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Research article

Ćirić type nonunique fixed point theorems in the frame of fuzzy metric spaces

Tatjana Došenović[1], Dušan Rakić[1], Stojan Radenović[2] and Biljana Carić[3,*]

[1] Faculty of Technology Novi Sad, Bulevar cara Lazara 1, University of Novi Sad, Serbia
[2] Faculty of Mechanical Engineering, University of Belgrade, Serbia
[3] Faculty of Technical Sciences, University of Novi Sad, Trg Dositeja Obradovića 6, University Novi Sad, Serbia

* **Correspondence:** Email:biljana@uns.ac.rs; Tel: +381214852293; Fax: +38121458133.

Abstract: The paper defines a new contractive condition within $\kappa-$orbitally complete fuzzy metric spaces $(\Theta, \mathcal{M}, \mathcal{T})$, as well as fixed point theorems for single-valued and multi-valued function on Θ which is not necessarily continuous. The contractive condition is motivated by an idea proposed in Ćirić's paper "On some maps with a nonunique fixed points". Continuity of mapping κ is replaced by $\kappa-$orbitally continuity property which provides the existence of the fixed point, but not necessarily uniqueness.

Keywords: fixed point; fuzzy metric space; t-norm; Ćiric type contraction; Cauchy sequence
Mathematics Subject Classification: 47H10, 54H25

1. Introduction

In 1965, Zadeh [30] introduced the concept of a fuzzy set that found great application in various branches of science and engineering, including mathematics as well as fixed point theory. Unlike a classical set, where an element either belongs or does not belong to a set in the theory of fuzzy sets, the elements belong to a set with a measure that it involves a continuous transition from non-belonging to full belonging. In a fuzzy set, each element of the set is assigned a value from $[0, 1]$ and that number (fuzzy number) represents the degree of belonging to the fuzzy set. Mathematically, if a set Θ is given, the fuzzy set is a mapping $\mathcal{A} : \Theta \to [0, 1]$. This mapping is called the membership function, whose value for a certain element in the set defines its degree of belonging.

Guided by the basic postulates of probability theory, the mathematician Karl Menger in 1942 defined probabilistic metric spaces as a generalization of the concept of metric spaces. The distance between two objects is not a fixed number but is assigned to points of space appropriate to the distribution function. A further generalization is given by Kramosil and Michalek [21] in 1975 where they use the following idea-the distance function does not have to be given by the distribution function but by the fuzzy set. Now the distance function represents the degree of certainty with which the points b and l are at a distance less than t. Next, George and Veeramani [6, 7], in order to obtain Pompeiu-Hausdorff metric modify the concept of fuzzy metrics defined by Kramosil and Michalek.

In the fixed point theory, the first significant result is published by Sehgal and Bharucha-Reid [3, 29] where they generalize the famous Banach's contraction principle [2] in probabilistic metric spaces using triangular norm $\mathcal{T}_M$. Following this result, scientists around the world publish papers that represent a generalization of Banach's contraction principle, both in probabilistic and fuzzy metric spaces. The novelty of this topic is confirmed by many recent papers, which includes the application of fixed point theory in solving various integral equations (see [19, 26]). Important generalizations of Banach's contraction principle in metric spaces were given by the following mathematicians: Edelstein-Nemitskii, Boyd and Wong, Meir and Keeler, Kannan, Chatterje, Zamfirescu, Reich, Hardy and Rogers, Geraghty, Ćirić which gives one of the most general contractive conditions (quasi-contraction) and many others. Later on, the mentioned contractive conditions and fixed point theorems related to a certain contractive condition are generalized in probabilistic, fuzzy, but also in other spaces that represent the generalization of metric spaces. Especially, Ćirić's quasi-contraction as one of the strongest generalization of Banach's contraction is frequently used in recent studies [23, 25, 27]. As Banach's contractive condition implies the continuity of the mapping κ, Kannan's work 1968 provides an answer to the question of whether there is a contractive condition sufficient for the existence of a fixed point, but that the continuity of the mapping f does not have to be implied. The question concerning the continuity of the mapping f is also raised in the paper of Ćirić from 1974 [5], with a new contractive condition in metric space $(\mathcal{X}, \varrho)$:

$$\min\{\varrho(\kappa b, \kappa l), \varrho(b, \kappa b), \varrho(l, \kappa l)\} - \min\{\varrho(b, \kappa l), \varrho(\kappa b, l)\} \leq q\varrho(b, l), \ b, l \in \mathcal{X}, \ q \in (0, 1),$$

where the existence of a fixed point is achieved by assuming that space $\mathcal{X}$ is $\kappa-$ orbitally complete and the mapping κ is orbitally continuous. Encouraged by this, in this paper we give a contractive condition within fuzzy metric spaces, where the existence of a fixed point is achieved by assuming that the mapping is f orbitally continuous. Both single-valued and multi-valued case is discussed in this paper.

Before the main results, we look at the known definitions and results that are necessary for this research.

Starting from the idea of the basic triangle inequality K. Menger [22] defined the term triangular norms. The first area where triangular norms played a significant role was the theory of probabilistic metric spaces. In addition, triangular norms are a significant operation in areas such as fuzzy sets and phase logic, the theory of generalized measures and the theory of nonlinear differential and differential equations. However, the original set of axioms was weak and B. Schweizer and A. Sklar [28] made changes and defined axioms for triangular norms that are still used today.

Definition 1.1. [20] A binary operation $\mathcal{T} : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous triangular norm if it satisfies the following conditions:

(t1) $\mathcal{T}$ is associative and commutative,
(t2) $\mathcal{T}$ is continuous,
(t3) $\mathcal{T}(p, 1) = p$, for all $p \in [0, 1]$,
(t4) $\mathcal{T}(p, q) \leq \mathcal{T}(r, s)$ whenever $p \leq r$ and $q \leq s$, for each $p, q, r, s \in [0, 1]$.

Typical examples of a continuous t-norm are $\mathcal{T}_{\mathcal{P}}(p, q) = p \cdot q$, $\mathcal{T}_{\mathcal{M}}(p, q) = \min\{p, q\}$ and $\mathcal{T}_{\mathcal{L}}(p, q) = \max\{p + q - 1, 0\}$.

