Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(1): 2137–2153.
DOI: 10.3934/math.2023110
Received: 23 August 2022
Revised: 03 October 2022
Accepted: 06 October 2022
Published: 28 October 2022

Research article

A novel numerical method for solution of fractional partial differential equations involving the ψ-Caputo fractional derivative

Amjid Ali[1], **Teruya Minamoto**[1], **Rasool Shah**[2] **and Kamsing Nonlaopon**[3,*]

[1] Faculty of Science and Engineering, Saga University, Saga 840-8502, Japan
[2] Department of Mathematics, Abdul Wali Khan University Mardan, Pakistan
[3] Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand

* **Correspondence:** Email: nkamsi@kku.ac.th.

Abstract: In this study, the ψ-Haar wavelets operational matrix of integration is derived and used to solve linear ψ-fractional partial differential equations (ψ-FPDEs) with the fractional derivative defined in terms of the ψ-Caputo operator. We approximate the highest order fractional partial derivative of the solution of linear ψ-FPDE using Haar wavelets. By combining the operational matrix and ψ-fractional integration, we approximate the solution and its other ψ-fractional partial derivatives. Then substituting these approximations in the given ψ-FPDEs, we obtained a system of linear algebraic equations. Finally, the approximate solution is obtained by solving this system. The simplicity and effectiveness of the proposed method as a mathematical tool for solving ψ-Fractional partial differential equations is one of its main advantages. The sparse nature of the operational matrices improves the ability of the proposed method to execute with less computation complexity. Numerical examples are provided to show the efficiency and effectiveness of the method.

Keywords: ψ-fractional partial differential equations; ψ-Caputo fractional derivatives; ψ-Haar wavelets operational matrices of integration
Mathematics Subject Classification: 26A33, 35A20, 35A35, 33B15, 33F05

1. Introduction

Fractional calculus is considered the generalization of classical calculus. Fractional differential equations have been widely employed in various science and engineering fields [1–3]. Many researchers have defined fractional order derivatives and integrals in various forms. New definitions of fractional differential operators and ψ-fractional derivatives and integrals have been considered by several researchers, such as the Riemann-Liouville, Caputo, Hilfer, Erdelyi-Kober, Hadamard [4–7].

Additionally, studies by Sousa et al. contain fascinating details concerning the ψ-Riemann-Liouville fractional partial integral, and the ψ-Caputo fractional partial derivative [8]. Furthermore, many interesting results from the qualitative analysis of fractional ordinary and partial differential equations involving different fractional derivatives have been recorded [9–16]. However, numerical solutions for fractional partial differential equations (FPDEs) involving the ψ-Caputo fractional partial derivatives have not been performed using the ψ-Haar wavelet operational matrix method. Thus, this study establishes a numerical technique for solving ψ-Caputo FPDEs.

The rest of the paper is organized as follows: Section 2 presents some fundamental definitions and results from ψ-Fractional calculus. Section 3 reviews Haar wavelets and their applications in function approximation. Furthermore, we derive the operational matrix of ψ-fractional integration of Haar wavelets. Section 4 presents a detailed numerical procedure for solving ψ-FPDEs using constant and variable coefficients. Finally, Section 5 presents the conclusion.

2. Basics of ψ-fractional calculus

This section highlights concepts, definitions, and basic conclusions from the ψ-fractional calculus that are important in later sections.

Let the function $f : [a, b] \to \mathbb{R}$ be integrable, α a positive real number, n a natural number and $\psi \in C^1([a, b])$ be an increasing function such that $\psi'(\varkappa) \neq 0$ for all $\varkappa \in [a, b]$.

Definition 1. [4, 5, 17] The ψ-Riemann-Liouvile (ψ-RL) fractional integral operator of order α is defined by

$$\mathcal{J}_a^{\alpha,\psi} f(\varkappa) = \frac{1}{\Gamma(\alpha)} \int_a^{\varkappa} \psi'(\Im)(\psi(\varkappa) - \psi(\Im))^{\alpha-1} f(\Im) d\Im. \tag{2.1}$$

The ψ-RL fractional differential operator is given by

$$D_a^{\alpha,\psi} f(\varkappa) = \left(\frac{1}{\psi'(\varkappa)} \frac{d}{d\varkappa}\right)^n \mathcal{J}_a^{n-\alpha,\psi} f(\varkappa) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(\varkappa)} \frac{d}{d\varkappa}\right)^n \int_a^{\varkappa} \psi^{'}(\Im)(\psi(\varkappa) - \psi(\Im))^{n-\alpha-1} f(\Im) d\Im,$$

where $n = \lfloor\alpha\rfloor + 1$.

Definition 2. [6] Let α be a positive real number, n a natural number and $f, \psi \in C^n([a, b])$ such that ψ is increasing and $\psi'(\varkappa) \neq 0$ for all $\varkappa \in [a, b]$. The ψ-Caputo differential operator of fractional order α is defined by

$${}^C D_a^{\alpha,\psi} f(\varkappa) = \frac{1}{\Gamma(n-\alpha)} \int_a^{\varkappa} \psi^{'}(\Im)(\psi(\varkappa) - \psi(\Im))^{n-\alpha-1} f_\psi^{[n]}(\Im) d\Im,$$

where $f_\psi^{[n]}(\varkappa) = \left(\frac{1}{\psi'(\varkappa)} \frac{d}{d\varkappa}\right)^n f(\varkappa)$, here $n = \lfloor\alpha\rfloor + 1$ for $\alpha \notin \mathbb{N}$, or

$${}^C D_a^{\alpha,\psi} f(\varkappa) = \mathcal{J}_a^{n-\alpha,\psi} f_\psi^{[n]}(\varkappa).$$

Also, the ψ-Caputo derivative can be defined as

$${}^C D_a^{\alpha,\psi} f(\varkappa) = D_a^{\alpha,\psi} \left[f(\varkappa) - \sum_{k=0}^{n-1} \frac{f_\psi^{[k]}(a)}{k!} (\psi(\varkappa) - \psi(a))^k \right],$$

where $n = \lceil\alpha\rceil$, whenever $\alpha \notin \mathbb{N}$ and for $\alpha \in \mathbb{N}, n = \alpha$.

Definition 3. The two-parameter Mittag–Leffler function $E_{\alpha,\beta}(\Im)$ is defined as

$$E_{\alpha,\beta}(\Im) = \sum_{k=0}^{\infty} \frac{\Im^k}{\Gamma(\alpha k + \beta)}, \quad \Im \in \mathbb{R}, \ \ \alpha, \, \beta > 0. \tag{2.2}$$

The Mittag-Leffler function can also be given for these exceptional cases:

(i) $E_{0,1}(\Im) = \dfrac{1}{1+\Im}$;
(ii) $E_{1,1}(\Im) = e^{\Im}$;
(iii) $E_{2,1}(-\Im^2) = \cos(\Im)$;
(iv) $E_{2,2}(-\Im^2) = \dfrac{\sin(\Im)}{\Im}$.

Properties of the ψ-fractional operators include:

Property 2.1. The following property holds:

$$\mathcal{J}_a^{\alpha,\psi} \mathcal{J}_a^{\beta,\psi} f(\Im) = \mathcal{J}_a^{\alpha+\beta,\psi} f(\Im).$$

Property 2.2. If $f(\Im) = (\psi(\Im) - \psi(a))^{\beta}$, where $\beta > n$ and $\alpha > 0$, then

$${}^{C}D_a^{\alpha,\psi} f(\Im) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} \left(\psi(\Im) - \psi(a)\right)^{\beta-\alpha}.$$

Property 2.3. The following property holds:

$${}^{C}D_a^{\alpha,\psi} \mathcal{J}_a^{\alpha,\psi} f(\Im) = f(\Im).$$

Proof. By definition, we have

$${}^{C}D_a^{\alpha,\psi} \mathcal{J}_a^{\alpha,\psi} f(\Im) = D_a^{\alpha,\psi} \left[\mathcal{J}_a^{\alpha,\psi} f(\Im) - \sum_{k=0}^{n-1} \frac{\left[\mathcal{J}_a^{\alpha,\psi} f\right]_{\psi}^{[k]}(a)}{k!} (\psi(\Im) - \psi(a))^k \right]. \tag{2.3}$$

