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Research article

On Cerone's and Bellman's generalization of Steffensen's integral inequality via conformable sense

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Abstract: By making use of the conformable integrals, we establish some new results on Cerone's and Bellman's generalization of Steffensen's integral inequality. In fact, we provide a variety of generalizations of Steffensen's integral inequality by using conformable calculus.

Keywords: conformable derivative and integral; convex functions; Steffensen's, Cerone's, Belman's, Hayashi's and Jensen's inequalities
Mathematics Subject Classification: 26D15, 26A33

1. Introduction

Fractional calculus is considered an attractive part of mathematical analysis due to its extensive application in a large range of fields [13–20, 22].

Recently, Khalil et al. [10], Katugampola [7], and Abdeljawad [12] defined novel well-behaved and simple like-wise fractional derivatives, referred to as the conformable derivatives, depending on the basic limit the definition of the derivative.

These derivatives seem to be similar in properties to the classical derivatives which makes them attractive for scientists to do research in many fields. Applications using conformable derivatives were spreading into fractional differential equations [21], inequalities [1–3, 8, 9], and other disciplines.

The integral inequality which compares the integrals (areas) over a whole interval and the integrals over a subset of this interval was established by J. F. Steffensen in 1918 (see [5]). Namely, if a and

b are real numbers such that $a < b$, f and g are integrable functions from $[a,b]$ into $\mathbb{R}$ such that f is monotone decreasing and for every $x \in [a,b]$, $0 \le g(x) \le 1$. Then

$$\int_{b-\lambda}^{b} f(t)\,dt \le \int_{a}^{b} f(t)\,g(t)\,dt \le \int_{a}^{a+\lambda} f(t)\,dt, \tag{1.1}$$

where $\lambda = \int_a^b g(t)\,dt$.

Bellman and other authors (see [11]) obtained some generalizations from Steffensen's inequality in L^p space. Whereas, Cerone [6] developed some generalizations of Steffensen's inequality which are dependent on a subinterval of the original interval. On the other hand, Mitrinovic et al. [4] obtained many integral inequalities related to Steffensen's inequality and Hayashi's inequality.

The convex and concave functions behaviors play an effective role in establishing a considerable number of useful inequalities (see the monograph [11] and references therein).

Recently, some authors have considered many generalizations of the Steffensen's inequality. For example, Abu Hany et al. [1–3] investigated many properties and extensions of Cerone's generalization of Steffensen's inequality as well as Hardy-Hilbert's type integral inequalities. Whereas, Sarikaya et al. [9] established some generalizations of Steffensen's type inequalities using the conformable integral.

In addition, Anderson et al. [8] have introduced Steffensen's inequality in the sense of conformable derivative and obtained some interested generalizations on these inequalities.

Motivated by these ideas, we investigate some new results concerning Cerone and Bellman's generalizations of Steffensen's inequality in the conformable sense. The article can be summarized as follows:

In section 2, we introduce some preliminaries of conformable derivative and basic inequalities. The Cerone's generalization of Steffensens' inequality via conformable integrals are discussed in section 3. The conformable mean is used in a connection with Steffensens' inequality in section 4. The last section is devoted to the results concerning the Bellman's generalization of Steffensens inequality using convex and concave functions.

2. Preliminaries

Some preliminary notes about conformable integral and derivative is recalled in this section. In addition, we introduce the basic inequalities that are related to Steffensens' inequality and facts about convex and concave functions.

Definition 2.1. *[10, 12] The conformable derivative of order $\alpha \in (0,1]$ of a function $f : [0,\infty) \to \mathbb{R}$ is defined by*

$$D^{\alpha}(f)(t) = \lim_{\epsilon \to 0} \frac{f(t+\epsilon t^{1-\alpha}) - f(t)}{\epsilon}, t > 0,$$

and

$$D^{\alpha}(f)(0) = \lim_{t \to 0^{+}} D^{\alpha}(f)(t).$$

If this limit exists, then f is said to be α-differentiable on $[0,\infty)$.

Definition 2.2. *[10, 12] The conformable integral of order $\alpha \in \mathbb{R}$ of a real-valued function f defined on $(a, \ t]$ is given by*

$$I_a^{\alpha}(f)(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx,$$

provided that this improper Riemann integral exists.

Definition 2.3. *[10, 12] Let $\alpha \in (0, \ 1]$ and $0 \leq a < b$. A function $f : [a, \ b] \to \mathbb{R}$ is α-conformable integrable on $[a, \ b]$ if the integral*

$$\int_a^b f(t) \, d_\alpha t := \int_a^b f(t) \ t^{\alpha-1} dt,$$

exists and finite.

Now, we introduce some basic inequalities in a connection of Steffensens' inequality.

Lemma 2.4. *[9] Let $\alpha \in (0, \ 1]$, $a, b \in \mathbb{R}$ with $0 \leq a < b$, and $g, h : [a, b] \to \mathbb{R}$, be α- integrable functions such that $0 \leq g \leq h$. If*

$$\lambda_\alpha = \frac{(b-a)}{\int_a^b h(t) \, d_\alpha t} \int_a^b g(t) \, d_\alpha t, \tag{2.1}$$

then $\lambda_\alpha \in [0, b-a]$ and

$$\int_{b-\lambda_\alpha}^b h(t) d_\alpha t \leq \int_a^b g(t) \, d_\alpha t \leq \int_a^{a+\lambda_\alpha} h(t) d_\alpha t. \tag{2.2}$$

This result generalizes that one obtained by Anderson et al. [8]. The next result is the conformable version of Hayashi's inequality which is a generalization of (2.2).

Theorem 2.5. *[9] Let $\alpha \in (0, \ 1]$, and $a, b \in \mathbb{R}$ with $0 \leq a < b$. Let $f, g, h : [a, b] \to \mathbb{R}$, be α-conformable integrable functions, such that $0 \leq g \leq h$.*

(i) If f is non-negative and monotone decreasing, then

$$\int_{b-\lambda_\alpha}^b h(t) f(t) \, d_\alpha t \leq \int_a^b f(t) \, g(t) \, d_\alpha t \leq \int_a^{a+\lambda_\alpha} h(t) f(t) \, d_\alpha t, \tag{2.3}$$

where λ_α is given by (2.1).

(ii) If f is non-positive and non-decreasing, then the inequalities in (2.3) are reversed.

Remark 2.6. *If $h = 1$ in (2.3), the result is reduced to the conformable version of Steffensen's inequality (1.1).*

Lemma 2.7. *Let $\alpha \in (0, \ 1]$ and $f, g : [a \ , \ b] \to \mathbb{R}$, be α-conformable integrable functions on $[a, b]$, such that f be nonnegative and monotone decreasing and $0 \leq g(t) \leq 1$. If $0 \leq \gamma \leq b-a$, then*

$$\int_{b-\gamma}^b f(t) d_\alpha t + \int_a^{b-\gamma} [f(t) - f(b)] \, g(t) d_\alpha t \tag{2.4}$$
$$\leq \int_a^b f(t) d_\alpha t - \left[\frac{(b-\gamma)^\alpha - a^\alpha}{\alpha}\right] f(b).$$

Proof. It is obvious, since

$$\begin{aligned}
& \int_{b-\gamma}^{b} f(t)d_\alpha t + \int_a^{b-\gamma} [f(t) - f(b)]\, g(t)d_\alpha t \\
\leq\ & \int_{b-\gamma}^{b} f(t)d_\alpha t + \int_a^{b-\gamma} f(t)d_\alpha t - f(b) \int_a^{b-\gamma} d_\alpha t \\
\leq\ & \int_a^b f(t)d_\alpha t - \left[\frac{(b-\gamma)^\alpha - a^\alpha}{\alpha}\right] f(b)\,.
\end{aligned}$$

This finishes the proof. □

Remark 2.8. *If $\alpha = 1$, we get the result as in [2].*

Definition 2.9. *A function φ is called convex if*

$$\varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y),$$

for $t \in [0,1]$, $x, y \in [a,b]$. If the inequality reverse, then φ is a concave function.