Very important class of triangular norms is given by O. Hadžić [9, 11].

Definition 1.2. [9] Let $\mathcal{T}$ be a triangular norm and $\mathcal{T}_n : [0,\ 1] \to [0,\ 1],\ n \in \mathbb{N}$. $\mathcal{T}$ is a triangular norm of Hadžić-type if the family $\{\mathcal{T}_n(b)\}_{n\in\mathbb{N}}$ defined in the following way:

$$\mathcal{T}_1(b) = \mathcal{T}(b, b),\ \mathcal{T}_{n+1}(b) = \mathcal{T}(\mathcal{T}_n(b), b),\ n \in \mathbb{N},\ b \in [0, 1],$$

is equi-continuous at $b = 1$.

Minimum triangular norm is trivial example of triangular norm of Hadžić-type, for nontrivial example readers are referred to the paper [9].

In the book Triangular norm writen by Klement, Mesiar and Pap is pointed out very useful statement that using associativity of triangular norms every t-norm $\mathcal{T}$ can be extended in a unique way to an n-ary operation taking for $(b_1, \ldots, b_n) \in [0, 1]^n$ the values

$$\mathcal{T}_{i=1}^{1} b_i = b_1,\ \mathcal{T}_{i=1}^{n} b_i = \mathcal{T}(\mathcal{T}_{i=1}^{n-1} b_i, b_n) = \mathcal{T}(b_1, b_2, \ldots, b_n).$$

Example 1. [13] n-ary extensions of the T_{min}, T_L and T_P t-norms:
$\mathcal{T}_{\mathcal{M}}(b_1, \ldots, b_n) = \min(b_1, \ldots, b_n)$,
$\mathcal{T}_{\mathcal{L}}(b_1, \ldots, b_n) = \max(\sum_{i=1}^{n} b_i - (n-1),\ 0)$,
$\mathcal{T}_{\mathcal{P}}(b_1, \ldots, b_n) = b_1 \cdot b_2 \cdot \ldots \cdot b_n$.

It has been shown in the paper [20] that the triangular norm $\mathcal{T}$ can be extended to a countable infinite operation taking for any sequence $(b_n)_{n\in\mathbb{N}}$ from $[0,\ 1]$ the value

$$\mathcal{T}_{i=1}^{+\infty} b_i = \lim_{n\to+\infty} \mathcal{T}_{i=1}^{n} b_i.$$

Since the sequence $(\mathcal{T}_{i=1}^{n} b_i)$, $n \in \mathbb{N}$ is non-increasing and bounded from below, the limit $\mathcal{T}_{i=1}^{+\infty} b_i$ exists.

In order to prove existence of fixed point the following condition is imposed [13, 14] investigate the classes of triangular norms $\mathcal{T}$ and sequences (b_n) from the interval $[0,\ 1]$ such that $\lim\limits_{n\to+\infty} b_n = 1$ and

$$\lim_{n\to+\infty} \mathcal{T}_{i=n}^{+\infty} b_i = \lim_{n\to+\infty} \mathcal{T}_{i=1}^{+\infty} b_{n+i} = 1. \tag{1.1}$$

The next proposition concerning triangular norms of Hadžić type is proved in [13].

Proposition 1.3. *Let $(b_n)_{n\in\mathbb{N}}$ be a sequence of numbers from $[0,\ 1]$ such that $\lim\limits_{n\to+\infty} b_n = 1$ and triangular norm $\mathcal{T}$ is of Hadžić type. Then $\lim\limits_{n\to+\infty} \mathcal{T}_{i=n}^{+\infty} b_i = \lim\limits_{n\to+\infty} \mathcal{T}_{i=1}^{+\infty} b_{n+i} = 1$.*

Definition 1.4. ([6, 7]) A 3−tuple $(\Theta, \mathcal{M}, \mathcal{T})$ is called a fuzzy metric space if Θ is an arbitrary (non-empty) set, $\mathcal{T}$ is a continuous triangular norm and $\mathcal{M}$ is a fuzzy set on $\Theta^2 \times (0, +\infty)$, satisfying the following conditions for each $b, l, z \in \Theta$ and $p, q > 0$,

(Fm-1) $\mathcal{M}(b, l, p) > 0$,
(Fm-2) $\mathcal{M}(b, l, p) = 1$ if and only if $b = l$,
(Fm-3) $\mathcal{M}(b, l, p) = \mathcal{M}(l, b, p)$,
(Fm-4) $\mathcal{T}(\mathcal{M}(b, l, p), \mathcal{M}(l, z, q)) \leq \mathcal{M}(b, z, p + q)$,
(Fm-5) $\mathcal{M}(b, l, \cdot) : (0, +\infty) \to [0, 1]$ is continuous.

Definition 1.5. ([6, 7]) Let $(\Theta,\ \mathcal{M},\ \mathcal{T})$ be a fuzzy metric space.

(i) A sequence $\{b_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\Theta,\ \mathcal{M},\ \mathcal{T})$ if for every $\delta \in (0,\ 1)$ there exists $i_0 \in \mathbb{N}$ such that $\mathcal{M}(b_j,\ b_k,\ p) > 1 - \delta,\ \ j,\ k \geq i_0,\ \ p > 0$.

(ii) A sequence $\{b_n\}_{n\in\mathbb{N}}$ converges to b in $(\Theta,\ \mathcal{M},\ \mathcal{T})$ if for every $\delta \in (0,\ 1)$ there exists $i_0 \in \mathbb{N}$ such that $\mathcal{M}(b_j,\ b,\ p) > 1 - \delta,\ \ j \geq i_0,\ \ p > 0$. Then, we say that $\{b_n\}_{n\in\mathbb{N}}$ is convergent.

(iii) A fuzzy metric space $(\Theta,\ \mathcal{M},\ \mathcal{T})$ is complete if every Cauchy sequence in $(\Theta,\ \mathcal{M},\ \mathcal{T})$ is convergent.

Lemma 1.6. *[8] $\mathcal{M}(b, l, \cdot)$ is non-decreasing for all $b, l \in \Theta$.*

Let Υ and Ω be two nonempty subsets of Θ, define the Hausdorff–Pompeiu fuzzy metric as

$$\mathcal{H}(\Upsilon, \Omega, p) = \min\{\inf_{b\in\Upsilon} \mathcal{E}(b, \Omega, p), \inf_{l\in\Omega} \mathcal{E}(l, \Upsilon, p)\}, p > 0,$$

where $\mathcal{E}(b, B, p) = \sup_{l\in\Omega} \mathcal{M}(b, l, p)$.

