Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(1): 2168–2190.
DOI: 10.3934/math.2023112
Received: 05 September 2022
Revised: 08 October 2022
Accepted: 11 October 2022
Published: 28 October 2022

Research article

A unified generalization for Hukuhara types differences and derivatives: Solid analysis and comparisons

Babak Shiri*

Data Recovery Key Laboratory of Sichuan Province, College of Mathematics and Information Science, Neijiang Normal University, Neijiang 641100, China

* **Correspondence:** Email: shire_babak@yahoo.com; Tel: 08322382097; Fax: 08322343159.

Abstract: Uncertain numbers, in a parallel definition of fuzzy numbers, are introduced. Model uncertainty and measurement uncertainty are our motivations for this study. A class of scalar multiplication and differences is proposed. Related algebra is investigated. A necessary and sufficient condition of the existence of the introduced differences is obtained. Then, the existing result for the derivative is studied. Many interestingly important results are obtained. For example, the Hukuhara derivative does not exist for any fuzzy function with the new viewpoint. Constructive conditions for the existence of the generalized Hukuhara derivative are introduced. Four possible categories for derivatives fall into two forms of the fuzzy derivative for the generalized Hukuhara derivative. Importantly, this bifurcation in the definition of the new generalized Hukuhara derivative does not happen. Finally, all definitions related to differences and derivatives of uncertain numbers are unified in one concrete form with concrete analysis. Some examples and counterexamples are provided to illustrate theories and theorems in detail.

Keywords: Hukuhara derivative; generalized Hukuhara derivative; fuzzy numbers; bifurcation in fuzzy derivative; measurement uncertainty; interval-valued functions
Mathematics Subject Classification: 26E50, 54A40

1. Introduction

All measurements have some degree of uncertainty that may come from various sources. There are many theories for presenting and analyzing measurement uncertainty. Statistical analysis, interval analysis and fuzzy analysis are some of them. Let $y(t)$ show the displacement at time t for a point of a building during an earthquake. To illustrate this uncertainty in measurement, we can use

$$y(t) = y_M(t) \pm \delta(t) \text{ (unit of measurement)},$$

where $y_M(t)$ is the measured value, and $\delta(t) \geq 0$ is the uncertainty at time t. Some literature uses the standard notation

$$y(t) = y_M(t) \pm \rho(t) y_M(t) \text{ (unit of measurement)}, \tag{1.1}$$

to cover uncertainty, where $\rho(t)$ is relative uncertainty. This means

$$y(t) \in [y_M(t) - \rho(t) y_M(t), y_M(t) + \rho(t) y_M(t)].$$

We want to investigate the effect of $\rho(t)$ on dynamical processes. Thus, we will use a Homotopy with uncertainty parameter $r \in [0, 1]$ such that $\rho(t, r)$ for $r = 0$ attains the largest bound for uncertainty, and for $r = 1$ it becomes a deterministic point (Figure 1). Thus, we impose on ρ the following condition:

$$\rho(t, 1) = 0, \text{ deterministic condition.} \tag{1.2}$$

Figure 1. Map from uncertainty toward determinism.

If we consider unsymmetrical uncertainty, we can consider the following uncertain interval with two different function ρ_1 and ρ_2:

$$y(t) \in [y_M(t) - \rho_1(r)|y_M(t)|, y_M(t) + \rho_2(r)|y_M(t)|].$$

Conclusively, interval analysis and fuzzy theory in connection to each other are important tools for describing uncertainty and its related dynamics in applied dynamical systems.

A challenging question is how to define the rate of change (derivative) of an uncertain function. To define a derivative only by Hukuhara difference, Bede and Gal introduced a strongly generalized derivative [1]. In this way, if one of the four types of derivatives exists, then it is a strongly defined derivative. Obviously, it is a solution for defining a derivative but not the best. The weaker form can be more applied and constructive in the means that all four types should exist. However, even accepting one form leads to restricting the space of differential functions for the Hukuhara derivative. In this paper, we investigate such restrictions in more detail. We think this is a constructive analysis to go further and use the correct form in the applied model.

Stefanini and Bede introduced a better solution for defining a derivative by generalizing Hukuhara's definition [2–4]. Since then, extensive studies with applications, characteristics, critics and more generalizations have emerged (for examples, see [5–11]). In this direction, we generalize some other differences and investigate their effects on defining a derivative.

The generalized Hukuhara derivative for an interval-valued function has been studied in [12, 13] in detail. We remark that the interval-valued functions are related to fuzzy-valued functions, but conceptually they are different. For recent developments on interval-valued functions, analysis and applications, one can consult [14–19].

In this paper, we introduce uncertain numbers by intervals as an equivalent concept of parametric fuzzy numbers. We propose various "differences" and investigate their properties. Surveying scalar derivatives, we introduce a new scalar derivative, such that the place of the lower and the upper uncertainty does not change. This leads to introducing a new generalized difference induced by this scalar multiplication. The new difference has the property that the uncertainties of similar parts make contributions to the resulting difference.

Then, from these differences, we extend a concrete definition for the scalar multiplication and differences that covers the previous and the new definition. A very detailed analysis of the existence of differences and their well-definiteness is carried out with theorems that completely clarify when such differences exist or do not exist.

Finally, we turn to the definition of the derivative, and we survey and analyze previous differences and derivatives.

In detail, we study the difficulties in the definitions of derivatives. For example, we will see that in the presence of uncertainty, the Hukuhara derivative for the fixed type of derivative cannot exist. The detailed analysis of the generalized Hukuhara derivative also shows not only bifurcation but also the restriction of the space of fuzzy derivative functions. However, we obtain better results with the new derivative.

Since information makes one decide what types of scalar multiplications, differences and derivatives are required in the modeling, we make a concrete analysis in the final sections for the general types of derivatives. We provide several examples in confirmation and clarification of this concrete analysis.

In Section 2, we define an uncertain number and investigate its connection to a fuzzy number. In Section 3, some new classes of scalar multiplication and Hukuhara differences are proposed. Finally, in Section 4 characterized theorems for various fuzzy derivatives are investigated.

2. Representation of the uncertainty by interval analysis and fuzzy theory

This section has two parts. In the first part, we introduce uncertain numbers (for the first time) as an equivalent tool in interval analysis relating to the parametric definition of fuzzy numbers. In the second part, we introduce the parametric form of fuzzy numbers and related equivalency theorems and equations.

2.1. Uncertain number

Throughout this paper, we denote by $\mathcal{K}$ the set of all closed real intervals $[a, b]$ such that $a \leq b$ and by $\mathcal{K}_{\mathcal{R}} \subset \mathcal{K}$ the set of degenerate intervals $[a, a]$ where $a \in \mathbb{R}$.

We regard $[a, a] = a$ as both an interval and a deterministic real number. Also, we define the space $C^M[a, b]$, consisting of all real valued functions

$$u : [a, b] \to \mathbb{R}$$

such that u is monotonically decreasing, left continuous on $(a, b]$ and right continuous at a. The restriction of this space to non-negative functions will be denoted by C^{M+}, i.e.,

$$C^{M+}[a, b] = \{u : [a, b] \to \mathbb{R}^+ : u \in C^M\}.$$

Definition 2.1. *An entirely uncertain number is a map $d_u : [0, 1] \to \mathcal{K}$ such that*

$$d_u(r) = [d - u_1(r), d + u_2(r)],\ r \in [0, 1], \tag{2.1}$$

where $d \in \mathbb{R}$, $u_1, u_2 \in C^{M+}[0,1]$. *Furthermore, if the condition*

$$u_1(1) = u_2(1) = 0 \tag{2.2}$$

holds, we say d_u *is an uncertain number. If* $u_1, u_2 \in C^m[0,1]$ *(*$m \in \mathbb{N} \cup \{0\}$*), we say* d_u *is a* C^m*-smooth (entirely) uncertain number. The interval* d *is the deterministic part, and* u_i *(*$i = 1, 2$*) are uncertain parts.*

Furthermore, an uncertain number can be regarded as an element of

$$\mathbb{R} \times C^M[0,1] \times C^M[0,1]$$

(a triple (d, u_1, u_2)) with their scalar multiplication and addition.

Let $D : \mathbb{R} \times C^{M+} \times C^{M+} \to [0,1]^{\mathcal{K}}$ be an operator with the following representation:

$$D(d, u_1, u_2) = [d - u_1(r), d + u_2(r)],\ r \in [0,1].$$

This operator creates a homomorphism between d_u and the triple (d, u_1, u_2) for uncertain numbers. Therefore, without loss of generality, we can use both definitions (see Figure 2).