Note that

$$\begin{aligned}
[\mathcal{J}_a^{\alpha,\psi} f]_{\psi}^{[k]}(\Im) &= \left(\frac{1}{\psi'(\Im)} \frac{d}{d\Im}\right)^k \mathcal{J}_a^{\alpha,\psi} f(\Im) \\
&= \left(\frac{1}{\psi'(\Im)} \frac{d}{d\Im}\right)^{k-1} \frac{1}{\psi'(\Im)} \frac{d}{d\Im} \int_a^{\Im} \frac{(\psi(\Im) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} \psi'(s) f(s) ds \\
&= \left(\frac{1}{\psi'(\Im)} \frac{d}{d\Im}\right)^{k-1} \frac{1}{\psi'(\Im)} \int_a^{\Im} \frac{(\alpha-1)\left(\psi(\Im) - \psi(s)\right)^{\alpha-2}}{\Gamma(\alpha)} \psi'(\Im) \psi'(s) f(s) ds \\
&= \left(\frac{1}{\psi'(\Im)} \frac{d}{d\Im}\right)^{k-1} \int_a^{\Im} \frac{(\psi(\Im) - \psi(s))^{\alpha-2}}{\Gamma(\alpha-1)} \psi'(s) f(s) ds \\
&= \left(\frac{1}{\psi'(\Im)} \frac{d}{d\Im}\right)^{k-1} \mathcal{J}_a^{\alpha-1} f(\Im).
\end{aligned}$$

Then, we have

$$\left[\mathcal{J}_a^{\alpha,\psi} f\right]_{\psi}^{[k]}(\mathfrak{I}) = [\mathcal{J}_a^{\alpha-1,\psi} f]_{\psi}^{[k-1]}(\mathfrak{I}).$$

Repeating the process k-times, we have

$$\left[\mathcal{J}_a^{\alpha,\psi} f\right]_{\psi}^{[k]}(\mathfrak{I}) = \mathcal{J}_a^{\alpha-k,\psi} f(\mathfrak{I}). \tag{2.4}$$

Now, substituting (2.4) into (2.3), we have

$$^{C}D_a^{\alpha,\psi}\mathcal{J}_a^{\alpha,\psi} f(\mathfrak{I}) = D_a^{\alpha,\psi}\left[\mathcal{J}_a^{\alpha,\psi} f(\mathfrak{I}) - \sum_{k=0}^{n-1}\frac{\mathcal{J}_a^{\alpha-k,\psi} f(a)}{k!}(\psi(\mathfrak{I}) - \psi(a))^k\right]. \tag{2.5}$$

Next, we show that $\mathcal{J}_a^{\alpha-k,\psi} f(a) = 0$. That is, we prove that $\lim_{\mathfrak{I}\to a}\mathcal{J}_a^{\alpha-k,\psi} f(\mathfrak{I}) = 0$. Now, we have

$$\begin{aligned}
\left\|\mathcal{J}_a^{\alpha,\psi} f(\mathfrak{I})\right\| &= \left\|\frac{1}{\Gamma(\alpha)}\int_a^{\mathfrak{I}} (\psi(\mathfrak{I}) - \psi(s))^{\alpha-1}\psi'(s)f(s)ds\right\| \\
&\le \frac{1}{\Gamma(\alpha)}\int_a^{\mathfrak{I}} \left\|(\psi(\mathfrak{I}) - \psi(s))^{\alpha-1}\psi'(s)f(s)\right\| ds \\
&\le \frac{\|f\|}{\Gamma(\alpha)}\int_a^{\mathfrak{I}} (\psi(\mathfrak{I}) - \psi(s))^{\alpha-1}\psi'(s)ds \\
&\le \|f\|\frac{(\psi(\mathfrak{I}) - \psi(a))^{\alpha}}{\Gamma(\alpha+1)},
\end{aligned}$$

since $\psi'(\mathfrak{I}) > 0$ and $\Gamma(\alpha+1) = (\alpha)\Gamma(\alpha)$. Hence, $\mathcal{J}_a^{\alpha,\psi} f(\mathfrak{I})$ tends to 0 as $\mathfrak{I}$ tends to a. Thus, from (2.5), we have

$$\begin{aligned}
^{C}D_a^{\alpha,\psi}\mathcal{J}_a^{\alpha,\psi} f(\mathfrak{I}) &= D_a^{\alpha,\psi}\mathcal{J}_a^{\alpha,\psi} f(\mathfrak{I}) \\
&= \left(\frac{1}{\psi'(\mathfrak{I})}\frac{d}{d\mathfrak{I}}\right)^n \mathcal{J}_a^{n-\alpha,\psi}\mathcal{J}_a^{\alpha,\psi} f(\mathfrak{I}) \\
&= \left(\frac{1}{\psi'(\mathfrak{I})}\frac{d}{d\mathfrak{I}}\right)^n \mathcal{J}_a^{n-\alpha+\alpha,\psi} f(\mathfrak{I}) \\
&= \left(\frac{1}{\psi'(\mathfrak{I})}\frac{d}{d\mathfrak{I}}\right)^n \mathcal{J}_a^{n,\psi} f(\mathfrak{I}).
\end{aligned}$$

Consequently, we have $^{C}D_a^{\alpha,\psi}\mathcal{J}_a^{\alpha,\psi} f(\mathfrak{I}) = f(\mathfrak{I})$. This complete the proof. □

By Leibnitz rule, we have

$$\begin{aligned}
\frac{1}{\psi'(\mathfrak{I})}\frac{d}{d\mathfrak{I}}\mathcal{J}_a^{1,\psi} f(\mathfrak{I}) &= \frac{1}{\psi'(\mathfrak{I})}\frac{d}{d\mathfrak{I}}\int_a^{\mathfrak{I}} (\psi(\mathfrak{I}) - \psi(s))^{1-1}\psi'(s)f(s)ds \\
&= \frac{1}{\psi'(\mathfrak{I})}\psi'(\mathfrak{I})f(\mathfrak{I}) \\
&= f(\mathfrak{I}).
\end{aligned}$$

Repeating the above process n-times we have

$$\left(\frac{1}{\psi'(\mathfrak{I})}\frac{d}{d\mathfrak{I}}\right)^n \mathcal{J}_a^{n,\psi} f(\mathfrak{I}) = f(\mathfrak{I}).$$

Lemma 1. The following property holds:

$$\mathcal{J}_a^{n,\psi} f_\psi^{[n]}(\Im) = f(\Im) - \sum_{k=0}^{n-1} \frac{f_\psi^{[n]}(a)}{k!} \left(\psi(\Im) - \psi(a)\right)^k .$$

Proof. For $n = 1$, we have

$$\begin{aligned} \mathcal{J}_a^{1,\psi} f_\psi^{[1]}(\Im) &= \int_a^\Im \frac{\left(\psi(\Im) - \psi(s)\right)^{1-1}}{\Gamma(1)} \psi'(s) \frac{1}{\psi'(s)} \frac{d}{ds} f(s) ds \\ &= \int_a^\Im \frac{d}{ds} f(s) ds = f(\Im) - f(a). \end{aligned} \tag{2.6}$$

For $n = 2$, we have

$$\begin{aligned} \mathcal{J}_a^{2,\psi} f_\psi^{[2]}(\Im) &= \mathcal{J}_a^{1,\psi} \mathcal{J}_a^{1,\psi} \left[f_\psi^{[1]} \right]_\psi^1 (\Im) = \mathcal{J}_a^{1,\psi} \left[f_\psi^{[1]} - f_\psi^{[1]} \right] \\ &= \mathcal{J}_a^{1,\psi} f_\psi^{[1]}(\Im) - \mathcal{J}_a^{1,\psi} f_\psi^{[1]}(a) \\ &= f(\Im) - f(a) - f_\psi^{[1]}(a) \int_a^\Im \left(\psi(\Im) - \psi(a)\right)^{1-1} \psi'(s) ds \\ &= f(\Im) - f(a) - f_\psi^{[1]}(a) (\psi(\Im) - \psi(a)). \end{aligned} \tag{2.7}$$

Repeating the above process n-times, we have

$$\mathcal{J}_a^{n,\psi} f_\psi^{[n]}(\Im) = f(\Im) - \sum_{k=0}^{n-1} \frac{f_\psi^{[k]}(a)}{k!} \left(\psi(\Im) - \psi(a)\right)^k .$$

This completes the proof. □

Lemma 2. The following property holds:

$$\mathcal{J}_a^{\alpha,\psi}\ {}^C D_a^{\alpha,\psi} f(\Im) = f(\Im) - \sum_{k=0}^{n-1} \frac{f_\psi^{[k]}(a)}{k!} (\psi(\Im) - \psi(a))^k.$$