Definition 1.7. [10, 12] Let $(\Theta,\ \mathcal{M},\ \mathcal{T})$ be a fuzzy metric space, $\emptyset \neq \mathcal{A} \subset \Theta$ and $\mathcal{F} : \mathcal{A} \to C(\mathcal{A})$, ($C(\mathcal{A}$ is the set of all closed subsets of $\mathcal{A}$). A mapping $\mathcal{F}$ is a *weakly demicompact if for every sequence $\{b_n\}_{n\in\mathbb{N}}$ from $\mathcal{A}$ such that* $\lim\limits_{n\to+\infty} \mathcal{M}(b_n, b_{n+1}, p) = 1,\ p > 0,\ b_{n+1} \in \mathcal{F}b_n,\ n \in \mathbb{N}$, *there exists a convergent subsequence* $\{b_{n_k}\}_{k\in\mathbb{N}}$.

Definition 1.8. [5] (i) An orbit of $\mathcal{F} : \Theta \to C(\Theta)$ at the point $b \in \Theta$ is a sequence $\{b_n : b_n \in \mathcal{F}b_{n-1}\}$, where $b_0 = b$.

(ii) A multivalued function $\mathcal{F}$ is orbitally upper-semicontinuous if $b_n \to u \in \Theta$ implies $u \in \mathcal{F}u$ whenever $\{b_n\}_{n\in\mathbb{N}}$ is an orbit of $\mathcal{F}$ at some $b \in \Theta$.

(iii) A space Θ is $\mathcal{F}$−orbitally complete if every orbit of $\mathcal{F}$ at some $b \in \Theta$ which is Cauchy sequence converges in Θ.

Lemma 1.9. *[17] Let $(\Theta, \mathcal{M}, \mathcal{T})$ be a fuzzy metric space and let $\{b_n\}$ be a sequence in Θ such that*

$$\lim_{p\to0^+} \mathcal{M}(b_n, b_{n+1}, p) > 0,\ \ n \in \mathbb{N}, \tag{1.2}$$

and

$$\lim_{n\to+\infty} \mathcal{M}(b_n, b_{n+1}, p) = 1,\ \ p > 0. \tag{1.3}$$

If $\{b_n\}$ is not a Cauchy sequence in $(\Theta, \mathcal{M}, \mathcal{T})$, then there exist $\varepsilon \in (0, 1)$, $p_0 > 0$, and sequences of positive integers $\{l_k\}$, $\{i_k\}$, $l_k > i_k > k$, $k \in \mathbb{N}$, such that the following sequences

$$\{\mathcal{M}(b_{i_k}, b_{l_k}, p_0)\}, \{\mathcal{M}(b_{i_k}, b_{l_k+1}, p_0)\}, \{\mathcal{M}(b_{i_k-1}, b_{l_k}, p_0)\},$$

$$\{\mathcal{M}(b_{i_k-1}, b_{l_k+1}, p_0)\}, \{\mathcal{M}(b_{i_k+1}, b_{l_k+1}, p_0)\},$$

tend to $1-\varepsilon$, *as* $k \to +\infty$.

Definition 1.10. [4, 5] A mapping κ is called a orbitally continuous if $\lim_{i\to+\infty} \kappa^{n_i} b = u$ implies $\lim_{i\to+\infty} \kappa\kappa^{n_i} b = \kappa u$, for every $u \in \Theta$. A space Θ is called a $\kappa-$orbitally complete if every Cauchy sequence of the form $\{\kappa^{n_i} b\}_{i=1}^{+\infty}$, $b \in \Theta$ converges in Θ.

Recently [24], orbitally continuous property is used to obtain common fixed points results in Menger probabilistic metric spaces.

2. Main results

Instigated by the paper [5] we give the following contractive condition.

Theorem 2.1. *Let* $(\Theta, \mathcal{M}, \mathcal{T})$ *be* $\kappa-$*orbitally complete fuzzy metric space such that* $\lim_{p\to+\infty} \mathcal{M}(b, l, p) = 1$ *and let* $\kappa : \Theta \to \Theta$ *be an orbitally continuous mapping on* Θ. *If* κ *satisfies the following condition:*

$$\max\{\mathcal{M}(\kappa b, \kappa l, p), \mathcal{M}(\kappa b, b, p), \mathcal{M}(\kappa l, l, p)\} + \min\{1 - \mathcal{M}(b, \kappa l, p), 1 - \mathcal{M}(\kappa b, l, p)\} \geq \mathcal{M}(b, l, \frac{p}{q}), \tag{2.1}$$

for some $q < 1$ *and for all* $b, l \in \Theta$, $p > 0$, *and if triangular norm* $\mathcal{T}$ *satisfies the following condition:*

For arbitrary $b_0 \in \Theta$ *and* $b_1 = \kappa b_0$ *there exists* $\varsigma \in (q, 1)$ *such that*

$$\lim_{n\to+\infty} \mathcal{T}_{i=n}^{+\infty} \mathcal{M}(b_0, b_1, \frac{p}{\varsigma^i}) = 1, \ p > 0, \tag{2.2}$$

then for each $b \in \Theta$ *the sequence* $\{\kappa^n b\}_{n=1}^{+\infty}$ *converges to a fixed point of* κ.

Proof. Let $b_0 \in \Theta$ is arbitrary. Now, we can construct a sequence $\{b_n\}_{n\in\mathbb{N}\cup\{0\}}$ such that $b_{n+1} = \kappa b_n$, for all $n \in \mathbb{N} \cup \{0\}$. If $b_{n_0} = b_{n_0+1}$ for some $n_0 \in \mathbb{N} \cup \{0\}$ then the proof is finished. So, suppose that $b_n \neq b_{n+1}$, for all $n \in \mathbb{N} \cup \{0\}$.

