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Research article

An inertial Mann algorithm for nonexpansive mappings on Hadamard manifolds

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Abstract: An inertial Mann algorithm will be presented in this article with the purpose of approximating a fixed point of a nonexpansive mapping on a Hadamard manifold. Any sequence that is generated by using the proposed approach, under suitable assumptions, converges to fixed points of nonexpansive mappings. The proposed method is also dedicated to solving inclusion and equilibrium problems. Lastly, we give a number of computational experiments that show how well the inertial Mann algorithm works and how it compares to other methods.

Keywords: fixed point problem; Hadamard manifold; inertial Mann method; nonexpansive mapping
Mathematics Subject Classification: 47H05, 47J25

1. Introduction

Fixed point problems for a nonexpansive mapping can be used for modeling a variety of problems that arise in several research fields, such as optimization problems [43, Corollary 17.5], monotone inclusion problems [43, Proposition 23.38], variational inequalities [43, Subchapter 25.5] and equilibrium problems [44]. Let us recall it in the following equation:

$$x = F(x), \tag{1.1}$$

where F is a nonexpansive self mapping defined on a subset of suitable spaces. The solutions of the problem (1.1) are said to be *fixed points* of the mapping F. The set of fixed points of the mapping F is represented by $\mathrm{Fix}(F)$. The construction of fixed points for nonexpansive mappings is of tremendous interest and importance since it has applications in a range of areas, including signal processing, image recovery and machine learning (see, for example [17–20, 33]) .

One of the iterative procedures that succeeds in approximating fixed points for nonexpansive-type mappings is, undoubtedly, the Mann iterative algorithm [8]. On the other hand, the Mann algorithm has a slow convergence rate, especially for large-scale problems (see [41, 42] for details). In 1964, Polyak [10] was the first to suggest the heavy ball method, which uses an inertial extrapolation technique to accelerate the method's convergence. It has proven to be a valuable resource for improving the method's performance and has great convergence properties. Numerous fast iterative algorithms have been developed using inertial methods such as the inertial proximal point methods [26, 27], the inertial forward-backward splitting methods [28, 29] and the inertial extragradient methods [30, 34]. Especially, Maingé [9] combined the Mann algorithm and the inertial extrapolation technique to construct the following inertial Mann algorithm:

$$\begin{cases} y_n = x_n + \lambda_n(x_n - x_{n-1}), \\ x_{n+1} = \gamma_n y_n + (1 - \gamma_n)F(y_n), \quad n \geq 1, \end{cases} \tag{1.2}$$

where $\{\lambda_n\} \in [0, 1)$ and $\{\gamma_n\} \in (0, 1)$. Under certain assumptions, the researcher [9] proved that the sequence $\{x_n\}$ converges weakly to fixed points of F.

During the last decade, many scholars have turned their attention to nonlinear problems on manifolds. This is due to the fact that the problems can't be posed in linear spaces and need a manifold structure (not necessary with a linear structure). The main advantages of these extensions are that constrained optimization problems can be considered as unconstrained from the perspective of Riemannian geometry, and that other optimization problems with non-convex objective functions can be equivalently transformed into convex ones by using an appropriate Riemannian metric. Note that several scholars have also developed optimization theory in a more general setting of a $\mathrm{CAT}(\kappa)$ space, which generalizes a Riemannian manifold with upper sectional curvature bounded by κ. However, without additional unnatural assumptions, the inertial step (which is the main interest of this paper) cannot be defined by such a general structure. Hence, we shall restrict ourselves only to the setting of a Hadamard manifold.

To further illustrate the motivation of extending from Hilbert space to manifold settings, we recall an optimization problem from [36, p. 1–3]. In this example, we would see two facts: (1) A nonconvex objective may be turned into a convex objective in the manifold setting; (2) An incomplete set can be turned into a complete metric space from the underlying manifold structure. Let M be a Riemannian manifold and $\varphi : [a, b] \to M$ be a geodesic. A function $g : M \to \mathbb{R}$ is a geodesic convex means that the function $g \circ \varphi : [a, b] \to \mathbb{R}$ is convex in the usual sense. The convex optimization problem on Riemannian manifolds for the geodesic convex objective function $g : M \to \mathbb{R}$ is defined by

$$\min_{x \in M} g(x). \tag{1.3}$$

Let $M = \mathbb{R}^m_{++} = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_i > 0, i = 1, \ldots, m\}$ be the Riemannian manifold endowed with the Riemannian metric $\langle \cdot | \cdot \rangle : TM \times TM \to \mathbb{R}$ defined by $\langle \upsilon | \nu \rangle := \upsilon^\top W(x) \nu$, for all $\upsilon, \nu \in T_x M$, where $T_x M$ is the tangent space of M at $x \in M$, TM is the tangent bundle of M, $\upsilon^\top$ is a transpose of υ and $W(x)$ is an $n \times n$ symmetric matrix defined by $W(x) = \text{diag}(x_1^{-2}, x_2^{-2}, \ldots, x_m^{-2})$. Then $(\mathbb{R}^m_{++}, \langle \cdot | \cdot \rangle)$ is a complete Riemannian manifold, as proved in [37, Corollary 10.1.5]. A geodesic joining $x = (x_1, \ldots, x_m) \in \mathbb{R}^m_{++}$ and $y = (y_1, \ldots, y_m) \in \mathbb{R}^m_{++}$ is given by

$$\varphi(s) := (x_1^{1-s} y_1^s, \ldots, x_m^{1-s} y_m^s), \quad \forall s \in \mathbb{R}.$$

This geodesic would be unique according to the Hopf-Rinow theorem.

Let $g : \mathbb{R}^m_{++} \to \mathbb{R}$ be a function defined by

$$g(x) := \sum_{i=1}^{m} \ln x_i, \quad \forall x = (x_1, \ldots, x_m) \in \mathbb{R}^m_{++}.$$

Then g is geodesic convex on $(\mathbb{R}^m_{++}, \langle \cdot | \cdot \rangle)$. Indeed,

$$\begin{aligned} (g \circ \varphi)(s) &= \sum_{i=1}^{m} \ln x_i^{1-s} y_i^s, \\ (g \circ \varphi)^{'}(s) &= \sum_{i=1}^{m} \frac{y_i}{x_i}, \end{aligned}$$

and

$$(g \circ \varphi)^{''}(s) = 0.$$

As a result, $(g \circ \varphi)$ is convex in the usual sense. Therefore g is geodesic convex, consults [37, Theorem 10.1.2] for more details.

On the other hand, g is not convex on $\mathbb{R}^m_{++}$ in the usual sense. Consider the Hessian of g on $\mathbb{R}^m_{++}$ which is given by

$$\text{Hess}\, g(x) = \begin{bmatrix} \frac{\partial^2 g(x)}{\partial x_1^2} & \frac{\partial^2 g(x)}{\partial x_1 x_2} & \cdots & \frac{\partial^2 g(x)}{\partial x_1 x_m} \\ \frac{\partial^2 g(x)}{\partial x_2 x_1} & \frac{\partial^2 g(x)}{\partial x_2^2} & \cdots & \frac{\partial^2 g(x)}{\partial x_2 x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 g(x)}{\partial x_m x_1} & \frac{\partial^2 g(x)}{\partial x_m x_2} & \cdots & \frac{\partial^2 g(x)}{\partial x_m^2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{x_1^2} & 0 & \cdots & 0 \\ 0 & -\frac{1}{x_2^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\frac{1}{x_m^2} \end{bmatrix}$$

which is not a positive semidefinite matrix on $\mathbb{R}^m_{++}$. Then by the fact that "a function on $\mathbb{R}^m$ is convex if and only if its Hessian is positive semidefinite", g is not convex on $\mathbb{R}^m_{++}$ in the usual sense. Therefore, the following non-convex optimization problem

$$\min_{x \in \mathbb{R}^m_{++}} g(x)$$

over an incomplete set becomes a (geodesic) convex optimization problem (1.3) on the complete Riemannian manifold $(\mathbb{R}^m_{++}, \langle \cdot | \cdot \rangle)$.

The above example shows that a non-convex optimization problem can be transformed into a (geodesic) convex optimization problem via the introduction of an appropriate Riemannian metric. Not only this, but non-monotone problems in Euclidean spaces may also be similarly transformed into monotone problems, see [25] for more details. As a result, many authors have focused on developing nonlinear problems on Riemannian manifolds, see, for instance, [11–14, 16, 21–25, 35, 38, 39].

Recently, several authors considered and studied the fixed point problem (1.1) in the Riemannian context. For instance; Li et al. [11] proposed the Mann algorithm and the Halpern algorithm for approximating fixed points of nonexpansive mappings in the setting of Hadamard manifolds. Chugh et al. [12] investigated the Ishikawa algorithm in the context of Hadamard manifolds. Huang [16] generalized an iterative method of the viscosity type with a weak contraction in Hadamard manifolds. The conjugate gradient technique was applied, as stated by Yao et al. [13], in order to speed up the Halpern algorithm. Sahu et al. [14] just recently presented the S-iterative technique as a method for approximating a common fixed point of two nonexpansive mappings.

The goal of this research is to develop an inertial Mann approach for Hadamard manifolds, motivated by the vast range of applications of the inertial Mann method. With the inertial extrapolation technique, our proposed method converges faster than some existing methods. We examine the inertial Mann method in terms of exponential mappings and illustrate how the proposed method can be utilized to solve inclusion problems and equilibrium problems.

