,t_{n+1})=\frac{\beta}{\Gamma(\alpha)}\sum_{j=0}^{n}f\left(x_i,t_j,u(x_i,t_j)\right)\int_{t_j}^{t_{j+1}}\tau^{\beta-1}(t_{n+1}-\tau)^{\alpha-1}\,d\tau, \tag{23}$$

$$=\frac{\beta}{\Gamma(\alpha)}\sum_{j=0}^{n}f\left(x_i,t_j,u(x_i,t_j)\right)t_{n+1}^{\alpha+\beta-1}\Big\{B\left(\frac{t_{j+1}}{t_{n+1}},\beta,\alpha\right)-B\left(\frac{t_j}{t_{n+1}},\beta,\alpha\right)\Big\}. \tag{24}$$

Therefore, the general scheme is given by

$$
\begin{aligned}
u_i^{n+1} = {} & \frac{\beta}{h\Gamma(\alpha)} \sum_{j=0}^{n-1} \Big(f\left(x_i, t_{j+1}, u(x_i, t_{j+1})\right) \\
& - f\left(x_i, t_j, u(x_i, t_j)\right)\Big) \left\{ t_{n+1}^{\alpha+\beta} \left(B\left(\frac{t_{j+1}}{t_{n+1}}, \beta+1, \alpha\right) - B\left(\frac{t_j}{t_{n+1}}, \beta+1, \alpha\right)\right) \right. \\
& \left. - t_{n+1}^{\alpha+\beta-1} \left(B\left(\frac{t_{j+1}}{t_{n+1}}, \beta, \alpha\right) - B\left(\frac{t_j}{t_{n+1}}, \beta, \alpha\right)\right)\right\} \\
& + \frac{\beta}{\Gamma(\alpha)} \sum_{j=0}^{n} f\left(x_i, t_j, u(x_i, t_j)\right) t_{n+1}^{\alpha+\beta-1} \left(B\left(\frac{t_{j+1}}{t_{n+1}}, \beta, \alpha\right) - B\left(\frac{t_j}{t_{n+1}}, \beta, \alpha\right)\right) \\
& + \frac{\beta}{h\Gamma(\alpha)} \left[f\left(x_i, t_{n+1}, \frac{\beta}{\Gamma(\alpha)} \sum_{j=0}^{n} f\left(x_i, t_j, u(x_i, t_j)\right) t_{n+1}^{\alpha+\beta-1} \left(B\left(\frac{t_{j+1}}{t_{n+1}}, \beta, \alpha\right) \right.\right.\right. \\
& \left.\left. - B\left(\frac{t_j}{t_{n+1}}, \beta, \alpha\right)\right)\right) \\
& \left. - f\left(x_i, t_n, u(x_i, t_n)\right) \right] \left[t_{n+1}^{\alpha+\beta} \left\{ B(1, \beta+1, \alpha) - B\left(\frac{t_n}{t_{n+1}}, \beta+1, \alpha\right)\right\} \right. \\
& \left. - t_{n+1}^{\alpha+\beta-1} \left\{ B(1, \beta, \alpha) - B\left(\frac{t_j}{t_{n+1}}, \beta, \alpha\right)\right\}\right].
\end{aligned} \tag{25}
$$

2.2. *Fractal-fractional partial differential equation with Mittag-Leffler kernel*

Modeling fractional order viscoelastic materials and the decay process of a human corpse are two examples of how the Mittag-Leffler function is used. When a human decomposes, blood and flesh cause quick degradation at first, followed by bones that decompose more slowly. Experimental studies into the time-dependent relaxation behavior of viscoelastic materials show that these materials exhibit a very rapid reduction in stress at the start of the process and an incredibly sluggish decline for a significant amount of time.