Let $n \in \mathbb{N}$ and $p > 0$. By (2.1) for $b = b_{n-1}$, $l = b_n$, we have

$$\begin{aligned} &\max\{\mathcal{M}(\kappa b_{n-1}, \kappa b_n, p), \mathcal{M}(\kappa b_{n-1}, b_{n-1}, p), \mathcal{M}(\kappa b_n, b_n, p)\} \\ +\ &\min\{1 - \mathcal{M}(b_{n-1}, \kappa b_n, p), 1 - \mathcal{M}(\kappa b_{n-1}, b_n, p)\} \geq \mathcal{M}(b_{n-1}, b_n, \frac{p}{q}), \end{aligned}$$

which means that

$$\max\{\mathcal{M}(b_n, b_{n+1}, p), \mathcal{M}(b_n, b_{n-1}, p)\} \geq \mathcal{M}(b_{n-1}, b_n, \frac{p}{q}). \tag{2.3}$$

Since $\frac{p}{q} > p$, by using Lemma 1.6, we get that

$$\mathcal{M}(b_n, b_{n+1}, p) \geq \mathcal{M}(b_{n-1}, b_n, \frac{p}{q}), \ n \in \mathbb{N}, \ p > 0. \tag{2.4}$$

In the following, we will show that the sequence $\{b_n\}$ is a Cauchy sequence.

Let $\vartheta \in (q, 1)$. Then, sum $\sum_{i=1}^{+\infty} \vartheta^i$ is convergent and there exists $i_0 \in \mathbb{N}$ such that $\sum_{i=n}^{+\infty} \vartheta^i < 1$, for every $j > i_0$. Let $j > k > i_0$. Since $\mathcal{M}$ is nondecreasing, by (Fm-4), for every $p > 0$ we have

$$\begin{aligned}\mathcal{M}(b_j, b_{j+k}, p) &\geq \mathcal{M}(b_j, b_{j+k}, p\sum_{s=j}^{j+k-1}\vartheta^s)\\ &\geq \mathcal{T}(\mathcal{M}(b_j, b_{j+1}, p\vartheta^j), \mathcal{M}(b_{j+1}, b_{j+k}, p\sum_{s=j+1}^{j+k-1}\vartheta^s))\\ &\geq \mathcal{T}(\mathcal{M}(b_j, b_{j+1}, p\vartheta^j), \mathcal{T}(\mathcal{M}(b_{j+1}, b_{j+2}, p\vartheta^{j+1},\\ \ldots &\quad \mathcal{M}(b_{j+k-1}, b_{j+k}, p\vartheta^{j+k-1})\ldots).\end{aligned}$$

By (2.4) follows that

$$\mathcal{M}(b_j, b_{j+1}, p) \geq \mathcal{M}(b_0, b_1, \frac{p}{q^n}),\ n \in \mathbb{N},\ p > 0,$$

and for $j > k > i_0$ we have

$$\begin{aligned}M(b_j, b_{j+k}, p) &\geq \mathcal{T}(\mathcal{M}(b_0, b_1, \frac{p\vartheta^i}{q^i}), \mathcal{T}(\mathcal{M}(b_0, b_1, \frac{p\vartheta^{i+1}}{q^{i+1}}),\\ \ldots &\quad \mathcal{M}(b_0, b_1, \frac{p\vartheta^{j+k-1}}{q^{j+k-1}})\ldots)\\ &= \mathcal{T}_{s=j}^{j+k-1} M(b_0, b_1, \frac{p\vartheta^s}{q^s})\\ &\geq \mathcal{T}_{s=j}^{+\infty} M(b_0, b_1, \frac{p}{\varsigma^s}),\ p > 0,\end{aligned}$$

where $\varsigma = \frac{q}{\vartheta}$. Since $\varsigma \in (q, 1)$ by (2.2) follows that $\{b_n\}$ is a Cauchy sequence.

Based on that Θ is $\kappa-$orbitally complete fuzzy metric space and $\{b_n\}$ is a Cauchy sequence it follows that $\lim_{n\to+\infty} \kappa^n b_0 = u \in \Theta$, and using orbital continuity of κ we have that $\kappa u = u$. □

Example 2. Let $\Theta = [0, 1]$, $k \in [\frac{1}{2}, 1)$, $\kappa b = \begin{cases} kb, & b \neq 0 \\ 1, & b = 0 \end{cases}$, $M(b, l, p) = e^{-\frac{|b-l|}{p}}$ and $\mathcal{T} = \mathcal{T}_{\mathcal{P}}$.

Further, condition (2.1), with $q = m$, will be checked.

Case 1. Let $b, l \neq 0$. Then

$$\begin{aligned}&\max\{\mathcal{M}(\kappa b, \kappa l, p), \mathcal{M}(\kappa b, b, p), \mathcal{M}(\kappa l, l, p)\}\\ &+ \min\{1 - \mathcal{M}(b, \kappa l, p), 1 - \mathcal{M}(\kappa b, l, p)\}\\ &\geq \max\{\mathcal{M}(\kappa b, \kappa l, p), \mathcal{M}(\kappa b, b, p), \mathcal{M}(\kappa l, l, p)\}\\ &\geq e^{-k\frac{|b-l|}{p}} = \mathcal{M}(b, l, \frac{p}{q}),\ p > 0.\end{aligned}$$

Case 2. Let $b = 0$ and $l \neq 0$. Then

$$\begin{aligned}&\max\{\mathcal{M}(\kappa b, \kappa l, p), \mathcal{M}(\kappa b, b, P), \mathcal{M}(\kappa l, l, p)\}\\ &+ \min\{1 - \mathcal{M}(b, \kappa l, p), 1 - \mathcal{M}(\kappa b, l, p)\}\\ &\geq e^{-(1-k)\frac{l}{p}}\end{aligned}$$

$$= e^{-(1-k)\frac{|b-l|}{p}}$$
$$\geq e^{-k\frac{|b-l|}{p}} = \mathcal{M}(b, l, \frac{p}{q}),\ p > 0.$$

Case 3. Let $b = l = 0$. Then

$$\begin{aligned}
&\max\{\mathcal{M}(\kappa b, \kappa l, p), \mathcal{M}(\kappa b, b, p), \mathcal{M}(\kappa l, l, p)\} \\
&+ \min\{1 - \mathcal{M}(b, \kappa l, p), 1 - \mathcal{M}(\kappa b, l, p)\} \\
&= 2 - e^{-\frac{1}{p}} \\
&\geq e^{k\frac{|b-l|}{p}} = \mathcal{M}(b, l, \frac{p}{q}),\ p > 0.
\end{aligned}$$

So, condition (2.1) is satisfied for all $b, l \in \Theta$ with $q = k$, but since κ is not orbital continuous then the statement of the Theorem 2.1 does not have to be true, and there is not a fixed point of the mapping κ.