Proof. Since

$${}^C D_a^{\alpha,\psi} f(\Im) = \mathcal{J}_a^{n-\alpha,\psi} f_\psi^{[n]}(\Im),$$

thus

$$\begin{aligned} \mathcal{J}_a^{\alpha,\psi C} D_a^{\alpha,\psi} f(\Im) &= \mathcal{J}_a^{\alpha,\psi} \mathcal{J}_a^{n-\alpha,\psi} f_\psi^{[n]}(\Im) \\ &= \mathcal{J}_a^{\alpha+n-\alpha,\psi} f_\psi^{[n]}(\Im) \\ &= \mathcal{J}_a^{n,\psi} f_\psi^{[n]}(\Im). \end{aligned}$$

Therefore, we have

$$\mathcal{J}_a^{\alpha,\psi C} D_a^{\alpha,\psi} f(\Im) = f(\Im) - \sum_{k=0}^{n-1} \frac{f_\psi^{[k]}(a)}{k!} \left(\psi(\Im) - \psi(a)\right)^k .$$

The proof is completed. □

3. Haar wavelets and function approximation

The Haar wavelet, invented by Hungarian mathematician Alfred Haar in 1909, is the most basic example of an orthogonal wavelet. The Haar mother wavelet is defined by a two-scale relation for the scaling function $\varphi(\Im) = \chi_{[0,1)}$ as:

$$h(\Im) = \varphi(2\Im) - \varphi(2\Im - 1) = \chi_{[0,\frac{1}{2})}(\Im) - \chi_{[\frac{1}{2},1)}(\Im). \tag{3.1}$$

Define

$$h_{j,k}(\Im) = 2^{\frac{j}{2}} h(2^j(\Im) - k), \quad 0 \le k < 2^j, \quad j = 0, 1, 2, \dots \,. \tag{3.2}$$

Then, the Haar system $\varphi, h_{j,k} : j \ge 0, 0 \le k < 2^j$ forms an orthonormal basis for the Hilbert space $L_2(0,1)$. Thus, for some fixed j, the inner product expansion of $f \in L_2[0,1]$ is given as

$$f(\Im) \approx \langle f, \varphi\rangle \varphi(\Im) \sum_{j=0}^{j-1} \sum_{k=0}^{2^j-1} \langle f, h_{j,k}\rangle h_{j,k}(\Im) = C^T H(\Im), \tag{3.3}$$

where C, determined by the inner product $c_i = \langle f(\Im), h_{j,k}(\Im)\rangle, \langle\cdot\rangle$ is a 1×2^j coefficient matrix and $H(\Im) = \left[\varphi, h, h_{1,0}, h_{1,1}, h_{2,0}, \dots, h_{2,3}, \dots, h_{j-1,0}, \dots, h_{j-1,2^j-1}\right]$ represents the vector of the Haar functions. For simplicity, consider $\phi = h_0, h_{0,0} = h = h_1$ and $h_i = h_{j,k}$, where $i = 0, 1, 2, \dots, m-1, m : 2^j$, then, equation (3.3) becomes

$$f(\Im) \approx \sum_{i=0}^{m-1} k_i h_i(\Im) = C^T H(\Im). \tag{3.4}$$

Also, a function of two variables, $y(\varkappa, \Im) \in L^2([0,1] \times [0,1])$ can be approximated using Haar wavelets as:

$$y(\varkappa, \Im) \approx \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} c_{i,j} h_i(\varkappa) h_j(\Im) = H^T(\varkappa) C H(\Im), \tag{3.5}$$

where C, a $2^j \times 2^j$ coefficient matrix, is computed using the inner product

$$c_{i,j} = \langle h_i(\varkappa), \langle y(\varkappa, \Im) h_j(\Im)\rangle\rangle.$$

ψ-Haar wavelets operational matrix:

The operational matrix of ψ-fractional integration of Haar wavelet is defined as

$$P_i^{\alpha,\psi}(\varkappa) = \frac{1}{\Gamma(\alpha)} \int_a^{\varkappa} \psi'(\Im)(\psi(\varkappa) - \psi(\Im))^{\alpha-1} h_i(\Im) d\Im. \tag{3.6}$$

Furthermore, the ψ-fractional integral can be generalized and approximated analytically as:

$$P_i^{\alpha,\psi}(\varkappa) = \begin{cases} 0, & \text{if } \varkappa < \zeta_1(i); \\ \frac{1}{\Gamma(\alpha+1)} \left[\psi(\varkappa) - \psi(\zeta_1(i))\right]^{\alpha}, & \text{if } \varkappa \in [\zeta_1(i), \zeta_2(i)); \\ \frac{1}{\Gamma(\alpha+1)} \left[(\psi(\varkappa) - \psi(\zeta_1(i)))^{\alpha} - 2\left(\psi(\varkappa) - \psi(\zeta_2(i))\right)^{\alpha}\right], & \text{if } \varkappa \in (\zeta_2(i), \zeta_3(i)]; \\ \frac{1}{\Gamma(\alpha+1)} \left[(\psi(\varkappa) - \psi(\zeta_1(i)))^{\alpha} - 2\left(\psi(\varkappa) - \psi(\zeta_2(i))\right)^{\alpha} \right. & \\ \left. + \left(\psi(\varkappa) - \psi(\zeta_3(i))\right)^{\alpha}\right], & \text{if } \varkappa > \zeta_3(i). \end{cases} \tag{3.7}$$

Equation (3.7) is true for $i > 1$ and for $i = 1$, we have

$$P^{\alpha,\psi}(\varkappa) = \frac{1}{\Gamma(\alpha+1)}\left[\psi(\varkappa) - \psi(a)\right]^{\alpha}. \tag{3.8}$$

Below is the operational matrix $P^{\alpha,\psi}$ of ψ-Haar wavelets computed for $\psi(\varkappa) = \sin(\varkappa)$ and $\alpha = 0.8$.

$$P^{\alpha,\psi} = \begin{bmatrix} 0.5606 & -0.2219 & -0.1403 & -0.0847 & -0.08164 & -0.0611 & -0.0484 & -0.0362 \\ 0.2769 & 0.0617 & -0.1403 & 0.1369 & -0.08164 & -0.0611 & 0.0896 & 0.0521 \\ 0.0656 & 0.0826 & 0.05017 & -0.0093 & -0.08164 & 0.0935 & -0.0092 & -0.0020 \\ 0.0689 & -0.0689 & 0 & 0.03492 & 0 & 0 & -0.0690 & 0.0696 \\ 0.0150 & 0.0188 & 0.04881 & -0.0011 & 0.03404 & -0.0054 & -0.0008 & -0.0003 \\ 0.0175 & 0.0231 & -0.04066 & -0.0044 & 0 & 0.0318 & -0.0048 & -0.0007 \\ 0.0178 & -0.0178 & 0 & 0.0401 & 0 & 0 & 0.0279 & -0.0041 \\ 0.0166 & -0.0166 & 0 & -0.0332 & 0 & 0 & 0 & 0.0224 \end{bmatrix}. \tag{3.9}$$

4. Numerical solution to ψ-FPDEs

This section discusses numerical solutions for linear ψ-FPDEs using a technique based on two-dimensional ψ-Haar wavelets.

4.1. ψ-FPDEs with constant coefficients

This section considers linear FPDEs with constant coefficients involving ψ-Caputo fractional derivative

$$\frac{\partial^{\alpha,\psi} y(\varkappa,\Im)}{\partial \Im^{\alpha,\psi}} + \lambda \frac{\partial^{\beta,\psi} y(\varkappa,\Im)}{\partial \Im^{\beta,\psi}} + \mu y(\varkappa,\Im) = \eta \frac{\partial^{\gamma,\psi} y(\varkappa,\Im)}{\partial \varkappa^{\gamma,\psi}} + f(\varkappa,\Im), \tag{4.1}$$

for $0 < \alpha \leq 2, 0 \leq \beta \leq 1, 1 \leq \gamma \leq 2$ and have non-homogeneous boundary and initial conditions given by

$$y(\varkappa,0) = \rho(\varkappa), \quad \left.\frac{\partial y(\varkappa,\Im)}{\partial \Im}\right|_{\Im=0} = \sigma(\varkappa), \quad y(0,\Im) = \xi(\Im), \quad y(1,\Im) = \zeta(\Im). \tag{4.2}$$

For $1 < \alpha \leq 2$ and $\lambda, \mu, \eta > 0$, then (4.1) reduces to the fractional telegraph equation. For special cases, it includes the heat, wave, and Poisson equations. The ψ-Haar wavelets technique provides numerical solutions. By approximating $\frac{\partial^{\alpha,\psi} y(\varkappa,\Im)}{\partial \Im^{\alpha,\psi}}$ using two-dimensional Haar wavelets, we have

$$\frac{\partial^{\alpha,\psi} y(\varkappa,\Im)}{\partial \Im^{\alpha,\psi}} = H_m^T(\varkappa) C_{m\times m} H_m(\Im). \tag{4.3}$$