The following is an overview of the paper's structure. Basic concepts and fundamental theorems in Riemannian geometry are presented in Section 2. We present the inertial Mann iterative method in Section 3 and show that a sequence generated by the suggested method converges to fixed points of nonexpansive mappings on Hadamard manifolds. In Section 4, we show how to use the proposed method to solve inclusion and equilibrium problems in the setting of Hadamard manifolds. Section 5 contains numerical examples to demonstrate the performance of the inertial Mann algorithm. Finally, Section 6 provides a concise overview of the paper.

2. Preliminaries

Let (M, g) be a finite-dimensional Riemannian manifold. The *tangent bundle* of M is denoted by $TM = \bigcup_{x\in M} T_xM$, where T_xM represents the *tangent space* of M at x and the zero section of TM is denoted by $\mathbf{0}$. For the sake of notational clarity, $\langle\cdot|\cdot\rangle \equiv g_x$ denotes the inner product on T_xM whose induced norm is written as $\|\cdot\|$. Let ∇ be the Riemannian connection induced by g. A Riemannian distance is the minimizing length of all such geodesics joining x to y, and it is denoted by $d(x, y)$.

If the geodesics of a Riemannian manifold are defined for any value of $s \in \mathbb{R}$, it is said to be *complete*. The Hopf-Rinow theorem says that any pair of points in M can be connected by a minimizing geodesic if M is complete. Also, the metric space denoted by (M, d) is a complete space, and subsets that are closed and bounded are compact.

Assuming that M is a complete Riemannian manifold, the exponential map $\exp_x : T_xM \to M$ at the point x is assigned by $\exp_x \upsilon = \varphi_\upsilon(1, x)$ for all $\upsilon \in T_xM$, where $\varphi(\cdot) = \varphi_\upsilon(\cdot, x)$ is the geodesic starting from x with velocity υ. Then we have $\exp_x s\upsilon = \varphi_\upsilon(s, x)$ and $\exp_x \mathbf{0} = \varphi_\upsilon(0, x) = x$ for any value s. The exponential map has inverse $\exp_x^{-1} : M \to T_xM$. Furthermore, for all $x, y \in M$, we have $d(x, y) = \|\exp_x^{-1} y\|$.

A *Hadamard manifold* is a complete, simply connected Riemannian manifold with non-positive sectional curvature. A finite-dimensional Hadamard manifold is referred to as M throughout the rest of this article. Given $x \in M$, the exponential mapping $\exp_x : T_xM \to M$ is a diffeomorphism. For any two points $x, y \in M$ there exists a unique normalized geodesic joining x to y, which is a minimizing geodesic [3, Theorem 4.1].

We now describe certain geometric properties of the finite dimensional Hadamard manifold M which are analogous to the settings of Euclidean space $\mathbb{R}^n$. Recall that a geodesic triangle $\triangle(x_1, x_2, x_3)$ of a Riemannian manifold M is a set consisting of three points x_1, x_2 and x_3, and three minimizing geodesics φ_i joining x_i to x_{i+1}, where $i = 1, 2, 3 \pmod 3$.

Proposition 1. [3] *Let $\triangle(x_1, x_2, x_3)$ be a geodesic triangle in M. Then*

$$d^2(x_1, x_2) + d^2(x_2, x_3) - 2\langle \exp_{x_2}^{-1} x_1 | \exp_{x_2}^{-1} x_3 \rangle \le d^2(x_3, x_1), \tag{2.1}$$

and

$$d^2(x_1, x_2) \le \langle \exp_{x_1}^{-1} x_3 | \exp_{x_1}^{-1} x_2 \rangle + \langle \exp_{x_2}^{-1} x_3 | \exp_{x_2}^{-1} x_1 \rangle. \tag{2.2}$$

Furthermore, if θ is the angle at x_1, then we get

$$\langle \exp_{x_1}^{-1} x_2 | \exp_{x_1}^{-1} x_3 \rangle = d(x_2, x_1) d(x_1, x_3) \cos\theta.$$

In [4] shows the relationship between geodesic triangles in Riemannian manifolds and triangles in $\mathbb{R}^2$.

Lemma 1. [4] *Let $\triangle(x_1, x_2, x_3)$ be a geodesic triangle in M. Then, there exists $\triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in $\mathbb{R}^2$ for $\triangle(x_1, x_2, x_3)$ such that*

$$d(x_1, x_2) = \|\overline{x_1} - \overline{x_2}\|, \quad d(x_2, x_3) = \|\overline{x_2} - \overline{x_3}\|, \quad d(x_2, x_3) = \|\overline{x_2} - \overline{x_3}\|.$$

The triangle $\triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ is called a *comparison triangle* of the geodesic triangle $\triangle(x_1, x_2, x_3)$, which is unique up to isometry of M. The points $\overline{x_1}$, $\overline{x_2}$, $\overline{x_3}$ are called *comparison points* to the points x_1, x_2, x_3, respectively.

Lemma 2. *Let $\triangle(x_1, x_2, x_3)$ be a geodesic triangle in M and $\triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ be its comparison triangle.*

(i) *Let $\theta_1, \theta_2, \theta_3$ (respectively, $\overline{\theta}_1, \overline{\theta}_2, \overline{\theta}_3$) be the angles of $\triangle(x_1, x_2, x_3)$ (respectively, $\triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$) at the vertices x_1, x_2, x_3 (respectively, $\overline{x_1}$, $\overline{x_2}$, $\overline{x_3}$). Then $\theta_1 \le \overline{\theta}_1$, $\theta_2 \le \overline{\theta}_2$ and $\theta_3 \le \overline{\theta}_3$.*

(ii) *Let p be a point on the geodesic connecting x_1 and x_2 and $\overline{p}$ its comparison point in the interval $[\overline{x_1}, \overline{x_2}]$. If $d(x_1, p) = \|\overline{x_1} - \overline{p}\|$ and $d(x_2, p) = \|\overline{x_2} - \overline{p}\|$, then $d(x_3, p) \le \|\overline{x_3} - \overline{p}\|$.*

Since every two points on a Hadamard manifold can be connected by a unique minimizing geodesic, we use the notation $\mathrm{P}_{y,x} : T_xM \to T_yM$ to denote the parallel translation along the minimizing geodesic connecting x and y for any two points $x, y \in M$.

The following results are critical to the proof of our main theorems.

Remark 1. [2] If $x, y \in M$ and $\upsilon \in T_xM$, then

$$\left\langle \upsilon | -\exp_x^{-1} y \right\rangle = \left\langle \upsilon | \mathrm{P}_{x,y} \exp_y^{-1} x \right\rangle = \left\langle \mathrm{P}_{y,x} \upsilon | \exp_y^{-1} x \right\rangle. \tag{2.3}$$

Lemma 3. [40] *Let M be an Hadamard manifold and $x, y, z \in M$, then*

$$\| \exp_x^{-1} z - \mathrm{P}_{x,y} \exp_y^{-1} z \| \le d(x, y). \tag{2.4}$$

Let Q be a nonempty subset of M. A set Q is said to be *geodesic convex* if, for any two points x and y in Q, the geodesic joining x to y is contained in Q. A function g from M to $\mathbb{R}$ is said to be *geodesic convex* if, for any geodesic $\varphi(s)$ ($s \in [0, 1]$) joining x to y in M, the function $g \circ \varphi$ is convex, that is,

$$g(\varphi(s)) \leq sg(\varphi(0)) + (1 - s)g(\varphi(s)) = sg(x) + (1 - s)g(y).$$

Proposition 2. [3] *Let M be an Hadamard manifold, then the following inequality holds:*

$$d(\exp_{x_1} s \exp_{x_1}^{-1} x_2, \exp_{y_1} s \exp_{y_1}^{-1} y_2) \leq (1 - s)d(x_1, y_1) + sd(x_2, y_2),$$

for all $x_1, x_2, y_1, y_2 \in M$ and $s \in [0, 1]$.

Specifically, for all $x \in M$, the function $d(\cdot, x) : M \to \mathbb{R}$ is a geodesic convex function.

Definition 1. [11, 23] A mapping $F : M \to M$ is said to be

(i) *nonexpansive* if

$$d(F(x), F(y)) \leq d(x, y), \quad \forall x, y \in M;$$

(ii) *firmly nonexpansive* if for any two points $x, y \in M$, the function $\xi : [0, 1] \to [0, +\infty]$ defined by

$$\xi(s) := d(\exp_x s \exp_x^{-1} F(x), \exp_y s \exp_y^{-1} F(y)),$$

for all $s \in [0, 1]$, is nonincreasing.

As can be seen from Definition 1, every firmly nonexpansive map is nonexpansive.

The concept described below is well-known.

Definition 2. [15] A mapping $F : M \to M$ is said to be *demiclosed* if, for any bounded sequence $\{x_n\}$ in M such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} d(x_n, F(x_n)) = 0$, then $F(x) = x$.

Remark 2. [15] It is easy to prove that each nonexpansive mapping $F : M \to M$ is demiclosed.

Let's end this section with some results that will be useful in the future.

Lemma 4. [31] *Let $\{a_n\}$ and $\{b_n\}$ are nonnegative real sequences, assume that $\eta \in [0, 1), \tau > 0$ and for all $n \in \mathbb{N}$, the following holds:*

$$a_{n+1} \leq \eta a_n + \tau b_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 5. [27] *Let $\{\zeta_n\}$, $\{\sigma_n\}$ and $\{\lambda_n\}$ are sequences in $[0, +\infty)$ satisfying*

$$\zeta_{n+1} \leq \zeta_n + \lambda_n(\zeta_n - \zeta_{n-1}) + \sigma_n, \quad \forall n \geq 1,$$

where $\sum_{n=1}^{\infty} \sigma_n < \infty$ and $0 \leq \lambda_n \leq \lambda < 1$ for all $n \geq 1$. Then the following hold:

(i) $\sum_{n=1}^{\infty}[\zeta_n - \zeta_{n-1}]_+ < \infty$ *with* $[t]_+ := \max\{t, 0\}$ *for any* $t \in \mathbb{R}$;

(ii) $\lim_{n\to\infty} \zeta_n = \zeta^* \in [0, \infty)$.

