Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(1): 1995–2017.
DOI: 10.3934/math.2023103
Received: 10 August 2022
Revised: 19 September 2022
Accepted: 23 September 2022
Published: 26 October 2022

Research article

On some classical integral inequalities in the setting of new post quantum integrals

Bandar Bin-Mohsin[1], **Muhammad Uzair Awan**[2,*], **Muhammad Zakria Javed**[2], **Sadia Talib**[2], **Hüseyin Budak**[3], **Muhammad Aslam Noor**[4] **and Khalida Inayat Noor**[4]

[1] Department of Mathematics College of Science King Saud University, Riyadh, Saudi Arabia
[2] Department of Mathematics, Government College University, Faisalabad, Pakistan
[3] Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey
[4] Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan

* **Correspondence:** Email: awan.uzair@gmail.com.

Abstract: In this article, we introduce the notion of ${}_a\bar{T}_{p,q}$-integrals. Using the definition of ${}_a\bar{T}_{p,q}$-integrals, we derive some new post quantum analogues of some classical results of Young's inequality, Hölder's inequality, Minkowski's inequality, Ostrowski's inequality and Hermite-Hadamard's inequality.

Keywords: convex functions; ${}_a\bar{T}_{p,q}$-integral; Young's inequality; Hölder's inequality; Minkowski's inequality; Ostrowski's inequality; Hermite-Hadamard's inequality
Mathematics Subject Classification: 26D10, 26D15, 26A51, 05A33

1. Introduction

Quantum calculus is one of the most fascinating subjects of mathematics. It serves as the bridge between mathematics and physics. The history of quantum calculus is old but in the past few decades it experienced rapid development. The concepts of quantum calculus have been used as tools in different areas of mathematics. For example Sudsutad et al. [13] and Tariboon and Ntouyas [14, 15] have utilized the concepts of quantum calculus in obtaining the quantum analogues of inequalities involving convexity. This motivated many researchers and consequently number of new quantum analogues of classical inequalities have been obtained in the literature. For instance Noor et al. [12] have obtained new quantum analogues of the Hermite-Hadamard type of inequalities. Du et al. [7] and Zhang et al. [19] obtained new generalized quantum integral identities and obtained several new associated quantum analogues of integral inequalities.

In 2017, Alp and Sarikaya [1] introduced another new definition of q-integral which they called as ${}_a\bar{T}_q$-integral. They discussed several basic properties pertaining to this so called ${}_a\bar{T}_q$-integral and also obtained new ${}_a\bar{T}_q$-analogue of Hermite-Hadamard's inequality. In another article [2] Alp and Sarikaya obtained the ${}_a\bar{T}_q$-analogues of Ostrowski's, Young's, Hölder's and Minkowski's inequalities. On the other hand in [10], Kara et al. introduced new quantum integral which is called ${}^b\bar{T}_q$-integral. The authors proved also the corresponding Hermite-Hadamard inequalities for ${}^b\bar{T}_q$-integrals.

Chakarabarti and Jagannathan [5] considered a new generalization of quantum calculus (also known as q-calculus) which is called as post-quantum calculus (also known as (p, q)-calculus). In q-calculus, we deal with a q–number with one base q, however, in (p, q)-calculus we have p- and q-numbers with two independent variables p and q. Kunt et al. [11] derived (p, q)-analogues of Hermite–Hadamard's inequality. Since then several new post quantum analogues of classical inequalities have been obtained in the literature. For example, Chu et al. [6] obtained new (p, q)–analogues of Ostrowski type of inequalities using new definitions of left–right (p, q)–derivatives and definite integrals. Yu et al. [18] obtained some new refinements of inequalities via (p, q)–calculus and also discussed their applications. For some more details and basic properties, see [16]. For more details see [20–24].