Operating both sides of (4.3) by $\mathcal{J}_{\Im}^{\alpha,\psi}$, we get

$$y(\varkappa,\Im) = H_m^T(\varkappa) C_{m\times m}\left(\int_0^{\Im} \psi'(\Im)\frac{(\psi(\Im)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} H_m(s)ds\right) + p(\varkappa)\Im + q(\varkappa). \tag{4.4}$$

Applying the initial conditions $y(\varkappa,0) = \rho(\varkappa)$ and $\frac{\partial y(\varkappa,\Im)}{\partial t}|_{\Im=0} = \sigma(\varkappa)$, from (4.2), we have $q(\varkappa) = \rho(\varkappa)$ and $p(\varkappa) = \sigma(\varkappa)$. Therefore, (4.4) becomes

$$y(\varkappa,\Im) = H_m^T(\varkappa) C_{m\times m} P_{m\times m}^{\alpha,\psi} H_m(\Im) + \sigma(\varkappa)\Im + \rho(\varkappa). \tag{4.5}$$

Applying $\dfrac{\partial^{\beta,\psi}}{\partial\Im^{\beta,\psi}}$ to (4.5), we obtain

$$\frac{\partial^{\beta,\psi}y(\varkappa,\Im)}{\partial\Im^{\beta,\psi}}=H_m^T(\varkappa)C_{m\times m}P_{m\times m}^{\alpha-\beta,\psi}H_m(\Im)+\sigma(\varkappa)\frac{\Im^{1-\beta}}{\Gamma(2-\beta)}. \tag{4.6}$$

By substituting (4.3), (4.5) and (4.6) in (4.1), we have

$$\begin{aligned}\eta\frac{\partial^{\gamma,\psi}y(\varkappa,\Im)}{\partial\varkappa^{\gamma,\psi}}&=H_m^T(\varkappa)C_{m\times m}H_m(\Im)+\lambda H_m^T(\varkappa)C_{m\times m}P_{m\times m}^{\alpha-\beta,\psi}H_m(\Im)\\&\quad+\mu H_m^T(\varkappa)C_{m\times m}P_{m\times m}^{\alpha,\psi}H_m(\Im)+g(\varkappa,\Im)\\&=H_m^T(\varkappa)\left(C_{m\times m}(I+\lambda P_{m\times m}^{\alpha-\beta,\psi}+\mu P_{m\times m}^{\alpha,\psi}+G_{m\times m})\right)H_m(\Im),\end{aligned} \tag{4.7}$$

where

$$g(\varkappa,\Im)=\sigma(\varkappa)\left(\frac{\lambda\Im^{1-\beta}}{\Gamma(2-\beta)}+\mu\Im\right)+\mu\rho(\varkappa)-f(\varkappa,\Im)=H_m^T(\varkappa)G_{m\times m}H_m(\Im).$$

Applying $\mathcal{J}_{\varkappa}^{\gamma,\psi}$ on both sides of (4.7), we have

$$\eta y(\varkappa,\Im)=\mathcal{J}_{\varkappa}^{\gamma,\psi}H_m^T(\varkappa)\left(C_{m\times m}(I+\lambda P_{m\times m}^{\alpha-\beta,\psi}+\mu P_{m\times m}^{\alpha,\psi}+G_{m\times m})\right)H_m(\Im)+x\phi_1(\Im)+\phi_2. \tag{4.8}$$

Implementing the condition$y(0,\Im)=\xi(\Im)$, we get $\phi_2(\Im)=\xi(\Im)$ and $y(1,\Im)=\zeta(\Im)$ gives

$$\phi_1(\Im)=\mathcal{J}_{\varkappa}^{\gamma,\psi}H_m^T(1)\left(C_{m\times m}(I+\lambda P_{m\times m}^{\alpha-\beta,\psi}+\mu P_{m\times m}^{\alpha,\psi}+G_{m\times m})\right)H_m(\Im)+\zeta(\Im)-\xi(\Im). \tag{4.9}$$

Substituting (4.9) in (4.8), we have

$$\begin{aligned}\eta y(\varkappa,\Im)&=H_m^T(\varkappa)\left((P_{m\times m}^{\gamma,\psi})^T-(Q_{m\times m}^{\gamma,\psi})^T\right)\\&\quad\times\left(C_{m\times m}(I+\lambda P_{m\times m}^{\alpha-\beta,\psi}+\mu P_{m\times m}^{\alpha,\psi}+G_{m\times m})\right)H_m(\Im)+x(\zeta(\Im)-\xi(\Im))+\xi(\Im),\end{aligned} \tag{4.10}$$

where

$$\mathcal{J}_{\varkappa}^{\gamma,\psi}H_m(\varkappa)=P_{m\times m}^{\gamma,\psi}H_m(\varkappa)=H_m^T(\varkappa)(P_{m\times m}^{\gamma,\psi})^T$$

and

$$x\mathcal{J}_{\varkappa}^{\gamma,\psi}H_m(1)=Q_{m\times m}^{\gamma,\psi}H_m(\varkappa).$$

From (4.5) and (4.10), we get the Sylvester equation

$$\begin{aligned}&\left((P_{m\times m}^{\gamma,\psi})^T-(Q_{m\times m}^{\gamma,\psi})^T\right)\left(C_{m\times m}(I+\lambda P_{m\times m}^{\alpha-\beta,\psi}+\mu P_{m\times m}^{\alpha,\psi}\right)-\eta C_{m\times m}P_{m\times m}^{\alpha,\psi}\\&=S_{m\times m}-\left((P_{m\times m}^{\gamma,\psi})^T-(Q_{m\times m}^{\gamma,\psi})^T\right)G_{m\times m},\end{aligned} \tag{4.11}$$

where

$$s(\varkappa,\Im)=\varkappa(\zeta(\Im)-\xi(\Im))+\xi(\Im)-\eta(\sigma(\varkappa)\Im+\varrho(\Im))=H_m^T(\varkappa)S_{m\times m}H_m(\Im).$$

Solving (4.11) for $C_{m\times m}$ and using (4.5) or (4.6), we can get the solution of the problem (4.1).

4.2. ψ-FPDEs with variable coefficients

This section discusses the procedure for numerical solutions of the following class of ψ-FPDEs.

$$\frac{\partial^{\gamma,\psi} y(\varkappa,\Im)}{\partial \Im^{\gamma,\psi}} - a(\varkappa)\frac{\partial^{\alpha,\psi} y(\varkappa,\Im)}{\partial \varkappa^{\alpha,\psi}} + b(\varkappa)\frac{\partial^{\beta,\psi} y(\varkappa,\Im)}{\partial \varkappa^{\beta,\psi}} + d(\varkappa)y(\varkappa,\Im) = f(\varkappa,\Im),\ 1 < \alpha \leq 2,\ \ 1 < \beta \leq 2,\ \ 0 < \gamma \leq 2, \tag{4.12}$$

with the initial conditions

$$y(\varkappa, 0) = \phi_1(\varkappa), \quad \left.\frac{y(\varkappa,\Im)}{\partial \Im}\right|_{\Im=0} = \psi_1(\varkappa),\ \text{or}\ y(\varkappa,0) = \phi_1(\varkappa), \quad y(\varkappa,1) = \psi_2(\varkappa), \tag{4.13}$$

and boundary conditions:

$$y(0,\Im) = \mu(\Im), \quad y(1,\Im) = \nu(\Im). \tag{4.14}$$

Here, we present a numerical technique based on ψ-Haar wavelets operational matrices for ψ-fractional integration. We approximate $\frac{\partial^{\alpha,\psi} y(\varkappa,\Im)}{\partial \varkappa^{\alpha,\psi}}$ with Haar wavelets as

$$\frac{\partial^{\alpha,\psi} y(\varkappa,\Im)}{\partial \varkappa^{\alpha,\psi}} = H_m^T(\varkappa) C_{m\times m} H_m(\Im). \tag{4.15}$$

Operating $\mathcal{J}_{\varkappa}^{\alpha,\psi}$ on (4.15), we get

$$y(\varkappa,\Im) = \mathcal{J}_{\varkappa}^{\alpha,\psi} H_m^T(\varkappa) C_{m\times m} H_m(\Im) + p(\Im)\varkappa + q(\Im). \tag{4.16}$$

Applying boundary conditions in (4.14) to (4.16), we have

$$q(\Im) = \mu(\Im), \ \ p(\Im) = -\mathcal{J}_{\varkappa}^{\alpha,\psi} H_m^T(\varkappa) C_{m\times m} H_m(\Im) + \nu(\Im) - \mu(\Im). \tag{4.17}$$