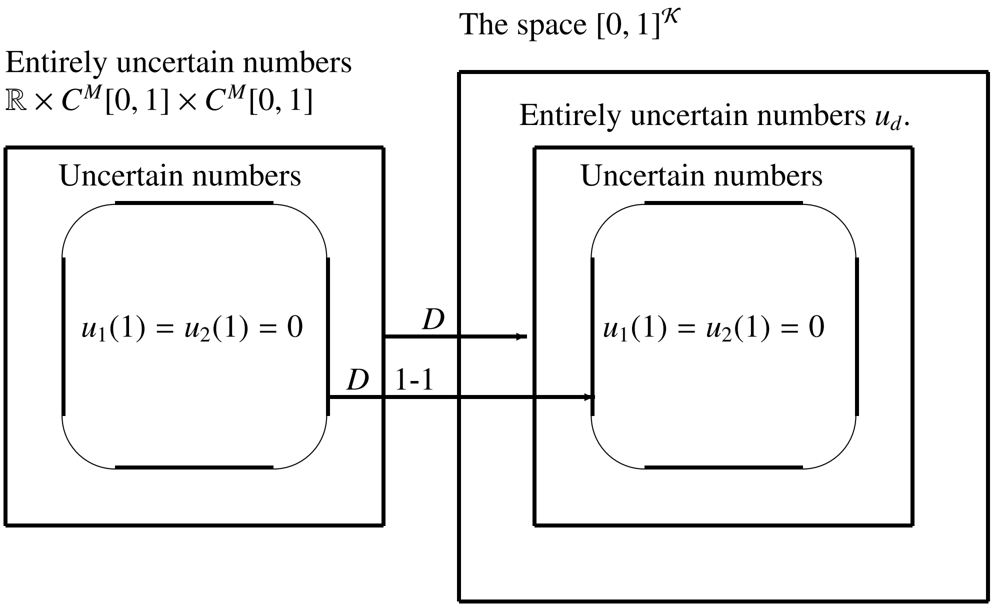

Figure 2. Homomorphism between two uncertain numbers.

Remark 1. *We recall that the condition 2.2 is a homotopy from uncertainty to determinism by the parameter* r*. Thus, we say it is an uncertain number. If this condition does not hold, we may not obtain a deterministic number by changing* r*, and thus we say it is an entirely uncertain number.*

Theorem 2.1. *Uncertain numbers are well-defined and have unique representations. Mathematically, this can be viewed as the map* D *restricted to uncertain numbers being one-to-one.*

Proof. Let (d, u_1, u_2) and (g, v_1, v_2) be two uncertain numbers that have the same representation: i.e.,

$$D(d, u_1, u_2) = D(g, v_1, v_2).$$

Then, for all $r \in [0, 1]$, we have

$$[d - u_1(r), d + u_2(r)] = [g - v_1(r), g + v_2(r)].$$

Putting $r = 1$, we get $d - u_1(1) = g - v_1(1)$. Thus, $d = g$, since $u_1(1) = v_1(1) = 0$. Therefore, $u_1(r) = v_1(r)$ follows from $d - u_1(r) = g - v_1(r) = d - v_1(r)$. Similarly, $u_2(r) = v_2(r)$. This implies $d_u = g_v$. □

From Theorem 2.1, one can conclude that the map D restricted to uncertain numbers is one-to-one (see Figure 2).

Remark 2. *Theorem 2.1 is not correct for entirely uncertain numbers. An entirely uncertain number may not have a unique representation.*

Remark 3. *The highest value of uncertainty is in the origin. The value of uncertainty in the origin* $u_i(0)$ *(*$i = 1, 2$*) is bounded, since* u_i *is a right continuous function at the point zero.*

2.2. The equivalency of an entirely uncertain number and a fuzzy number

Definition 2.2. *(LU-representation) [20] A fuzzy number (fn) on* $\mathbb{R}$ *with a parametric representation is an ordered pair* $f = (f_1, f_2)$ *of two real functions* $f_1, f_2 : [0, 1] \to \mathbb{R}$*, which satisfy the following requirements:*
(i) f_1 *is bounded, monotonically increasing, left continuous on* $(0, 1]$ *and right continuous at 0.*
(ii) f_2 *is bounded, monotonically decreasing, left continuous on* $(0, 1]$ *and right continuous at 0.*
(iii) $f_1(1) \le f_2(1)$*.*
Here, $f_1(r)$ *and* $f_2(r)$ *are called the left and right r-cut boundaries, respectively, where* $0 \le r \le 1$*.*

Definition 2.3. *Let* f *be a fuzzy number. Then, the deterministic part of* f *is defined by*

$$d(f) = \frac{f_1(1) + f_2(1)}{2}, \tag{2.3}$$

and uncertain parts of f *are defined by*

$$u_1(f)(r) := d(f) - f_1(r), \text{ lower or left uncertain part,} \tag{2.4}$$

and

$$u_2(f)(r) := f_2(r) - d(f), \text{ upper or right uncertain part.} \tag{2.5}$$

The following theorem connects the fuzzy theory to the uncertain theory.

Theorem 2.2. *An entirely uncertain number is fuzzy number and a fuzzy number is an entirely uncertain number*

Proof. Let f be a fuzzy number. It follows from (2.3)–(2.5), that

$$f_1(r) = d(f) - u_1(f)(r), \tag{2.6}$$

and

$$f_2(r) = d(f) + u_2(f)(r), \tag{2.7}$$

and hence f can be written as

$$f = [d(f) - u_1(f)(r), d(f) + u_2(f)(r)]. \tag{2.8}$$

We note that

$$u_1(f)(r) = \frac{f_1(1) + f_2(1)}{2} - f_1(r) = \frac{2(f_1(1) - f_1(r)) + f_2(1) - f_1(1)}{2}. \tag{2.9}$$

By the first property of Definition 2.2, $f_1(1) - f_1(r) \geq 0$, and $f_1(1) - f_1(r)$ is a monotonically decreasing function, left continuous on $(0, 1]$ and right continuous at 0. By the third property of Definition 2.2, $f_2(1) - f_1(1) \geq 0$. Thus, Eq (2.9) proves that $u_1(f)(r) \geq 0$ is monotonically decreasing, left continuous on $(0, 1]$ and right continuous at 0, so $u_1(f) \in C^{M+}[0, 1]$. Similarly, noting that

$$u_2(f)(r) = f_2(r) - \frac{f_1(1) + f_2(1)}{2} = \frac{2(f_2(r) - f_2(1)) + f_2(1) - f_1(1)}{2}, \tag{2.10}$$

we can conclude that $u_2(f) \in C^{M+}[0, 1]$.

Now, let $d_u(r) = [d - u_1(r), d + u_2(r)]$ be a representation of an entirely uncertain number. Setting $f_1(r) = d - u_1(r)$ and $f_2(r) = d + u_2(r)$, we find that f_1 and f_2 satisfy conditions (i)–(iii) of Definition 2.2, and $f = (f_1, f_2)$ is a fuzzy number. □

Equation (2.8) is a representation of a fuzzy number by an uncertain number. Thus, the fuzzy number and the entirely uncertain number are the same concept with different representations. We use both representations according to what we need.

3. Basic arithmetic operations for uncertain numbers

Let $f = (f_1, f_2)$ and $g = (g_1, g_2)$ be fuzzy numbers and k be a scalar number. We recall that addition and scalar multiplication are defined by

$$f + g = (f_1 + g_1, f_2 + g_2), \tag{3.1}$$

and

$$kf = \begin{cases} (kf_1, kf_2), & \text{if } k > 0, \\ 0, & \text{if } k = 0, \\ (kf_2, kf_1), & \text{if } k < 0, \end{cases} \tag{3.2}$$

respectively. It is an intriguing idea to define scalar multiplication for an entirely uncertain number, represented as follows:

$$kf = [kd(f) - |k|u_1(f)(r), kd(f) + |k|u_2(f)(r)]. \tag{3.3}$$

Obviously, the right hand side of (3.3) is an interval. For $k > 0$, both definitions are equal. However, for $k < 0$, we have

$$\begin{aligned} kd(f) - |k|u_1(f)(r) &= k(d(f) + u_1(f)(r)) \\ &= k(d(f) + (d(f) - f_1(r))) \\ &= k(2d(f) - f_1(r)) \\ &= k(f_1(1) + f_2(1) - f_1(r)) \neq kf_2(r). \end{aligned} \tag{3.4}$$

Thus, this is a new scalar multiplication, and (3.3) is not equivalent to (3.2).