3. Main results

We present the inertial Mann method that is described below. This algorithm is an extension of algorithm (1.2) from Hilbert spaces to Hadamard manifolds.

Algorithm 1. Let M be an Hadamard manifold and $F : M \to M$ a mapping. Choose $x_0, x_1 \in M$. Define a sequence $\{x_n\}$ by the following iterative scheme:

$$y_n := \exp_{x_n}(-\lambda_n \exp_{x_n}^{-1} x_{n-1}), \tag{3.1}$$
$$x_{n+1} := \exp_{y_n}(1-\gamma_n)\exp_{y_n}^{-1} F(y_n), \tag{3.2}$$

where $\{\lambda_n\} \subset [0, \infty)$ and $\{\gamma_n\} \subset (0, 1)$ satisfy the following conditions:

(C1) $0 \le \lambda_n \le \lambda < 1, \quad \forall n \ge 1$;
(C2) $\sum_{n=1}^{\infty} \lambda_n d^2(x_n, x_{n-1}) < \infty$;
(C3) $0 < \gamma_1 \le \gamma_n \le \gamma_2 < 1, \quad \forall n \ge 1$;
(C4) $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Remark 3. We can get the following particular conclusions from Algorithm 1:

(i) If $M = \mathbb{R}^n$, then Algorithm 1 reduces to iterative process (1.2) which introduced by Maingé [9].
(ii) The posterior condition (C2) for constructing λ_n can be implemented from the known value of $d^2(x_n, x_{n-1})$ following the rules [27]:

$$0 \le \lambda_n \le \overline{\lambda_n}, \quad \overline{\lambda_n} = \begin{cases} \min\left\{\dfrac{\epsilon_n}{d^2(x_n, x_{n-1})}, \lambda\right\}, & \text{if } x_n \ne x_{n-1}, \\ \lambda, & \text{othewise}, \end{cases} \tag{3.3}$$

where $\{\epsilon_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} \epsilon_n < \infty$.
(iii) If $\lambda_n \equiv 0$, then we obtain the Mann iteration introduced by Li et al. [11].
(iv) We provide some prototypes that the sequences γ_n satisfies conditions (C3) and (C4).

$$\gamma_n := \frac{1}{(n+1)^2}, \frac{1}{e^n}, \frac{1}{2^n}, \quad n \ge 1.$$

Next, we present and prove the convergence theorem for Algorithm 1.

Theorem 1. *Let M be an Hadamard manifold and $F : M \to M$ be a nonexpansive mapping such that* $\mathrm{Fix}(F) \ne \emptyset$. *Then the sequence $\{x_n\}$ defined by Algorithm 1 converges a fixed point of F.*

Proof. Fix $n \in \mathbb{N}$, let $z \in \mathrm{Fix}(F)$ and $\triangle(y_n, F(y_n), z) \subseteq M$ be a geodesic triangle with vertices y_n, $F(y_n)$ and z, and $\triangle\left(\overline{y_n}, \overline{F(y_n)}, \overline{z}\right) \subseteq \mathbb{R}^2$ be the corresponding comparison triangle. In accordance with Lemma 1, we have $d(y_n, F(y_n)) = \left\|\overline{y_n} - \overline{F(y_n)}\right\|$, $d(y_n, z) = \|\overline{y_n} - \overline{z}\|$ and $d(F(y_n), z) = \left\|\overline{F(y_n)} - \overline{z}\right\|$. Let $\overline{x_{n+1}} = \gamma_n \overline{y_n} + (1-\gamma_n)\overline{F(y_n)}$ be the comparison point of x_{n+1}. Using Lemma 2 (ii) together with the nonexpansiveness of F,

$$\begin{aligned} d^2(x_{n+1}, z) &\le \|\overline{x_{n+1}} - \overline{z}\|^2 \\ &= \left\|\gamma_n \overline{y_n} + (1-\gamma_n)\overline{F(y_n)} - \overline{z}\right\|^2 \end{aligned}$$

$$
\begin{aligned}
&= \gamma_n\|\overline{y_n}-\overline{z}\|^2+(1-\gamma_n)\left\|\overline{F(y_n)}-\overline{z}\right\|^2-\gamma_n(1-\gamma_n)\left\|\overline{y_n}-\overline{F(y_n)}\right\|^2 \\
&\le \gamma_n d^2(y_n,z)+(1-\gamma_n)d^2(y_n,z)-\gamma_n(1-\gamma_n)d^2(y_n,F(y_n)) \qquad (3.4)\\
&= d^2(y_n,z). \qquad (3.5)
\end{aligned}
$$

Consider the geodesic triangle $\triangle(y_n,x_n,z)$. Then, by Lemma 1, there exists the corresponding comparison $\triangle(\overline{y_n},\overline{x_n},\overline{z})\subseteq\mathbb{R}^2$ such that

$$
d(y_n,x_n)=\|\overline{y_n}-\overline{x_n}\|,\quad d(y_n,z)=\|\overline{y_n}-\overline{z}\|\quad\text{and}\quad d(x_n,z)=\|\overline{x_n}-\overline{z}\|.
$$

Now,

$$
\begin{aligned}
d^2(y_n,z) &= \|\overline{y_n}-\overline{z}\|^2\\
&= \|\overline{y_n}-\overline{x_n}+\overline{x_n}-\overline{z}\|^2\\
&= \|\overline{y_n}-\overline{x_n}\|^2+\|\overline{x_n}-\overline{z}\|^2+2\langle\overline{y_n}-\overline{x_n}|\overline{x_n}-\overline{z}\rangle\\
&= \|\overline{y_n}-\overline{x_n}\|^2+\|\overline{x_n}-\overline{z}\|^2+2\langle\overline{y_n}-\overline{x_n}|\overline{x_n}-\overline{z}\rangle+2\|\overline{x_n}-\overline{z}\|^2-2\|\overline{x_n}-\overline{z}\|^2\\
&= d^2(y_n,x_n)+d^2(x_n,z)+2\langle\overline{y_n}-\overline{z}|\overline{x_n}-\overline{z}\rangle-2d^2(x_n,z). \qquad (3.6)
\end{aligned}
$$

Let the angles at the vertices z and $\overline{z}$ be denoted by θ and $\overline{\theta}$ respectively. We can obtain $\theta\le\overline{\theta}$ by using (i) of Lemma 2. Applying Proposition 1, we arrive at the conclusion that

$$
\begin{aligned}
\langle\overline{y_n}-\overline{z}|\overline{x_n}-\overline{z}\rangle &= \|\overline{y_n}-\overline{z}\|\|\overline{x_n}-\overline{z}\|\cos\overline{\theta}\\
&= d(y_n,z)d(z,x_n)\cos\overline{\theta}\\
&\le d(y_n,z)d(z,x_n)\cos\theta\\
&= \left\langle \exp_z^{-1}y_n|\exp_z^{-1}x_n\right\rangle. \qquad (3.7)
\end{aligned}
$$

Inserting (3.7) in (3.6), gives

$$
d^2(y_n,z)\le d^2(y_n,x_n)+d^2(x_n,z)+2\left\langle\exp_z^{-1}y_n|\exp_z^{-1}x_n\right\rangle-2d^2(x_n,z).
$$

From Eq (3.1), we can deduce that $\exp_{x_n}^{-1}y_n = -\lambda_n\exp_{x_n}^{-1}x_{n-1}$ and we also know that $d(x_n,y_n)=\lambda_n d(x_n,x_{n-1})$, where $\lambda_n\in[0,1)$. Then the last inequality becomes

$$
\begin{aligned}
d^2(y_n,z) &\le \lambda_n^2 d^2(x_n,x_{n-1})+d^2(x_n,z)+2\left\langle\exp_z^{-1}y_n|\exp_z^{-1}x_n\right\rangle-2d^2(x_n,z)\\
&\le \lambda_n d^2(x_n,x_{n-1})+d^2(x_n,z)+2\left\langle\exp_z^{-1}y_n|\exp_z^{-1}x_n\right\rangle-2d^2(x_n,z).
\end{aligned}
$$

Taking into consideration Remark 1 and Lemma 3 in the above inequality, we obtain

$$
\begin{aligned}
d^2(y_n,z) &\le \lambda_n d^2(x_n,x_{n-1})+d^2(x_n,z)-2d^2(x_n,z)\\
&\quad +2\left\langle\exp_z^{-1}y_n-\mathrm{P}_{z,x_n}\exp_{x_n}^{-1}y_n+\mathrm{P}_{z,x_n}\exp_{x_n}^{-1}y_n|\exp_z^{-1}x_n\right\rangle\\
&= \lambda_n d^2(x_n,x_{n-1})+d^2(x_n,z)-2d^2(x_n,z)\\
&\quad +2\left\langle\exp_z^{-1}y_n-\mathrm{P}_{z,x_n}\exp_{x_n}^{-1}y_n|\exp_z^{-1}x_n\right\rangle-2\left\langle\exp_{x_n}^{-1}y_n|\exp_{x_n}^{-1}z\right\rangle\\
&\le \lambda_n d^2(x_n,x_{n-1})+d^2(x_n,z)-2d^2(x_n,z)+2\|\exp_z^{-1}y_n-\mathrm{P}_{z,x_n}\exp_{x_n}^{-1}y_n\|\|\exp_z^{-1}x_n\|
\end{aligned}
$$

$$
\begin{aligned}
& -2\left\langle \exp_{x_n}^{-1} y_n | \exp_{x_n}^{-1} z \right\rangle \\
\leq\ & \lambda_n d^2(x_n, x_{n-1}) + d^2(x_n, z) - 2d^2(x_n, z) + 2d(z, x_n)d(z, x_n) - 2\left\langle \exp_{x_n}^{-1} y_n | \exp_{x_n}^{-1} z \right\rangle \\
=\ & \lambda_n d^2(x_n, x_{n-1}) + d^2(x_n, z) - 2\left\langle \exp_{x_n}^{-1} y_n | \exp_{x_n}^{-1} z \right\rangle . \qquad (3.8)
\end{aligned}
$$