Example 3. Let $\Theta = [0, 1]$, $k \in [\frac{1}{2}, 1)$, $\kappa b = \begin{cases} kb, & b \text{ rational} \\ b, & b \text{ irrational.} \end{cases}$, $\mathcal{M}(b, l, p) = e^{-\frac{|b-l|}{p}}$ and $\mathcal{T} = \mathcal{T}_{\mathcal{P}}$.

Then, condition (2.1) is satisfied for all $b, l \in \Theta$ with $q = k$, κ is not continuous but it is orbital continuous and by Theorem 2.1 κ has fixed point. Moreover, κ has infinitely many fixed points.

Remark 1.

(i) Using Proposition 1.3 we conclude that the Theorem 2.1 holds if instead of condition (2.2) we use a triangular norm of Hadžić type.

(ii) Let $(b_n)_{n\in\mathbb{N}}$ be a sequence from (0, 1) such that:

a) $\sum_{n=1}^{+\infty}(1 - b_n)^\lambda$, $\lambda \in (0, +\infty)$ convergent. Then $\lim_{n\to+\infty}(\mathcal{T}_\lambda^\star)_{i=n}^{+\infty} b_i = 1$, $\star \in \{D, AA\}$.

b) $\sum_{n=1}^{+\infty}(1 - b_n)$, $\lambda \in (-1, +\infty]$ convergent. Then $\lim_{n\to+\infty}(\mathcal{T}_\lambda^{SW})_{i=n}^{+\infty} b_n = 1$.

Therefore, Theorem 2.1 is valid if instead of an arbitrary triangular norm that satisfies the condition (2.2) one uses the Dombi $(\mathcal{T}_\lambda^D)_{\lambda\in(0,+\infty)}$, Aczél-Alsina $(\mathcal{T}_\lambda^{AA})_{\lambda\in(0,+\infty)}$ or Sugeno-Weber $(\mathcal{T}_\lambda^{SV})_{\lambda\in[-1,+\infty)}$ family of triangular norms with additional conditions: $\sum_{i=1}^{+\infty}(1 - \frac{1}{\varsigma^i})^\lambda$ for Dombi and Aczél-Alsina family of triangular norms, as well as $\sum_{i=1}^{+\infty}(1 - \frac{1}{\varsigma^i})$ for the Sugeno-Weber family of triangular norms. Namely, this statement is obvious by using the proposition given in [13].

For more information on the mentioned families of triangular norms, readers are referred to the book [13].

Theorem 2.2. *Let* $(\Theta, \mathcal{M}, \mathcal{T})$ *be a fuzzy metric such that* $\lim_{p\to+\infty}\mathcal{M}(b, l, p) = 1$. *Let* $\mathcal{F} : \Theta \to C(\Theta)$ *be orbitally upper-semicontinuous,* Θ *is* $\mathcal{F}$*−orbitally complete and* $\mathcal{F}$ *satisfies the following:*

For every b, $l \in \Theta$, $u \in \mathcal{F}b$ *and* $0 < \delta < 1$, *there exists* $v \in \mathcal{F}b$ *such that*

$$\max\{\mathcal{M}(u, v, p), \mathcal{M}(u, b, p), \mathcal{M}(v, l, p)\} + \min\{1 - \mathcal{M}(u, l, p), 1 - \mathcal{M}(v, b, p)\} \geq \mathcal{M}(b, l, \frac{p-\delta}{q}), \quad (2.5)$$

for some $q \in (0, 1)$ *and every* $p > \max\{\frac{q}{1-q}, \delta\}$. *Also, one of the conditions (i) or (ii) is satisfied:*

(i) $\mathcal{F}$ is weakly demicompact

or

(ii) there exists b_0, $b_1 \in \Theta$, $b_1 \in \mathcal{F}b_0$ and $\varsigma \in (q, 1)$ such that triangular norm $\mathcal{T}$ satisfies:

$$\lim_{n\to+\infty} \mathcal{T}_{i=n}^{+\infty} \mathcal{M}(b_0, b_1, \frac{p}{\varsigma^i}) = 1.$$

Then there exist $b \in \Theta$ such that $b \in \mathcal{F}b$.

Proof. Let b_0, $b_1 \in \Theta$ such that $b_1 \in \mathcal{F}b_0$. Using (2.5), for $u = b_1$, $b = b_0$, $l = b_1$ and $\delta = q$ there exist $b_2 \in \Theta$ such that $b_2 \in \mathcal{F}b_1$ and

$$\begin{aligned} &\max\{\mathcal{M}(b_1, b_2, p), \mathcal{M}(b_0, b_1, p), \mathcal{M}(b_1, b_2, p)\} \\ + \ &\min\{1 - \mathcal{M}(b_0, b_2, p), 1 - \mathcal{M}(b_1, b_1, p)\} \geq \mathcal{M}(b_0, b_1, \frac{p-q}{q}). \end{aligned}$$

Then,

$$\max\{\mathcal{M}(b_1, b_2, p), \mathcal{M}(b_0, b_1, p)\} \geq \mathcal{M}(b_0, b_1, \frac{p-q}{q}).$$

Suppose that $\max\{\mathcal{M}(b_1, b_2, p), \mathcal{M}(b_0, b_1, p)\} = \mathcal{M}(b_0, b_1, p)$, we get $\mathcal{M}(b_0, b_1, p) \geq \mathcal{M}(b_0, b_1, \frac{p-q}{q})$, which means that $p \geq \frac{p-q}{q}$. Since this is contradictory with assumption ($p > \frac{q}{1-q}$), we conclude that

$$\mathcal{M}(b_1, b_2, p) \geq \mathcal{M}(b_0, b_1, \frac{p-q}{q}). \tag{2.6}$$