Remark 4. *Most interesting uncertain functions and numbers have symmetric uncertain representation, which means that* $u_1(f) = u_2(f)$*. In this case, for* $k < 0$*, we have*

$$\begin{aligned} kd(f) - |k|u_1(f)(r) &= k(d(f) + u_1(f)(r)) \\ &= k(d(f) + u_2(f)(r)) \\ &= kf_2(r), \end{aligned} \tag{3.5}$$

and similarly $kd(f) + |k|u_2(f)(r) = kf_1(r)$*. Consequently, for symmetric uncertain numbers,* (3.3) *is equivalent to* (3.2).

Now, let us turn to the definition of the difference operator. Defining difference by $f - g = f + (-1)g$ or $f - g = f + (-1).g$ leads to some unorthodox properties, such as $f - f \neq \{0\}$. To fix these problems, Hukuhara [21] and later Stefanini [3] introduced the H-difference and gH- difference as

$$f \ominus g = h \;\Leftrightarrow\; f = g + h \tag{3.6}$$

and

$$f \ominus_g g = h \;\Leftrightarrow\; \begin{cases} f = g + h, \\ \text{or } g = f + (-1)h, \end{cases} \tag{3.7}$$

respectively. The important properties of these definitions are that, if they exist, then

$$f \ominus g = (f_1 - g_1, f_2 - g_2) \tag{3.8}$$

and

$$f \ominus_g g = (\min\{f_1 - g_1, f_2 - g_2\}, \max\{f_1 - g_1, f_2 - g_2\}). \tag{3.9}$$

With uncertain number representation, if they exist, then

$$f \ominus g = [d(f) - d(g) - (u_1(f) - u_1(g)), d(f) - d(g) + (u_2(f) - u_2(g))], \tag{3.10}$$

and

$$\begin{aligned} f \ominus_g g = (&d(f) - d(g) - \max\{(u_1(f) - u_1(g)), -(u_2(f) - u_2(g))\}, \\ &d(f) - d(g) + \max\{-(u_1(f) - u_1(g)), (u_2(f) - u_2(g))\}). \end{aligned} \tag{3.11}$$

In a similar manner, we generalize the new scalar definition, as

$$f \ominus_g g = h, \;\Leftrightarrow\; \begin{cases} f = g + h, \\ \text{or } g = f + (-1)h. \end{cases} \tag{3.12}$$

It is straightforward to see that

$$f \ominus_g g = [d(f) - d(g) - |u_1(f) - u_1(g)|, d(f) - d(g) + |u_2(f) - u_2(g)|]. \tag{3.13}$$

To guide this result, we note that if $f = g+h$, then $u_i(f) = u_i(g)+u_i(h)$, and hence $u_i(h) = u_i(f)-u_i(g)$, $(i = 1, 2)$. These functions can be uncertain parts of $f \ominus_g g$ if $u_i(f) - u_i(g) \geq 0$. Otherwise, we should suppose $g = f + (-1)h$. Thus, $u_i(g) = u_i(f) + u_i(h)$, and $u_i(h) = u_i(g) - u_i(f) \geq 0$. Conclusively, $u_i(h) = |u_i(g) - u_i(f)|$.

Remark 5. *The interesting feature of* (3.13) *is that the uncertainty of the left boundary of the difference is obtained by the uncertainty of the left boundaries of f and g, while in* (3.11)*, both right and left boundaries are involved in the uncertainty of left boundaries of difference. A similar feature holds for the uncertainty of the right boundaries involved in the fuzzy difference.*

The rate of change is defined by the difference in a tiny time. The rate of change of position of an object in any direction is speed. Obviously, we have uncertainty in measuring the distance. Thus, what is the uncertainty of evaluating speed? This is an important difficult question since the difference in a tiny time for uncertain numbers has many definitions. Which one should we choose? We can choose the best definition by adding more information. If we accept that the uncertainty of one boundary in the rate of change is just affected by the uncertainty of that boundary, our new definition Eq (3.13) is reasonable to use for defining such changes. However, if the uncertainty is vaguer, and both boundaries of motion contribute to each boundary of the change, then the previous definition of the generalized Hukuhara differences is recommended.

It is tempting to consider a more general definition of the difference described by

$$\begin{aligned} f \ominus_G g &= d(f) - d(g) \\ &+ [-G_1(u_1(f) - u_1(g), u_2(f) - u_2(g)), G_2(u_1(f) - u_1(g), u_2(f) - u_2(g)], \end{aligned}$$

where $G_i : \mathbb{R}^2 \to \mathbb{R}$ $(i = 1, 2)$ are two-variable functions (We call them uncertain norms). If $G_i : \mathbb{R}^2 \to \mathbb{R}^+$, for all $r \in [0, 1]$, then $[-G_1(,), G_2(,)]$ is a well defined interval. This definition should support a reverse formula

$$f \ominus_G G = h, \quad \Leftrightarrow \quad \begin{cases} f = g + (1)_G h, \\ \text{or } g = f + (-1)_G h, \end{cases} \tag{3.14}$$

where $(k)_G : \mathbb{R}_F \to \mathbb{R}_F$ is an scalar operation such that

$$(k)_G(f) = [k(d(f)) - (k)_{G_1}(u_1(f), u_2(f)), k(d(f)) + (k)_{G_2}(u_1(f), u_2(f))],$$

and $(k)_{G_i} : (\mathbb{R}^+)^2 \to \mathbb{R}^+$ $(i = 1, 2)$ have one of the following properties:

$$\begin{aligned} &(1)_{G_i}(G_1(u_1(f) - u_1(g), u_2(f) - u_2(g)), G_2(u_1(f) - u_1(g), u_2(f) - u_2(g))) \\ &\quad + u_i(g) = u_i(f), \end{aligned} \tag{3.15}$$

or

$$\begin{aligned} &u_i(g) = u_i(f) + \\ &(-1)_{G_i}(G_1(u_1(f) - u_1(g), u_2(f) - u_2(g)), G_2(u_1(f) - u_1(g), u_2(f) - u_2(g))), \end{aligned} \tag{3.16}$$

for $i = 1, 2$, where g and f are fuzzy numbers.

Example 1. *For H difference,*

$$G_i(v_1, v_2) = v_i,$$

with

$$(k)_{G_i}(v_1, v_2) = kv_i.$$

For gH difference,

$$G_i(v_1, v_2) = \max\{(-1)^{i+1} v_1, (-1)^i v_2\},$$

with

$$(k)_{G_i}(v_1, v_2) = \begin{cases} kv_i, & \text{if } k \geq 0, \\ kv_{3-i}, & \text{if } k < 0. \end{cases}$$

For g, H difference,

$$G_i(v_1, v_2) = |v_i|,$$

with

$$(k)_{G_i}(v_1, v_2) = |k|v_i.$$

Also, for $p = 2s$, $s \in \mathbb{N}$,

$$G_i(v_1, v_2) = v_i^p,$$

$$(k)_{G_i}(v_1, v_2) = kv_i^{1/p}, \tag{3.17}$$

and for $p = 2s + 1$,

$$G_i(v_1, v_2) = |v_i|^p,$$

with

$$(k)_{G_i}(v_1, v_2) = \begin{cases} kv_i, & \text{if } k \geq 0, \\ kv_{3-i}, & \text{if } k < 0. \end{cases}$$

or

$$G_i(v_1, v_2) = \max\{(-1)^{i+1}v_1, (-1)^i v_2\}^p,$$

with

$$(k)_{G_i}(v_1, v_2) = \begin{cases} kv_i, & \text{if } k \geq 0, \\ kv_{3-i}, & \text{if } k < 0, \end{cases}$$

can be other suitable uncertain norms.

We note that Eqs (3.10), (3.11) and (3.13) are valid if the corresponding differences exist. As far as we know, a comprehensive characteristic theorem for the existence of even previously defined differences does not exist. The following theorem covers such characteristics.

Theorem 3.1. *Let f and g be two fuzzy numbers. The differences (I) $f \ominus g$, (II) $f \ominus_g g$, (III) $f \ominus_{g.} g$ and (IV) $f \ominus_G g$ exist if*

(I) $u_i(f) - u_i(g) \in C^{M+}[0, 1]$ *for* $i = 1, 2$;

(II) $\max\{(u_1(g) - u_1(f)), -(u_2(g) - u_2(f))\} \in C^{M+}[0, 1]$ *and* $\max\{-(u_1(g) - u_1(f)), (u_2(g) - u_2(f))\} \in C^{M+}[0, 1]$;

(III) $|u_i(f) - u_i(g)| \in C^M[0, 1]$ *for* $i = 1, 2$;

(IV) $G_i\left(u_1(f) - u_1(g), u_2(f) - u_2(g)\right) \in C^{M+}[0, 1]$ *for* $i = 1, 2$,

respectively.