Therefore, (4.16) can be written as

$$\begin{aligned} y(\varkappa,\Im) = {} & \mathcal{J}_{\varkappa}^{\alpha,\psi} H_m^T(\varkappa) C_{m\times m} H_m(\Im) - \varkappa \mathcal{J}_{\varkappa}^{\alpha,\psi} H_m^T(\varkappa) C_{m\times m} H_m(\Im) \\ & + \varkappa(nu(\Im) - \mu(\Im)) + \mu(\Im). \end{aligned} \tag{4.18}$$

Since $\mathcal{J}_{\varkappa}^{\alpha,\psi} H_m(\varkappa) = P_{m\times m}^{\alpha,\psi} H_m(\varkappa)$ and $\phi^1(\varkappa)\mathcal{J}_{\varkappa}^{\alpha,\psi} H_m(\varkappa) = Q_{m\times m}^{\alpha,1}$, where $\phi^1(\varkappa) = \varkappa$. Therefore, (4.18) takes the form

$$\begin{aligned} y(\varkappa,\Im) = {} & H_m^T(\varkappa)(P_{m\times m}^{\alpha,\psi})^T C_{m\times m} H_m(\Im) - H_m^T(\varkappa)(Q_{m\times m}^{\alpha,1})^T C_{m\times m} H_m(\Im) \\ & + \varkappa(\nu(\Im) - \mu(\Im)) + \mu(\Im). \end{aligned} \tag{4.19}$$

Applying the ψ-Caputo operator $\frac{\partial^{\beta,\psi}}{\partial \varkappa^{\beta}}$ on (4.18), we have

$$\begin{aligned} \frac{\partial^{\beta,\psi} y(\varkappa,\Im)}{\partial \varkappa^{\beta}} = {} & \mathcal{J}_{\varkappa}^{\alpha,\psi} H_m^T(\varkappa) C_{m\times m} H_m(\Im) - \frac{\varkappa^{1-\beta}}{\Gamma(2-\beta)} \mathcal{J}_{\varkappa}^{\alpha,\psi} H_m^T(\varkappa) C_{m\times m} H_m(\Im) \\ & + \frac{\varkappa^{1-\beta}}{\Gamma(2-\beta)}(nu(\Im) - \mu(\Im)) + \mu(\Im). \end{aligned} \tag{4.20}$$

For simplicity, we introduced some convenient notations.

$$
\begin{aligned}
\phi_2 &= \frac{b(\varkappa)\varkappa^{1-\beta}}{\Gamma(2-\beta)},\ \phi_3 = \varkappa d(\varkappa),\\
r(\varkappa,\Im) &= \frac{\varkappa^{1-\beta}}{\Gamma(2-\beta)}(nu(\Im)-\mu(\Im)) + d(\varkappa)q(\Im) + (\nu(\Im)-\mu(\Im))\varkappa d(\varkappa),\\
s(\varkappa,\Im) &= \varkappa(\nu(\Im)-\mu(\Im)) + \mu(\Im) + \psi_1(\varkappa)\Im + \phi_1(\varkappa),\\
g(\varkappa,\Im) &= \varkappa(\nu(\Im)-\mu(\Im)) - \mu(\Im) + (\psi_2(\varkappa)-\phi_1(\varkappa))\Im + \phi_1(\varkappa)\\
d(\varkappa)\mathcal{J}_\varkappa^{\alpha,\psi}H_m(\varkappa) &= \bar{P}^{\alpha,\psi}_{m\times m}H_m(\varkappa),\\
b(\varkappa)\mathcal{J}_\varkappa^{\alpha,\psi}H_m(\varkappa) &= \bar{P}^{\alpha,\psi}_{m\times m}H_m(\varkappa).
\end{aligned}
$$

Substituting (4.15), (4.18) and (4.20) in (4.12), we have

$$
\begin{aligned}
\frac{\partial^{\gamma,\psi}y(\varkappa,\Im)}{\partial\Im^{\gamma,\psi}} &= \Big(a(\varkappa)H_m^T(\varkappa) - H_m^T(\varkappa)(\bar{P}^{\alpha-\beta}_{m\times m})^T + H_m^T(\varkappa)(Q^{2,\alpha}_{m\times m})^T - H_m^T(\varkappa)(\bar{P}^{\alpha}_{m\times m})^T + H_m^T(\varkappa)(Q^{\alpha,3}_{m\times m})^T\Big)\\
&\quad\times C_{m\times m}H_m(\Im) + H_m^T(\varkappa)R_{m\times m}H_m(\Im).
\end{aligned}
$$

Applying $\mathcal{J}_\varkappa^{\gamma,\psi}$ on previous equation, we get

$$
\begin{aligned}
y(\varkappa,\Im) &= \Big(a(\varkappa)H_m^T(\varkappa) - H_m^T(\varkappa)(P^{\alpha-\beta}_{m\times m})^T + H_m^T(\varkappa)(Q^{2,\alpha}_{m\times m})^T - H_m^T(\varkappa)(\bar{P}^{\alpha}_{m\times m})^T + H_m^T(\varkappa)(Q^{\alpha,3}_{m\times m})^T\Big)\\
&\quad\times C_{m\times m}\mathcal{J}_\varkappa^{\gamma,\psi}H_m(\Im) + H_m^T(\varkappa)R_{m\times m}\mathcal{J}_\varkappa^{\gamma,\psi}H_m(\Im) + w(\varkappa)\Im + \omega(\varkappa).
\end{aligned}
$$

Using the initial conditions, we get $\omega(\varkappa) = \phi_1(\varkappa)$ and $w(\varkappa) = \psi_1(\varkappa)$. Therefore,

$$
\begin{aligned}
y(\varkappa,\Im) &= \Big(a(\varkappa)H_m^T(\varkappa) - H_m^T(\varkappa)(\bar{P}^{\alpha-\beta}_{m\times m})^T + H_m^T(\varkappa)(Q^{2,\alpha}_{m\times m})^T - H_m^T(\varkappa)(\bar{P}^{\alpha}_{m\times m})^T + H_m^T(\varkappa)(Q^{\alpha,3}_{m\times m})^T\Big)\\
&\quad\times C_{m\times m}\mathcal{J}_\varkappa^{\gamma,\psi}H_m(\Im) + H_m^T(\varkappa)R_{m\times m}\mathcal{J}_\varkappa^{\gamma,\psi}H_m(\Im) + \psi_1(\varkappa)\Im + \phi_1(\varkappa). \qquad (4.21)
\end{aligned}
$$

Now, we employ the boundary conditions to get $w(\varkappa) = \phi_1(\varkappa)$ and

$$
\begin{aligned}
v(\varkappa) = \Big[\Big(&a(\varkappa)H_m^T\varkappa - H_m^T\varkappa(\hat{P}^{\alpha-\beta,\psi}_{m\times m})^T + H_m^T\varkappa(Q^{\alpha,\psi,2}_{m\times m})^T - H_m^T\varkappa(\bar{P}^{\alpha,\psi}_{m\times m})^T\\
&+H_m^T\varkappa(Q^{\alpha,\psi,3}_{m\times m})^T\Big)C_{m\times m} + H_m^T\varkappa R_{m\times m}\Big]\mathcal{J}_t^{\gamma}H_m(1) + \psi_2\varkappa - \phi_1(\varkappa). \qquad (4.22)
\end{aligned}
$$

Therefore, (4.21) becomes

$$
\begin{aligned}
y(\varkappa,\Im) = \Big[\Big(&a(\varkappa)\Psi_M^T(\varkappa) - H_m^T(\varkappa)(\hat{P}^{\alpha-\beta,\psi}_{m\times m})^T + H_m^T(\varkappa)(Q^{\alpha,\psi,2}_{m\times m})^T - H_m^T(\varkappa)(\bar{P}^{\alpha,\psi}_{m\times m})^T\\
&+H_m^T(\varkappa)(Q^{\alpha,\psi,3}_{m\times m})^T\Big)C_{m\times m} + H_m^T(\varkappa)R_{m\times m}\Big](P^{\gamma}_{m\times m} - Q^{\gamma}_{m\times m})\,\Psi_m(\Im)\\
&+ (\psi_2(\varkappa) - \phi(\varkappa))\Im + \phi_1(\varkappa). \qquad (4.23)
\end{aligned}
$$