Remark 2.10. *The Jensen's inequality for a convex function φ is given as*

$$\varphi\left(\int_a^b f(t)\, d_\alpha\, t\right) \leq \int_a^b \varphi(f(t))\, d_\alpha t,$$

for any α-conformable integrable function f. The inequality is reversed if the function φ is concave. As example of a convex function is $\varphi(x) = x^p$, $p \geq 1, x \geq 0$, and it is concave if $0 < p \leq 1$. In these two cases, the Jensen's inequalities are given respectively

$$\left(\int_a^b f(t)\, d_\alpha\, t\right)^p \leq \int_a^b (f(t))^p\, d_\alpha t, p \geq 1,$$

and

$$\int_a^b (f(t))^p\, d_\alpha t \leq \left(\int_a^b f(t)\, d_\alpha\, t\right)^p, 0 < p \leq 1.$$

The Cerone's generalization of Steffensens' inequality is given by the next result.

Lemma 2.11. *[6] Let $f, g : [a, b] \to \mathbb{R}$, be integrable functions on $[a,b]$. Further, let $[c,\ d] \subseteq [a,\ b]$ such that $\int_a^b g(t)\, dt = d - c$. Then*

$$\begin{aligned}
& \int_c^d f(t)\, dt - \int_a^b f(t) g(t)\ dt \\
=\ & \int_a^c [f(d) - f(t)]\, g(t)\, dt + \int_c^d [f(t) - f(d)]\, [1 - g(t)]\ dt \\
& + \int_d^b [f(d) - f(t)]\, g(t)\, dt,
\end{aligned}$$

and

$$\int_a^b f(t) g(t)\ dt - \int_c^d f(t)\, dt$$

$$= \int_a^c [f(t) - f(c)]\, g(t)\, dt + \int_c^d [f(c) - f(t)]\, [1 - g(t)]\; dt$$
$$+ \int_d^b [f(t) - f(c)]\, g(t)\, dt.$$

The Bellman's generalization of Steffensen's inequality is given by the next result.

Theorem 2.12. *[11] Assume that $f \in L^p[a, b]$, $p > 1$, is nonnegative monotone decreasing, and g is nonnegative in $[a, b]$, such that $\int_a^b (g(t))^q\, dt \leq 1, q = \frac{p}{p-1}$. Then*

$$\int_a^{a+\eta} (f(t))^p\, dt \geq \left(\int_a^b f(t)g(t)dt \right)^p,$$

where $\eta = \left(\int_a^b g(t)dt \right)^p$.

3. Cerone's generalization of Steffensen's inequality via α-conformable integrable functions

In this section, we introduce the α-conformable version of Cerone's generalization of Steffensen's inequality [6]. Namely, we extend the ideas of Steffensen's inequality to finite number of subintervals of $[a,\; b]$.

Lemma 3.1. *Let $\alpha \in (0,\; 1]$ and $f, g : [a\;,\; b] \to \mathbb{R}$, be α-conformable integrable functions on $[a, b]$. Further, let $[c,\; d] \subseteq [a,\; b]$ such that $\alpha \int_a^b\; g(t)\, d_\alpha t = d^\alpha - c^\alpha$. Then*

$$\int_c^d f(t)\, d_\alpha t - \int_a^b f(t) g\,(t)\; d_\alpha t \tag{3.1}$$
$$= \int_a^c [f(d) - f(t)]\, g(t)\, d_\alpha t + \int_c^d [f(t) - f(d)]\, [1 - g(t)]\; d_\alpha t$$
$$+ \int_d^b [f(d) - f(t)]\, g(t)\, d_\alpha t,$$

and

$$\int_a^b f(t) g\,(t)\; d_\alpha t - \int_c^d f(t)\, d_\alpha t \tag{3.2}$$
$$= \int_a^c [f(t) - f(c)]\, g(t)\, d_\alpha t + \int_c^d [f(c) - f(t)]\, [1 - g(t)]\; d_\alpha t$$
$$+ \int_d^b [f(t) - f(c)]\, g(t)\, d_\alpha t.$$

Proof. Let $a \leq c < d \leq b$, and

$$\lambda_{c,a}^{d,b} = \int_c^d f(t)\, d_\alpha t - \int_a^b f(t) g\,(t)\; d_\alpha t. \tag{3.3}$$

Then

$$\begin{aligned}\lambda_{c,a}^{d,b} &= \int_c^d [1-g(t)]\, f(t)\, d_\alpha t - \left[\int_a^c f(t)g(t)\, d_\alpha t + \int_d^b f(t)g(t)\, d_\alpha t\right] \\ &= \int_c^d [1-g(t)]\,[f(t)-f(d)]\, d_\alpha t + f(d)\int_c^d [1-g(t)]\ d_\alpha t \\ &\quad + \int_a^c [f(d)-f(t)]\ g(t)\, d_\alpha t - f(d)\int_a^c g(t)\, d_\alpha t \\ &\quad + \int_d^b [f(d)-f(t)]\ g(t) d_\alpha t - f(d)\int_d^b g(t)\, d_\alpha t.\end{aligned}$$

The identity (3.1) is readily obtained on noting that

$$f(d)\left[\int_c^d d_\alpha t - \int_a^b g(t)\, d_\alpha t\right] = 0.$$

Similar arguments can be applied to obtain (3.2) using (3.1) and (3.3). This finishes the proof. □

Remark 3.2. *If $\alpha = 1$, we obtain a result as in [6]. Furthermore, if $a = d$, and $\alpha = 1$, then the result agrees with the obtained by Mitrinovic [4].*

Theorem 3.3. *Let $\alpha \in (0,\ 1]$ and $f, g : [a\ ,\ b] \to \mathbb{R}$, $b > a \geq 0$, be α-conformable integrable functions on $[a, b]$, and f be monotone decreasing. Further, let $0 \leq g(t) \leq 1$ and $[c_i,\ d_i] \subseteq [a,\ b]$ such that $\alpha\int_a^b g(t)\, d_\alpha t = d_i^\alpha - c_i^\alpha$, $i = 1, 2$. Then*

$$\int_{c_2}^{d_2} f(t)\, d_\alpha t - \underline{\mu}_2 \leq \int_a^b f(t)g(t)\, d_\alpha t \leq \int_{c_1}^{d_1} f(t)\, d_\alpha t + \overline{\mu}_1, \tag{3.4}$$

where

$$\overline{\mu}_1 = \int_a^{c_1} [f(t) - f(d_1)]\ g(t)\, d_\alpha t \geq 0,$$

and

$$\underline{\mu}_2 = \int_{d_2}^b [\,f(c_2) - f(t)]\ g(t)\, d_\alpha t \geq 0.$$