Proof. We proof the case (I). Other cases are similar. Let assumption (I) hold. Assume

$$c = [d(f) - d(g) - (u_1(f) - u_1(g)), d(f) - d(g) + (u_2(f) - u_2(g))].$$

Then, from (I) c is an entirely uncertain number and thus it is a fuzzy number by Theorem 2.2. We note that $g + c = (d(f) - u_1(f), d(f) + u_2(f)) = (f_1, f_2) = f$. It follows by the definition of H-difference that $c = f \ominus g$.

Now, let $c = f \ominus g$. We prove the case (I). Since c is a fuzzy number, thus c is an entirely uncertain number by Theorem 2.2. Therefore $u_1(c) \geq 0$ and $u_2(c) \geq 0$ are monotonically decreasing, left continuous on $(0, 1]$ and right continuous at 0, i. e.: $u_i(c) \in C^{M+}[0, 1]$ for $i = 1, 2$. We show $u_1(c) = u_1(f) - u_1(g)$ and $u_2(c) = u_2(f) - u_2(g)$. Since $g + c = f$ thus $(g_1 + c_1, g_2 + c_2) = (f_1, f_2)$. Immediately it follows $c_1 = f_1 - g_1$ and $c_2 = f_2 - g_2$. From (2.3) and (2.4),

$$\begin{aligned} u_1(c) &= \frac{c_1(1) + c_2(1)}{2} - c_1(r) \\ &= \frac{f_1(1) - g_1(1) + f_2(1) - g_2(1)}{2} - (f_1(r) - g_1(r)) \\ &= \frac{f_1(1) + f_2(1)}{2} - f_1(r) - \left(\frac{g_1(1) + g_2(1)}{2} - g_1(r)\right) \\ &= u_1(f) - u_1(g). \end{aligned}$$

Similarly, $u_2(c) = u_2(f) - u_2(g)$, and this completes the proof. □

Conditions of the case (ii) lead to the following categorizing.

Corollary 1. *Let $h_i = u_i(f) - u_i(g)$ $(i = 1, 2)$. Assume the fuzzy difference $f \ominus_g g$ exists. Then, $h_1(r)h_2(r) \geq 0$ for all $r \in [0, 1]$.*

Since the detail of this proof is important, we refer to the proof of this theorem in the following remark.

Remark 6. *There are two possibilities,* $\max\{-h_1, h_2\} = h_2$ *or* $\max\{-h_1, h_2\} = -h_1$. *For each case, we can distinguish the following characterizations:*

(i) $\max\{-h_1, h_2\} = h_2$; *then,* $\max\{h_1, -h_2\} = h_1$. *By condition (II), $h_i \in C^{M+}[0, 1]$, and that means that $h_1 \geq 0$ and $h_2 \geq 0$. From Eq* (3.11)*, we have*

$$f \ominus_g g = (d(f) - d(g) - h_1, d(f) - d(g) + h_2).$$

(ii) $\max\{-h_1, h_2\} = -h_1$; *then,* $\max\{h_1, -h_2\} = -h_2$. *From Condition (II), we obtain $-h_i \in C^{M+}[0, 1]$. Thus, $h_2 \leq 0$, and $h_1 \leq 0$. From Eq* (3.11)*, we have*

$$f \ominus_g g = (d(f) - d(g) - (-h_2), d(f) - d(g) - h_1).$$

Example 2. *Does $f \ominus_g g$ exist for all $f, g \in \mathbb{R}_f$? The answer is negative. In fact, Corollary 1 sheds light on how to find such numbers. Indeed, if $h_1(r)h_2(r) < 0$ for some $r \in [0, 1]$, then the fuzzy difference $f \ominus_g g$ does not exist. For example, set $f = (-3 + 3r, 2.5 - 2.5r)$ and $g = (-1 + r, 4 - 4r)$. Supposing the existence of $f \ominus_g g$, by Eq* (3.9) *we should have*

$$\begin{aligned} f \ominus_g g &= [\min\{-2 + 2r, -1.5 + 1.5r\}, \max\{-2 + 2r, -1.5 + 1.5r\}] \\ &= (-2 + 2r, -1.5 + 1.5r). \end{aligned} \tag{3.18}$$

Obviously, the pair $(-2 + 2r, -1.5 + 1.5r)$ is not a fuzzy number.

I do not think Definition 4 of [22] can remedy this problem.

4. Definition of rate of changes

A fuzzy-valued function is a function whose value is fuzzy. Generally, in the rest of this paper, we consider $f : [a, b] \to \mathbb{R}_f$ where $\mathbb{R}_f$ stands for the set of fuzzy numbers. The limit of $f(t)$ when $t \to t_0$ is a fuzzy number $l = (l_1, l_2)$ such that $\lim_{t\to t_0} f_i(r, t) = l_i(r)$ for all $r \in [0, 1]$, point-wise. Some authors prefer uniform convergence with distance:

$$D(f(t), l) = sup_{r\in[0,1]} \max\{|f_1(t, r) - l_1(r)|, |f_2(t, r) - l_2(r)|\}.$$

Then, $\lim_{t\to t_0} f(t) = l$ uniformly if $\lim_{t\to t_0} D(f(t), l) = 0$. Clearly, uniform limit delivers the point-wise limit. The inverse may not be correct. In this paper, we consider point-wise convergence.

The definition of the derivative depends on the definition of the difference. So, we can generalize this definition as follows.

Definition 4.1. *Let $f : [a, b] \to \mathbb{R}_f$ be a fuzzy valued function. Then, the right (left) fuzzy derivative of f exits on $t \in (a, b)$ if the limits*

$$D^+ f(t) = \lim_{h\to 0^+} \frac{f(t + h)\tilde{\ominus} f(t)}{h}, \tag{4.1}$$

and

$$\begin{aligned} D^- f(t) &= \lim_{h\to 0^-} \frac{f(t + h)\tilde{\ominus} f(t)}{h} \\ &= \lim_{h\to 0^+} \frac{f(t - h)\tilde{\ominus} f(t)}{-h}, \end{aligned} \tag{4.2}$$

exist and are fuzzy, respectively. We say the fuzzy derivative of f exists if $D^+ f(t) = D^- f(t)$, and we show it with $Df = D^+ f(t) = D^- f(t)$. We use D, D_g, $D_{g.}$ and D_G for $\tilde{\ominus} = \ominus$, $\tilde{\ominus} = \ominus_g$, $\tilde{\ominus} = \ominus_{g.}$ and $\tilde{\ominus} = \ominus_{g.}$, respectively.

We note that in the realm of fuzzy theory, the backward difference $f(t)\tilde{\ominus} f(t \pm h)$ may change the definition of the derivative. This will add two more definitions. Thus, the definitions of derivatives are categorized into four types [1]. Here, we study only one type, since the "or" condition in their definition makes the analysis poor.

4.1. Characteristic theorems for Hukuhara derivative

Again, to the best of our knowledge, we do not have a comprehensive, solid characteristic theorem that points out the optimal conditions for the existence of a fuzzy derivative.

Theorem 4.1. *Let $f : [a, b] \to \mathbb{R}_F$. Let*

$$f(r, t) = [d(f)(t) - u_1(f)(r, t), d(f)(t) + u_2(f)(r, t)],$$

for $(r, t) \in [0, 1] \times [a, b]$, be a fuzzy valued function with a representation of an uncertain function. Assume the following:

(i) Scalar derivatives of $d(f)(t)$, $u_1(f)(r, t)$ and $u_2(f)(r, t)$ exist on $N_\epsilon(t_0) \times [0, 1]$ for a given $\epsilon > 0$ where $N_\epsilon(t_0) := (t_0 - \epsilon, t_0 + \epsilon)$;

(ii) $\frac{du_i(f)}{dt}(r,t) \geq 0$ *(*$i = 1,2$*) on* $N_\epsilon(t_0) \times [0,1]$*;*

(iii) There exists $\hat{h} > 0$ *such that for all* $0 < h < \hat{h}$*, the difference* $u_i(f)(r, t_0 + h) - u_i(f)(r, t_0)$ *(*$i = 1,2$*) is a monotonically decreasing function with respect to r.*