Combining (4.19) and (4.21) gives the Sylvester equation

$$
\begin{aligned}
&\Big((P^{\alpha,\psi}_{m\times m})^T - (Q^{\alpha,\psi,1}_{m\times m})^T\Big)C_{m\times m}\\
&-\Big(\frac{\eta}{m}\Psi_{m\times m}A_{m\times m}H^T_{m\times m} - (\hat{P}^{\alpha-\beta,\psi}_{m\times m}) - (\bar{P}^{\alpha,\psi}_{m\times m})^T + (Q^{\alpha,\psi,2}_{m\times m})^T + (Q^{\alpha,\psi,3}_{m\times m})^T\Big)C_{m\times m}P^{\gamma}_{m\times m}\\
&= R_{m\times m}P^{\gamma}_{m\times m} + S_{m\times m}, \qquad (4.24)
\end{aligned}
$$

where $A_{m\times m} := diag[(a(\varkappa_i))]$, $x_i = \frac{2i-1}{2m}$, $i = 1, 2, 3, \ldots, m$. Also, using (4.19) and (4.23), we obtain the following matrix form:

$$\begin{aligned}&\Big((P^{\alpha,\psi}_{m\times m})^T - (Q^{\alpha,\psi,1}_{m\times m})^T\Big) C_{m\times m}\\ &\quad - \Big(\frac{\eta}{m}\Psi_{m\times m}A_{m\times m}H^T_{m\times m} - (\hat{P}^{\alpha-\beta,\psi}_{m\times m}) - (\bar{P}^{\alpha,\psi}_{m\times m})^T + (Q^{\alpha,\psi,2}_{m\times m})^T + (Q^{\alpha,\psi,3}_{m\times m})^T\Big) C_{m\times m}\,(P^{\gamma}_{m\times m} - Q^{\gamma}_{m\times m})\\ &\quad = R_{m\times m}\,(P^{\gamma}_{m\times m} - Q^{\gamma}_{m\times m}) + G_{m\times m}. \end{aligned} \tag{4.25}$$

By solving (4.25) for $G_{m\times m}$ and substituting it into (4.19), we get the approximate solution of problem (4.12).

To solve various ψ-FPDEs, we use the ψ-Haar wavelets technique. Additionally, we compared the graphical results obtained using the proposed method with the exact solutions. For the first two examples, we use the technique discussed in subsection 4.1 and for the other two examples, we follow the procedure discussed in subsection 4.2.

Example 1. Consider the time-fractional telegraph equation with ψ-Caputo fractional derivative

$$\begin{aligned}&\frac{\partial^{\alpha,\psi} y(\varkappa,\Im)}{\partial \Im^{\alpha,\psi}} + \frac{\partial^{\alpha-1,\psi} y(\varkappa,\Im)}{\partial \Im^{\alpha-1,\psi}} + y(\varkappa,\Im)\\ &= \frac{\partial^2 y(\varkappa,\Im)}{\partial \varkappa^2} + \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}\left(1 + \frac{\psi(\Im)}{\alpha+1}\right)(\psi(\Im))^{\alpha,\psi}\cos(7\varkappa) + 50(\psi(\Im))^{2\alpha}\cos(7\varkappa)\end{aligned} \tag{4.26}$$

satisfying the initial and boundary conditions

$$y(\varkappa,0) = 0, \quad \left.\frac{\partial y(\varkappa,\Im)}{\partial \Im}\right|_{\Im=0} = 0, \quad y(0,\Im) = (\psi(\Im))^{2\alpha}, \quad y(1,\Im) = 0.7539022(\psi(\Im))^{2\alpha}.$$

The exact solution for the problem (4.26) is given by

$$y(\varkappa,\Im) = (\psi(\Im))^{2\alpha}\cos(7\varkappa).$$

Exact and approximate solutions of the problem (4.26) and their absolute error are plotted in Figure 1.

Also absolute error for problem (4.26) is given in Table 1 for various choices of the parameters $\alpha, J, \Im$ and $\varkappa$.

Table 1. Absolute error for $\psi(\varkappa) = \sin(\varkappa)$.

$\Im$	$\varkappa$	α	$J = 3$	$J = 4$	$J = 5$	$J = 6$	$J = 7$
0.25	0.20	1.5	3.3780×10^{-3}	7.4702×10^{-4}	2.4498×10^{-5}	8.0425×10^{-6}	2.6428×10^{-6}
	0.50	1.6	2.1513×10^{-3}	6.5313×10^{-4}	1.9862×10^{-5}	6.0507×10^{-6}	1.8462×10^{-6}
	0.80	1.7	2.0818×10^{-3}	5.8754×10^{-4}	1.6603×10^{-5}	4.6985×10^{-6}	1.3317×10^{-6}
0.50	0.20	1.8	2.0718×10^{-3}	5.4730×10^{-4}	1.4458×10^{-5}	3.8213×10^{-6}	1.0106×10^{-6}
	0.50	1.9	2.0363×10^{-3}	5.3432×10^{-4}	1.3359×10^{-5}	3.3400×10^{-6}	8.3502×10^{-7}
	0.80	2.0	2.0363×10^{-4}	5.3432×10^{-4}	1.3359×10^{-5}	3.3400×10^{-6}	8.3502×10^{-7}

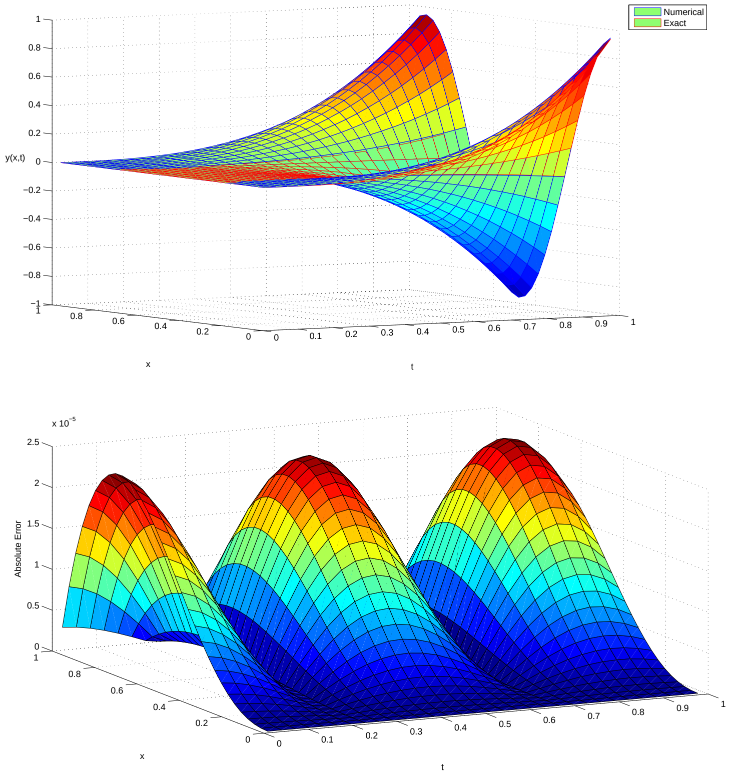

Figure 1. Approximate and exact solutions of (4.26) and their absolute error.

Example 2. Consider the ψ-FPDE given by

$$\frac{\partial^{\alpha,\psi} y(\varkappa,\Im)}{\partial \Im^{\alpha,\psi}} - \lambda \frac{\partial^2 y(\varkappa,\Im)}{\partial \varkappa^2}$$
$$= \left(\frac{(\psi(\Im))^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)} (\psi(\Im))^{2\alpha} \right) + 144\lambda\psi(\Im)\left[1 - (\psi(\Im))^{3\alpha-1}\right]\sin(12\varkappa), \quad 0 \le \alpha \le 1 \tag{4.27}$$

with the initial and boundary conditions

$$y(\varkappa, 0) = y(0, \Im) = 0, \;\; y(1, \Im) = -0.536573\psi(\Im)\left[1 - (\psi(\Im))^{3\alpha-1}\right].$$

The exact solution of problem (4.27) is given by

$$y(\varkappa, \Im) = \sin(12\varkappa)\psi(\Im)\left[1 - (\psi(\Im))^{3\alpha-1}\right].$$

Approximate and exact solution of the problem (4.27) and their absolute error are plotted in Figure 2.

Also the maximum absolute error is presented in the Table 2.

Table 2. Maximum absolute error for $\psi(\varkappa) = \varkappa^2$ and different values of J and α.