We now consider a general partial differential equation.

$$
\begin{cases}
{}^{FFM}_{\ \ 0}D_t^{\alpha,\beta} u(x,t) = f\left(x, t, u(x,t)\right), \\
u(x,0) = l(x), \\
u(0,t) = g(t).
\end{cases} \tag{26}
$$

The conditions presented earlier are still valid here. To proceed, we covert this into

$$
{}^{ABR}_{\ \ 0}D_t^{\alpha} u(x,t) = \beta t^{\beta-1} f\left(x, t, u(x,t)\right). \tag{27}
$$

Then,

$$u(x,t) - u(x,0) = (1-\alpha)\beta t^{\beta-1} f\big(x,t,u(x,t)\big) + \frac{\alpha\beta}{\Gamma(\alpha)} \int_0^t \tau^{\beta-1} (t-\tau)^{\alpha-1} f\big(x,\tau,u(x,\tau)\big) d\tau, \tag{28}$$

$$u(x_i,t_{n+1}) - u(x_i,0) = (1-\alpha)\beta t_{n+1}^{\beta-1} f\big(x_i,t_{n+1},u^p(x_i,t_{n+1})\big) + \frac{\alpha\beta}{\Gamma(\alpha)} \int_0^{t_{n+1}} \tau^{\beta-1} (t_{n+1}-\tau)^{\alpha-1} f\big(x_i,\tau,u(x_i,\tau)\big) d\tau, \tag{29}$$

$$u(x_i,t_{n+1}) = u(x_i,0) + (1-\alpha)\beta t_{n+1}^{\beta-1} f\big(x_i,t_{n+1},u^p(x_i,t_{n+1})\big) + \frac{\alpha\beta}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \tau^{\beta-1} (t_{n+1}-\tau)^{\alpha-1} f\big(x_i,\tau,u(x_i,\tau)\big) d\tau, \tag{30}$$

$$u(x_i,t_{n+1}) \approx u(x_i,0) + (1-\alpha)\beta t_{n+1}^{\beta-1} f\big(x_i,t_{n+1},u^p(x_i,t_{n+1})\big) + \frac{\alpha\beta}{\Gamma(\alpha)} \sum_{j=1}^{n} \int_{t_j}^{t_{j+1}} \left[\frac{\tau - t_{j-1}}{h} f\left(x_i,t_j,u(x_i,t_j)\right) - \frac{\tau - t_j}{h} f\left(x_i,t_{j-1},u(x_i,t_{j-1})\right)\right] (t_{n+1}-\tau)^{\alpha-1} \tau^{\beta-1} d\tau, \tag{31}$$

$$u(x_i,t_{n+1}) \approx u_i^{n+1} = u_i^0 + \frac{\alpha\beta}{h\Gamma(\alpha)} \sum_{j=1}^{n} f\left(x_i,t_j,u(x_i,t_j)\right) \int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{\alpha-1} \tau^{\beta-1} (\tau - t_j) d\tau - \frac{\alpha\beta}{h\Gamma(\alpha)} \sum_{j=1}^{n} f\left(x_i,t_{j-1},u(x_i,t_{j-1})\right) \int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{\alpha-1} \tau^{\beta-1} (\tau - t_{j-1}) d\tau + (1-\alpha) f\big(x_i,t_{n+1},u^p(x_i,t_{n+1})\big) \beta t_{n+1}^{\beta-1}. \tag{32}$$

However,

$$\int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{\alpha-1} (\tau - t_j) \tau^{\beta-1} d\tau = \int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{\alpha-1} \tau^{\beta} d\tau - t_j \int_{t_j}^{t_{j+1}} \tau^{\beta-1} (t_{n+1}-\tau)^{\alpha-1} d\tau, \tag{33}$$

$$\int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{\alpha-1}(\tau - t_{j-1})\tau^{\beta-1} d\tau$$
$$= \int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{\alpha-1}\tau^{\beta} d\tau - t_{j-1}\int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{\alpha-1}\tau^{\beta-1} d\tau. \tag{34}$$

Therefore,

$$\int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{\alpha-1}(\tau - t_j)\tau^{\beta-1} d\tau = t_{n+1}^{\alpha+\beta}\left[B\left(\frac{t_{j+1}}{t_{n+1}},\beta+1,\alpha\right) - B\left(\frac{t_j}{t_{n+1}},\beta+1,\alpha\right)\right]$$
$$- t_j t_{n+1}^{\alpha+\beta-1}\left[B\left(\frac{t_{j+1}}{t_{n+1}},\beta,\alpha\right) - B\left(\frac{t_j}{t_{n+1}},\beta,\alpha\right)\right], \tag{35}$$