From the fact that $\exp_{x_n}^{-1} y_n = -\lambda_n \exp_{x_n}^{-1} x_{n-1}$ and using Remark 1 and Lemma 3, then Eq (3.8) becomes

$$
\begin{aligned}
d^2(y_n, z) \leq\ & \lambda_n d^2(x_n, x_{n-1}) + d^2(x_n, z) + 2\lambda_n \left\langle \exp_{x_n}^{-1} x_{n-1} | \exp_{x_n}^{-1} z \right\rangle \\
=\ & \lambda_n d^2(x_n, x_{n-1}) + d^2(x_n, z) + 2\lambda_n \left\langle \exp_{x_n}^{-1} x_{n-1} - \mathrm{P}_{x_n,z} \exp_z^{-1} x_{n-1} + \mathrm{P}_{x_n,z} \exp_z^{-1} x_{n-1} | \exp_{x_n}^{-1} z \right\rangle \\
=\ & \lambda_n d^2(x_n, x_{n-1}) + d^2(x_n, z) + 2\lambda_n \left\langle \exp_{x_n}^{-1} x_{n-1} - \mathrm{P}_{x_n,z} \exp_z^{-1} x_{n-1} | \exp_{x_n}^{-1} z \right\rangle \\
& -2\lambda_n \left\langle \exp_z^{-1} x_{n-1} | \exp_z^{-1} x_n \right\rangle \\
\leq\ & \lambda_n d^2(x_n, x_{n-1}) + d^2(x_n, z) + 2\lambda_n \| \exp_{x_n}^{-1} x_{n-1} - \mathrm{P}_{x_n,z} \exp_z^{-1} x_{n-1} \| \| \exp_{x_n}^{-1} z \| \\
& -2\lambda_n \left\langle \exp_z^{-1} x_{n-1} | \exp_z^{-1} x_n \right\rangle \\
\leq\ & \lambda_n d^2(x_n, x_{n-1}) + d^2(x_n, z) + 2\lambda_n d(x_n, z)d(x_n, z) - 2\lambda_n \left\langle \exp_z^{-1} x_{n-1} | \exp_z^{-1} x_n \right\rangle \\
=\ & \lambda_n d^2(x_n, x_{n-1}) + d^2(x_n, z) + 2\lambda_n d^2(x_n, z) - 2\lambda_n \left\langle \exp_z^{-1} x_{n-1} | \exp_z^{-1} x_n \right\rangle . \qquad (3.9)
\end{aligned}
$$

Let $\triangle(x_n, x_{n-1}, z)$ be a geodesic triangle. It is stated in Proposition 1 that

$$
-2\left\langle \exp_z^{-1} x_{n-1} | \exp_z^{-1} x_n \right\rangle \leq d^2(x_n, x_{n-1}) - d^2(x_{n-1}, z) - d^2(x_n, z). \qquad (3.10)
$$

By combining (3.9) and (3.10), we get

$$
\begin{aligned}
d^2(y_n, z) \leq\ & \lambda_n d^2(x_n, x_{n-1}) + d^2(x_n, z) + 2\lambda_n d^2(x_n, z) + \lambda_n d^2(x_n, x_{n-1}) - \lambda_n d^2(x_{n-1}, z) - \lambda_n d^2(x_n, z) \\
=\ & d^2(x_n, z) + \lambda_n(d^2(x_n, z) - d^2(x_{n-1}, z)) + 2\lambda_n d^2(x_n, x_{n-1}). \qquad (3.11)
\end{aligned}
$$

Substitution (3.11) into (3.5) yields

$$
d^2(x_{n+1}, z) \leq d^2(x_n, z) + \lambda_n(d^2(x_n, z) - d^2(x_{n-1}, z)) + 2\lambda_n d^2(x_n, x_{n-1}). \qquad (3.12)
$$

Since $\sum_{n=1}^{\infty} \lambda_n d^2(x_n, x_{n-1}) < \infty$, where $0 \leq \lambda_n < \lambda < 1$. By combining this last estimate with the previous one (3.12) and utilizing Lemma 5, we deduce that the sequence $\{d(x_n, z)\}$ is convergent (as a result, $\{x_n\}$ is bounded). We may see that the sequence $\{y_n\}$ is also bounded in view of (3.11). Again from (3.12) and Lemma 5, we get $\sum_{n=1}^{\infty}[d^2(x_n, z) - d^2(x_{n-1}, z)]_+ < \infty$. Since $d(y_n, x_n) = \lambda_n d(x_n, x_{n-1})$, this implies that $\lim_{n\to\infty} d(x_n, y_n) = 0$, because $\lim_{n\to\infty} \lambda_n d(x_n, x_{n-1}) = 0$ by the assumption $\sum_{n=1}^{\infty} \lambda_n d^2(x_n, x_{n-1}) < \infty$.

Next, we show that $\lim_{n\to\infty} d(y_n, F(y_n)) = 0$. By combining (3.4) and (3.11), then

$$
\begin{aligned}
\gamma_1(1-\gamma_2)d^2(y_n, F(y_n)) \leq\ & \gamma_n(1-\gamma_n)d^2(y_n, F(y_n)) \\
\leq\ & d^2(x_n, z) - d^2(x_{n+1}, z) + \lambda_n(d^2(x_n, z) - d^2(x_{n-1}, z)) + 2\lambda_n d^2(x_n, x_{n-1}).
\end{aligned}
$$

Since $\lambda_n \in [0, 1)$, $\lim_{n\to\infty} d^2(x_n, z)$ exists and $\lim_{n\to\infty} \lambda_n d(x_n, x_{n-1}) = 0$, we get

$$
\lim_{n\to\infty} d(y_n, F(y_n)) = 0.
$$

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $\{x_{n_k}\}$ converges to some x^* in M (as $k \to \infty$). Since $\lim_{n\to\infty} d(x_n, y_n) = 0$, we get $\{y_{n_k}\}$ converges to x^*. We already have $\lim_{k\to\infty} d(y_{n_k}, F(y_{n_k})) = 0$, and from the fact that F is demiclosed, this implies that $x^* \in \mathrm{Fix}(F)$.

Now, we prove that $\{x_n\}$ converges to an element in $\mathrm{Fix}(F)$. By combining (3.4) and (3.11), then

$$\begin{aligned} d^2(x_{n+1}, x^*) &\leq (1-\gamma_n)[d^2(x_n, x^*) + \lambda_n(d^2(x_n, x^*) - d^2(x_{n-1}, x^*)) + 2\lambda_n d^2(x_n, x_{n-1})] + \gamma_n d^2(y_n, x^*) \\ &\leq (1-\gamma_1)d^2(x_n, x^*) + \lambda(1-\gamma_1)[d^2(x_n, x^*) - d^2(x_{n-1}, x^*)]_+ + 2(1-\gamma_1)\lambda_n d^2(x_n, x_{n-1}) \\ &\quad + \gamma_n d^2(y_n, x^*), \end{aligned}$$

which implies that

$$a_{n+1} \leq \eta a_n + \tau b_n,$$

where

$$\begin{aligned} a_n &= d^2(x_n, x^*), \\ b_n &= \lambda(1-\gamma_1)[d^2(x_n, x^*) - d^2(x_{n-1}, x^*)]_+ + 2(1-\gamma_1)\lambda_n d^2(x_n, x_{n-1}) + \gamma_n d^2(y_n, x^*), \\ \eta &= 1-\gamma_1, \\ \tau &= 1. \end{aligned}$$

By applying

$$\sum_{n=1}^{\infty}[d^2(x_n, x^*) - d^2(x_{n-1}, x^*)]_+ < \infty, \sum_{n=1}^{\infty} \lambda_n d^2(x_n, x_{n-1}) < \infty$$

and (C4), and utilizing Lemma 4, we conclude that $\lim_{n\to\infty} d(x_n, x^*) = 0$. Thus, the sequence $\{x_n\}$ converges to an element in $\mathrm{Fix}(F)$. Therefore, the proof is completed. □

Next, we study the convergence theorem for Algorithm 1 by eliminating the condition (C4). To show the next convergence theorem, we require the lemma stated below.

Lemma 6. [32, Theorem 3.2] *Let $\{x_n\}$ be a sequence in M such that there exists a nonempty set $Q \subset M$ satisfying:*

(i) *For all $x^* \in Q$, $\lim_{n\to\infty} d(x_n, x^*)$ exists.*
(ii) *Any cluster point of $\{x_n\}$ belongs to Q.*

Then, there exists $\tilde{x} \in Q$ such that $\{x_n\}$ converges to $\tilde{x}$.

