Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(1): 2083–2092.
DOI:10.3934/math.2023107
Received: 16 August 2022
Revised: 13 October 2022
Accepted: 19 October 2022
Published: 27 October 2022

Research article

L_∞-norm minimum distance estimation for stochastic differential equations driven by small fractional Lévy noise

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Abstract: This paper is concerned with L_∞-norm minimum distance estimation for stochastic differential equations driven by small fractional Lévy noise. By applying the Gronwall-Bellman lemma, Chebyshev's inequality and Taylor's formula, the minimum distance estimator is established and the consistency and asymptotic distribution of the estimator are derived when a small dispersion coefficient $\varepsilon \to 0$.

Keywords: L_∞-norm minimum distance estimation; stochastic differential equations; small fractional Lévy noise; consistency, asymptotic distribution
Mathematics Subject Classification: 60H10, 62F12

1. Introduction

Almost all systems are affected by noise and exhibit certain random characteristics. Therefore, it is reasonable and interesting to use random systems to model actual systems. When modeling or optimizing a stochastic system, due to the complexity of the internal structure and the uncertainty of the external environment, parameters of the system are unknown. It is necessary to use theoretical tools to estimate the parameters of the system. In the past few years, some authors studied the parameter estimation problem for stochastic models ([1, 2]). For example, Ji et al. ([3]) considered the parameter estimation problems of two-input single-output Hammerstein finite impulse response systems. Prakasa Rao ([4]) discussed estimation of parameters for models governed by a stochastic differential equation driven by a mixed fractional Brownian motion with Gaussian random effects based on discrete observations. Wang et al. ([5]) studied the parameter estimation issues of a class of multivariate output-error systems. Xu et al. ([6]) investigated the problem of parameter estimation for frequency response signals. When the system is observed partially, Wei ([7]) analyzed state and parameter estimation for nonlinear stochastic systems by applying extended Kalman filtering. Wei ([8]) discussed the strong consistency and asymptotic normality of the maximum likelihood estimator for partially observed stochastic differential equations driven by fractional Brownian motion. Zhang and Ding ([9]) designed a state filter for a time-delay state-space system with unknown parameters from noisy observation information. The parameter estimation for diffusion processes with small noise is well developed as well ([10–12]).

In practical applications, most of the system noise is non-Gaussian. Non-Gaussian noise can more accurately reflect the practical random perturbation. Therefore, fractional Lévy noise, as a kind of important non-Gaussian noise, has attracted many authors' attention ([13, 14]). For instance, Bishwal ([15]) analyzed the quasi-likelihood estimator of the drift parameter in the stochastic partial differential equations driven by a cylindrical fractional Lévy process when the process is observed at the arrival times of a Poisson process. Prakasa Rao ([16]) discussed nonparametric estimation of the linear multiplier in a trend coefficient in models governed by a stochastic differential equation driven by a fractional Lévy process with small noise. Xu et al. ([17]) used an integral transform method to investigate an averaging principle for fractional stochastic differential equations with Lévy motion. Yang ([18]) studied the existence and uniqueness of (weighted pseudo) almost automorphic solutions in distribution for fractional stochastic differential equations driven by Lévy noise.

The minimum distance methodology can be applied to the estimation of locally stationary moving average processes. This novel approach allows for the analysis of time series data exhibiting non-stationary behavior. The main advantages of this method are that it does not depend on the distribution of the process, can handle missing data and is computationally efficient. Some authors studied minimum distance estimation and use the method to estimate the parameter for stochastic differential equations. For example, Chen et al. ([19]) derived new estimators for the generalized Pareto distribution by the minimum distance estimation and the M-estimation in the linear regression. Hajargasht and Griffiths ([20]) described the efficient methods of estimation and inference based on two data generating mechanisms and derived several results useful for comparing the two methods of inference. Vicuna et al. ([21]) investigated some large sample properties of the new estimator, established its consistency and asymptotic normality. Although minimum distance estimation has been used by some authors to study parameter estimation problem, the stochastic differential equations are driven by Gaussian noise. As Non-Gaussian noise can more accurately reflect the practical random perturbation. Moreover, the financial empirical research showed that volatility in financial asset prices shows long-range dependence and self-similarity and the fractional Lévy noise could be used to exhibit these properties. Therefore, it is necessary to investigate the stochastic system driven by fractional Lévy noise. Inspired by the aforementioned works, in this paper, we consider L_∞-norm minimum distance estimation for stochastic differential equations driven by small fractional Lévy noise. The minimum distance estimator is established, the consistency and asymptotic distribution of the estimator are derived when a small dispersion coefficient $\varepsilon \to 0$.