$$\int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{\alpha-1}\tau^{\beta-1}(\tau - t_{j-1}) d\tau$$
$$= t_{n+1}^{\alpha+\beta}\left[B\left(\frac{t_{j+1}}{t_{n+1}},\beta+1,\alpha\right) - B\left(\frac{t_j}{t_{n+1}},\beta+1,\alpha\right)\right]$$
$$- t_{j-1} t_{n+1}^{\alpha+\beta-1}\left[B\left(\frac{t_{j+1}}{t_{n+1}},\beta,\alpha\right) - B\left(\frac{t_j}{t_{n+1}},\beta,\alpha\right)\right], \tag{36}$$

Substituting the above into the original equation yields

$$u_i^{n+1} = u_i^0 + \frac{\alpha\beta}{h\Gamma(\alpha)}\sum_{j=1}^{n} f\left(x_i, t_j, u(x_i,t_j)\right)\left[t_{n+1}^{\alpha+\beta}\left\{B\left(\frac{t_{j+1}}{t_{n+1}},\beta+1,\alpha\right)\right.\right.$$
$$\left.\left. - B\left(\frac{t_j}{t_{n+1}},\beta+1,\alpha\right)\right\} - t_j t_{n+1}^{\alpha+\beta-1}\left\{B\left(\frac{t_{j+1}}{t_{n+1}},\beta,\alpha\right) - B\left(\frac{t_j}{t_{n+1}},\beta,\alpha\right)\right\}\right]$$
$$- \frac{\alpha\beta}{h\Gamma(\alpha)}\sum_{j=1}^{n} f\left(x_i, t_{j-1}, u(x_i,t_{j-1})\right)\left[t_j t_{n+1}^{\alpha+\beta-1}\left\{B\left(\frac{t_{j+1}}{t_{n+1}},\beta,\alpha\right)\right.\right.$$
$$\left.\left. - B\left(\frac{t_j}{t_{n+1}},\beta,\alpha\right)\right\}\right] + \beta t_{n+1}^{\beta-1}(1-\alpha) f\left(x_i, t_{n+1}, u^p(x_i,t_{n+1})\right), \tag{37}$$

where

$$u^p_{(x_i,t_{n+1})} = \frac{\beta}{\Gamma(\alpha)}\sum_{j=0}^{n} f\left(x_i, t_j, u(x_i,t_j)\right) t_{n+1}^{\alpha+\beta-1}\left\{B\left(\frac{t_{j+1}}{t_{n+1}},\beta,\alpha\right) - B\left(\frac{t_j}{t_{n+1}},\beta,\alpha\right)\right\}. \tag{38}$$

In an application to some partial differential equation, we consider here a fractal-fractional heat equation:

$$\begin{cases} {}^{FFP}_{\ \ 0}D_t^{\alpha,\beta} u(x,t) = a\dfrac{\partial^2 u(x,t)}{\partial x^2}, \\ u(x,0) = f(x), \\ u(0,t) = 0 = u(L,t) \ \ \forall t > 0, \end{cases} \tag{39}$$

$$\begin{cases} {}^{FFM}_{\ 0}D_t^{\alpha,\beta} u(x,t) = a\dfrac{\partial^2 u(x,t)}{\partial x^2}, \\ u(x,0) = f(x), \\ u(0,t) = 0 = u(L,t) \ \ \forall t > 0, \end{cases} \tag{40}$$

We start with the first equation.

$$\begin{cases} {}^{RL}_{\ 0}D_t^{\alpha} u(x,t) = \beta t^{\beta-1} a\dfrac{\partial^2 u(x,t)}{\partial x^2}, \\ u(x,0) = f(x), \\ u(0,t) = 0 = u(L,t) \ \ \forall t > 0, \end{cases} \tag{41}$$

$$\begin{cases} u(x,t) = u(x,0) - \dfrac{\beta}{\Gamma(\alpha)} a \displaystyle\int_0^t \tau^{\beta-1} \dfrac{\partial^2 u(x,t)}{\partial x^2} \ (t-\tau)^{\alpha-1} d\tau, \\ u(x,0) = f(x), \\ u(0,t) = 0 = u(L,t) \ \ \forall t > 0. \end{cases} \tag{42}$$