Then, the right fuzzy derivative of f at t_0 *in the sense of Hukuhara exists, and*

$$D^+ f(t_0) = (f_1'(t_0), f_2'(t_0)).$$

Proof. Since $\frac{du_i(f)}{dt}(r,t) \geq 0$, $u_i(f)(r,t)$ are increasing functions, and for all $0 < h < \epsilon$, we have

$$\varpi_i(r) = u_i(f)(r, t_0 + h) - u_i(f)(r, t_0) \geq 0, \quad i = 1,2.$$

It is trivial that each $\varpi_i(r)$ is left continuous on $(0,1]$ and right continuous at 0. Further, each $\varpi_i(r)$ is a monotonically decreasing function by Assumption 3 for all $0 < h < \hat{h}$. Therefore, the assumptions of Theorem 3.1 hold, and $f(t_0 + h) \ominus f(t_0)$ exists for all $0 < h < \min\{\hat{h}, \epsilon\}$. From (3.8),

$$f(t_0 + h) \ominus f(t_0) = (f_1(t_0 + h) - f_1(t_0), f_2(t_0 + h) - f_2(t_0)).$$

It follows from the definition of the Hukuhara derivative that

$$\begin{aligned} D^+ f(t_0) &= \lim_{h \to 0^+} \frac{f(t_0 + h) \ominus f(t_0)}{h} \\ &= \lim_{h \to 0^+} \frac{(f_1(r, t_0 + h) - f_1(r, t_0), f_2(r, t_0 + h) - f_2(r, t_0))}{h} \\ &= \left(\lim_{h \to 0^+} \frac{f_1(r, t_0 + h) - f_1(r, t_0)}{h}, \lim_{h \to 0^+} \frac{f_2(r, t_0 + h) - f_2(r, t_0))}{h} \right) \\ &= \left(\lim_{h \to 0^+} \frac{f_1(r, t_0 + h) - f_1(r, t_0)}{h}, \lim_{h \to 0^+} \frac{f_2(r, t_0 + h) - f_2(r, t_0))}{h} \right) \\ &= (f_1'(r, t_0), f_2'(r, t_0)). \end{aligned} \tag{4.3}$$

□

Similarly, for the existence of D^- with the Hukuhara derivative, we have the following theorem.

Theorem 4.2. *Let* $f : [a,b] \to \mathbb{R}_f$*. Let*

$$f(r,t) = [d(f)(t) - u_1(f)(r,t), d(f)(t) + u_2(f)(r,t)]$$

for $(r,t) \in [0,1] \times [a,b]$ *be a fuzzy valued function. Assume the following:*

(i) Scalar derivatives of $d(f)(t)$*,* $u_1(f)(r,t)$ *and* $u_2(f)(r,t)$ *exist on* $N_\epsilon(t_0) \times [0,1]$*, for a given* $\epsilon > 0$ *where* $N_\epsilon(t_0) := (t_0 - \epsilon, t_0 + \epsilon)$*;*

(ii) $\frac{du_i(f)}{dt}(r,t) \leq 0$ *(*$i = 1,2$*) on* $N_\epsilon(t_0) \times [0,1]$*;*

(iii) There exists $\hat{h} > 0$ *such that for all* $0 < h < \hat{h}$*, the difference* $u_i(f)(r, t_0 - h) - u_i(f)(r, t_0)$ *(*$i = 1,2$*) is a monotonically decreasing function with respect to r.*

Then, the left fuzzy derivative of f at t_0 *in the sense of Hukuhara exists, and*

$$D^- f(t_0) = (f_2'(t_0), f_1'(t_0)).$$

Proof. Since $\frac{du_i(f)}{dt}(r,t) \leq 0$, uncertain parts of f are decreasing functions. Thus, for $0 < h < \epsilon$,

$$\varpi_i(r) = u_i(f)(r, t_0 - h) - u_i(f)(r, t_0) \geq 0, \ i = 1, 2.$$

Similar to the proof of Theorem 4.1, the functions $\varpi_i(r)$ for $i = 1, 2$ satisfy the assumptions of Theorem 3.1 for $0 < h < \min\{\epsilon, \hat{h}\}$. Therefore, $f(t_0 - h) \ominus f(t_0)$ exists, and from (3.8) we get

$$f(t_0 - h) \ominus f(t_0) = (f_1(t_0 - h) - f_1(t_0), f_2(t_0 - h) - f_2(t_0)).$$

By the definition of the derivative, we obtain

$$\begin{aligned} D^- f(t_0) &= \lim_{h \to 0^+} \frac{f(t_0 - h) \ominus f(t_0)}{-h} \\ &= \lim_{h \to 0^+} \frac{(f_1(r, t_0 - h) - f_1(r, t_0), f_2(r, t_0 - h) - f_2(r, t_0))}{-h} \\ &= \left(\lim_{h \to 0^+} \frac{f_2(r, t_0 - h) - f_2(r, t_0)}{h}, \lim_{h \to 0^+} \frac{f_1(r, t_0 - h) - f_1(r, t_0))}{h} \right) \\ &= \left(\lim_{h \to 0^+} \frac{f_2(r, t_0 - h) - f_2(r, t_0)}{-h}, \lim_{h \to 0^+} \frac{f_1(r, t_0 - h) - f_1(r, t_0))}{-h} \right) \\ &= \left(f_2'(r, t_0), f_1'(r, t_0) \right). \end{aligned} \tag{4.4}$$

□

Example 3. *Consider the fuzzy valued number $f(t) = e^{-t}(-1 + r, 1 - r)$. According to Eqs* (2.3)–(2.5), *we have $d(f) = 0$ and $u_1(f) = u_2(f) = e^{-t}(1 - r)$. Assumption 2 of Theorem 4.1 does not hold since $\frac{u_i(f)}{dt} = -e^{-t}(1 - r)$. Obviously,*

$$e^{-(t+h)}(-1 + r, 1 - r) \ominus e^{-t}(-1 + r, 1 - r)$$

is not well defined for any $h > 0$. Thus, D^+ is not well defined. However, all assumptions of Theorem 4.2 hold. Thus, D^- is well defined, and

$$D^- f(t) = -e^{-t}(-1 + r, 1 - r) = e^{-t}(-1 + r, 1 - r) = f(t).$$

Remark 7. *Condition (ii) of Theorem 4.1 is in contrast with condition (ii) of Theorem 4.2. These conditions not only are necessary conditions but also are sufficient conditions, since $f(t_0 - h) \ominus f(t_0)$ and $f(t_0 + h) \ominus f(t_0)$ should be well defined in some neighborhood of h. Thus, Hukuhara's definition is not well-defined except for functions that satisfy $\frac{du_i(f)}{dt}(r,t) = 0$.*

Corollary 2. *Let $f(t) : [a, b] \to \mathbb{R}_F$ be a fuzzy function such that its left and right r-cut boundaries (f_1 and f_2) are differentiable, and ($f_1 \neq f_2$). Then, $Df(t)$ in a Hukuhara sense does not exist.*

Proof. Let $Df(t_0) = (l_1, l_2) = l$. Thus,

$$D^+ f(t_0) = \lim_{h \to o^+} \frac{f(t_0 + h) \ominus f(t_0)}{h} = (l_1, 1_2).$$

From the definition of the limit, there exists $\tilde{h}$ such that for all $0 < h < \tilde{h}$, $\frac{u(t_0+h)\ominus f(t_0)}{h}$ is a fuzzy number and converges to l. Now, (3.8) implies

$$\frac{f(t_0+h)\ominus f(t_0)}{h} = (\frac{f_1(t_0+h)-f_1(t_0)}{h}, \frac{f_2(t_0+h)-f_2(t_0)}{h}).$$

Immediately, this leads to $f_1'(t_0) = l_1$ and $f_2'(t_0) = l_2$. Similarly, since

$$D^{-}f(t_0) = \lim_{h\to o^+} \frac{f(t_0-h)\ominus f(t_0)}{-h} = (l_1, 1_2),$$

(3.8) implies

$$\frac{f(t_0-h)\ominus f(t_0)}{-h} = (\frac{f_2(t_0-h)-f_2(t_0)}{-h}, \frac{f_1(t_0-h)-f_1(t_0)}{-h})$$

is a fuzzy number and converges to l. Thus, we get $f_2'(t_0) = l_1$ and $f_1'(t_0) = l_2$. Conclusively, $f_2'(t_0) = f_1'(t_0)$, and hence $l_1 = l_2$. Obviously, this is in contradiction with the fact that l is a fuzzy number. □

4.2. Characteristic theorems for generalized Hukuhara derivative

Fortunately, the generalized Hukuhara derivative is well-defined for a vast range of functions. Mostly, with the strong definition, they are categorized into two cases. This bifurcation is also observed with our definition without "or" conditions.