Proof. It is obvious that $\overline{\mu}_1$, and $\underline{\mu}_2$ are nonnegative. In accordance with (3.1), we obtain

$$\begin{aligned}&\int_{c_1}^{d_1} f(t)\, d_\alpha t - \int_a^b f(t)g(t)\ d_\alpha t + \int_a^{c_1} [f(t) - f(d_1)\,]\, g(t)\, d_\alpha t \\ = &\int_{c_1}^{d_1} [f(t) - f(d_1)\,]\,[1 - g(t)]\, d_\alpha t + \int_{d_1}^b [f(d_1) - f(t)\,]\, g(t)\, d_\alpha t \geq 0.\end{aligned}$$

Hence

$$\int_a^b f(t)\, g(t)\, d_\alpha t \leq \int_{c_1}^{d_1} f(t)\, d_\alpha t + \int_a^{c_1} [f(t) - f(d_1)\,]\, g(t)\, d_\alpha t,$$

and thus, the right inequality of (3.4) is valid. Now, using (3.2), we get

$$\begin{aligned} &\int_a^b f(t)g(t)\, d_\alpha t - \int_{c_2}^{d_2} f(t)\, d_\alpha t + \int_{d_2}^{b} [f(c_2) - f(t)]\, g(t)\, d_\alpha t \\ = &\int_{c_2}^{d_2} [f(c_2) - f(t)]\,[1 - g(t)]\, d_\alpha t + \int_a^{c_2} [f(t) - f(c_2)]\, g(t)\, d_\alpha t \geq 0. \end{aligned}$$

Thus

$$\int_a^b f(t)g(t)\, d_\alpha t \geq \int_{c_2}^{d_2} f(t) d_\alpha t - \int_{d_2}^{b} [f(c_2) - f(t)]\, g(t)\, d_\alpha t,$$

giving the left inequality of (3.4). This finishes the proof. □

Remark 3.4. *The following special cases are easily obtained:*

(i) If $g(t) = 0$, the inequality (3.4) reduces to $-\underline{\mu} \leq 0 \leq \overline{\mu}$, and if $g(t) = 1$, the inequality (3.4) reduces to $\int_a^b f(t)\, d_\alpha t \leq \int_a^b f(t) d_\alpha t \leq \int_a^b f(t)\, d_\alpha t$, where $\overline{\mu} = \underline{\mu} = 0$, $d_2 = d_1 = b$, and $a = c_1 = c_2$.

(ii) Put $\alpha = 1$, $d_2 = b$ and $c_1 = a$ in Theorem 3.3, then $\overline{\mu} = \underline{\mu} = 0$, $c_2 = b - \int_a^b g(t)\, dt$, and $d_1 = a + \int_a^b g(t)\, dt$, we get the original Steffensen's inequality (1.1).

(iii) We can obtain many copies of (3.4) such as

$$\int_{c_1}^{d_1} f(t)\, d_\alpha t - \underline{\mu}_1 \leq \int_a^b f(t)g(t)\, d_\alpha t \leq \int_{c_2}^{d_2} f(t)\, d_\alpha t + \overline{\mu}_2,$$

and

$$\begin{aligned} &\int_{c_1}^{d_1} f(t)\, d_\alpha t + \int_{c_2}^{d_2} f(t)\, d_\alpha t - \left(\underline{\mu}_1 + \underline{\mu}_2\right) \\ \leq\ & 2\int_a^b f(t)g(t)\, d_\alpha t \leq \int_{c_1}^{d_1} f(t)\, d_\alpha t + \int_{c_2}^{d_2} f(t)\, d_\alpha t + (\overline{\mu}_1 + \overline{\mu}_2)\,. \end{aligned}$$

Theorem 3.5. *Let the conditions of Theorem 3.3 hold. Then*

$$\int_{c_2}^{b} f(t)\, d_\alpha t - \left(\frac{b^\alpha - {d_2}^\alpha}{\alpha}\right) f(c_2) \leq \int_a^b f(t)g(t)\, d_\alpha t \leq \int_a^{d_1} f(t) d_\alpha t - \frac{({c_1}^\alpha - a^\alpha)}{\alpha} f(d_1). \tag{3.5}$$

Proof. The result of Theorem 2.7 and the fact that $0 \leq g(t) \leq 1$, give

$$\begin{aligned} \underline{\mu} &= \int_{d_2}^{b} [\, f(c_2) - f(t)]\, g(t)\, d_\alpha t \\ &\leq \int_{d_2}^{b} [\, f(c_2) - f(t)]\, d_\alpha t \\ &= \left(\frac{b^\alpha - d_2^\alpha}{\alpha}\right) f(c_2) - \int_{d_2}^{b} f(t) d_\alpha t. \end{aligned}$$

Hence

$$\int_{c_2}^{d_2} f(t)d_\alpha t - \underline{\mu} \geq \int_{c_2}^{d_2} f(t)d_\alpha t + \int_{d_2}^{b} f(t)d_\alpha t - \left(\frac{b^\alpha - d_2^\alpha}{\alpha}\right) f(c_2) = \int_{c_2}^{b} f(t)d_\alpha t - \left(\frac{b^\alpha - d_2^\alpha}{\alpha}\right) f(c_2).$$

The left inequality of (3.5) follows. Similarly, we obtain

$$0 \leq \overline{\mu} = \int_a^{c_1} [f(t) - f(d_1)]\, g(t)\, d_\alpha t \leq \int_a^{c_1} f(t)d_\alpha t - \left(\frac{c_1^\alpha - a^\alpha}{\alpha}\right) f(d_1),$$

which leads to

$$\int_{c_1}^{d_1} f(t)\, d_\alpha t + \overline{\mu} \leq \int_a^{d_1} f(t)g(t)\, d_\alpha t - \left(\frac{c_1^\alpha - a^\alpha}{\alpha}\right) f(d_1).$$

Hence the right inequality follows. This finishes the proof. □

Remark 3.6. *The following special cases are followed directly:*

(i) If $\alpha = 1$, we get the same result as in [6].

(ii) If we take $\alpha = 1$, $c_1 = a$, and $d_2 = b$, then $d_1 = a + \int_a^b g(t)\, dt$ and $c_2 = b - \int_a^b g(t)\, dt$. Thus (3.5) reduces to the original Steffensen's inequality (1.1).

A generalization of Theorem 3.3, with n subintervals are given in the next result. Actually, one can obtain many copies of the next bounds.