Further, for $\delta = q^2$ there exists $b_3 \in \mathcal{F}b_2$ such that by (2.5) we have

$$\begin{aligned} &\max\{\mathcal{M}(b_2, b_3, p), \mathcal{M}(b_1, b_2, p), \mathcal{M}(b_2, b_3, p)\} \\ + \ &\min\{1 - \mathcal{M}(b_1, b_3, p), 1 - \mathcal{M}(b_2, b_2, p)\} \geq \mathcal{M}(b_1, b_2, \frac{p-q^2}{q}). \end{aligned}$$

which implies that $\max\{\mathcal{M}(b_2, b_3, p), \mathcal{M}(b_1, b_2, p)\} = \mathcal{M}(b_2, b_3, p)$. So,

$$\mathcal{M}(b_2, b_3, p) \geq \mathcal{M}(b_1, b_2, \frac{p-q^2}{q}) \geq \mathcal{M}(b_0, b_1, \frac{p-2q^2}{q^2}).$$

Continuing, for $\delta = q^3$, $\delta = q^4, \ldots$ we can construct sequence $\{b_n\}_{n\in\mathbb{N}}$ from Θ such that the following conditions are satisfied:

(a) $b_{n+1} \in \mathcal{F}b_n$,

(b) $\mathcal{M}(b_n, b_{n+1}, p) \geq \mathcal{M}(b_{n-1}, b_n, \frac{p-q^n}{q})$, $n \in \mathbb{N}$.

Using (b) we have that

$$\mathcal{M}(b_n, b_{n+1}, p) \geq \mathcal{M}(b_1, b_0, \frac{p-nq^n}{q^n}), \ n \in \mathbb{N}.$$

Since, $\lim_{n\to+\infty} \mathcal{M}(b_1, b_0, pq^{-n} - n) = 1$ we conclude that

$$\lim_{n\to+\infty} \mathcal{M}(b_n, b_{n+1}, p) = 1. \tag{2.7}$$

If we suppose that $\mathcal{F}$ is weakly demicompact (condition (i)), using (2.7) and $b_{n+1} \in \mathcal{F}b_n$ we conclude that there exists a convergent subsequence $(b_{n_k})_{k\in\mathbb{N}}$ of the sequence $(b_n)_{n\in\mathbb{N}}$.

It remains to be proved that a sequence $(b_n)_{n\in\mathbb{N}}$ is convergent if triangular norm $\mathcal{T}$ satisfies condition (ii).

Let $\vartheta = \frac{q}{\varsigma}$. As $\vartheta \in (q, 1)$ follows that $\sum\limits_{i=1}^{+\infty} \vartheta^i$ is convergent, and there exists $m_0 \in \mathbb{N}$ such that $\sum\limits_{i=m_0}^{+\infty} \vartheta^i < 1$. So, for all $m > m_0$ and $s \in \mathbb{N}$ we have

$$p > p\sum_{i=m_0}^{+\infty} \vartheta^i > p\sum_{i=m}^{m+s} \vartheta^i.$$

Then,

$$\begin{aligned}
\mathcal{M}(b_{m+s+1}, b_m, p) &\geq \mathcal{M}(b_{m+s+1}, b_m, p\sum_{i=m}^{m+s}\vartheta^i)\\
&\geq \underbrace{\mathcal{T}(\mathcal{T}(\ldots\mathcal{T}}_{s-times}(\mathcal{M}(b_{m+s+1}, b_{m+s}, p\vartheta^{m+s}),\\
&\quad \mathcal{M}(b_{m+s}, b_{m+s-1}, p\vartheta^{m+s-1})), \ldots, \mathcal{M}(b_{m+1}, b_m, p\vartheta^m))\\
&\geq \underbrace{\mathcal{T}(\mathcal{T}(\ldots\mathcal{T}}_{s-times}(\mathcal{M}(b_1, b_0, \frac{p\vartheta^{m+s} - (m+s)q^{m+s}}{q^{m+s}}),\\
&\quad \mathcal{M}(b_1, b_0, \frac{p\vartheta^{m+s-1} - (m+s-1)q^{m+s-1}}{q^{m+s-1}}),\\
\ldots\\
&\quad \mathcal{M}(b_1, b_0, \frac{p\vartheta^m - mq^m}{q^m}))\\
&= \underbrace{\mathcal{T}(\mathcal{T}(\ldots\mathcal{T}}_{s-times}(\mathcal{M}(b_1, b_0, \frac{p}{(\frac{q}{\vartheta})^{m+s}} - (m+s)),\\
&\quad \mathcal{M}(b_1, b_0, \frac{p}{(\frac{q}{\vartheta})^{m+s-1}} - (m+s-1)),\\
\ldots\\
&\quad \mathcal{M}(b_1, b_0, \frac{p}{(\frac{q}{\vartheta})^m} - m))\\
&= \mathcal{T}_{i=m}^{m+s}\mathcal{M}(b_1, b_0, \frac{p}{\varsigma^i} - i).
\end{aligned}$$

Since, $\varsigma \in (q, 1)$, there exist $m_1(t) > m_0$ such that $\frac{p}{\varsigma^m} - m > \frac{p}{2\varsigma^m}$, for every $m > m_1(p)$. Now, for all $s \in \mathbb{N}$ we have

$$\begin{aligned}
\mathcal{M}(b_{m+s+1}, b_m, p) &\geq \mathcal{T}_{i=m}^{m+s}\mathcal{M}(b_1, b_0, \frac{p}{2\varsigma^i})\\
&\geq \mathcal{T}_{i=m}^{+\infty}\mathcal{M}(b_1, b_0, \frac{p}{2\varsigma^i}).
\end{aligned}$$

Using assumption that $\lim\limits_{m\to+\infty} \mathcal{T}_{i=m}^{+\infty}\mathcal{M}(b_1, b_0, \frac{1}{\varsigma^i}) = 1$, we conclude that $\lim\limits_{m\to+\infty} \mathcal{T}_{i=m}^{+\infty}\mathcal{M}(b_1, b_0, \frac{p}{2\varsigma^i}) = 1$, for every $p > 0$. So, for every $p > 0$, $\lambda \in (0, 1)$, there exists $m_2(p, \lambda) > m_1(p)$ such that $\mathcal{M}(b_{m+s+1}, b_m, p) > 1 - \lambda$, for all $m > m_2(p, \lambda)$ and every $s \in \mathbb{N}$.