Applying the presented procedure yields

$$\begin{aligned} u_i^{n+1} & \\ = \frac{\beta}{\Gamma(\alpha)} \sum_{j=0}^{n-1} & f\left(x_i, t_{j+1}, u(x_i, t_{j+1})\right) \\ - f\left(x_i, t_j, u(x_i, t_j)\right) & \left\{ t_{n+1}^{\alpha+\beta} \left(B\left(\frac{t_{j+1}}{t_{n+1}}, \beta+1, \alpha\right) - t_{n+1}^{\alpha+\beta} B\left(\frac{t_j}{t_{n+1}}, \beta+1, \alpha\right)\right)\right. \\ - t_{n+1}^{\alpha+\beta-1} B & \left.\left(\frac{t_{j+1}}{t_{n+1}}, \beta, \alpha\right) + t_{n+1}^{\alpha+\beta-1} B\left(\frac{t_j}{t_{n+1}}, \beta, \alpha\right)\right\} \\ + \frac{\beta}{\Gamma(\alpha)} \sum_{j=0}^{n} & f\left(x_i, t_j, u(x_i, t_j)\right) t_{n+1}^{\alpha+\beta-1} \left(B\left(\frac{t_{j+1}}{t_{n+1}}, \beta, \alpha\right) - B\left(\frac{t_j}{t_{n+1}}, \beta, \alpha\right)\right) \\ + \frac{\beta}{h\Gamma(\alpha)} & \left[f\left(x_i, t_{n+1}, \frac{\beta}{\Gamma(\alpha)} \sum_{j=0}^{n} f\left(x_i, t_j, u(x_i, t_j)\right)\right) t_{n+1}^{\alpha+\beta-1} \left(B\left(\frac{t_{j+1}}{t_{n+1}}, \beta, \alpha\right)\right.\right. \\ - B & \left.\left.\left(\frac{t_j}{t_{n+1}}, \beta, \alpha\right)\right) - t_{n+1}^{\alpha+\beta-1} \left\{ B(1,\beta,\alpha) - B\left(\frac{t_j}{t_{n+1}}, \beta, \alpha\right)\right\}\right]. \end{aligned} \tag{43}$$

where,

$$f\left(x_i, t_j, u(x_i, t_j)\right) = \alpha \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{(\Delta x)^2}. \tag{44}$$

For the second equation, we have

$$\begin{cases} {}^{ABR}_{\ 0}D_t^{\alpha} u(x,t) = \beta t^{\beta-1} \dfrac{\partial^2 u(x,t)}{\partial x^2}, \\ u(x,0) = f(x), \\ u(0,t) = 0 = u(L,t) \ \ \forall t > 0\,, \end{cases} \tag{45}$$

$$\begin{cases} u(x,t) = (1-\alpha)\beta t^{\beta-1} \dfrac{\partial^2 u(x,t)}{\partial x^2} + \dfrac{\alpha\beta}{\Gamma(\alpha)} \displaystyle\int_0^t \dfrac{\partial^2 u(x,\tau)}{\partial x^2} \tau^{\beta-1} (t-\tau)^{\alpha-1} d\tau, \\ \qquad u(x,0) = f(x), \\ \qquad u(0,t) = 0 = u(L,t), \forall t > 0. \end{cases} \tag{46}$$

Applying the procedure presented earlier yields

$$\begin{aligned} u_i^{n+1} = (1-\alpha)\beta t_{n+1}^{\beta-1} &\left[a \frac{u_{i+1}^{p_{n+1}} - u_i^{p_{n+1}} + u_{i-1}^{p_{n+1}}}{\Delta x} \right] \\ &+ \frac{\alpha\beta}{h\Gamma(\alpha)} \sum_{j=1}^{n} \frac{u_{i+1}^{j} - 2u_i^{j} + u_{i-1}^{j}}{(\Delta x)^2} \left[t_{n+1}^{\beta+\alpha} \left(B\left(\frac{t_{j+1}}{t_{n+1}}, \beta+1, \alpha \right) \right.\right. \\ &\left.\left. - B\left(\frac{t_j}{t_{n+1}}, \beta+1, \alpha \right) \right) - t_{n+1}^{\alpha+\beta-1} t_j \left(B\left(\frac{t_{j+1}}{t_{n+1}}, \beta, \alpha \right) - B\left(\frac{t_j}{t_{n+1}}, \beta, \alpha \right) \right) \right] \\ &- \frac{\alpha\beta}{h\Gamma(\alpha)} \sum_{j=1}^{n} \frac{u_{i+1}^{j-1} - 2u_i^{j-1} + u_{i-1}^{j-1}}{(\Delta x)^2} \left[t_j t_{n+1}^{\alpha+\beta-1} \left\{ B\left(\frac{t_{j+1}}{t_{n+1}}, \beta, \alpha \right) \right.\right. \\ &\left.\left. - B\left(\frac{t_j}{t_{n+1}}, \beta, \alpha \right) \right\} \right]. \end{aligned} \tag{47}$$