The paper is organized as follows. In Section 2, we define the minimum distance estimator and give some assumptions. In Section 3, we give some lemmas and derive the consistency and asymptotic distribution of the estimator. The conclusion is given in Section 4.

2. Problem formulation and preliminaries

Definition 1. *([22]) Let $L = (L(t))_{t\in\mathbb{R}}$ be a zero-mean two sided Lévy process with $\mathbb{E}[L(1)^2] < \infty$ and without a Brownian component. For fractional integration parameter $d \in (0, \frac{1}{2})$, a stochastic process*

$$L_t^d := \frac{1}{\Gamma(d+1)} \int_{-\infty}^{\infty} [(t-s)_+^d - (-s)_+^d] L(ds), \quad t \in \mathbb{R},$$

is called a fractional Lévy process, where $x_+ = x \vee 0$.

In this paper, we consider the following stochastic differential equations driven by small Lévy noise:

$$\begin{cases} dX_t = f(X_t, \theta)dt + \varepsilon dL_t^d, \quad t \in [0, T], \\ X_0 = x_0, \end{cases} \tag{2.1}$$

where $\theta \in \Theta$ is an unknown parameter, Θ is an open bounded set in R^d, $d \geq 1$, $\varepsilon \in (0, 1]$.

Let P_θ^ε be the probability measure induced by the process $\{X_t, 0 \leq t \leq T\}$.

Let $x_t(\theta)$ be the solution of the differential equation:

$$dx_t = f(x_t, \theta)dt, \quad t \in [0, T]. \tag{2.2}$$

Suppose the following conditions hold:

Assumption 1. $|f(x, \theta)| \leq K(1 + |x|)$ *for all $t \in [0, T]$ where $K > 0$ is constant.*

Assumption 2. $|f(x, \theta) - f(y, \theta)| \leq K_1(|x - y|)$ *for all $t \in [0, T]$ where $K_1 > 0$ is constant.*

Assumption 3. *For any $\eta > 0$, $a(\eta) = \inf_{|\theta-\theta_0|\geq\eta} \sup_{0\leq t\leq T} |x_t(\theta) - x_t(\theta_0)| > 0$, where θ_0 is the true value of the parameter, $x_t(\theta)$ is the solution of (2.2).*

Remark 1. *It is known that under the Assumptions 1–2, there exists a unique solution of (2.1).*

Define the minimum distance estimator

$$\theta_\varepsilon^* = \arg\min_{\theta\in\Theta} \sup_{0\leq t\leq T} |X_t - x_t(\theta)|. \tag{2.3}$$

3. Main results

Before giving the theorems, we need to establish some preliminary results.

Lemma 1. *([22]) Let $|f|, |g| \in H$, H is the completion of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with respect to the norm $\|g\|_H^2 = \mathbb{E}[L^2(1)] \int_R (I_{-g}^d)^2(u)du$. Then,*

$$\mathbb{E}[\int_R f(s)dL_s^d \int_R g(s)dL_s^d] = \frac{\Gamma(1-2d)\mathbb{E}[L^2(1)]}{\Gamma(d)\Gamma(1-d)} \int_R \int_R f(t)g(s)|t-s|^{2d-1}dsdt.$$

Lemma 2. *([22]) For any $0 < b_2 \leq b_1 \leq a_1$, $0 < b_2 \leq a_2 \leq a_1$, and $b_1 - b_2 = a_1 - a_2$, there exists a constant C only depend on r and d, such that*

$$|\int_{b_2}^{b_1} \int_{a_2}^{a_1} e^{r(u+v)}|u-v|^{2d-1}dudv|$$

$$\leq \begin{cases} C|e^{r(a_1+b_1)} - e^{r(a_2+b_2)}||a_1 - b_2|^{2d}, & if r \neq 0, \\ C|a_1 - b_2|^{2d}, & if r = 0, \end{cases}$$

where r denotes a constant and d is the fractional integration parameter of fractional Lévy process.