Theorem 2. *Let M be an Hadamard manifold and $F : M \to M$ be a nonexpansive mapping such that $\mathrm{Fix}(F) \neq \emptyset$. Suppose that $\{x_n\}$ be a sequence generated by Algorithm 1 and $\{\lambda_n\} \subset [0, \infty)$, $\{\gamma_n\} \subset (0, 1)$ satisfy conditions (C1)–(C3). Then the sequence $\{x_n\}$ converges to a fixed point of F.*

Proof. In accordance with the proof of Theorem 1, we have that $\lim_{n\to\infty} d(x_n, z)$ exists, where $z \in \mathrm{Fix}(F)$, and any cluster point of $\{x_n\}$ that belongs to $\mathrm{Fix}(F)$. Using Lemma 6, we can see that $\{x_n\}$ converges to an element in $\mathrm{Fix}(F)$. Therefore, the proof is completed.

□

4. Applications

In this section, we discuss two implementations of the inertial algorithm we proposed in Hadamard manifolds: monotone inclusion problems and equilibrium problems.

4.1. Inclusion problems

We denote by $\Psi(M)$ the set of all multivalued vector fields $A : M \to 2^{TM}$ such that $A(x) \subseteq T_xM$ for all $x \in M$, and $D(A)$ the domain of A defined by $D(A) = \{x \in M : A(x) \neq \emptyset\}$. In this subsection, we consider the problem of finding $x^* \in M$ such that

$$\mathbf{0} \in A(x^*). \tag{4.1}$$

The point x^* is called a *singularity* of A and the set of all singularities of A is denoted by $A^{-1}(\mathbf{0}) = \{x \in M : \mathbf{0} \in A(x)\}$.

The notion of monotonicity for multivalued vector fields on Hadamard manifolds is then discussed.

Definition 3. [5] A multivalued vector field $A \in \Psi(M)$ is said to be

(i) *monotone* if for any $x, y \in D(A)$, we have

$$\langle \upsilon | \exp_x^{-1} y\rangle + \langle \nu | \exp_y^{-1} x\rangle \leq 0, \quad \forall \upsilon \in A(x) \text{ and } \forall \nu \in A(y);$$

(ii) *maximal monotone* if it is monotone and the following implication holds for any $x \in M$ and $\upsilon \in T_xM$:

$$\langle \upsilon | \exp_x^{-1} y\rangle + \langle \nu | \exp_y^{-1} x\rangle \leq 0, \quad \forall y \in D(A) \text{ and } \forall \nu \in A(y) \Longrightarrow \upsilon \in A(x).$$

Li et al. [23] proposed a resolvent of multivalued vector fields on Hadamard manifolds as well as a relation between nonexpansiveness and monotonicity.

Definition 4. [23] For a given $\mu > 0$, the resolvent of a multivalued vector field $A \in \Psi(M)$ of order μ is a multivalued map $J_\mu^A : M \to 2^M$ defined by

$$J_\mu^A(x) := \left\{ z \in M : \frac{1}{\mu} \exp_z^{-1} x \in A(z) \right\}, \quad \forall x \in M.$$

Remark 4. [23] For $\mu > 0$, the range of resolvent J_μ^A contained the domain of A and $\mathrm{Fix}(J_\mu^A) = A^{-1}(\mathbf{0})$.

Theorem 3. [23] *Let a vector field $A \in \Psi(M)$. Then, for all $\mu > 0$, the vector field A is monotone if and only if J_μ^A is single-valued and firmly nonexpansive.*

It was shown by Li et al. [2] that the subdifferential of a proper, lower semicontinuous and geodesic convex function is a maximal monotone vector field.

Definition 5. [3] Let $g : M \to \mathbb{R}$ be a geodesic convex function. The *subdifferential* $\partial g(x)$ of g at $x \in M$ is defined by

$$\partial g(x) := \{\upsilon \in T_xM : \langle \upsilon | \exp_x^{-1} y\rangle \leq g(y) - g(x), \quad \forall y \in M\}. \tag{4.2}$$

It is easy to check that $\partial g(x)$ is closed and geodesic convex.

Lemma 7. [2] *Let* $g : M \to \mathbb{R} \cup \{+\infty\}$ *be a proper, lower semicontinuous and geodesic convex function and* $D(g) = M$. *Then the subdifferential* ∂g *of* g *is a maximal monotone vector field.*

We can understand this by looking at Remark 4, which tells us that the problem of finding singularities of A becomes the problem of finding fixed points of the mapping J_μ^A. Following this, we will implement Algorithm 1 in order to find singularities of monotone multivalued vector fields.

Theorem 4. *Suppose that* $A \in \Psi(M)$ *is a monotone multivalued vector field, and* J_μ^A *is the resolvent of* A *for* $\mu > 0$ *such that* $A^{-1}(\mathbf{0}) \neq \emptyset$. *Let* $x_0, x_1 \in M$ *and a sequence* $\{x_n\}$ *is defined by*

$$y_n := \exp_{x_n}(-\lambda_n \exp_{x_n}^{-1} x_{n-1}),$$
$$x_{n+1} := \exp_{y_n}(1-\gamma_n)\exp_{y_n}^{-1} J_\mu^A(y_n),$$

where $\{\lambda_n\} \subset [0,\infty)$, $\{\gamma_n\} \subset (0,1)$ *satisfy conditions (C1)–(C4). Then the sequence* $\{x_n\}$ *converges to an element in* $A^{-1}(\mathbf{0})$.

Proof. By taking $F = J_\mu^A$. From Theorem 3, F is single-valued and firmly nonexpansive mapping. As a result, it is nonexpansive with $\text{Fix}(F) = A^{-1}(\mathbf{0})$. From the hypothesis, $\text{Fix}(F) = A^{-1}(\mathbf{0}) \neq \emptyset$. As a consequence of this, the desired results are achieved according to Theorem 1. □

Theorem 5. *Suppose that* $A \in \Psi(M)$ *is a monotone multivalued vector field, and* J_μ^A *is the resolvent of* A *for* $\mu > 0$ *such that* $A^{-1}(\mathbf{0}) \neq \emptyset$. *Let* $x_0, x_1 \in M$ *and a sequence* $\{x_n\}$ *is defined by*

$$y_n := \exp_{x_n}(-\lambda_n \exp_{x_n}^{-1} x_{n-1}),$$
$$x_{n+1} := \exp_{y_n}(1-\gamma_n)\exp_{y_n}^{-1} J_\mu^A(y_n),$$

where $\{\lambda_n\} \subset [0,\infty)$, $\{\gamma_n\} \subset (0,1)$ *satisfy conditions (C1)–(C3). Then the sequence* $\{x_n\}$ *converges to an element in* $A^{-1}(\mathbf{0})$.

Proof. By taking $F = J_\mu^A$. From Theorem 3, F is single-valued and firmly nonexpansive mapping. As a result, it is nonexpansive with $\text{Fix}(F) = A^{-1}(\mathbf{0})$. From the hypothesis, $\text{Fix}(F) = A^{-1}(\mathbf{0}) \neq \emptyset$. As a consequence of this, the desired results are achieved according to Theorem 2. □

4.2. Equilibrium problems

Let Q be a nonempty, closed and geodesic convex of M, and let $\psi : Q \times Q \to \mathbb{R}$ a bifunction that satisfies the equation $\psi(x,x) = 0$ for all $x \in Q$. Calao et al. [6] conducted research on an equilibrium problem in the setting of Hadamard manifolds. Let us recall it in the following problem: Find $x^* \in Q$ such that

$$\psi(x^*, y) \geq 0, \quad \forall y \in Q. \tag{4.3}$$

The solution of the equilibrium problem (4.3) is said to be an *equilibrium point*, and $EP(\psi)$ stands for the set of all equilibrium points.

In order to study the equilibrium problem (4.3), let us suppose that ψ satisfies the following assumptions

(H1) $\psi(x,x) \geq 0$ for all $x \in Q$.

(H2) ψ is monotone bifunction, that is, $\psi(x, y) + \psi(y, x) \le 0$ for all $x, y \in Q$.
(H3) For all $y \in Q$, $x \mapsto \psi(x, y)$ is upper semicontinuous.
(H4) For all $x \in Q$, $y \mapsto \psi(x, y)$ is geodesic convex and lower semicontinuous.

Calao et al. [6] have presented the notion of resolvent for the bifunction on Hadamard manifolds as follows: Let $\psi : Q \times Q \to \mathbb{R}$, the resolvent of a bifunction ψ is a multivalued mapping $R_\mu^\psi : M \to 2^Q$ such that for all $x \in M$

$$R_\mu^\psi(x) = \left\{ z \in Q : \psi(z, y) - \frac{1}{\mu} \langle \exp_z^{-1} x | \exp_z^{-1} y \rangle \ge 0, \ \forall y \in Q \right\}.$$

The following theorem concerns the bifunction $\psi : Q \times Q \to M$ that is defined in Hadamard manifolds:

Theorem 6. [6, 7] *Let $\psi : Q \times Q \to \mathbb{R}$ be a bifunction satisfying the following conditions:*

(1) *ψ is monotone;*
(2) *For all $\mu > 0$, R_μ^ψ is properly defined, that is, the domain $D(R_\mu^\psi) \neq \emptyset$.*

Then for any $\lambda > 0$,

(i) *the resolvent R_μ^ψ is single-valued and firmly nonexpansive;*
(ii) *the fixed point set of R_μ^ψ is the equilibrium point set of ψ,*

$$\mathrm{Fix}(R_\mu^\psi) = EP(\psi).$$

Moreover, if ψ satisfying conditions (H1)–(H4). Then $D(R_\mu^\psi) = M$.

In light of Theorem 6, we can see that the equilibrium problem (4.3) were working on transforms into the problem of finding fixed points of R_μ^ψ. Next, we apply Algorithm 1 to find equilibrium points.