The above solution can be used for numerical simulations.

3. Comparison

In this section, we compare the power law and the extended Mittag-Leffler function by applying them to a given function and evaluating their Bode diagrams to gratify readers who are unfamiliar with these concepts and show their impacts on a certain dynamical process. First, it should be noted that, as shown in the pictures below, the power law is scale invariant, whereas the generalized Mittag-Leffler function exhibits crossover behaviors from stretched exponential to power law.

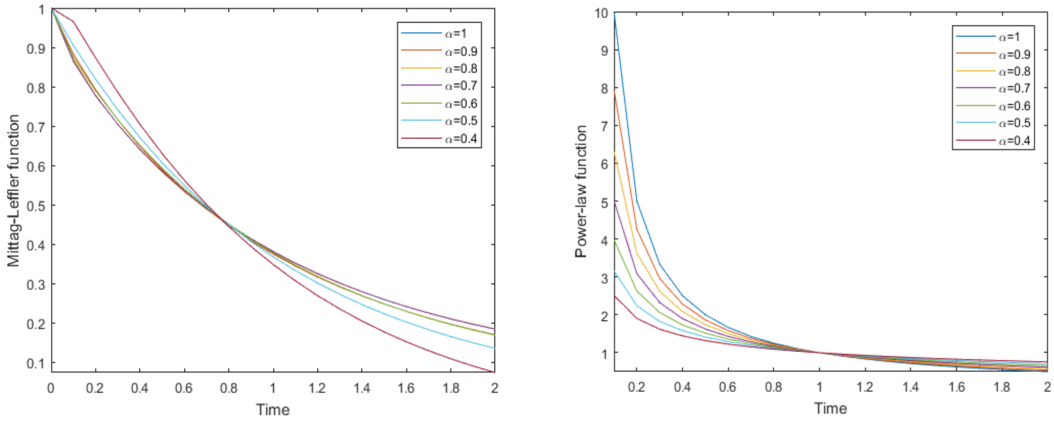

Figure 1. Mittag-Leffler function and power law.

In the figure below, we show their corresponding magnitude diagrams obtained via their respective Laplace transform for different values of fractional order.