Since, the sequence $(b_n)_{n\in\mathbb{N}}$ is a Cauchy and the space Θ is $\mathcal{F}-$orbitally complete we have that $\lim_{n\to+\infty} b_n$ exists.

So, in both cases (i) and (ii) there exists a subsequence $(b_{n_k})_{k\in\mathbb{N}}$ such that

$$u = \lim_{k\to+\infty} b_{n_k} \in \Theta.$$

The upper semi-continuity of $\mathcal{F}$ implies that $u \in \mathcal{F}u$. □

Theorem 2.3. *Let $(\Theta, \mathcal{M}, \mathcal{T})$ be a fuzzy metric such that $\mathcal{M}$ is increasing by t and $\lim_{p\to+\infty} \mathcal{M}(b, y, p) = 1$. Let $\mathcal{F} : \Theta \to C(\Theta)$ be orbitally upper-semicontinuous. If Θ is $\mathcal{F}-$orbitally complete and $\mathcal{F}$ satisfies the following:*

There exists $q \in (0, 1)$ such that for every $b, l \in \Theta,\ p > 0$,

$$\max\{\mathcal{H}(\mathcal{F}b, \mathcal{F}l, p), \mathcal{E}(\mathcal{F}b, b, p), \mathcal{E}(\mathcal{F}l, l, p)\} + (\min\{1 - \mathcal{E}(\mathcal{F}b, l, p), 1 - \mathcal{E}(b, \mathcal{F}l, p)\}) \geq \mathcal{M}(b, l, \frac{p}{q}). \tag{2.8}$$

and if one of the conditions (i) or (ii) from the Theorem 2.2 is satisfied, then there exists $b \in \Theta$ such that $b \in \mathcal{F}b$.

Proof. Let $a > 0$ be an arbitrary small real number less then 1 and let $t_0 > 0$ be arbitrary. Since $\mathcal{M}(b, l, p)$ is increasing by p, for every $b \in \Theta$ there exists $l \in \mathcal{F}b$, ($l \neq b$, otherwise b is a fixed point) such that:

$$\mathcal{M}(b, l, p_0) \geq \mathcal{E}(b, \mathcal{F}b, q^a p_0). \tag{2.9}$$

Let $\kappa : \Theta \to \Theta$ be a function such that $\kappa b = l,\ b \in \Theta$.

We take arbitrary $b \in \Theta$ and consider a orbit of κ defined as $\kappa b_{n-1} = b_n,\ n \in \mathbb{N}$ where $b_0 = b$. Note that $b_n \in \mathcal{F}b_{n-1}$ implies $\mathcal{E}(b_n, \mathcal{F}b_n, p) \geq \mathcal{H}(\mathcal{F}b_{n-1}, \mathcal{F}b_n, p)$ and $\mathcal{E}(b_n, \mathcal{F}b_{n-1}, p) = 1,\ p > 0$. Now, we have using (2.8) for $b = b_{n-1}$ and $l = b_n,\ n \in \mathbb{N}$:

$$\begin{aligned}
& \max\{\mathcal{E}(\mathcal{F}b_{n-1}, b_{n-1}, p), \mathcal{E}(\mathcal{F}b_n, b_n, p)\} \\
= \ & \max\{\mathcal{H}(\mathcal{F}b_{n-1}, \mathcal{F}b_n, p), \mathcal{E}(\mathcal{F}b_{n-1}, b_{n-1}, p), \mathcal{E}(\mathcal{F}b_n, b_n, p)\} \\
- \ & (\min\{1 - \mathcal{E}(\mathcal{F}b_{n-1}, b_n, p), 1 - \mathcal{E}(b_{n-1}, \mathcal{F}b_n, p)\}) \\
\geq \ & \mathcal{M}(b_{n-1}, b_n, \frac{p}{q}),\ p > 0.
\end{aligned}$$

Then, by (2.9), we have

$$\begin{aligned}
\max\{\mathcal{M}(b_n, b_{n-1}, p_0), \mathcal{M}(b_{n+1}, b_n, p_0)\} \ &\geq \ \max\{\mathcal{E}(\mathcal{F}b_{n-1}, b_{n-1}, q^a p_0), \mathcal{E}(\mathcal{F}b_n, b_n, q^a p_0)\} \\
&\geq \ \mathcal{M}(b_{n-1}, b_n, \frac{p_0}{q^{1-a}}),
\end{aligned}$$

which implies that

$$\mathcal{M}(b_{n+1}, b_n, p_0) \geq \mathcal{M}(b_{n-1}, b_n, \frac{p_0}{q^{1-a}}),\ n \in \mathbb{N}.$$

Finally, we conclude that

$$\mathcal{M}(b_{n+1}, b_n, p) \geq \mathcal{M}(b_{n-1}, b_n, \frac{p}{q^{1-a}}),\ n \in \mathbb{N},\ p > 0.$$

Let $q_1 = q^{1-a}$. Then $q_1 \in (0,1)$ and we can use the same technique as in Theorem 2.1 to conclude that the sequence $\{b_n\}$ is a Cauchy sequence and by the assumption that he mapping $\mathcal{F}$ is orbitally complete we conclude that there exists $b \in \Theta$ such that $b \in \mathcal{F}b$. □

Let Φ be collection of all continuous mappings $\varphi : [0,1] \to [0,1]$ such that $\varphi(p) > p,\ p \in (0,1),\ \varphi(1) = 1$.

Such type of collection Φ is, together with contraction condition proposed by Ćirić in [5], used in [1] to obtain nonunique fixed point results in $b-$metric space. In the following theorem collection Φ within slightly modified contraction condition is observed in fuzzy metric spaces and existence of unique fixed point is proved. Open question: is it possible to obtain nonunique result with original Ćirić's condition, using collection Φ, as it is done in [1].