Theorem 4.3. *Let* $f : [a,b] \to \mathbb{R}_F$, $\varpi_i(h) := u_i(f(t+h)) - u_i(f(t))$, $i = 1,2$, *and let the left and right r-cut boundaries of f be differentiable with respect to t for all* $[r,t] \in [0,1]\times[a,b]$. *Then, the generalized Hukuhara derivative exists if the following hold:*

(i) There exists $h_0 > 0$ *such that for all* $0 < h < h_0$,

$$\begin{aligned} \max\{\varpi_1(h), -\varpi_2(h)\} &= \varpi_1(h), \\ \max\{\varpi_1(-h), -\varpi_2(-h)\} &= -\varpi_2(-h), \end{aligned} \tag{4.5}$$

and

$$\{\varpi_1(h), -\varpi_1(-h), \varpi_2(h), -\varpi_2(-h)\} \subset C^M[0,1].$$

(ii) There exists $h_0 > 0$ *such that for all* $0 < h < h_0$,

$$\begin{aligned} \max\{\varpi_1(h), -\varpi_2(h)\} &= -\varpi_2(h), \\ max\{\varpi_1(-h), -\varpi_2(-h)\} &= \varpi_1(-h), \end{aligned} \tag{4.6}$$

and

$$\{-\varpi_1(h), \varpi_1(-h), -\varpi_2(h), \varpi_2(-h)\} \subset C^M[0,1].$$

(iii) There exists $h_0 > 0$ *such that for all* $0 < h < h_0$,

$$\begin{aligned} \max\{\varpi_1(h), -\varpi_2(h)\} &= \varpi_1(h), \\ \max\{\varpi_1(-h), -\varpi_2(-h)\} &= \varpi_1(-h), \\ \{\varpi_1(\pm h), \varpi_2(\pm h)\} &\subset C^M[0,1], \end{aligned} \tag{4.7}$$

and

$$f_1 = f_2 + c(r), \tag{4.8}$$

where $c : [0,1] \to \mathbb{R}$ *is a function that does not depend on t.*

(iv) There exists $h_0 > 0$ *such that for all* $0 < h < h_0$,

$$\max\{\varpi_1(-h), -\varpi_2(-h)\} = -\varpi_2(-h),$$
$$\max\{\varpi_1(h), -\varpi_2(h)\} = -\varpi_2(h), \tag{4.9}$$
$$\{-\varpi_1(\pm h), -\varpi_2(\pm h)\} \subset C^M[0,1],$$

and (4.8) *holds, where* $c : [0,1] \to \mathbb{R}$ *is a function that does not depend on* t.

Furthermore, for case (i), we have

$$D_g f = (f_1', f_2').$$

For case (ii), we have

$$D_g f = (f_2', f_1').$$

For cases (iii) and (iv), we have

$$D_g f = (f_2', f_1') = (f_1', f_2') = \textit{deterministic function}.$$

Proof. Let $\varpi_i(h) = u_i(f(t+h)) - u_i(f(t))$. According to the sign of $\pm\varpi_i(\pm h)$, four classes can be distinguished.

(i) We have $h_0 > 0$ such that for all $0 < h < h_0$, $\max\{\varpi_1(h), -\varpi_2(h)\} = \varpi_1(h)$, $\max\{-\varpi_1(h), \varpi_2(h)\} = \varpi_2(h)$, and both functions belong to $C^M[0,1]$. Thus, for all $0 < h < h_0$, the difference $f(t+h) \ominus_g f(t)$ is well defined, and

$$\begin{aligned} D_g^+ f(t) &= \lim_{h\to0^+} \frac{f(t+h)\ominus_g f(t)}{h} \\ &= \lim_{h\to0^+} \frac{[d(f(t+h)) - d(f(t)) - \varpi_1, d(f(t+h)) - d(f(t)) + \varpi_2]}{h} \\ &= [\lim_{h\to0^+} \frac{d(f(t+h)) - d(f(t)) - \varpi_1}{h}, \lim_{h\to0^+} \frac{d(f(t+h)) - d(f(t)) + \varpi_2}{h}] \\ &= [d'(f(t)) - u_1'(f(t)), d'f(t) + u_2'(f(t))] \\ &= (f_1'(t), f_2'(t)). \end{aligned} \tag{4.10}$$

(ii) We have $h_0 > 0$ such that for all $0 < h < h_0$, $\max\{\varpi_1(-h), -\varpi_2(-h)\} = -\varpi_2(-h)$, and hence $\max\{-\varpi_1(-h), \varpi_2(-h)\} = -\varpi_1(-h)$, and the functions $-\varpi_2(-h)$ and $-\varpi_1(-h)$ belong to $C^M[0,1]$. Thus, for all $0 < h < h_0$, the difference $f(t-h) \ominus_g f(t)$ is well defined, and

$$D_g^- f(t) = (f_1'(t), f_2'(t)). \tag{4.11}$$

(iii) We have $h_0 > 0$ such that for all $0 < h < h_0$, $\max\{\varpi_1(h), -\varpi_2(h)\} = -\varpi_2(h)$, and hence $\max\{-\varpi_1(h), \varpi_2(h)\} = -\varpi_1(h)$, and the functions $-\varpi_2$ and $-\varpi_1$ belong to $C^M[0,1]$. Thus, for all $0 < h < h_0$, the difference $f(t+h) \ominus_g f(t)$ is well defined, and

$$\begin{aligned} D_g^+ f(t) &= \lim_{h\to0^+} \frac{f(t+h)\ominus_g f(t)}{h} \\ &= \lim_{h\to0^+} \frac{[d(f(t+h)) - d(f(t)) + \varpi_2, d(f(t+h)) - d(f(t)) - \varpi_1]}{h} \\ &= [\lim_{h\to0^+} \frac{d(f(t+h)) - d(f(t)) + \varpi_2}{h}, \lim_{h\to0^+} \frac{d(f(t+h)) - d(f(t)) - \varpi_1}{h}] \\ &= [d'(f(t)) + u_2'(f(t)), d'f(t) - u_1'(f(t))] \\ &= (f_2'(t), f_1'(t)). \end{aligned} \tag{4.12}$$

(iv) We have $h_0 > 0$ such that $\max\{\varpi_1(-h), -\varpi_2(-h)\} = \varpi_1(-h)$ for all $0 < h < h_0$, and hence $\max\{-\varpi_1(-h), \varpi_2(-h)\} = \varpi_2(-h)$, and the functions $\varpi_2(-h)$ and $\varpi_1(-h)$ belong to $C^M[0, 1]$. Then, for all $0 < h < h_0$, the difference $f(t-h) \ominus_g f(t)$ is well defined, and

$$D_g^- f(t) = (f_2'(t), f_1'(t)). \tag{4.13}$$

Categories (i) and (ii) provide the conditions of case (*i*). Categories (iii) and (iv) provide the conditions of case (*ii*). Categories (i) and (iv) provide the conditions of case (*iii*) and finally, categories (ii) and (iii) provide the conditions of case (*iv*). □

Remark 8. *These four categories of conditions fall into two categories of result, which are distinguished by switching right and left cut derivatives.*

In some literature, the existence of a generalized fuzzy derivative is based on the length of a fuzzy function. Let the length of a fuzzy number $f = (f_1, f_2)$ be defined by

$$l(f) := f_2 - f_1 = u_1(f) + u_2(f),$$

where f is an l-increasing function if it is increasing with respect to t.

Corollary 3. *Let $f(r,t) = y(t)x(r)$, $y : [a,b] \to \mathbb{R}$ be a real valued differentiable function and x be a fuzzy number.*

(i) If f is an l-increasing function, $D_g f = (f_1', f_2') = y'x$.
(ii) If f is an l-decreasing function, $D_g f = (f_2', f_1') = y'x$.

Proof. We prove case (*ii*). A similar proof can be done for case (*i*). Since $l(f) = f_2 - f_1 = y(x_2 - x_1)$, and $x_2 - x_1$ is non-negative for all $r \in [0, 1]$, y is a decreasing function, and $y'(t) \leq 0$ on $[a, b]$. Taking into account that

$$\varpi_i(h) = (y(t+h) - y(t))u_i(x), \quad i = 1, 2,$$

we conclude that $\varpi_1(h) \leq 0$, $\varpi_1(-h) \geq 0$, $-\varpi_2(-h) \leq 0$, $-\varpi_2(h) \geq 0$, and thus

$$\{-\varpi_1(h), \varpi_1(-h), \varpi_2(-h), -\varpi_2(h)\} \subset C^M.$$

Also, $\max\{\varpi_1(h), -\varpi_2(h)\} = -\varpi_2(h)$ and $\max\{\varpi_1(-h), -\varpi_2(-h)\} = \varpi_1(-h)$. Thus, conditions of Theorem 4.3 (case (*ii*)) hold, and we have

$$D_g f = (f_2', f_1') = (y'x_2, y'x_1) = y'(x_1, x_2) = y'x.$$

□

4.3. Characteristic theorems for new generalized Hukuhara derivative

An interesting facet of the new generalized Hukuhara derivative is that it is not bifurcated. Accordingly, it is a good replacement for well-defined differential equations in uncertain dynamical processes.