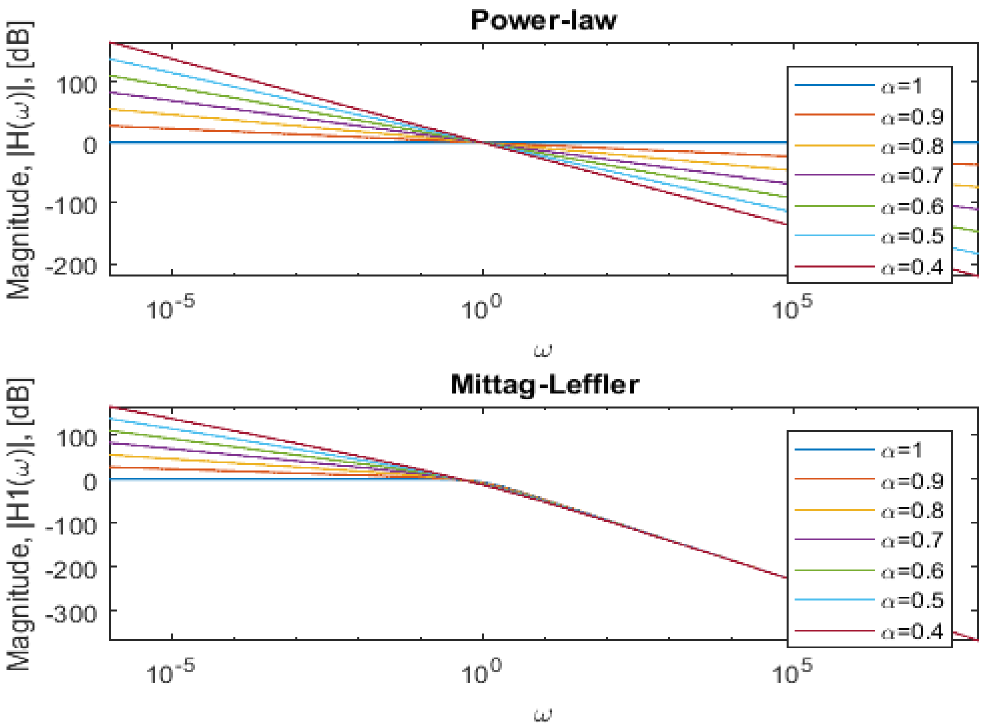

Figure 2. Diagrams associated with power law and generalized-Mittag-Leffler functions.

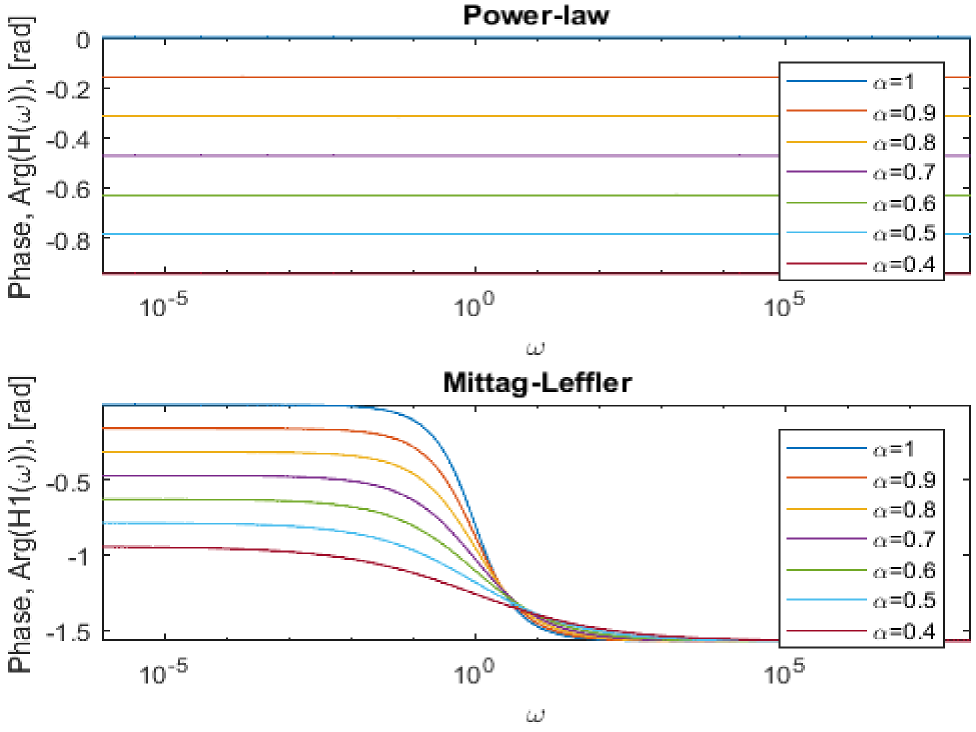

Figure 3. Phase diagrams associated with power law and generalized-Mittag-Leffler functions.