Lemma 3. *([23]) (Gronwall-Bellman lemma) Let c_0, c_1 and c_2 be nonnegative constants, $\mu(t)$ be a nonnegative bounded function and $\nu(t)$ be a nonnegative integrable function on $[0, 1]$ such that*

$$\mu(t) \leq c_0 + c_1 \int_0^t \nu(s)\mu(s)ds + c_2 \int_0^t \nu(s)[\int_0^s \mu(t)dK(t)]ds,$$

where $K(\cdot)$ is a nondecreasing right continuous function with $0 \leq K(s) \leq 1$. Then

$$\mu(t) \leq c_0 \exp\{(c_1 + c_2) \int_0^t \nu(s)ds\}, 0 \leq t \leq 1.$$

In the following theorem, the consistency of the minimum distance estimator is proved.

Theorem 1. *Under Assumptions 1-3, when $\varepsilon \to 0$,*

$$\mathbb{P}^{\varepsilon}_{\theta_0}(|\theta^*_{\varepsilon} - \theta_0| \geq \eta) \leq \frac{2CT^{2d}K_1\varepsilon e^T}{a(\eta)}.$$

Proof. Note that

$$X_t - x_t(\theta_0) = \int_0^t (f(X_s, \theta_0) - f(x_s, \theta_0))ds + \varepsilon L^d_t. \tag{3.1}$$

Then,

$$\begin{aligned} &|X_t - x_t(\theta_0)| \\ &= |\int_0^t (f(X_s, \theta_0) - f(x_s, \theta_0))ds + \varepsilon L^d_t| \\ &\leq \int_0^t |f(X_s, \theta_0) - f(x_s, \theta_0)|ds + \varepsilon|L^d_t|. \end{aligned}$$

By using the Gronwall-Bellman lemma and Assumption 2, it can be checked that

$$\sup_{0\leq t\leq T} |X_t - x_t(\theta_0)| \leq K_1\varepsilon e^T \sup_{0\leq t\leq T} |L^d_t|. \tag{3.2}$$

Let $||\cdot||$ be the uniform norm and

$$G_0 = G_0(\eta) = \{\omega : \inf_{|\theta-\theta_0|<\eta} ||X - x(\theta)|| < \inf_{|\theta-\theta_0|\geq\eta} ||X - x(\theta)||\}. \tag{3.3}$$

Thus, for all $\omega \in G_0$, the minimum distance estimator $\theta^*_{\varepsilon} \in \{\theta : |\theta - \theta_0| < \eta\}$.
Since

$$\inf_{|\theta-\theta_0|<\eta} ||x(\theta) - x(\theta_0)|| = 0, \tag{3.4}$$

together with (3.2) and Lemmas 1–2, we can obtain that

$$\mathbb{P}^{\varepsilon}_{\theta_0}(|\theta^*_{\varepsilon} - \theta_0| > \eta) = P^{\varepsilon}_{\theta_0}(G^c_0)$$

$$\begin{aligned}
&\leq \mathbb{P}^{\varepsilon}_{\theta_0}(\inf_{|\theta-\theta_0|<\eta}(\|X - x(\theta_0)\| + \|x(\theta) - x(\theta_0)\|) \\
&\geq \inf_{|\theta-\theta_0|\geq\eta}(\|x(\theta) - x(\theta_0)\| - \|X - x(\theta_0)\|)) \\
&\leq \mathbb{P}^{\varepsilon}_{\theta_0}(\|X - x(\theta_0)\| \geq a(\eta) - K_1\varepsilon e^T \sup_{0\leq t\leq T}|L^d_t|) \\
&\leq P(2K_1\varepsilon e^T \sup_{0\leq t\leq T}|L^d_t| \geq a(\eta)) \\
&\leq P(\sup_{0\leq t\leq T}|L_t| \geq \frac{a(\eta)}{2K_1\varepsilon e^T}).
\end{aligned}$$