$\mathfrak{I}$	$\varkappa$	α	$J = 3$	$J = 4$	$J = 5$	$J = 6$	$J = 7$
0.25	0.20	0.5	4.3007×10^{-2}	4.1167×10^{-4}	4.5672×10^{-5}	3.2361×10^{-5}	3.4349×10^{-6}
	0.50	0.6	6.3553×10^{-3}	6.2332×10^{-4}	5.4130×10^{-5}	4.2321×10^{-6}	2.6340×10^{-7}
	0.80	0.7	4.6571×10^{-3}	2.7212×10^{-5}	4.2014×10^{-6}	3.1478×10^{-6}	4.3216×10^{-7}
0.50	0.20	0.8	6.5786×10^{-4}	6.7634×10^{-5}	6.3132×10^{-6}	4.7324×10^{-7}	3.6210×10^{-7}
	0.50	0.9	2.1714×10^{-4}	5.3452×10^{-6}	3.0884×10^{-6}	4.2703×10^{-7}	5.7381×10^{-8}
	0.80	1.0	3.2738×10^{-5}	1.2753×10^{-6}	2.8801×10^{-7}	4.6721×10^{-8}	8.5382×10^{-9}

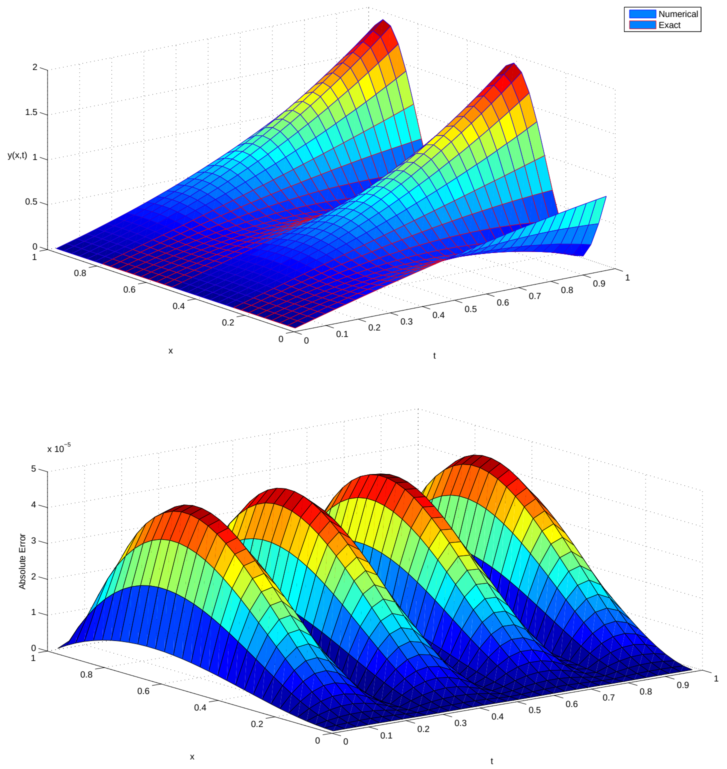

Figure 2. Approximate and exact solutions of (4.27) and their absolute error.

Example 3. Consider the linear fractional diffusion equation with ψ-Caputo derivative

$$\frac{\partial y(\varkappa, \mathfrak{I})}{\partial \mathfrak{I}} = a(\varkappa)\frac{\partial^{1.8,\psi} y(\varkappa, \mathfrak{I})}{\partial \varkappa^{1.8,\psi}} + f(\varkappa, \mathfrak{I}) \tag{4.28}$$

with initial and boundary conditions

$$y(\varkappa, 0) = (\psi(\varkappa))^2(1 - \psi(\varkappa)) \text{ and } y(0, \mathfrak{I}) = 0, \quad y(1, \mathfrak{I}) = 0.$$

For $a(\varkappa) = \Gamma(1.2)(\psi(\varkappa))^{1.8}$ and $f(\varkappa, \mathfrak{I}) = (6\psi(\varkappa) - 3)(\psi(\varkappa))^2 e^{-\mathfrak{I}}$, the problem (4.28) has the exact solution as

$$y(\varkappa, \mathfrak{I}) = ((\psi(\varkappa))^2 - (\psi(\varkappa))^3)e^{-\mathfrak{I}}.$$

Numerical and exact solutions using ψ-Haar wavelets technique and their absolute error for $\alpha = 1.8$ and $J = 5$ are shown in Figure 3. Also absolute error for problem (4.28) is given in Table 3 for various choices of the parameters $\alpha, J, \mathfrak{I}$ and $\varkappa$.

Table 3. Absolute error for $\varkappa = 0.25, \varkappa = 0.5$, different values of J, α and $\psi(\varkappa) = \varkappa^3$.

$\mathfrak{I}$	$\varkappa$	α	$J = 3$	$J = 4$	$J = 5$	$J = 6$	$J = 7$
0.25	0.20	1.5	1.2733×10^{-3}	6.2471×10^{-4}	3.0934×10^{-4}	1.5391×10^{-4}	7.6766×10^{-5}
	0.50	1.6	1.2910×10^{-3}	6.3216×10^{-4}	3.1273×10^{-4}	1.5552×10^{-4}	7.7551×10^{-5}
	0.80	1.7	1.1161×10^{-3}	5.4369×10^{-4}	2.6824×10^{-4}	1.3321×10^{-4}	6.6382×10^{-5}
0.50	0.20	1.8	7.1349×10^{-4}	3.4173×10^{-4}	1.6710×10^{-4}	8.2612×10^{-5}	4.1071×10^{-5}
	0.50	2.0	6.1030×10^{-5}	1.5258×10^{-5}	3.8146×10^{-6}	9.5367×10^{-7}	2.3841×10^{-7}

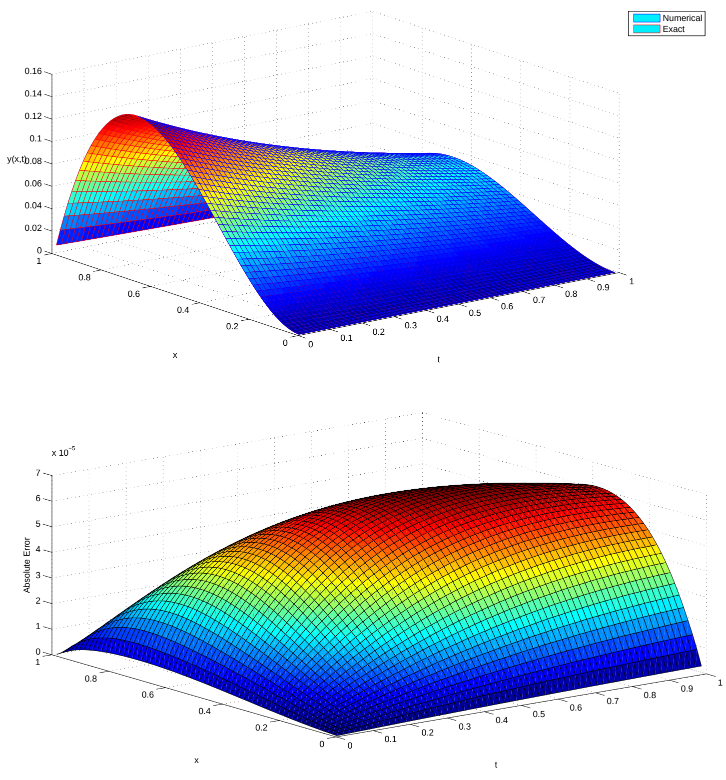

Figure 3. Approximate and exact solutions of (4.28) and their absolute error.

Example 4. Consider the convection-diffusion equation with ψ-Caputo fractional derivative:

$$\frac{\partial^{\gamma,\psi} y(\varkappa, \mathfrak{I})}{\partial \mathfrak{I}^{\gamma,\psi}} = -a\varkappa \frac{\partial^{\alpha,\psi} y(\varkappa, \mathfrak{I})}{\partial \varkappa^{\alpha,\psi}} + b\varkappa \frac{\partial^{\beta,\psi} y(\varkappa, \mathfrak{I})}{\partial \varkappa^{\beta,\psi}} + f(\varkappa, \mathfrak{I}),\ 1 < \alpha \le 2,\ 0 < \beta \le 1,\ 0 < \gamma \le 2 \quad (4.29)$$

with initial and boundary conditions

$$y(\varkappa, 0) = y(\varkappa, 1) = 0,\ y(0, \mathfrak{I}) = 0,\ y(1, \mathfrak{I}) = 0.$$

We solve this problem with

$$a(\varkappa) = \Gamma(\beta + 2)\Gamma(5 - \{\alpha + \beta\})\psi(\varkappa)^{\beta},\ \ b(\varkappa) = \Gamma(2\beta + 2 - \alpha)\Gamma(5 - 2\alpha)\psi(\varkappa)^{\alpha},$$

$$f(\varkappa,\Im) = (2\pi\psi(\varkappa)^{2\beta+1} - \psi(\varkappa)^{4-\alpha})\psi(\Im)^{1-\gamma}E_{2,2-\gamma(-(2\pi\psi(\Im))^2)}$$
$$+\Big(\Gamma(2\beta+2)\Gamma(5-\{\alpha+\beta\}) - \Gamma(5-2\alpha)\psi(\varkappa)^{2\beta+1}$$
$$+\Gamma(5-2\alpha)(\Gamma(2\beta+2-\alpha) - \Gamma(\beta+2))\psi(\varkappa)^{4-\alpha}\Big)\sin(2\pi\psi(\Im)).$$

The exact solution of the problem (4.29) is

$$y(\varkappa,\Im) = (\psi(\varkappa)^{2\beta+1} - \psi(\varkappa)^{4-\alpha})\sin(2\pi\psi(\Im)).$$

Exact and approximate solutions of problem 4.29 and their absolute error is plotted in Figure 4.