Theorem 3.7. *Let $\alpha \in (0, 1]$ and $f, g : [a, b] \to \mathbb{R}$, be α-conformable integrable functions on $[a, b]$ such that f is nonnegative and monotone decreasing and $0 \leq g(t) \leq 1$. If $[c_i, d_i] \subseteq [a, b]$, such that $\alpha \int_a^b g(t)\, d_\alpha t = d_i^\alpha - c_i^\alpha$, $i = 1, 2, \ldots, n$. Then*

$$\sum_{i=1}^{n} \int_{c_i}^{d_i} f(t)\, d_\alpha t - \underline{\mu}_n \leq n \int_a^b f(t)g(t)\, d_\alpha t \leq \sum_{i=1}^{n} \int_{c_i}^{d_i} f(t)\, d_\alpha t + \overline{\mu}_n, \tag{3.6}$$

where

$$\overline{\mu}_n = \sum_{i=1}^{n} \int_a^{c_i} [f(t) - f(d_i)]\, g(t)\, d_\alpha t \geq 0,$$

and

$$\underline{\mu}_n = \sum_{i=1}^{n} \int_{d_i}^{b} [f(c_i) - f(t)]\, g(t)\, d_\alpha t \geq 0.$$

Proof. It is obvious that $\overline{\mu}_n$ and $\underline{\mu}_n$ are nonnegative. Let

$$\lambda_{c_i,a}^{d_i,b} = \int_{c_i}^{d_i} f(t)\, d_\alpha t - \int_a^b f(t)g(t)\, d_\alpha t,$$

$i = 1, 2, ..., n$. In accordance with (3.1), we obtain

$$\begin{aligned}&\lambda_{c_i,a}^{d_i,b}+\int_a^{c_i}\left[f(t)-f(d_i)\right]\,g(t)d_\alpha t\\=&\int_{c_i}^{d_i}\left[f(t)-f(d_i)\right]\left[1-g(t)\right]\,d_\alpha t+\int_{d_i}^{b}\left[f(d_i)-f(t)\right]g(t)\,d_\alpha t\geq 0.\end{aligned}$$

Then, we obtain

$$\sum_{i=1}^{n}\int_{c_i}^{d_i}f(t)\,d_\alpha t\;-n\int_a^b f(t)g(t)\,d_\alpha t+\sum_{i=1}^{n}\int_a^{c_i}\left[f(t)-f(d_i)\right]\,g(t)d_\alpha t\;\geq 0.$$

Hence, the right side of (3.6) follows. Similar arguments can be applied as in (3.2) to obtain

$$\begin{aligned}&\int_a^b f(t)g\,(t)\;d_\alpha t-\int_{c_i}^{d_i}f(t)\,d_\alpha t+\int_{d_i}^{b}\left[f(c_i)-f(t)\right]g(t)\,d_\alpha t\\=&\int_a^{c_i}\left[f(t)-f(c_i)\right]g(t)\,d_\alpha t+\int_{c_i}^{d_i}\left[f(c_i)-f(t)\right]\left[1-g(t)\right]\,d_\alpha t\geq 0,\end{aligned}$$

for $i=1,2,...,n$. It follows that

$$n\int_a^b f(t)g(t)\,d_\alpha t-\sum_{i=1}^{n}\int_{c_i}^{d_i}f(t)\,d_\alpha t+\sum_{i=1}^{n}\int_{d_i}^{b}\left[\,f(c_i)-f(t)\right]\,g(t)\,d_\alpha t\geq 0.$$

Hence, the left side inequality of (3.6) follows. This finishes the proof. □

Lemma 3.8. *Let the conditions of Theorem 3.7 hold. Then*

$$\begin{aligned}&\sum_{i=1}^{n}\int_{c_i}^{b}f(t)\,d_\alpha t-\sum_{i=1}^{n}\left[\frac{b^\alpha-d_i{}^\alpha}{\alpha}\right]\,f\,(c_i)\\\leq\;&n\int_a^b f(t)g(t)\,d_\alpha t\leq\sum_{i=i}^{n}\int_a^{d_i}f(t)\,d_\alpha t-\sum_{i=1}^{n}\left[\frac{c_i{}^\alpha-a^\alpha}{\alpha}\right]f\,(d_i)\,.\end{aligned}\tag{3.7}$$

Proof. In virtue of Theorem 2.11 and the fact that $0\leq g\,(t)\;\leq 1$, we deduce that

$$\begin{aligned}0\;&\leq\;\underline{\mu}_n=\sum_{i=1}^{n}\int_{d_i}^{b}\left[\,f(c_i)-f(t)\right]\,g(t)\,d_\alpha t\\&\leq\;\sum_{i=1}^{n}\int_{d_i}^{b}\left[\,f(c_i)-f(t)\right]\,d_\alpha t=\sum_{i=1}^{n}\left(\frac{b^\alpha-d_i{}^\alpha}{\alpha}\right)f(c_i)-\sum_{i=1}^{n}\int_{d_i}^{b}f(t)d_\alpha t,\end{aligned}$$

which by (3.6) leads to

$$\begin{aligned}&n\int_a^b f(t)g(t)d_\alpha t\\\geq\;&\sum_{i=1}^{n}\int_{c_i}^{d_i}f(t)d_\alpha t-\underline{\mu}_n\end{aligned}$$

$$\geq \sum_{i=1}^{n}\int_{d_i}^{b} f(t)d_\alpha t + \sum_{i=1}^{n}\int_{c_i}^{d_i} f(t)d_\alpha t - \sum_{i=1}^{n}\left(\frac{b^\alpha - {d_i}^\alpha}{\alpha}\right)f(c_i)$$
$$= \sum_{i=1}^{n}\int_{c_i}^{b} f(t)d_\alpha t - \sum_{i=1}^{n}\left(\frac{b^\alpha - {d_i}^\alpha}{\alpha}\right)f(c_i).$$

Thus, the left inequality of (3.7) follows. Similarly, we can obtain

$$n\int_{a}^{b} f(t)g(t)\,d_\alpha t$$
$$\leq \sum_{i=1}^{n}\int_{c_i}^{d_i} f(t)\,d_\alpha t + \overline{\mu}_n$$
$$\leq \sum_{i=i}^{n}\int_{a}^{d_i} f(t)\,d_\alpha t - \sum_{i=1}^{n}\left[\frac{{c_i}^\alpha - a^\alpha}{\alpha}\right]f(d_i),$$

which gives the right inequality of (3.7). This finishes the proof. □

4. Steffensen's inequalities via the α-conformable integral mean

The idea of the α-conformable integral mean of f over a subinterval $[c,\ d] \subseteq [a,\ b]$ is introduced in this section. The α-conformable integral mean of the function f over $[c\,, d]$ is defined as [8]

$$\mu_\alpha(f; c, d) = \frac{\alpha}{d^\alpha - c^\alpha}\int_{c}^{d} f(t)\,d_\alpha t.$$

The classical integral mean of f over $[c\,, d]$ is $\mu_1(f; c, d) = \frac{1}{d-c}\int_c^d f(t)\,dt$.

The first result on the α-conformable integral mean of f over the $[c, d]$ is given next.