Applying Chebyshev's inequality, we have

$$\begin{aligned}
&\mathbb{P}^{\varepsilon}_{\theta_0}(|\theta^*_\varepsilon - \theta_0| > \eta) \\
&\leq \mathbb{E}\sup_{0\leq t\leq T}|L^d_t|\frac{2K_1\varepsilon e^T}{a(\eta)} \\
&\leq \frac{2CT^{2d}K_1\varepsilon e^T}{a(\eta)},
\end{aligned}$$

where C is a constant and d is the fractional integration parameter of fractional Lévy process.

The proof is complete. □

Remark 2. *When $\varepsilon \to 0$, it is easy to check that $\theta^*_\varepsilon \xrightarrow{P} \theta_0$.*

We consider a special case to investigate the limit distribution of $\varepsilon^{-1}(\theta^*_\varepsilon - \theta_0)$.

We suppose that

$$f(X_t, \theta) = M(X_t, \theta) + \int_0^t N(X_s, \theta)ds, \tag{3.5}$$

where $M(x, \theta)$ and $N(x, \theta)$ have two continuous bounded derivatives with respect to x and θ.

Let $\dot{x}_t(\theta)$ denotes the vector of derivatives of $x_t(\theta)$ with respect to θ. It can be checked that the derivative exists.

It is supposed that

$$\inf_{\theta\in\Theta}\inf_{|e|=1}\sup_{0\leq t\leq T}(e, \dot{x}_t(\theta)\dot{x}_t(\theta)^T e) > 0, \tag{3.6}$$

where e is a unit vector in R^d and $(\cdot,\cdot)$ denotes the inner product.

We introduce a stochastic differential equation:

$$\begin{cases} dX^{(1)}_t = [M_x(X_t, \theta)X^{(1)}_t + \int_0^t N_x(X_s, \theta)X^{(1)}_s ds]dt + dL^d_t, \quad t \in [0, T], \\ X^{(1)}_0 = 0, \end{cases} \tag{3.7}$$

where M_x and N_x are the derivatives of $M(x, \theta)$ and $N(x, \theta)$ with respect to x.

Define $\zeta = \zeta(\theta_0)$ by the relation

$$\|X^{(1)} - (\zeta, \dot{x}(\theta_0))\| = \inf_{\mu\in R^d}\|X^{(1)} - (\mu, \dot{x}(\theta_0))\|. \tag{3.8}$$

It is assumed that (3.8) has a unique solution ζ with probability one.

Theorem 2. *Under Assumptions 1–3, when $\varepsilon \to 0$,*

$$\varepsilon^{-1}(\theta^*_\varepsilon - \theta_0) \xrightarrow{d} \zeta.$$

Proof. Let

$$\eta = \eta_\varepsilon = \varepsilon\lambda_\varepsilon \to 0, \tag{3.9}$$

where $\lambda_\varepsilon \to \infty$ when $\varepsilon \to 0$.

Note that $|\theta^*_\varepsilon - \theta_0| < \eta_\varepsilon$ whenever $\omega \in G_0$.

Let

$$H(\mu) = \sup_{0\le t\le T} |x_t(\theta_0 + \mu) - x_t(\theta_0)|^2. \tag{3.10}$$

It is obvious that

$$\sup_{0\le t\le T} |x_t(\theta_0 + \mu) - x_t(\theta_0) - (\mu, \dot{x}_t(\theta_0))| = O(|\mu|^2). \tag{3.11}$$

Define

$$k_0 = \inf_{|e|=1} \sup_{0\le t\le T} (e, \dot{x}_t(\theta)\dot{x}_t(\theta)^T e), \tag{3.12}$$

then $k_0 > 0$.