Also absolute error is given in Table 4 for various values of $\alpha, \varkappa$ and J at $\Im = 0.25$ and $\Im = 0.50$.

Table 4. Absolute error for $\psi(\varkappa) = \varkappa^2$.

$\Im$	$\varkappa$	α	$J=3$	$J=4$	$J=5$	$J=6$	$J=7$
0.25	0.20	1.5	4.3854×10^{-4}	1.4252×10^{-4}	4.6203×10^{-5}	1.4996×10^{-5}	4.8789×10^{-6}
	0.50	1.6	3.3031×10^{-4}	1.0001×10^{-4}	3.0183×10^{-5}	9.1122×10^{-6}	2.7562×10^{-6}
	0.80	1.7	2.4252×10^{-4}	6.8593×10^{-5}	1.9314×10^{-5}	5.4339×10^{-6}	1.5301×10^{-6}
0.50	0.20	1.8	1.7673×10^{-4}	4.6930×10^{-5}	1.2396×10^{-5}	3.2674×10^{-6}	8.6081×10^{-7}
	0.50	2.0	1.3575×10^{-4}	3.4133×10^{-5}	8.5580×10^{-6}	2.1426×10^{-6}	5.3605×10^{-7}

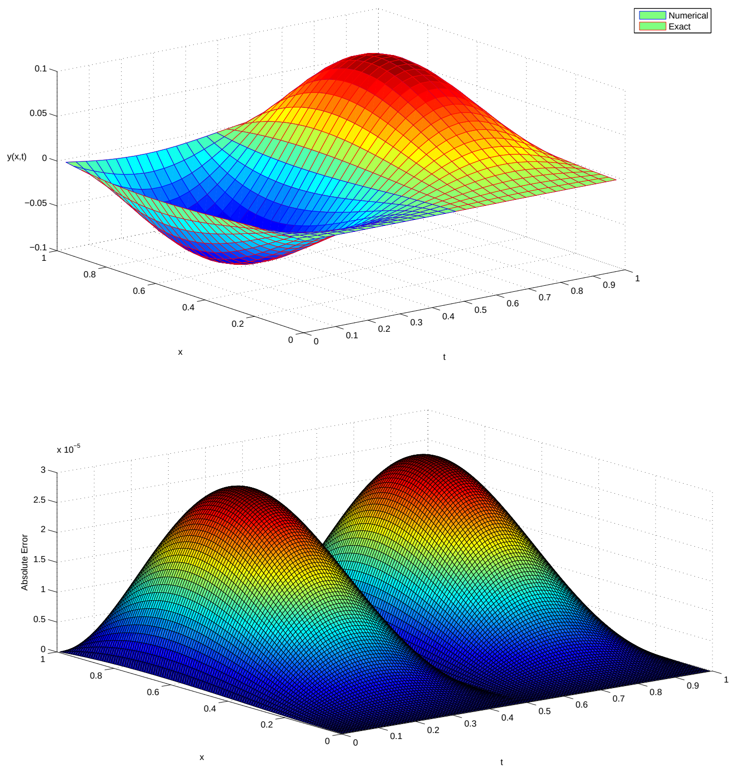

Figure 4. Approximate and exact solutions of (4.29) and their absolute error.

5. Conclusions

We developed and used the ψ-Haar wavelets operational matrix of integration of fractional order for the first time for the numerical solution of ψ-FPDEs. The numerical results of the proposed method are compared to the exact solutions and illustrated along with their absolute error in the figures. Furthermore, the absolute errors are presented in tables, indicating that our method agrees well with the exact solutions. The proposed method can also be applied to other wavelet bases, such as Legendre, Chebyshev, and Gegenbauer wavelets, and can also be applied to nonlinear ψ-FPDEs.

Conflict of interest

The authors declare no conflicts of interest.

References

1. V. E. Tarasov, *Handbook of fractional calculus with applications*, Boston, Berlin: de Gruyter, **5** (2019).
2. L. Debnath, Recent applications of fractional calculus to science and engineering, *Int. J. Math. Math. Sci.*, **2003** (2003), 753601. https://doi.org/10.1155/S0161171203301486
3. H. G. Sun, Y. Zhang, D. Baleanu, W. Chen, Y. Q. Chen, A new collection of real world applications of fractional calculus in science and engineering, *Commun. Nonlinear Sci.*, **64** (2018), 213–231. https://doi.org/10.1016/j.cnsns.2018.04.019
4. R. Almeida, A. B. Malinowska, T. Odzijewicz, An extension of the fractional Gronwall inequality, In: *Advances in non-tnteger order calculus and its applications*, Springer, 2018, 20–28.
5. R. Almeida, Fractional differential equations with mixed boundary conditions, *B. Malays. Math. Sciences So.*, **42** (2019), 1687–1697.
6. R. Almeida, A Caputo fractional derivative of a function with respect to another function, *Commun. Nonlinear Sci.*, **44** (2017), 460–481. https://doi.org/10.1016/j.cnsns.2016.09.006
7. C. Derbazi, Z. Baitiche, M. S. Abdo, T. Abdeljawad, Qualitative analysis of fractional relaxation equation and coupled system with ψ-Caputo fractional derivative in Banach spaces, *AIMS Mathematics*, **6** (2021), 2486–2509. https://doi.org/10.3934/math.2021151
8. J. V. da C. Sousa, E. C. de Oliveira, On the stability of a hyperbolic fractional partial differential equation, *Differ. Equat. Dyn. Sys.*, 2019, 1–22. https://doi.org/10.48550/arXiv.1805.05546
9. N. Adjimi, A. Boutiara, M. S. Abdo, M. Benbachir, Existence results for nonlinear neutral generalized Caputo fractional differential equations, *J. Pseudo-Differ. Oper.*, **12** (2021), 1–17. https://doi.org/10.1007/s11868-021-00400-3
10. S. Abbas, M. Benchohra, N. Hamidi, J. Henderson, Caputo-Hadamard fractional differential equations in Banach spaces, *Fract. Calc. Appl. Anal.*, **21** (2018), 1027–1045. https://doi.org/10.1515/fca-2018-0056
11. D. Vivek, E. M. Elsayed, K. Kanagarajan, Theory and analysis of partial differential equations with a ψ-Caputo fractional derivative, *Rocky Mt. J. Math.*, **49** (2019), 1355–1370. https://doi.org/10.1216/RMJ-2019-49-4-1355

12. A. Suechoei, P. S. Ngiamsunthorn, Existence uniqueness and stability of mild solutions for semilinear ψ-Caputo fractional evolution equations, *Adv. Differ. Equ.*, **2020** (2020), 114. https://doi.org/10.1186/s13662-020-02570-8
13. Y. Yang, M. H. Heydari, Z. Avazzadeh, A. Atangana, Chebyshev wavelets operational matrices for solving nonlinear variable-order fractional integral equations, *Adv. Differ. Equ.*, **2020** (2020), 611. https://doi.org/10.1186/s13662-020-03047-4
14. M. Bilal, A. R. Seadawy, M. Younis, S. T. R. Rizvi, H. Zahed, Dispersive of propagation wave solutions to unidirectional shallow water wave Dullin-Gottwald-Holm system and modulation instability analysis, *Math. Method. Appl. Sci.*, **44** (2021), 4094–4104. https://doi.org/10.1002/mma.7013
15. A. R. Seadawy, A. Ali, W. A. Albarakati, Analytical wave solutions of the (2+1)-dimensional first integro-differential Kadomtsev-Petviashivili hierarchy equation by using modified mathematical methods, *Results. Phys.*, **15** (2019), 102775. https://doi.org/10.1016/j.rinp.2019.102775
16. A. R. Seadawy, M. Arshad, D. Lu, The weakly nonlinear wave propagation of the generalized third-order nonlinear Schrodinger equation and its applications, *Wave. Random. Complex.*, **32** (2022), 819–831. https://doi.org/10.1080/17455030.2020.1802085
17. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations, *Elsevier Sci. Limited.*, **204** (2006), 1–523.