Lemma 4.1. *Assume that*

(B1) $f : [a\,, b] \to \mathbb{R}$ is differentiable,

(B2) $g : [a\,, b] \to \mathbb{R}$ is α- conformable integrable function on $[a, b]$, $\alpha \in (0,\ 1]$, and

(B3) $G\,(t) = \int g(t)\,d_\alpha t$ is α- conformable integrable function on $[a, b]$,

(B4) $[c, d] \subseteq [a, b]$, such that $\alpha\int_a^b g(t)\,d_\alpha t = d^\alpha - c^\alpha$. Then

$$\int_{a}^{b} f(t)g\,(t)\ d_\alpha t - \int_{c}^{d} f(t)\,d_\alpha t$$
$$= (f(b) - \mu_\alpha(f; c, d))\,(G\,(b) - G(a)) - \int_{a}^{b}(G(t) - G\,(a))\,df(t), \tag{4.1}$$

$$\begin{aligned}
&\int_c^d f(t)\,d_\alpha t-\int_a^b f(t)g\,(t)\,d_\alpha t\\
=\;&(\mu_\alpha(f;c,d)-f(a))\,(G\,(b)-G(a))+\int_a^b (G(t)-G(b))\,df(t),
\end{aligned}\tag{4.2}$$

and

$$\begin{aligned}
&\int_a^b f(t)\,d_\alpha t-\int_c^d f(t)g\,(t)\,d_\alpha t\\
=\;&\left[\int_a^c \frac{f(t)}{G\,(b)-G(a)}d_\alpha t+\int_d^b \frac{f(t)}{G\,(b)-G(a)}\,d_\alpha t-\,f(d)\right]G(d)\\
&+\left[f(c)-\int_a^c \frac{f(t)}{G\,(b)-G(a)}\,d_\alpha t-\int_d^b \frac{f(t)}{G\,(b)-G(a)}\,d_\alpha t\right]G(c)\\
&+\int_c^d G(t)df(t)+\mu_\alpha(f;c,d)\,(G\,(b)-G(a))\,.
\end{aligned}$$

Proof. It is obvious that $G\,(b)-G(a)=\int_a^b g(t)\,d_\alpha t\neq 0$, and $\alpha\,(G\,(b)-G(a))=\alpha\int_a^b g(t)\,d_\alpha t=d^\alpha-c^\alpha$. Define

$$L:=\int_a^b f(t)g\,(t)\,d_\alpha t-\int_c^d f(t)\,d_\alpha t.$$

Then, L can be rewritten as

$$\begin{aligned}
L\;&=\;\int_a^b f(t)g\,(t)\;d_\alpha t-\frac{\alpha}{(d^\alpha-c^\alpha)}\int_a^b g(t)\,d_\alpha t\int_c^d f(t)\,d_\alpha t\\
&=\;\int_a^b g\,(t)\,[f(t)-\mu_\alpha(f;c,d)]\;d_\alpha t.
\end{aligned}$$

Integration by parts gives

$$\begin{aligned}
L\;&=\;[f(b)-\mu_\alpha(f;c,d)]\,G(b)-[f(a)-\mu_\alpha(f;c,d)]\,G(a)\\
&\quad-\int_a^b G(t)d\,[f(t)-\mu_\alpha(f;c,d)]\\
&=\;f(b)G(b)-f(a)G(a)-\mu_\alpha(f;c,d)\,(G\,(b)-G(a))\\
&\quad-\int_a^b G(t)df(t)\\
&=\;(f(b)-\mu_\alpha(f;c,d))\,(G\,(b)-G(a))-\int_a^b (G(t)-G\,(a))\,df(t).
\end{aligned}$$

The second identity is readily obtained by using the fact

$$\int_c^d f(t)\,d_\alpha t-\int_a^b f(t)g\,(t)\,d_\alpha t=-L.$$

For the last identity, let

$$U := \int_c^d f(t)g(t)\,d_\alpha t - \int_a^b f(t)\,d_\alpha t,$$

then

$$\begin{aligned} U &= \int_c^d \left[f(t) - \frac{\alpha}{(d^\alpha - c^\alpha)} \int_a^c f(t)\,d_\alpha t - \frac{\alpha}{(d^\alpha - c^\alpha)} \int_d^b f(t)\,d_\alpha t \right] g(t)\,d_\alpha t \\ &\quad - \frac{\mu_\alpha(f;c,d)\,(d^\alpha - c^\alpha)}{\alpha}. \end{aligned}$$

Integration by parts gives

$$\begin{aligned} U &= \left[f(d) - \frac{\alpha}{(d^\alpha - c^\alpha)} \int_a^c f(t)\,d_\alpha t - \frac{\alpha}{(d^\alpha - c^\alpha)} \int_d^b f(t)\,d_\alpha t \right] G(d) \\ &\quad - \left[f(c) - \frac{\alpha}{(d^\alpha - c^\alpha)} \int_a^c f(t)\,d_\alpha t - \frac{\alpha}{(d^\alpha - c^\alpha)} \int_d^b f(t)\,d_\alpha t \right] G(c) \\ &\quad - \int_c^d G(t) df(t) - \frac{\mu_\alpha(f;c,d)\,(d^\alpha - c^\alpha)}{\alpha}. \end{aligned}$$

This finishes the proof. □

Theorem 4.2. *Assume that (B1)-(B3) hold, and*

(B5) $[c_i,\ d_i] \subseteq [a,\ b]$ such that $\alpha \int_a^b g(t)\,d_\alpha t = d_i^\alpha - c_i^\alpha$, $i = 1,\ 2$. Then

$$\begin{aligned} &\int_{c_2}^{d_2} f(t)\,d_\alpha t - (G(b) - G(a))\,(\mu_\alpha(f;c_2,d_2) - f(b)) \\ \leq\ &\int_a^b f(t)g(t)\,d_\alpha t \leq \int_{c_1}^{d_1} f(t)\,d_\alpha t + (G(b) - G(a))\,(f(a) - \mu_\alpha(f;c_1,d_1)), \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &\int_{c_1}^{d_1} f(t)\,d_\alpha t + \int_{c_2}^{d_2} f(t)\,d_\alpha t \\ &- (G(b) - G(a))\,(\mu_\alpha(f;c_1,d_1) + \mu_\alpha(f;c_2,d_2) - 2f(b)) \\ \leq\ &2\int_a^b f(t)g(t)\,d_\alpha t \leq \int_{c_1}^{d_1} f(t)\,d_\alpha t + \int_{c_2}^{d_2} f(t)\,d_\alpha t \\ &+ (G(b) - G(a))\,(2f(a) - \mu_\alpha(f;c_1,d_1) - \mu_\alpha(f;c_2,d_2)), \end{aligned}$$

which are equivalent to

$$f(b)\,(G(b) - G(a)) \leq \int_a^b f(x)g(x)\,d_\alpha x \leq f(a)\,(G(b) - G(a)). \tag{4.4}$$

Proof. The hypotheses indicate that

$$\int_a^b (G(t) - G(a))\, df(t) \le 0,$$

and

$$\int_a^b (G(t) - G(b))\, df(t) \ge 0.$$

Therefore, using (4.1) and the identity $\alpha\,(G(b) - G(a)) = d_i^\alpha - c_i^\alpha, i = 1, 2$, we obtain

$$\begin{aligned}
\int_a^b f(t) g(t)\; d_\alpha t &\ge \int_{c_i}^{d_i} f(t)\, d_\alpha t + (f(b) - \mu_\alpha(f; c_i, d_i))\,(G(b) - G(a)) \\
&= \int_{c_i}^{d_i} f(t)\, d_\alpha t + (G(b) - G(a))\, f(b) \\
&\quad - \frac{\alpha\,(G(b) - G(a))}{d_i^\alpha - c_i^\alpha} \int_{c_i}^{d_i} f(t)\, d_\alpha t \\
&= f(b)\,(G(b) - G(a)).
\end{aligned}$$