Theorem 4.4. *Let $f : [a,b] \to \mathbb{R}_F$, $\varpi_i(h) := u_i(f(t+h)) - u_i(f(t))$, $i = 1, 2$, and let the left and right r-cut boundaries of f be differentiable with respect to t for all $[r,t] \in [0,1] \times [a,b]$. Then, the new generalized Hukuhara derivative exists iff there exists h_0 such that for all $0 < h < h_0$, both $|\varpi_i(h)|$ and $|\varpi_i(-h)|$ belong to $C^M[0,1]$. In this case, we will have*

$$D_g.f = [d'(f) - |u_1'(f)|, d'f + |u_2'(f)|].$$

Proof. Obviously, $|\varpi_i(h)| \in C^{M+}[0,1]$, and hence for all $0 < h < h_0$, the difference $f(t+h) \ominus_{g.} f(t)$ is well defined by Theorem 3.1. Therefore,

$$\begin{aligned} D_g^+ f &= \lim_{h\to 0^+} \frac{f(t+h)\ominus_{g.} f(t)}{h} \\ &= \lim_{h\to 0^+} \frac{1}{h}.[d(f(t+h)) - d(f(t)) - |g_1|, d(f(t+h)) - d(f(t)) + |g_2|] \\ &= [\lim_{h\to 0^+} \frac{d(f(t+h)) - d(f(t)) - |\varpi_1|}{h}, \lim_{h\to 0^+} \frac{d(f(t+h)) - d(f(t)) + |\varpi_2|}{h}] \\ &= [d'(f(t)) - \lim_{h\to 0^+} |\frac{\varpi_1}{h}|, d'f(t) + \lim_{h\to 0^+} |\frac{\varpi_2}{h}|] \\ &= [d'(f(t)) - |u_1'(f)|, d'f(t) + |u_2'(f)|]. \end{aligned} \tag{4.14}$$

Similarly, since $|\varpi_i(-h)|$ belongs to $C^{M+}[0,1]$ for all $0 < h < h_0$, the difference $f(t-h) \ominus_{g.} f(t)$ is well-defined, and similarly

$$D_{g.}^- f = [d'(f(t)) - |u_1'(f)|, d'f(t) + |u_2'(f)|]. \tag{4.15}$$

Thus, $D_{g.}^- f = D_{g.}^+ f$, and hence $D_{g.} f$ exists, and

$$D_{g.} f = [d'(f(t)) - |u_1'(f)|, d'f(t) + |u_2'(f)|].$$

□

5. Derivative with more generalized abstract definition

Now, it is time to turn to the concrete form of the derivative. Let $\ominus_G$ be a generalized difference with an induced scalar multiplication $|.|_G : \mathbb{R}_F \to \mathbb{R}_F$. Let $f : [a,b] \to \mathbb{R}_F$ be a fuzzy valued function, and $t \in (a,b)$. By Definition 4.1, we have

$$\begin{aligned} D_G^+ f(t) &= \lim_{h\to 0^+} \frac{f(t+h)\tilde{\ominus}_G f(t)}{h} \\ &= \lim_{h\to 0^+} \left|\frac{1}{h}\right|_G \Big(d(f)(t+h) - d(f)(t) \\ &\quad + [-G_1(u_{12}(f)(t+h) - u_{12}(f)(t)), G_2(u_{12}(f)(t+h) - u_{12}(f)(t))]\Big) \\ &= \lim_{h\to 0^+} \Big(\frac{d(f)(t+h) - d(f)(t)}{h} \\ &\quad + [-\left|\frac{1}{h}\right|_{G_1} G_{12}(u_{12}(f)(t+h) - u_{12}(f)(t)), \left|\frac{1}{h}\right|_{G_2} G_{12}(u_{12}(f)(t+h) - u_{12}(f)(t))]\Big) \\ &= d(f)' \\ &\quad + [-\left|\frac{1}{h}\right|_{G_1} G_{12}(u_{12}(f)(t+h) - u_{12}(f)(t)), \left|\frac{1}{h}\right|_{G_2} G_{12}(u_{12}(f)(t+h) - u_{12}(f)(t))]\Big), \end{aligned} \tag{5.1}$$

where, for brevity,

$$u_{12}(f)(t+h) - u_{12}(f)(t) = [u_1(f)(t+h) - u_1(f)(t), u_2(f)(t+h) - u_2(f)(t)],$$

and

$$G_{12}(u_{12}(f)(t+h) - u_{12}(f)(t))$$
$$= [G_1(u_{12}(f)(t+h) - u_{12}(f)(t)), G_2(u_{12}(f)(t+h) - u_{12}(f)(t))]. \tag{5.2}$$

Similarly,

$$\begin{aligned}
D_G^- f(t) &= \lim_{h\to 0^+} \frac{f(t-h)\tilde{\ominus}_G f(t)}{h} \\
&= \lim_{h\to 0^+} \left|\frac{1}{-h}\right|_G \Big(d(f)(t-h) - d(f)(t) \\
&\quad + [-G_1(u_{12}(f)(t-h) - u_{12}(f)(t)), G_2(u_{12}(f)(t-h) - u_{12}(f)(t)]\Big) \\
&= + \lim_{h\to 0^+} \Big(\frac{d(f)(t-h) - d(f)(t)}{h} \\
&\quad + [-\left|\frac{1}{-h}\right|_{G_1} G_1(u_{12}(f)(t-h) - u_{12}(f)(t)), \\
&\quad \left|\frac{1}{-h}\right|_{G_2} G_2(u_{12}(f)(t-h) - u_{12}(f)(t))]\Big) \\
&= d(f)' \\
&\quad + [-\lim_{h\to 0^+} \left|\frac{1}{-h}\right|_{G_1} G_{12}(u_{12}(f)(t-h) - u_{12}(f)(t), \\
&\quad \lim_{h\to 0^+} \left|\frac{1}{-h}\right|_{G_2} G_{12}(u_{12}(f)(t-h) - u_{12}(f)(t)]\Big).
\end{aligned} \tag{5.3}$$

Theorem 5.1. *Let $t \in [a, b]$. Suppose there exists $h_0 > 0$ such that for all $0 < h < h_0$,*

$$w_i(t, h) := \left|\frac{1}{h}\right|_{G_i} G_{12}(u_{12}(f)(t+h) - u_{12}(f)(t)) \in C^{M+}[0, 1],$$

and

$$z_i(t, h) := \left|\frac{1}{-h}\right|_{G_i} G_{12}(u_{12}(f)(t-h) - u_{12}(f)(t)) \in C^{M+}[0, 1],$$

for $i = 1, 2$. Let f be a fuzzy valued function on $[a, b]$, such that its r-cuts are differential with respect to t. Then, $D_G^+ f$ and $D_G^- f$ exist on $t \in (a, b)$ if $\lim_{h\to 0^+} w_i(t, h)$ and $\lim_{h\to 0^+} z_i(t, h)$ exist, respectively. Furthermore, if

$$\lim_{h\to 0^+} w_i(t, h) = \lim_{h\to 0^+} z_i(t, h),$$

then $D_G f(t)$ exists, and

$$\begin{aligned}
D_G f(t) &= d(f)'(t) + [\lim_{h\to 0^+} w_1(t, h), \lim_{h\to 0^+} w_2(t, h)] \\
&= d(f)'(t) + [\lim_{h\to 0^+} z_1(t, h), \lim_{h\to 0^+} z_2(t, h)].
\end{aligned} \tag{5.4}$$

Proof. The proof is straightforward considering the representation (5.1)–(5.3). □

Example 4. *Let* $p = 2s$, $s \in \mathbb{N}$.

$$\begin{aligned} G_i(v_1, v_2) &= v_i^p, \\ (k)_{G_i}(v_1, v_2) &= |k| v_i^{1/p}. \end{aligned} \tag{5.5}$$