4. Numerical simulations

We present some numerical simulations of the heat equation using the suggested numerical solution. To perform this simulation, the following are considered:

$$u(x,0) = \exp(-x), \qquad \Delta x = 0.05, \Delta t = 0.01$$

Numerical simulations are depicted in Figures 4, 5 and 6 below

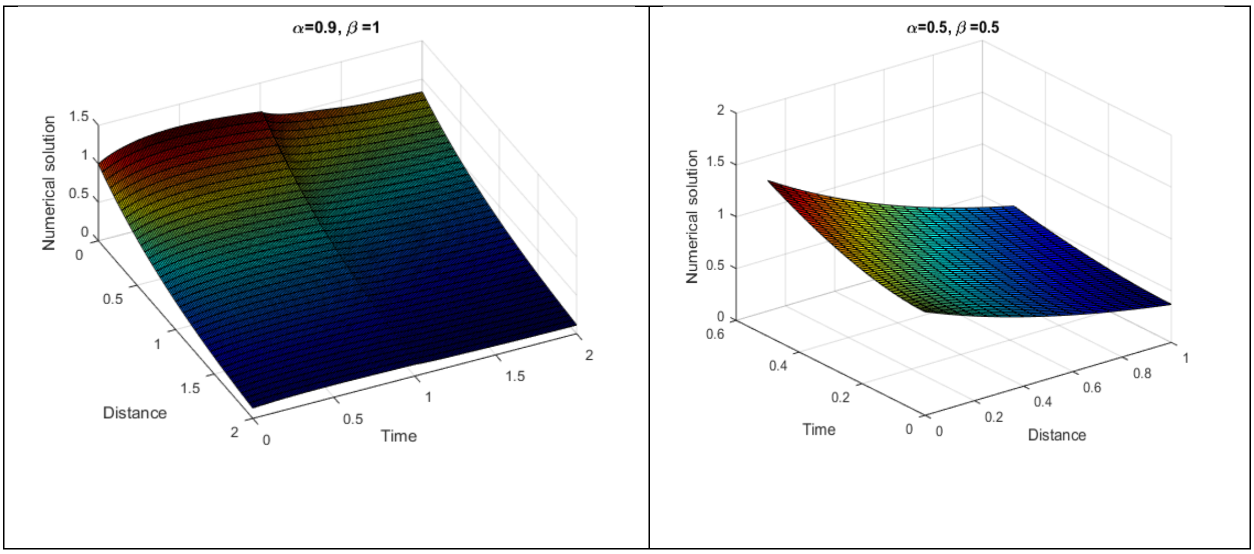

Figure 4. Numerical simulation of the model with the generalized-Mittag-Leffler kernel.