Thus, the left inequalities are obtained. Similarly, from (4.2), we obtain

$$\begin{aligned}
\int_a^b f(t) g(t)\, d_\alpha t &\le \int_{c_i}^{d_i} f(t)\, d_\alpha t + (G(b) - G(a))\,(f(a) - \mu_\alpha(f; c_i, d_i)) \\
&= \int_{c_i}^{d_i} f(t)\, d_\alpha t - \frac{\alpha\,(G(b) - G(a))}{d_i^\alpha - c_i^\alpha} \int_{c_i}^{d_i} f(t)\, d_\alpha t + f(a)\,(G(b) - G(a)) \\
&= f(a)\,(G(b) - G(a)).
\end{aligned}$$

Hence, the right inequalities are obtained. This finishes the proof. □

The inequality (4.4) can be obtained directly by using the facts that $f(b) = \inf\{f(t) : t \in [a, b]\}$, $f(a) = \sup\{f(t) : t \in [a, b]\}$, and $\int_a^b g(x)\, d_\alpha x = G(b) - G(a)$, by which we obtain

$$\left(\inf_{x\in[a,\,b]} f(x)\right) \int_a^b g(x)\, d_\alpha x \le \int_a^b f(x) g(x)\, d_\alpha x \le \left(\sup_{x\in[a,\,b]} f(x)\right) \int_a^b g(x)\, d_\alpha x.$$

If $g(t) \ge 0$ on $[a, b]$, then we can obtain (4.4) by noting that

$$\int_a^b g(t)\,[f(t) - f(b)]\; d_\alpha t \ge 0,$$

and

$$\int_a^b g(t)\,[f(a) - f(t)]\; d_\alpha x \ge 0.$$

An extension to Theorem 4.2 is given in the next result.

Theorem 4.3. *Assume that (B1)–(B3), and (B5) hold. If G is monotone increasing, then*

$$\begin{aligned} &\sum_{k=1}^{m}\int_{c_{i_k}}^{d_{i_k}} f(t)\,d_\alpha t-(G(b)-G(a))\left(\sum_{k=1}^{m}\mu_\alpha(f;c_{i_k},d_{i_k})-mf(b)\right) \\ \le\ & m\int_a^b f(t)g(t)\,d_\alpha t \\ \le\ & \sum_{k=1}^{m}\int_{c_{i_k}}^{d_{i_k}} f(t)\,d_\alpha t+(G(b)-G(a))\left(mf(a)-\sum_{k=1}^{m}\mu_\alpha(f;c_{i_k},d_{i_k})\right), \end{aligned} \tag{4.5}$$

or

$$\begin{aligned} &\frac{1}{m_1}\sum_{k=1}^{m_1}\int_{c_{i_k}}^{d_{i_k}} f(t)\,d_\alpha t-\frac{1}{m_1}(G(b)-G(a))\left(\sum_{k=1}^{m_1}\mu_\alpha(f;c_{i_k},d_{i_k})-m_1f(b)\right) \\ \le\ & \int_a^b f(t)g(t)\,d_\alpha t \\ \le\ & \frac{1}{m_2}\sum_{k=1}^{m_2}\int_{c_{j_k}}^{d_{j_k}} f(t)\,d_\alpha t+\frac{1}{m_2}(G(b)-G(a))\left(m_2f(a)-\sum_{k=1}^{m_2}\mu_\alpha(f;c_{j_k},d_{j_k})\right), \end{aligned} \tag{4.6}$$

where $\{i_1,i_2,...,i_{m_1}\}$, *and* $\{j_1,j_2,...,j_{m_2}\}$ *are subsets of* $\{1,2,...,n\}$. *Moreover, both of the inequalities (4.5) and (4.6) are equivalent to*

$$f(b)(G(b)-G(a))\le\int_a^b f(t)g(t)\,d_\alpha t\le f(a)(G(b)-G(a)).$$

Proof. In virtue of (4.3), we obtain

$$\begin{aligned} &\int_{c_i}^{d_i} f(t)\,d_\alpha t-(G(b)-G(a))(\mu_\alpha(f;c_i,d_i)-f(b)) \\ \le\ & \int_a^b f(t)g(t)\,d_\alpha t \\ \le\ & \int_{c_j}^{d_j} f(t)\,d_\alpha t+(G(b)-G(a))\left(f(a)-\mu_\alpha(f;c_j,d_j)\right), \end{aligned} \tag{4.7}$$

for all $i,j=1,2,...,n$. If $i=j$, we take the summation over any subset $\{i_1,i_2,...,i_m\}, m\le n$, of $\{1,2,...,n\}$, then

$$\begin{aligned} &\sum_{k=1}^{m}\int_{c_{i_k}}^{d_{i_k}} f(t)\,d_\alpha t-(G(b)-G(a))\left(\sum_{k=1}^{m}\mu_\alpha(f;c_{i_k},d_{i_k})-mf(b)\right) \\ \le\ & m\int_a^b f(t)g(t)\,d_\alpha t \\ \le\ & \sum_{k=1}^{m}\int_{c_{i_k}}^{d_{i_k}} f(t)\,d_\alpha t+(G(b)-G(a))\left(mf(a)-\sum_{k=1}^{m}\mu_\alpha(f;c_{i_k},d_{i_k})\right), \end{aligned}$$

which establishes the inequality (4.5). In the case that $i \neq j$ in (4.7), we take the summations over the two different subsets $\{i_1, i_2, ..., i_{m_1}\}$, and $\{j_1, j_2, ..., j_{m_2}\}$, we obtain

$$\begin{aligned} & \frac{1}{m_1}\sum_{k=1}^{m_1}\int_{c_{i_k}}^{d_{i_k}} f(t)\,d_\alpha t - \frac{1}{m_1}\left(G(b)-G\left(a\right)\right)\left(\sum_{k=1}^{m_1}\mu_\alpha(f;c_{i_k},d_{i_k}) - m_1 f(b)\right) \\ \leq & \int_a^b f(t)g(t)\,d_\alpha t \\ \leq & \frac{1}{m_2}\sum_{k=1}^{m_2}\int_{c_{j_k}}^{d_{j_k}} f(t)\,d_\alpha t + \frac{1}{m_2}\left(G(b)-G\left(a\right)\right)\left(m_2 f(a) - \sum_{k=1}^{m_2}\mu_\alpha(f;c_{j_k},d_{j_k})\right). \end{aligned}$$

This establishes the inequality (4.6). The inequality (4.6) is also true if the subsets $\{i_1, i_2, ..., i_{m_1}\}$, and $\{j_1, j_2, ..., j_{m_2}\}$ are intersected. Therefore, the inequality (4.6) can be restricted to the inequality (4.5). Now, using the identity $\alpha\left(G\left(b\right)-G(a)\right) = d_{i_k}^\alpha - c_{i_k}^\alpha$, we obtain

$$\sum_{k=1}^{m_1}\mu_\alpha(f;c_{i_k},d_{i_k}) = \sum_{k=1}^{m_1}\frac{\alpha}{d_{i_k}^\alpha - c_{i_k}^\alpha}\int_{c_{i_k}}^{d_{i_k}} f(t)\,d_\alpha t = \frac{1}{\left(G\left(b\right)-G(a)\right)}\sum_{k=1}^{m_1}\int_{c_{i_k}}^{d_{i_k}} f(t)\,d_\alpha t.$$