Theorem 7. *Suppose that Q is a nonempty, closed and geodesic convex subset of M. Let $\psi : Q \times Q \to \mathbb{R}$ be a bifunction satisfying (H1)–(H4), and R_μ^ψ be the resolvent of ψ for $\mu > 0$ such that $EP(\psi) \neq \emptyset$. Let $x_0, x_1 \in M$ and a sequence $\{x_n\}$ is defined by*

$$y_n := \exp_{x_n}(-\lambda_n \exp_{x_n}^{-1} x_{n-1}),$$
$$x_{n+1} := \exp_{y_n}(1 - \gamma_n) \exp_{y_n}^{-1} R_\mu^\psi(y_n),$$

where $\{\lambda_n\} \subset [0, \infty)$, $\{\gamma_n\} \subset (0, 1)$ satisfy conditions (C1)–(C4). Then the sequence $\{x_n\}$ converges to an element in $EP(\psi)$.

Proof. By taking $F = R_\mu^\psi$. From Theorem 6, F is single-valued and firmly nonexpansive mapping. As a result, it is nonexpansive with $\mathrm{Fix}(F) = EP(\psi)$. From the hypothesis, $\mathrm{Fix}(F) = EP(\psi) \neq \emptyset$. As a consequence of this, the desired results are achieved according to Theorem 1. □

Theorem 8. *Suppose that Q is a nonempty, closed and geodesic convex of M. Let $\psi : Q \times Q \to \mathbb{R}$ be a bifunction satisfying (H1)–(H4), and R_μ^ψ be the resolvent of ψ for $\mu > 0$ such that $EP(\psi) \neq \emptyset$. Let $x_0, x_1 \in M$ and a sequence $\{x_n\}$ is defined by*

$$y_n := \exp_{x_n}(-\lambda_n \exp_{x_n}^{-1} x_{n-1}),$$
$$x_{n+1} := \exp_{y_n}(1 - \gamma_n) \exp_{y_n}^{-1} R_\mu^\psi(y_n),$$

where $\{\lambda_n\} \subset [0, \infty)$, $\{\gamma_n\} \subset (0, 1)$ satisfy conditions (C1)–(C3). Then the sequence $\{x_n\}$ converges to an element in $EP(\psi)$.

Proof. By taking $F = R_{\mu}^{\psi}$. From Theorem 6, F is single-valued and firmly nonexpansive mapping. As a result, it is nonexpansive with $\mathrm{Fix}(F) = EP(\psi)$. From the hypothesis, $\mathrm{Fix}(F) = EP(\psi) \neq \emptyset$. As a consequence of this, the desired results are achieved according to Theorem 2. □

5. Numerical examples

We give three numerical experiments to illustrate the computational performance of the inertial Mann algorithm and compare it with other existence algorithms. All programs was coded in Matlab Program, and the computations were done on a personal computer with an `Intel(R) Core(TM) i7 @1.80 GHz`, together with `8 GB 1600 MHz DDR3`.

Example 1. Let $M := \mathbb{R}^m_{++} = \{x \in \mathbb{R}^m : x_i > 0, i = 1, \ldots, m\}$ and $\mathbb{R}^m_{+} = \{x \in \mathbb{R}^m : x_i \geq 0, i = 1, \ldots, m\}$. As [25], let $(\mathbb{R}^m_{++}, \langle\cdot|\cdot\rangle)$ be the Riemannian manifold with the Riemannian metric $\langle\cdot|\cdot\rangle$ defined by $\langle \upsilon|\nu\rangle := \upsilon^T W(x)\nu$ for all $x \in \mathbb{R}^m_{++}$ and $\upsilon, \nu \in T_x\mathbb{R}^m_{++}$ where $W(x)$ is a diagonal metrix defined by $W(x) = \mathrm{diag}\left(x_1^{-2}, x_2^{-2}, \ldots, x_m^{-2}\right)$. Tangent space at $x \in \mathbb{R}^m_{++}$, denoted by $T_x\mathbb{R}^m_{++}$. Additional, the mapping $\sigma : \mathbb{R}^m \to \mathbb{R}^m_{++}$ given by $\sigma(x) = (e^{x_1}, e^{x_2}, \ldots, e^{x_m})$ is isometry between the Euclidean space $\mathbb{R}^m$ and the Riemannian manifold $(\mathbb{R}^m_{++}, \langle\cdot|\cdot\rangle)$. Then the Riemannian distance $d : \mathbb{R}^m_{++} \times \mathbb{R}^m_{++} \to \mathbb{R}^m_{+}$ is defined by

$$d(x, y) := \left|\sigma^{-1}(x) - \sigma^{-1}(y)\right| = \sqrt{\sum_{i=1}^{m} \ln^2 \frac{x_i}{y_i}}, \quad \forall x, y \in \mathbb{R}^m_{++}.$$

Thus, $(\mathbb{R}^m_{++}, \langle\cdot|\cdot\rangle)$ is a Hadamard manifold. The exponential map on $\mathbb{R}^m_{++}$ is given by

$$\exp_x s\upsilon = \left(x_1 e^{\frac{\upsilon_1 s}{x_1}}, x_2 e^{\frac{\upsilon_2 s}{x_2}}, \ldots, x_m e^{\frac{\upsilon_m s}{x_m}}\right).$$

for $x \in \mathbb{R}^m_{++}$ and $\upsilon \in T_x\mathbb{R}^m_{++}$. The inverse of the exponential map is assigned by

$$\exp_x^{-1} y = \left(x_1 \ln \frac{y_1}{x_1}, x_2 \ln \frac{y_2}{x_2}, \ldots, x_m \ln \frac{y_m}{x_m}\right), \quad \forall x, y \in \mathbb{R}^m_{++}.$$

Test 1. We verify the usefulness of Algorithm 1 in $M = (\mathbb{R}_{++}, \langle\cdot|\cdot\rangle)$. Let $F : \mathbb{R}_{++} \to \mathbb{R}_{++}$ be a nonexpansive mapping defined by

$$F(x) := e^{\ln x/2}, \quad \forall x \in \mathbb{R}_{++}.$$

Then $\mathrm{Fix}(F) = \{1\}$. We set $\gamma_n = 1/(n+1)^2$ and let the inertial parameter λ_n be updated by (3.3) where $\epsilon_n = (1/2)^n$ and $\lambda \in \{0, 0.1, 0.3, 0.5, 0.7, 0.9\}$. Denoted $D_n = d(x_n, 1) < 10^{-6}$ as the stopping criterion. The initial values x_0, x_1 are randomly generated in MATLAB program. We perform a parameter analysis on the proposed Algorithm 1. The numerical results are shown in Table 1 and Figure 1.

Table 1. Computation results for Test 1 of Example 1.

λ	0	0.1	0.3	0.5	0.7	0.9
Iteration	19	16	13	19	22	33
Time(s)	0.0075	0.0049	0.0033	0.0049	0.0076	0.0077

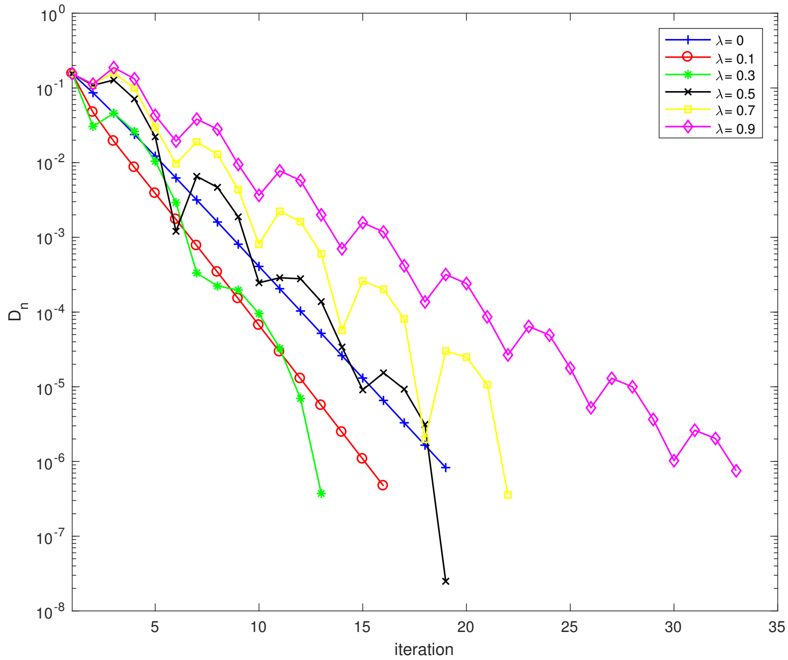


Figure 1. Numerical behavior of $\{D_n\}$ for Test 1 of Example 1.

Remark 5. (i) Table 1 shows that Algorithm 1 with six different parameter choices is efficient and simple to implement. The most essential point is that Algorithm 1 converges quickly when the control parameter $0 < \lambda \leq 0.5$. Also, we can see that the parameter values we choose have no effect on our computational results.

(ii) The speed of our proposed Algorithm 1 with the parameter $\lambda = 0.3$ is clearly faster than others, as can be seen in Figure 1 .