The aim of this paper is to introduce ${}_a\bar{T}_{p,q}$-integrals. We derive new ${}_a\bar{T}_{p,q}$-analogues of certain classical inequalities. For example, we obtain ${}_a\bar{T}_{p,q}$-Young's inequality, ${}_a\bar{T}_{p,q}$-Hölder's inequality, ${}_a\bar{T}_{p,q}$-Minkowski's inequality, ${}_a\bar{T}_{p,q}$-Ostrowski's inequality and ${}_a\bar{T}_{p,q}$-Hermite-Hadamard's inequality respectively. To the best of our knowledge these results are new in the literature. We hope that the ideas and techniques of this paper will inspire interested readers working in this field.

2. Preliminaries

In this section, for the sake of completeness, we now recall some basic concepts from quantum and post quantum calculus.

2.1. *q-calculus and some inequalities*

In this part, we give some of the necessary explanations and related inequalities regarding q-calculus. Also, here and further we use the following notation (see [9]):

$$[\tilde{\varpi}]_q = \frac{1 - q^{\tilde{\varpi}}}{1 - q} = 1 + q + q^2 + \ldots + q^{\tilde{\varpi}-1}, \ \ q \in (0, 1).$$

In [8], Jackson gave the q-Jackson integral from 0 to b for $0 < q < 1$ as follows:

$$\int_0^b \bar{\Xi}(x) \ \mathrm{d}_q x = (1 - q)\, b \sum_{\tilde{\varpi}=0}^{\infty} q^{\tilde{\varpi}} \bar{\Xi}\left(b q^{\tilde{\varpi}}\right), \tag{2.1}$$

provided the sum converges absolutely.

Jackson [8] gave the q-Jackson integral in a generic interval $[a, b]$ as:

$$\int_a^b \bar{\Xi}(x) \ \mathrm{d}_q x = \int_0^b \bar{\Xi}(x) \ \mathrm{d}_q x - \int_0^a \bar{\Xi}(x) \ \mathrm{d}_q x.$$

Definition 2.1 ([14]). *Let $\bar{\Xi} : [a,b] \to \mathbb{R}$ be a continuous function. Then, the $\tilde{\mathfrak{q}}_{2\tilde{\mathfrak{a}}_1}$-definite integral on $[a,b]$ is defined as*

$$\int_a^b \bar{\Xi}(x)\ {}_a\mathsf{d}_q x = (1-q)(b-a)\sum_{\tilde{\varpi}=0}^{\infty} q^{\tilde{\varpi}}\bar{\Xi}\left(q^{\tilde{\varpi}}b + \left(1-q^{\tilde{\varpi}}\right)a\right) = (b-a)\int_0^1 \bar{\Xi}((1-\bar{\varrho})a + \bar{\varrho}b)\ \mathsf{d}_q\bar{\varrho}. \quad (2.2)$$

In [3], Alp et al. proved the following $\tilde{\mathfrak{q}}_{2\tilde{\mathfrak{a}}_1}$-Hermite-Hadamard inequalities for convex functions in the setting of quantum calculus:

Theorem 2.1. *Let $\bar{\Xi} : [a,b] \to \mathbb{R}$ be a convex differentiable function on $[a,b]$ and $0 < q < 1$. Then q-Hermite-Hadamard inequalities are as follows:*

$$\bar{\Xi}\left(\frac{qa+b}{1+q}\right) \le \frac{1}{b-a}\int_a^b \bar{\Xi}(x)\ {}_a\mathsf{d}_q x \le \frac{q\bar{\Xi}(a) + \bar{\Xi}(b)}{1+q}. \quad (2.3)$$

Definition 2.2 ([14,15]). *Let $\bar{\Xi} : J := [a,b] \subset \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Then the q–derivative of $\bar{\Xi}$ on J at $\bar{\varrho}$, is defined*

$${}_aD_q\bar{\Xi}(\bar{\varrho}) = \frac{\bar{\Xi}(\bar{\varrho}) - \bar{\Xi}(q\bar{\varrho} + (1-q)a)}{(1-q)(\bar{\varrho}-a)},\ \bar{\varrho} \ne a \ \text{ and } \ D_q\bar{\Xi}(a) = \lim_{\bar{\varrho}\to a} D_q\bar{\Xi}(\bar{\varrho}),$$

where $0 < q < 1$ is a constant.