Theorem 2.4. *Let $(\Theta, \mathcal{M}, \mathcal{T})$ be $\kappa-$orbitally complete fuzzy metric space and $\kappa : \Theta \to \Theta$ be an orbitally continuous mapping on Θ such that $\lim\limits_{p\to+\infty} \mathcal{M}(b,l,p) = 1$ and $\lim_{t\to 0^+} \mathcal{M}(b_n, b_{n+1}, p) > 0,\ n \in \mathbb{N}$. If κ satisfies the following condition*

$$\mathcal{M}(\kappa b, \kappa l, p) + \min\{1 - \mathcal{M}(\kappa b, b, p), 1 - \mathcal{M}(\kappa l, l, p), 1 - \mathcal{M}(b, \kappa l, p), 1 - \mathcal{M}(\kappa b, l, p)\} \geq \varphi(\mathcal{M}(b,l,p)). \tag{2.10}$$

$\varphi \in \Phi,\ b, l \in \Theta,\ p > 0$, *then for each $b \in \Theta$ the sequence $\{\kappa^n b\}_{n=1}^{+\infty}$ converges to a fixed point of κ.*

Proof. Let $b_0 \in \Theta$ is arbitrary. Now, we can construct a sequence $\{b_n\}_{n\in\mathbb{N}\cup\{0\}}$ such that $b_{n+1} = b_n$, for all $n \in \mathbb{N}\cup\{0\}$. If $b_{n_0} = b_{n_0+1}$, for some $n_0 \in \mathbb{N}\cup\{0\}$, then the proof is finished. So, suppose that $b_n \neq b_{n+1}$, for all $n \in \mathbb{N}\cup\{0\}$.

By (2.10), for $b = b_{n-1},\ l = b_n,\ n \in \mathbb{N}$, we have

$$\mathcal{M}(b_{n+1}, b_n, p) \geq \varphi(\mathcal{M}(b_n, b_{n-1}, p)) > \mathcal{M}(b_n, b_{n-1}, p),\ p > 0,$$

and conclude that

$$\mathcal{M}(b_{n+1}, b_n, p) > \mathcal{M}(b_n, b_{n-1}, p) > \cdots > \mathcal{M}(b_1, b_0, p),\ n \in \mathbb{N},\ p > 0.$$

Therefore, since the sequence $\{\mathcal{M}(b_n, b_{n+1}, p)\}$, $n \in \mathbb{N}$ is monotone increasing we have that there exists $a \leq 1$ such that

$$\lim_{n\to+\infty} \mathcal{M}(b_n, b_{n+1}, p) = a,\ p > 0.$$

Suppose that $a < 1$, then using (2.10) we have contradiction

$$a \geq \varphi(a) > a,$$

and conclude that $a = 1$.

Further, we need to prove that $\{b_n\}$ is a Cauchy sequence. Suppose that is not true and by Lemma 1.9, we have that there exist $\varepsilon \in (0,1),\ p_0 > 0$ and sequences $\{b_{m_k}\}$ and $\{b_{n_k}\}$ such that $\lim\limits_{k\to+\infty} M(b_{m_k}, b_{n_k}, p_0) = 1 - \varepsilon$. By (2.10)

$$\begin{aligned}&\mathcal{M}(b_{m_k}, b_{n_k}, p_0) + \min\{1 - \mathcal{M}(b_{m_k-1}, b_{m_k}, p_0), 1 - \mathcal{M}(b_{n_k-1}, b_{n_k}, p_0),\\ &1 - \mathcal{M}(b_{m_k}, b_{n_k-1}, p_0), 1 - \mathcal{M}(b_{n_k}, b_{m_k-1}, p_0)\} \geq \varphi(\mathcal{M}(b_{m_k-1}, b_{n_k-1}, p_0)),\end{aligned}$$

and when $k \to +\infty$ we get contradiction

$$1-\varepsilon \geq \varphi(1-\varepsilon) > 1-\varepsilon.$$

So, $\{b_n\}$ is a Cauchy sequence.

Since $(\Theta, \mathcal{M}, \mathcal{T})$ is complete there exists $u \in \Theta$ such that $\lim_{n\to+\infty} b_n = u$. By condition (2.10), with $b = u, l = b_n$, we have

$$\begin{aligned}&\mathcal{M}(\kappa u, b_{n+1}, p) + \min\{1-\mathcal{M}(u, \kappa u, p), 1-\mathcal{M}(b_n, b_{n+1}, p),\\ &1-\mathcal{M}(u, b_{n+1}, p), 1-\mathcal{M}(\kappa u, b_n, p)\} \geq \varphi(\mathcal{M}(u, b_n, p)),\ n \in \mathbb{N},\ p > 0.\end{aligned}$$

If we take $n \to +\infty$ it follows that u is a fixed point for κ:

$$\mathcal{M}(\kappa u, u, p) \geq \varphi(M(u, u, p)) = \varphi(1) = 1.$$

Moreover, u is the unique fixed point for κ. Suppose that different u and v are fixed points and take (2.10) with $b = u, l = v$:

$$\begin{aligned}&\mathcal{M}(\kappa u, \kappa v, p) + \min\{1-\mathcal{M}(u, \kappa u, p), 1-\mathcal{M}(v, \kappa v, p),\\ &1-\mathcal{M}(u, \kappa v, p), 1-\mathcal{M}(\kappa u, v, p)\} \geq \varphi(\mathcal{M}(u, v, p)),\ p > 0.\end{aligned}$$

So, we have contradiction

$$\mathcal{M}(\kappa u, \kappa v, p) \geq \varphi(\mathcal{M}(u, v, p)) > \mathcal{M}(u, v, p) = \mathcal{M}(\kappa u, \kappa v, p).$$

□

Example 4. Let $\Theta = \mathbb{R}$, $\kappa u = \frac{u}{2}$, $M(b, l, p) = e^{-\frac{|b-l|}{p}}$ and $\mathcal{T} = \mathcal{T}_{\mathcal{P}}$ and $\varphi(p) = \sqrt{p}$. Then all conditions of Theorem 2.4 are satisfied and 0 is the unique fixed point.

3. Conclusions

Using the countable extension of the triangular norm, we were able to prove theorems about the fixed point for the single-valued and multi-valued cases within fuzzy metric spaces. The mapping is not assumed to be continuous, but orbitally continuous. A fixed point is not necessarily unique as illustrated by an example. Potentially, obtained results could be applied for solving different integral equations and integral operators as it is done, for example, in [15, 16, 18].

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Conflict of interest

The authors declare that they have no conflicts of interest.

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