Then,

$$\begin{aligned} w_i(t,h) &= \left|\frac{1}{h}\right|_{G_i} (u_{12}(f)(t+h) - u_{12}(f)(t))^p \\ &= \frac{1}{|h|} ((u_i(f)(t+h) - u_i(f)(t))^p)^{\frac{1}{p}} \\ &= \frac{|u_i(f)(t+h) - u_i(f)(t)|}{|h|}, \end{aligned} \tag{5.6}$$

and similarly,

$$\begin{aligned} z_i(t,h) &= \left|\frac{1}{-h}\right|_{G_i} (u_{12}(f)(t-h) - u_{12}(f)(t))^p \\ &= \frac{1}{|-h|} ((u_i(f)(t-h) - u_i(f)(t))^p)^{\frac{1}{p}} \\ &= \frac{|u_i(f)(t-h) - u_i(f)(t)|}{|-h|}. \end{aligned} \tag{5.7}$$

Thus, by Theorem 5.1, the derivative exists if there exists $h_0 > 0$ *such that for all* $0 < h < h_0$,

$$|u_i(f)(t-h) - u_i(f)(t)|,$$

and

$$|u_i(f)(t+h) - u_i(f)(t)|,$$

for $i = 1, 2$ *are monotonically decreasing, left continuous on* $(0, 1]$ *and right continuous at 0. Furthermore,*

$$D_G f(t) = d(f)'(t) + [-|u_1(f)'(t)|, |u_2(f)'(t)|].$$

It is remarkable to note that the derivative with this difference and scalar multiplication is exactly the same as in Theorem 4.4.

Example 5. *Now, let* $p = 2s + 1$, $s \in \mathbb{N}$.

$$G_i(v_1, v_2) = |v_i|^p,$$

with

$$(k)_{G_i}(v_1, v_2) v_i = k v_i^{1/p}.$$

Since

$$\begin{aligned} z_i(t,h) &= \left|\frac{1}{-h}\right|_{G_i} (u_{12}(f)(t-h) - u_{12}(f)(t))^p \\ &= \frac{1}{-h} ((u_i(f)(t-h) - u_i(f)(t))^p)^{\frac{1}{p}} \\ &= \frac{|u_i(f)(t-h) - u_i(f)(t)|}{-h}, \end{aligned} \tag{5.8}$$

obviously, $z_i(t,h)$ *is always negative. This prevents the definition of* D_G^-. *Thus,* D^- *as a Hukuhara derivative does not exist.*

Example 6. *Let* $p = 2s + 1$, $s \in \mathbb{N}$.

$$G_i(v_1, v_2) = \max\{(-1)^{i+1}v_1, (-1)^i v_2\}^p,$$

with

$$(k)_{G_i}(v_1, v_2) = \begin{cases} kv_i, & \text{if } k \geq 0, \\ kv_{3-i}, & \text{if } k < 0. \end{cases}$$

Then,

$$\begin{aligned} w_1(t,h) &= \left|\frac{1}{h}\right|_{G_1} [\max\{(\varpi_1(h))^p, -(\varpi_2(h))^p\}, \max\{-(\varpi_1(h))^p, (\varpi_2(h))^p\}] \\ &= \frac{1}{h}(\max\{(\varpi_1(h))^p, -\varpi_2(h)^p\})^{1/p}, \end{aligned} \tag{5.9}$$

$$\begin{aligned} w_2(t,h) &= \left|\frac{1}{h}\right|_{G_2} [\max\{(\varpi_1(h))^p, -(\varpi_2(h))^p\}, \max\{-(\varpi_1(h))^p, (\varpi_2(h))^p\}] \\ &= \frac{1}{h}(\max\{(-\varpi_1(h))^p, \varpi_2(h)^p\})^{1/p}, \end{aligned} \tag{5.10}$$

where $\varpi_i(h) = u_i(f)(t+h) - u_i(f)(t)$ *(*$i = 1, 2$*). Similarly,*

$$\begin{aligned} z_1(t,h) &= \left|\frac{1}{-h}\right|_{G_1} [\max\{(\varpi_1(-h))^p, -\varpi_2(-h)^p\}, \max\{-(\varpi_1(-h))^p, \varpi_2(-h)^p\})^{1/p}] \\ &= \frac{1}{-h}(\max\{-(\varpi_1(-h))^p, \varpi_2(-h)^p\})^{1/p}, \end{aligned} \tag{5.11}$$

$$\begin{aligned} z_2(t,h) &= \left|\frac{1}{-h}\right|_{G_2} [\max\{(\varpi_1(-h))^p, -\varpi_2(-h)^p\}, \max\{-(\varpi_1(-h))^p, \varpi_2(-h)^p\})^{1/p}] \\ &= \frac{1}{-h}(\max\{(\varpi_1(-h))^p, -\varpi_2(-h)^p\})^{1/p}. \end{aligned} \tag{5.12}$$

It is clear that the same result as it is pointed out in Theorem 4.3 holds. We just study case (i) for clarification. Others are similar. Suppose there exists $h_0 > 0$ *such that for all* $0 < h < h_0$,

$$\max\{\varpi_1(h), -\varpi_2(h)\} = \varpi_1(h),$$

$$\max\{\varpi_1(-h), -\varpi_2(-h)\} = -\varpi_2(-h),$$

and the functions $\varpi_1(h)$, $-\varpi_1(-h)$, $\varpi_2(h)$, $-\varpi_2(-h)$ *belong to* $C^M[0, 1]$. *Then*

$$\max\{\varpi_1(h)^p, -\varpi_2(h)^p\} = \varpi_1(h)^p \Leftrightarrow \max\{-\varpi_1(h)^p, \varpi_2(h)^p\} = \varpi_2(h)^p,$$

and

$$\max\{\varpi_1(-h)^p, -\varpi_2(-h)^p\} = -\varpi_2(-h)^p \Leftrightarrow \max\{-\varpi_1(-h)^p, \varpi_2(-h)^p\} = -\varpi_1(-h)^p.$$

(We note that p is odd, and we do not care about the minus sign). Thus,

$$w_1(t,h) = \frac{\varpi_1(h)}{h},$$

$$w_2(t,h) = \frac{\varpi_2(h)}{h},$$
$$z_1(t,h) = \frac{-\varpi_1(-h)}{-h},$$
$$z_2(t,h) = \frac{-\varpi_2(h)}{-h}.$$

Obviously,

$$\lim_{h\to 0^+} w_1(t,h) = u_1(f)',$$
$$\lim_{h\to 0^+} w_2(t,h) = u_2(f)',$$
$$\lim_{h\to 0^+} z_1(t,h) = u_1(f)',$$
$$\lim_{h\to 0^+} z_2(t,h) = u_2(f)'.$$

Thus,

$$D_G f(t) = [d(f)'(t) + -u_1(f)', d(f)'(t) + u_2(f)'] = [f_1', f_2'].$$

Almost identically, with different differences of the generalized Hukuhara type, we obtained exactly the same derivative and results as in Theorem 4.3.

6. Conclusions

We introduced uncertain numbers as an alternative representation of parametric fuzzy numbers concerning interval analysis. We surveyed, introduced, and analyzed several scalar multiplications, differences, and derivatives for fuzzy-valued functions. This analysis helps to entirely determine when the generalized differences exist and compare them. For example, this analysis reveals that if

$$(u_1(f) - u_1(g))(u_2(f) - u_2(g)) < 0,$$

then the gH difference does not exist. Based on introduced differences, corresponding derivatives are introduced and analyzed. One of the interesting results is that even when we have a right or left derivative of the Hukuhara type, the Hukuhara derivative does not exist at all for fuzzy numbers. We obtained the conditions for the existence of classical generalized Hukuhara differences. It revealed that such existence results conditioned the left uncertainty part on the right for the existence of a derivative. With a detailed analysis, we obtained the bifurcation result for a large class of functions. If the derivative exists, it has two forms. We obtained the necessary and sufficient results for such existence. For the special case of separable fuzzy numbers, the result can be converted to the previously stated theorem related to l-decreasing or increasing functions.

Another important result is that we can avoid the inconvenient bifurcation property with a new generalized Hukuhara derivative. Finally, we made concrete all the definitions in one form and investigated the existences of the corresponding difference and derivative.

There are many possible directions for future investigations. For example, how could we introduce numerical methods for such derivatives? What can be corresponding integrals and their related theorems? Another important direction is investigating uncertain dynamical processes described by partial or differential equations with new definitions of uncertain derivatives. Obviously, extending the uncertainty analysis and definitions for uncertain fractional operators also are important. We expect a flow of research and publications in this respect in the very near future.

Acknowledgements

This work is supported by Neijiang Normal University.

Conflict of interest

The author declares no conflict of interest.

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