Test 2. In this test, we evaluate the performance of Algorithm 1 in $M = (\mathbb{R}^m_{++}, \langle\cdot|\cdot\rangle)$ to solve the inclusion problem (4.1). Let a function $g : \mathbb{R}^m_{++} \to \mathbb{R}$ defined by

$$g(x) := \sum_{i=1}^{m} g_i(x_i), \quad g_i(x_i) := a_i \ln\left(x_i^{d_i} + b_i\right) - c_i \ln(x_i), \quad i = 1, \ldots, m,$$

where $a_i, b_i, c_i, d_i \in \mathbb{R}_{++}$ satisfy $c_i < a_i d_i$ and $d_i \geq 2$ for $i = 1, \ldots, m$. The minimizer of g is $x^* = (x_1^*, x_2^*, \ldots, x_m^*)$, where $x_i^* = \sqrt[d_i]{b_i c_i/(a_i d_i - c_i)}$, for $i = 1, \ldots, m$. The function g is not Euclidian convex. However, Ferreira et al. [24] showed that g is geodesic convex function in $(\mathbb{R}^m_{++}, \langle\cdot|\cdot\rangle)$. In view of Definition 5, we have

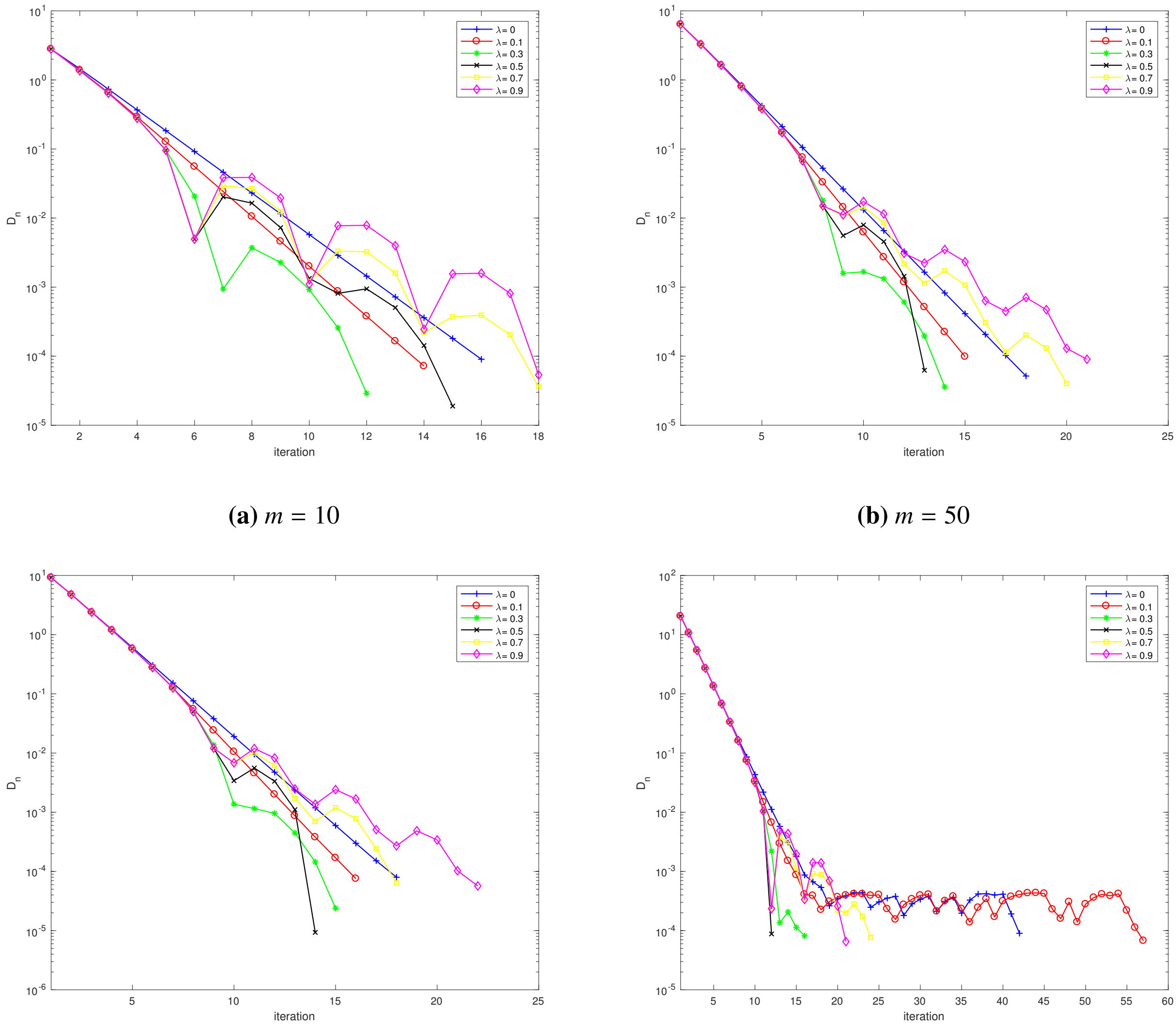
$$\partial g(x) = \left\{ v \in T_x\mathbb{R}^m_{++} \mid g(y) \geq g(x) + \langle v|\exp_x^{-1} y\rangle \right\}, \quad \forall y \in \mathbb{R}^m_{++}.$$

The subdifferential ∂g of g is a maximal monotone vector field, as shown by Lemma 7 . We consider the maximal monotone vector field A by ∂g in (4.1). Moreover, we have

$$J_\mu^{\partial g}(x) = \operatorname*{arg\,min}_{y \in \mathbb{R}^m_{++}} \left\{ g(y) + \frac{1}{2\mu} d^2(y, x) \right\}, \ \forall \mu > 0.$$

It is easy to see that $x^* \in \min_{\mathbb{R}^m_{++}} g \Leftrightarrow \mathbf{0} \in \partial g(x^*)$, where $\min_{\mathbb{R}^m_{++}} g = \{x \in \mathbb{R}^m_{++} : g(x) \leq g(y), \forall y \in \mathbb{R}^m_{++}\}$ is the set of minimizers of g. Hence, $\partial g^{-1}(\mathbf{0}) = \{x^* = (x_1^*, x_2^*, \ldots, x_m^*)$, where $x_i^* = \sqrt[d_i]{b_i c_i/(a_i d_i - c_i)}$, for $i = 1, \ldots, m\}$.

We set $\mu = 1$ and our parameters are the same as Test 1. Let $a_i = b_i = c_i = 1$ and $d_i = 2$, for $i = 1, \ldots, m$, then $x^* = 1$, where $x_i^* = 1$ for $i = 1, \ldots, m$. We take into account of the various of numbers for dimension m and parameters. Denoted $D_n = d(x_n, x^*) < 10^{-4}$ as the stopping criterion. The initial values x_0, x_1 are randomly generated in MATLAB. Our numerical results are reported in Table 2 and Figure 2.



(a) $m = 10$ **(b)** $m = 50$

(c) $m = 100$ **(d)** $m = 500$

Figure 2. Numerical behavior of $\{D_n\}$ for Test 2 of Example 1.

Table 2. Computation results for Test 2 of Example 1.

λ	m							
	10		50		100		500	
	Iteration	Time(s)	Iteration	Time(s)	Iteration	Time(s)	Iteration	Time(s)
0	16	0.7073	18	1.1911	18	2.6154	42	15.9526
0.1	14	0.5878	15	0.9592	16	2.1549	57	21.5351
0.3	12	0.5829	14	0.8666	15	1.8511	16	6.0665
0.5	15	0.6268	13	0.6365	14	1.6361	12	4.4499
0.7	18	0.7743	20	1.2781	18	2.1459	24	8.7960
0.9	18	0.6962	21	1.1829	22	2.5327	21	7.5839

Remark 6. (i) According to the results shown in Table 2, the numerical experiments show that the proposed Algorithm 1 with different dimensional values and parameters converges to a singularity of the maximal monotone multivalued vector field. Our method is efficient and simple to implement for solving the inclusion problem (4.1). Furthermore, the number of iterations required by Algorithm 1 is unaffected by the dimension selection; in fact, the number of iterations required by the proposed method is only slightly affected by the dimension leaping change.

(ii) When the dimension changes, the Algorithm 1 with $\lambda = 0.5$ has a faster convergence rate, as seen in Figure 2.

(iii) Since g is non-convex in the Euclidean sense, which implies that the Euclidean methods [9, 33] can not be applied to solve the inclusion problem (4.1).

Example 2. Let $M = (\mathbb{R}^3, \langle\cdot|\cdot\rangle)$ be an Hadamard manifolds with Riemannian metric $\langle \upsilon|\nu\rangle = \upsilon^\top W(x)\nu$ for all $\upsilon, \nu \in T_x\mathbb{R}^3$ and $x = (x_1, x_2, x_3) \in M$, where $W(x)$ is 3×3 matrix defined by

$$W(x) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + 4x_2^2 & -2x_2 \\ 0 & -2x_2 & 1 \end{pmatrix}, \quad \forall x \in \mathbb{R}^3.$$

The Riemannian distance is defined by

$$d^2(x, y) = \sum_{i=1}^{2} (x_i - y_i)^2 + (x_2^2 - x_3 - y_2^2 + y_3)^2, \quad \forall x, y \in \mathbb{R}^3.$$

More information can be found in [25]. The geodesic that joins the points $\varphi(0) = x$ and $\varphi(1) = y$ is given by

$$\varphi(s) := (\varphi_1(s), \varphi_2(s), \varphi_3(s)), \quad \forall s \in [0, 1],$$

where $\varphi_i(s) = x_i + s(y_i - x_i)$ for all $i = 1, 2$ and

$$\varphi_3(s) = x_3 + s((y_3 - x_3) - 2(y_2 - x_2)^2) + 2s^2(y_2 - x_2)^2.$$

Therefore, $\exp_x(s\upsilon) = \varphi(s)$, where $\varphi : \mathbb{R} \to \mathbb{R}^3$ is the unique geodesic starting from $\varphi(0) = x$ with $\upsilon = \varphi'(0) \in T_x\mathbb{R}^3$. The inverse exponential mapping is assigned by

$$\exp_x^{-1} y = \left(y_1 - x_1, y_2 - x_2, y_3 - x_3 - (y_2 - x_2)^2\right).$$