Hence, the general inequality (4.6) can be reduced to

$$f(b) \leq \frac{\int_a^b f(t)g(t)\,d_\alpha t}{G(b)-G\left(a\right)} \leq f(a).$$

This finishes the proof. □

5. Bellman's generalization of Steffensen's inequality via α-conformable integral

Consider the space $L_\alpha^p[a,b]$ of all Lebesque $\alpha-$conformable integrable functions f over $[a,b]$ such that $\int\limits_a^b |f(t)|^p d_\alpha t < \infty$. The dual space of $L_\alpha^p[a,b]$ is $L_\alpha^q[a,b]$, where $q = \frac{p}{1-p}$ (For more details see [12]). The theoretical results of this section depends mainly on the property of convexity or concavity of a function φ that is used on Belman's generalization of Steffensen's inequality in terms of conformable integrals.

The first result is concerning with a generalization of Steffensen's inequality using convex function.

Theorem 5.1. *Assume that*

(i) f is nonnegative monotone decreasing and $\alpha-$conformable integrable on $[a,b]$, $\alpha \in (0,1]$,

(ii) g is nonnegative $\alpha-$conformable integrable on $[a,b]$, and $0 \leq g < A$,

(iii) φ is nonnegative continuous on $[0,\infty)$ and monotone increasing such that $\varphi(0) = 0$,

(iv) B, η are real numbers such that $0 \leq B \leq A^{-1}$, and $B\int\limits_a^b g(t)d_\alpha t \leq \min\left\{\frac{(a+\eta)^\alpha - a^\alpha}{\alpha}, AB\left(\frac{b^\alpha - a^\alpha}{\alpha}\right)\right\}$. Then

$$\int\limits_a^{a+\eta}(\varphi\circ f)(t)d_\alpha t \geq B\int\limits_a^b(\varphi\circ f)(t)g(t)d_\alpha t. \tag{5.1}$$

Particularly, if $f \in L_\alpha^p[a,b]$, $p>1$, then

$$\int_a^{a+\eta} (f(t))^p \, d_\alpha t \geq \frac{1}{A^{p+1}} \left(\frac{\alpha}{b^\alpha - a^\alpha} \right)^{p-1} \left(\int_a^b f(t)g(t) d_\alpha t \right)^p . \tag{5.2}$$

Proof. It is obvious that the composite function $\varphi \circ f$ is monotone decreasing on $[a,b]$. Furthermore, $0 \leq Bg(t) \leq 1$, $B \int_a^b g(t) d_\alpha t \leq \frac{(a+\eta)^\alpha - a^\alpha}{\alpha}$, and then $0 \leq \eta \leq b-a$. We obtain

$$\begin{aligned}
& \int_a^{a+\eta} (\varphi \circ f)(t) d_\alpha t - B \int_a^b (\varphi \circ f)(t) g(t) d_\alpha t \\
= & \int_a^{a+\eta} (\varphi \circ f)(t) d_\alpha t - B \int_a^{a+\eta} (\varphi \circ f)(t) g(t) d_\alpha t - B \int_{a+\eta}^b (\varphi \circ f)(t) g(t) d_\alpha t \\
= & \int_a^{a+\eta} (\varphi \circ f)(t) (1 - Bg(t)) \, d_\alpha t - B \int_{a+\eta}^b (\varphi \circ f)(t) g(t) d_\alpha t \\
\geq & (\varphi \circ f)(a+\eta) \left(\int_a^{a+\eta} (1 - Bg(t)) \, d_\alpha t - B \int_{a+\eta}^b g(t) d_\alpha t \right) \\
= & (\varphi \circ f)(a+\eta) \left(\frac{(a+\eta)^\alpha - a^\alpha}{\alpha} - B \int_a^b g(t) d_\alpha t \right) \geq 0.
\end{aligned}$$

This leads to the inequality (5.1). In particular, if $\varphi(x) = x^p$, and $B = \frac{1}{A^{p+1}} \left(\frac{\alpha}{b^\alpha - a^\alpha} \int_a^b (g(t))^q \, d_\alpha t \right)^{p-1}$, where $q = 1 + qp^{-1}$, then, the hypotheses are satisfied. In this case, we deduce

$$\int_a^{a+\eta} (f(t))^p \, d_\alpha t \geq \frac{1}{A^{p+1}} \left(\frac{\alpha}{b^\alpha - a^\alpha} \int_a^b (g(t))^q \, d_\alpha t \right)^{p-1} \int_a^b (f(t))^p \, g(t) d_\alpha t. \tag{5.3}$$

Since, $f \in L_\alpha^p[a,b]$, and $g \in L_\alpha^q[a,b]$. Thus, by Holder's inequality, we have

$$\left(\int_a^b f(t)g(t) d_\alpha t \right)^p \leq \left(\int_a^b (g(t))^q \, d_\alpha t \right)^{\frac{p}{q}} \left(\int_a^b (f(t))^p \, d_\alpha t \right).$$

This with inequality (5.3), we obtain (5.2). This finishes the proof. □

Remark 5.2. *The classical identity can be obtained, when* $\alpha = 1 = A = B$, *and* $\eta = \left(\int_a^b g(t) dt \right)^p$.

The next shows that we still obtain the concerning result using a degreasing φ function.

Lemma 5.3. *Assume that*

(i) f is nonnegative monotone increasing and $\alpha-$conformable integrable on $[a,b]$, $\alpha \in (0,1]$,

(ii) g is nonnegative $\alpha-$conformable integrable on $[a,b]$, and $0 \le g < A$,

(iii) φ is strictly positive continuous on $(0,1)$ and monotone decreasing such that $\varphi(1)=0$,

(iv) η is a real number such that $\eta^{\varphi(p)} \le A^{-1}$, $t \in [a,b]$, and

$$\eta^{\varphi(p)} \int_a^b g(t) d_\alpha t \le \min\left\{ \frac{(a+\eta)^\alpha - a^\alpha}{\alpha}, A\eta^{\varphi(p)} \left(\frac{b^\alpha - a^\alpha}{\alpha} \right) \right\}.$$

Then

$$\int_a^{a+\eta} (\varphi \circ f)(t) d_\alpha t \ge \eta^{\varphi(p)} \int_a^b (\varphi \circ f)(t) g(t) d_\alpha t.$$

Particularly, we have

$$\int_a^{a+\eta} \frac{d_\alpha t}{f(t)} \ge \eta^{\frac{1}{p}-1} \int_a^b \frac{g(t)}{f(t)} d_\alpha t.$$

Proof. The first inequality can be established similar as the proof of Theorem 5.1 with $B = \eta^{\varphi(p)}$. The second can be obtained, if we put $\varphi(p) = \frac{1}{p} - 1$, $p \in (0,1]$. Then $A\eta^{\frac{1}{p}-1} \le 1$, and we obtain

$$\int_a^{a+\eta} \left(\frac{1}{f(t)} - 1 \right) d_\alpha t \ge \eta^{\frac{1}{p}-1} \int_a^b \left(\frac{1}{f(t)} - 1 \right) g(t) d_\alpha t.$$

Simplifying and using the condition $\eta^{\frac{1}{p}-1} \int_a^b g(t) d_\alpha t \le \frac{(a+\eta)^\alpha - a^\alpha}{\alpha}$, we obtain

$$\int_a^{a+\eta} \frac{d_\alpha t}{f(t)} + \eta^{\frac{1}{p}-1} \int_a^b g(t) d_\alpha t \ge \eta^{\frac{1}{p}-1} \int_a^b g(t) d_\alpha t + \eta^{\frac{1}{p}-1} \int_a^b \frac{g(t)}{f(t)} d_\alpha t.$$

This finishes the proof. □

The next result indicates the role of concave function in the Steffensen's like inequalities.