Hence, there exists a neighborhood V of zero such that

$$\inf_{\mu\in V} \frac{H(\mu)}{|\mu|^2} \ge \frac{1}{2}k_0, \tag{3.13}$$

and for $\mu \in V$

$$H(\mu) \ge \frac{1}{2}k_0|\mu|^2. \tag{3.14}$$

According to Assumption 3, we have $H(\mu) > 0$ for $\mu \notin V$. Thus, for all $\mu \in \Theta - \{\theta_0\}$, there exists $k > 0$ such that

$$H(\mu) \ge k|\mu|^2. \tag{3.15}$$

Then, we obtain

$$\inf_{|\mu|>\eta_\varepsilon} \sup_{0\le t\le T} |x_t(\theta_0 + \mu) - x_t(\theta_0)|^2 \ge k\eta_\varepsilon^2. \tag{3.16}$$

Thus,

$$a(\eta) \ge \sqrt{k}\eta_\varepsilon. \tag{3.17}$$

Together with (3.17) and Theorem 1, when $\varepsilon \to 0$, we have

$$\begin{aligned} &\mathbb{P}^\varepsilon_{\theta_0}(|\theta^*_\varepsilon - \theta_0| \ge \eta_\varepsilon) \\ &\le \frac{2CT^{2d}K_1\varepsilon e^T}{\sqrt{k}\eta_\varepsilon} \\ &= \frac{2CT^{2d}K_1 e^T}{\sqrt{k}\lambda_\varepsilon} \\ &\to 0. \end{aligned}$$

Let $\theta = \theta_0 + \varepsilon\mu$, we have

$$
\begin{aligned}
&\varepsilon^{-1}\|X - x(\theta)\| \\
&= \|\frac{X - x(\theta_0)}{\varepsilon} - \frac{x(\theta) - x(\theta_0)}{\varepsilon}\| \\
&= \|x^{(1)} - (\mu, \dot{x}(\theta_0)) - (\frac{x(\theta_0 + \varepsilon\mu) - x(\theta_0)}{\varepsilon} - (\mu, \dot{x}(\theta_0))) + (\frac{X - x(\theta_0)}{\varepsilon} - x^{(1)})\|.
\end{aligned}
$$

Applying Taylor's formula, we have

$$
\begin{aligned}
&\sup_{0\le t\le T} |\frac{x_t(\theta_0 + \varepsilon\mu) - x_t(\theta_0)}{\varepsilon} - (\mu, \dot{x}_t(\theta_0))| \\
&= \sup_{0\le t\le T} |(\mu, (\dot{x}_t(\theta_0^*) - \dot{x}_t(\theta_0)))| \\
&\le |\mu| \sup_{0\le t\le T} |\dot{x}_t(\theta_0^*) - \dot{x}_t(\theta_0)| \\
&\le C\varepsilon|\mu|^2,
\end{aligned}
$$

where C is a constant.

Thus, we have

$$
\sup_{|\mu|\le\lambda_\varepsilon} \sup_{0\le t\le T} |\frac{x_t(\theta_0 + \varepsilon\mu) - x_t(\theta_0)}{\varepsilon} - (\mu, \dot{x}_t(\theta_0))| \le C\varepsilon\lambda_\varepsilon^2. \tag{3.18}
$$