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a nonexpansive mapping defined by

$$F(x) = \left(\frac{x_1}{2}, \frac{x_2}{3}, \frac{x_3}{2} + \frac{x_2^2}{2}\right), \quad \forall x \in \mathbb{R}^3.$$

Then $\mathrm{Fix}(F) = \{(0,0,0)\}$. In order to show the effectiveness of Algorithm 1, we compare it to three other algorithms: the Mann algorithm [11], the Halpern algorithm [11], and the Ishikawa algorithm [12]. In Mann algorithm and Halpern algorithm, we set $\alpha_n = 1/(n+3)^2$. In Ishikawa algorithm, we take $\alpha_n = 0.9 - 1/(n+4)$ and $\beta_n = (1/2)^n$. In the proposed Algorithm 1, let inertial parameter λ_n be updated by (3.3) where $\epsilon_n = (1/2)^n$ and $\lambda = 0.3$, and the parameter $\gamma_n = 1/100(n+1)^2$. Denoted $D_n = d(x_n, (0,0,0)) < 10^{-6}$ as the stopping criterion. The initial values x_0, x_1 are randomly generated in MATLAB. The numerical results of our investigation are shown in Table 3, as well as Figure 3.

Table 3. Computation results of Example 2.

Algorithm	Iteration	Time(s)
Algorithm 1	15	0.0030
Mann [11]	20	0.0064
Ishikawa [12]	19	0.0090
Halpern [11]	1277	0.0328

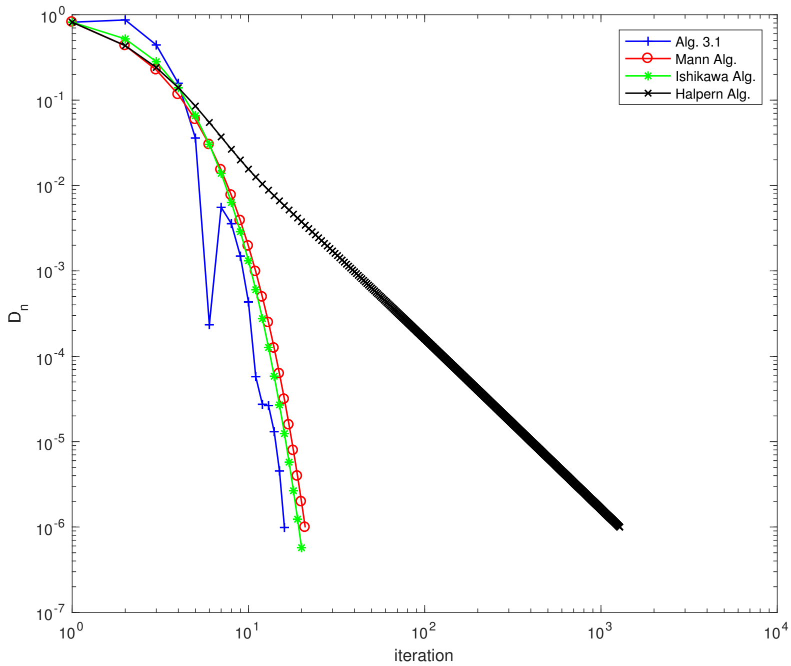


Figure 3. Numerical behavior of $\{D_n\}$ for Example 2.

Example 3. Let $M := \mathbb{H}^3 = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \langle x, x \rangle = -1, x_4 > 0\}$ be the 3-dimensional hyperbolic space endowed the symmetric bilinear form (which is called the Lorentz metric) defined by

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4, \quad x = (x_1, x_2, x_3, x_4),\ y = (y_1, y_2, y_3, y_4) \in \mathbb{H}^3.$$

For richer detail, see [1]. Then, $\mathbb{H}^3$ is a Hadamard manifold with sectional curvature -1. Furthermore, the normalized geodesic $\varphi : \mathbb{R} \to \mathbb{H}^3$ starting from $x \in \mathbb{H}^3$ is given by

$$\varphi(s) = (\cosh s)x + (\sinh s)\upsilon,$$

where $s > 0$ and $\upsilon \in T_x\mathbb{H}^3$ is unit vector. This means that the expression $\exp_x s\upsilon = (\cosh s)x+(\sinh s)\upsilon$. From [1], one can check the inverse exponential map is given by

$$\exp_x^{-1} y = \operatorname{arccosh}(-\langle x, y\rangle)\frac{y + \langle x, y\rangle x}{\sqrt{\langle x, y\rangle^2 - 1}}, \quad \forall x, y \in \mathbb{H}^3.$$

The Riemannian distance $d : \mathbb{H}^3 \times \mathbb{H}^3 \to \mathbb{R}$ is defined by $d(x, y) = \operatorname{arccosh}(-\langle x, y\rangle)$.

Let $F : \mathbb{H}^3 \to \mathbb{H}^3$ be a nonexpansive mapping defined by

$$F(x_1, x_2, x_3, x_4) = (-x_1, -x_2, -x_3, x_4), \quad \forall x = (x_1, x_2, x_3, x_4) \in \mathbb{H}^3.$$

Then, $\mathrm{Fix}(F) = \{(0, 0, 0, 1)\}$. To demonstrate the effectiveness of our proposed method, we compare it to the Mann and Halpern algorithms in [11], as well as the Ishikawa algorithm in [12]. In Mann algorithm and Halpern algorithm, we set $\alpha_n = 1/(n+3)^2$. In Ishikawa algorithm, we take $\alpha_n = 1/(n+3)$ and $\beta_n = 1/(n + 3)^2$. In the proposed Algorithm 1, let inertial parameter λ_n be updated by (3.3) with $\epsilon_n = (1/2)^n$ and $\lambda = 0.5$, and the parameter $\gamma_n = (n+1)/(2n+4)$. Denoted $D_n = d(x_n, (0, 0, 0, 1)) < 10^{-6}$ as the stopping criterion. The initial values x_0, x_1 are chosen from [1]

$$\begin{aligned} x_0 &= (0.82054041398189, 1.78729906182707, 0.11578260956854, 2.20932797928782),\\ x_1 &= (0.93181457846166, 0.46599434167542, 0.41864946772751, 1.50356127641534). \end{aligned}$$

Table 4 and Figure 4 provide the numerical results that we have obtained.

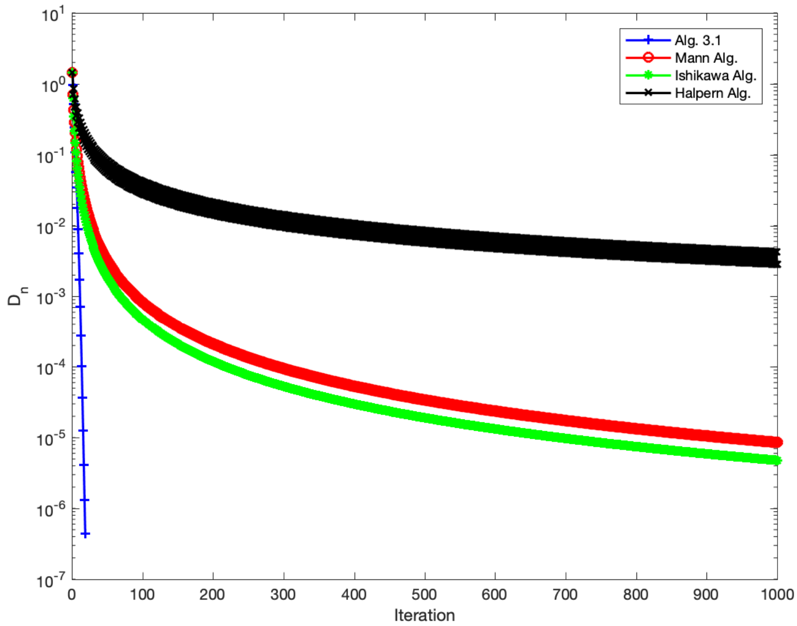


Figure 4. Numerical behavior of $\{D_n\}$ for Example 3.

Table 4. Computation results of Example 3.

Algorithm	Iteration	Time(s)
Algorithm 1	18	0.0015
Mann [11]	2930	0.0365
Ishikawa [12]	2190	0.0480
Halpern [11]	2630744	25.9806

Remark 7. (i) According to Tables 3 and 4, we can observe that all of the algorithms converge to a fixed point of the given nonexpansive mapping. The numerical results indicate that our proposed method performs better than the other three algorithms in terms of both the amount of time required to compute and the number of iterations.

(ii) Figures 3 and 4 depicts the convergence history of Algorithm 1, the Mann algorithm, the Ishikawa algorithm and the Halpern algorithm for Examples 2 and 3, respectively. It is noted from Figures 3 and 4 that the convergence rate of Algorithm 1 to a fixed point of the given nonexpansive mapping is quite faster than the convergence rate of the Mann, Halpern and Ishikawa methods.

6. Conclusions

The problem of finding fixed points of nonexpansive mappings on Hadamard manifolds is the subject of this article. Regarding the solution to this problem, an inertial Mann algorithm is suggested. The proposed method has been shown to converge under certain assumptions. The effectiveness of the proposed method is shown by numerical examples. The proposed algorithm's convergence rate will be taken into account in future studies.

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Conflict of interest

The authors declare that they have no conflict interests.

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