Theorem 5.4. *Assume that*

(i) f is nonnegative monotone increasing and $\alpha-$conformable integrable on $[a,b]$,

(ii) g is nonnegative $\alpha-$conformable integrable on $[a,b]$, and $0 \le g < A$,

(iii) φ is nonnegative, continuous, monotone increasing, and concave on $[0,\infty)$ such that $\varphi(0)=0$,

(iv) $\eta \geq 1$ is real number such that $\eta^{1-\frac{1}{p}} g(t) \leq 1$, $p \geq 1$, and $(a+\eta)^\alpha - a^\alpha \leq \alpha \eta^{1-\frac{1}{p}} \int_a^b g(t) d_\alpha t$. Then

$$\int_a^{a+\eta} (\varphi \circ f)(t) d_\alpha t \leq \varphi\left(\eta^{1-\frac{1}{p}} \int_a^b f(t)g(t)d_\alpha t\right).$$

Particularly, for $0 \leq q \leq 1$, we have

$$\int_a^{a+\eta} (f(t))^q \, d_\alpha t \leq \eta^{q-\frac{q}{p}} \left(\int_a^b f(t)g(t)d_\alpha t\right)^q .$$

Proof. The subadditivity of the concave function and Jensen's inequality lead to

$$\begin{aligned}
& \int_a^{a+\eta} (\varphi \circ f)(t) d_\alpha t - \varphi\left(\eta^{1-\frac{1}{p}} \int_a^b f(t)g(t)d_\alpha t\right) \\
\leq\ & \int_a^{a+\eta} (\varphi \circ f)(t) d_\alpha t - \eta\varphi\left(\eta^{-\frac{1}{p}} \int_a^b f(t)g(t)d_\alpha t\right) \\
\leq\ & \int_a^{a+\eta} (\varphi \circ f)(t) d_\alpha t - \eta^{1-\frac{1}{p}}\left(\int_a^b (\varphi \circ f)(t)g(t)d_\alpha t\right) \\
=\ & \int_a^{a+\eta} (\varphi \circ f)(t)\left(1-\eta^{1-\frac{1}{p}}g(t)\right) d_\alpha t - \eta^{1-\frac{1}{p}} \int_{a+\eta}^b (\varphi \circ f)(t)g(t)d_\alpha t \\
\leq\ & (\varphi \circ f)(a+\eta)\left(\int_a^{a+\eta} \left(1-\eta^{1-\frac{1}{p}}g(t)\right) d_\alpha t - \eta^{1-\frac{1}{p}} \int_{a+\eta}^b g(t)d_\alpha t\right) \\
=\ & (\varphi \circ f)(a+\eta)\left(\frac{(a+\eta)^\alpha - a^\alpha}{\alpha} - \eta^{1-\frac{1}{p}} \int_a^b g(t)d_\alpha t\right) \leq 0.
\end{aligned}$$

Thus, the first result follows. The second is obtained by assuming $\varphi(x) = x^q, x \geq 0, q \in [0,1]$. This finishes the proof. □

The next shows that we still obtain the concerning result using a degreasing f function.

Theorem 5.5. *Assume that*

(i) f is nonnegative monotone decreasing $\alpha-$conformable integrable on $[a,b]$,

(ii) g is nonnegative $\alpha-$conformable integrable on $[a,b]$, and $0 \leq g < A$,

(iii) φ is nonnegative, continuous, and monotone increasing on $[0,\infty)$ such that $\varphi(0) = 0$,

(iv) h is nonnegative $\alpha-$conformable integrable on $[a, b]$, and η is a real number such that $h(t) \geq \eta^{-\frac{1}{p}}\varphi(\eta^{\frac{1}{p}})g(t)$, $p \geq 1$, and $\eta^{\frac{1}{p}} \int_a^{a+\eta} h(t)d_\alpha t \geq \varphi(\eta^{\frac{1}{p}}) \int_a^b g(t)d_\alpha t$. Then

$$\int_a^{a+\eta} (\varphi \circ f)(t)h(t)d_\alpha t \geq \eta^{-\frac{1}{p}}\varphi(\eta^{\frac{1}{p}}) \int_a^b (\varphi \circ f)(t)g(t)d_\alpha t.$$

Proof. The hypotheses of the theorem lead to

$$\begin{aligned}
&\int_a^{a+\eta} (\varphi \circ f)(t)h(t)d_\alpha t - \eta^{-\frac{1}{p}}\varphi(\eta^{\frac{1}{p}}) \int_a^b (\varphi \circ f)(t)g(t)d_\alpha t \\
= &\int_a^{a+\eta} (\varphi \circ f)(t)\left(h(t) - \eta^{-\frac{1}{p}}\varphi(\eta^{\frac{1}{p}})g(t)\right)d_\alpha t - \eta^{-\frac{1}{p}}\varphi(\eta^{\frac{1}{p}}) \int_{a+\eta}^b (\varphi \circ f)(t)g(t)d_\alpha t \\
\geq &(\varphi \circ f)(a+\eta)\left(\int_a^{a+\eta} \left(h(t) - \eta^{-\frac{1}{p}}\varphi(\eta^{\frac{1}{p}})g(t)\right)d_\alpha t - \eta^{-\frac{1}{p}}\varphi(\eta^{\frac{1}{p}}) \int_{a+\eta}^b g(t)d_\alpha t\right) \\
= &(\varphi \circ f)(a+\eta)\left(\int_a^{a+\eta} h(t)d_\alpha t - \eta^{-\frac{1}{p}}\varphi(\eta^{\frac{1}{p}}) \int_a^b g(t)d_\alpha t\right) \geq 0.
\end{aligned}$$

This finishes the proof. □

6. Conclusions

In the context of this paper, we introduced a new α-conformable version of Steffensen's inequality and we developed the ideas of Cerone and Bellman's generalizations of Steffensen's inequality by using the conformable integral. We obtained some new results regarding these generalizations. In addition, we proved some new results of the conformable integral mean, which is related to Steffensens' inequality. The applications of these inequalities may be helpful for proving the existence and uniqueness of the solution of some fractional differential models, as well as important tools of the comparative analysis. Future research can aim to generalize these ideas on other inequalities and use very recent local and nonlocal fractional derivatives.

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Conflict of interest

The authors declare that they have no conflicts of interests.

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