Then, we obtain

$$
\begin{aligned}
&|\frac{X_t - x_t(\theta_0)}{\varepsilon} - x_t^{(1)}| \\
&= |\int_0^t [\frac{f_\eta(X_t, \theta_0) - f_\eta(x_t, \theta_0)}{\varepsilon} - M_x(x_\eta, \theta_0)x_\eta^{(1)} - \int_0^\eta N_x(x_h, \theta_0)x_h^{(1)}dh]d\eta| \\
&\le \int_0^t |\frac{M(X_s, \theta_0) - M(x_s, \theta_0)}{\varepsilon} - M_x(x_s, \theta_0)x_s^{(1)}|ds \\
&+ \int_0^t \int_0^s |\frac{N(X_s, \theta_0) - N(X_\eta, \theta_0)}{\varepsilon} - N_x(x_\eta, \theta_0)x_\eta^{(1)}|d\eta ds \\
&\le \int_0^t |M_x(\widetilde{X}_s, \theta_0)\frac{(X_s - x_s)}{\varepsilon} - M_x(x_s, \theta_0)x_s^{(1)}|ds \\
&+ \int_0^t \int_0^s |N_x(\widehat{X}_\eta, \theta_0)\frac{(X_\eta - x_\eta)}{\varepsilon} - N_x(x_\eta, \theta_0)x_\eta^{(1)}|d\eta ds \\
&\le \int_0^t |M_x(\widetilde{X}_s, \theta_0)||\frac{(X_s - x_s)}{\varepsilon} - x_s^{(1)}|ds + \int_0^t |M_x(\widetilde{X}_s, \theta_0) - M_x(x_s, \theta_0)||x_s^{(1)}|ds \\
&+ \int_0^t \int_0^s |N_x(\widehat{X}_\eta, \theta_0)||\frac{(X_\eta - x_\eta)}{\varepsilon} - x_\eta^{(1)}|d\eta ds \\
&+ \int_0^t \int_0^s |N_x(\widehat{X}_\eta, \theta_0) - N_x(x_\eta, \theta_0)||x_\eta^{(1)}|d\eta ds \\
&\le L_1 \int_0^t |\frac{(X_s - x_s(\theta_0))}{\varepsilon} - x_s^{(1)}|ds + L_2 \int_0^t \int_0^s |\frac{(X_\eta - x_\eta(\theta_0))}{\varepsilon} - x_\eta^{(1)}|d\eta ds \\
&+ L_3\varepsilon \sup_{0\le t\le T} |L_t^d| \sup_{0\le t\le T} |x_t^{(1)}|,
\end{aligned}
$$

where L_1, L_2, L_3 are positive constants.

Then, we have

$$\sup_{0\le t\le T}|\frac{X_t - x_t(\theta_0)}{\varepsilon} - x_t^{(1)}| \le C\varepsilon \sup_{0\le t\le T}|L_t^d|^2. \tag{3.19}$$

Therefore,

$$\begin{aligned}
&\sup_{|\mu|<\lambda_\varepsilon}\sup_{0\le t\le T}|\frac{X_t - x_t(\theta_0+\varepsilon\mu)}{\varepsilon} - (x_t^{(1)} - (\mu, \dot{x}_t(\theta_0)))| \\
&\le \sup_{|\mu|<\lambda_\varepsilon}\sup_{0\le t\le T}\{|\frac{X_t - x_t(\theta_0)}{\varepsilon} - x_t^{(1)}| + |\frac{x_t(\theta_0+\varepsilon\mu) - x_t(\theta_0)}{\varepsilon} - (\mu, \dot{x}_t(\theta_0))|\} \\
&\le C\varepsilon \sup_{0\le t\le T}|L_t^d|^2 + C\varepsilon\lambda_\varepsilon^2.
\end{aligned}$$

When $\varepsilon \to 0$ and $\varepsilon\lambda_\varepsilon^2 \to 0$, we have

$$\sup_{|\mu|<\lambda_\varepsilon}\frac{\|X - x(\theta_0+\varepsilon\mu)\|}{\varepsilon} - \|x^{(1)} - (\mu, \dot{x}(\theta_0))\| \xrightarrow{d} \zeta. \tag{3.20}$$

The proof is complete.

□

4. Conclusions

The aim of this paper is to study L_∞-norm minimum distance estimation for stochastic differential equations driven by small fractional Lévy noise. The consistency and asymptotic distribution of the estimator have been investigated by applying the Gronwall-Bellman lemma, Chebyshev's inequality and Taylor's formula. Further research topics will include minimum distance estimation for partially observed stochastic differential equations driven by small fractional Lévy noise.

Acknowledgments

This work was supported in part by the key research projects of universities under Grant 22A110001.

Conflict of interest

The authors declare that there are no conflicts of interest.

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