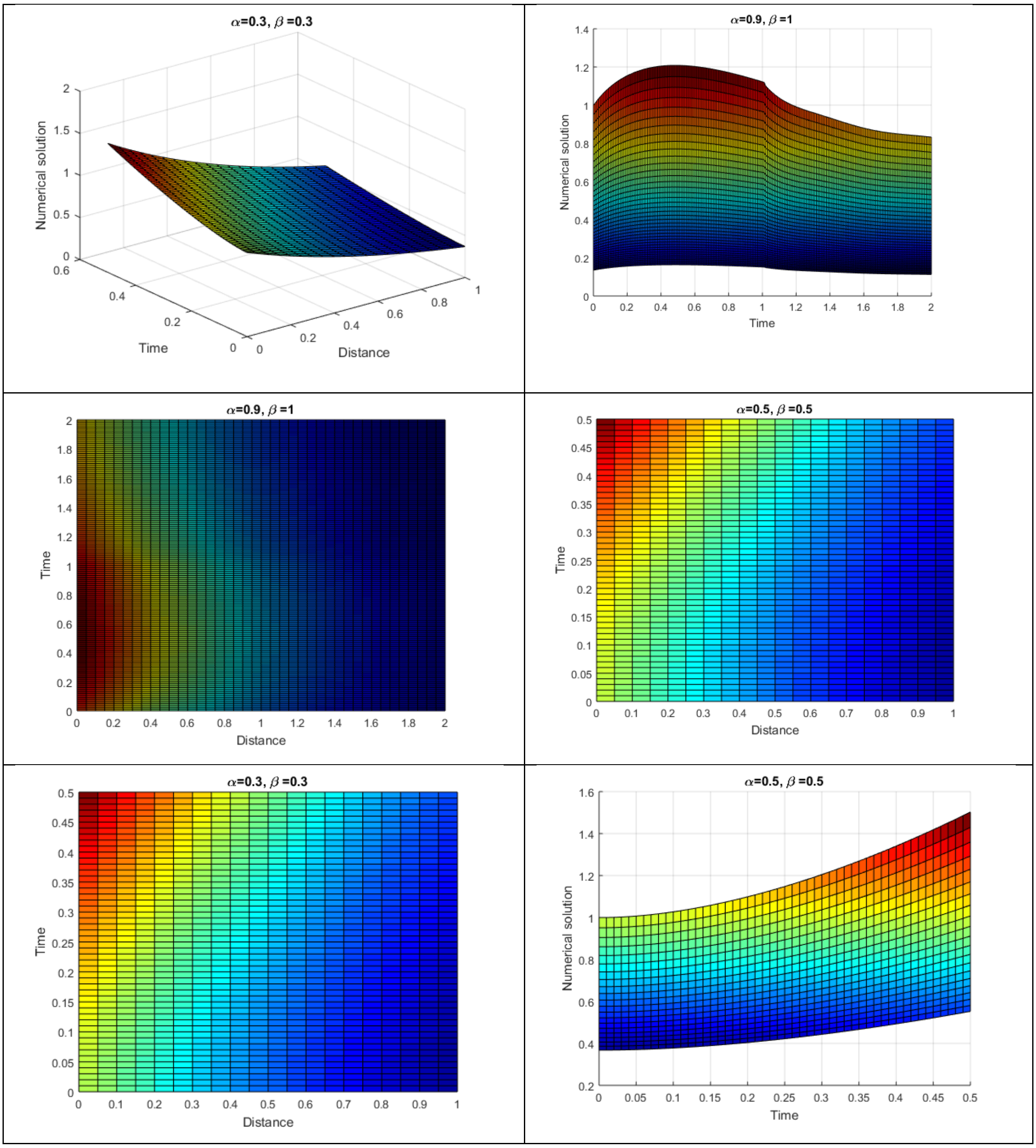

Figure 5. Numerical simulation of the model with power law.

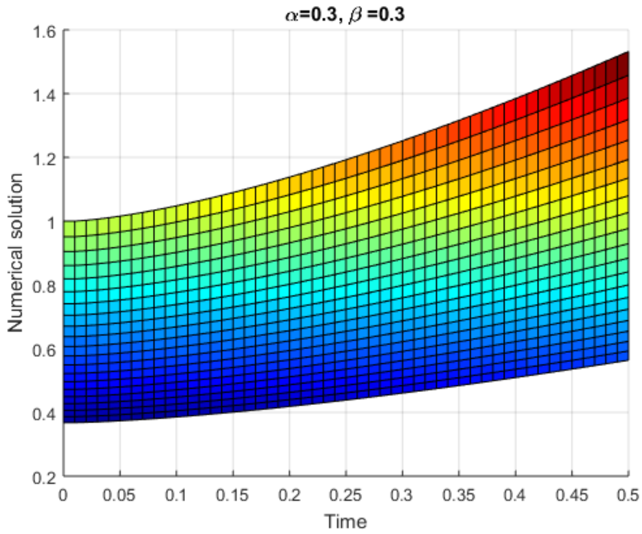

Figure 6. Numerical simulation for model with power law.

From the obtained figures one can see that the component of fractal dimension adds some features especially to the contour plot mapping.

5. Conclusions

Given that the idea was just suggested a few years ago, partial differential equations with fractal-fractional derivatives have naturally received little attention in recent years. Particularly, unlike in the case of ordinary differential equations with fractal-fractional derivatives, numerical approaches for solving these problems have not been thoroughly developed and validated. With the help of the generalized Mittag-Leffler kernel and the power law, an attempt was made to provide a numerical method for solving generic fractal-fractional partial differential equations in this study. The work did not discuss theoretical ideas like convergence and stability. The fractal-fractional heat equation, however, was considered and solved using the recommended technique and several simulations. It was shown that, for the case of the generalized Mittag-Leffler function, there is a trend of crossover behaviors characterizing the passage from stretched exponential to power law, with some similarities occurring due to the fractal dimension. This can be seen as the movement of the plume from matrix rocks to fracture, where within the matrix rock, we have fading memory behaviors, and in fracture we have long-range behaviors. On the other hand, for the power law case, one can see a trend of long-range dependency throughout, which is to be interpretated as plume, moving along preferential paths known as fractures.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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