On the other hand, Bermudo et al. gave the following new definition and related Hermite-Hadamard type inequalities:

Definition 2.3 ([4]). *Let $\bar{\Xi} : [a,b] \to \mathbb{R}$ be a continuous function. Then, the q^b-definite integral on $[a,b]$ is defined as*

$$\int_a^b \bar{\Xi}(x)\ {}^b\mathsf{d}_q x = (1-q)(b-a)\sum_{\tilde{\varpi}=0}^{\infty} q^{\tilde{\varpi}}\bar{\Xi}\left(q^{\tilde{\varpi}}a + \left(1-q^{\tilde{\varpi}}\right)b\right) = (b-a)\int_0^1 \bar{\Xi}(ta + (1-\bar{\varrho})b)\ \mathsf{d}_q\bar{\varrho}. \quad (2.4)$$

Theorem 2.2 ([4]). *Let $\bar{\Xi} : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$ and $0 < q < 1$. Then, q-Hermite-Hadamard inequalities are as follows:*

$$\bar{\Xi}\left(\frac{a+qb}{1+q}\right) \le \frac{1}{b-a}\int_a^b \bar{\Xi}(x)\ {}^b\mathsf{d}_q x \le \frac{\bar{\Xi}(a) + q\bar{\Xi}(b)}{1+q}. \quad (2.5)$$

2.2. (p,q)-calculus and some inequalities

In this part, we review some fundamental notions and notations of (p,q)-calculus.

The $[\tilde{\varpi}]_{p,q}$ is said to be (p,q) integers and is expressed as:

$$[\tilde{\varpi}]_{p,q} = \frac{p^{\tilde{\varpi}} - q^{\tilde{\varpi}}}{p-q},$$

with $0 < q < p \le 1$.

Definition 2.4 ([16]). *The definite $(p,q)_a$-integral of mapping $\bar{\Xi}:[a,b]\to\mathbb{R}$ on $[a,b]$ is stated as:*

$$\int_a^x \bar{\Xi}(\bar{\varrho})\ {}_a\mathrm{d}_{p,q}\bar{\varrho} = (p-q)(x-a)\sum_{\tilde{\varpi}=0}^{\infty}\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}\bar{\Xi}\left(\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}x+\left(1-\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}\right)a\right), \tag{2.6}$$

for $x\in[a,\ qb+(1-p)a]$ and $0<q<p\le 1$.

Definition 2.5 ([16]). *Let $\bar{\Xi}:J\to\mathbb{R}$ be a continuous function and let $x\in J$ and $0<q<p\le 1$. Then the (p,q)-derivative on J of function $\bar{\Xi}$ at x is defined as*

$$\mathrm{D}_{p,q}\bar{\Xi}(x) = \frac{\bar{\Xi}(px+(1-p)a)) - \bar{\Xi}(qx+(1-q)a)}{(p-q)(x-a)}, x\neq a.$$

Definition 2.6. *From [17], the definite $(p,q)^b$-integral of mapping $\bar{\Xi}:[a,b]\to\mathbb{R}$ on $[a,b]$ is stated as:*

$$\int_x^b \bar{\Xi}(\bar{\varrho})\ {}^b\mathrm{d}_{p,q}\bar{\varrho} = (p-q)(b-x)\sum_{\tilde{\varpi}=0}^{\infty}\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}\bar{\Xi}\left(\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}x+\left(1-\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}\right)b\right), \tag{2.7}$$

for $x\in[pa+(1-p)b,\ b]$ and $0<q<p\le 1$.

Remark 2.1. *It is evident that if we pick $p=1$ in (2.6) and (2.7), then the equalities (2.6) and (2.7) change into (2.2) and (2.4), respectively.*

In [11], Kunt et al. proved the following Hermite-Hadamard type inequalities for convex functions via $(p,q)_a$ integral:

Theorem 2.3. *For a convex mapping $\bar{\Xi}:[a,b]\to\mathbb{R}$ which is differentiable on $[a,b]$, the following inequalities hold for $(p,q)_a$ integral:*

$$\bar{\Xi}\left(\frac{qa+pb}{[2]_{p,q}}\right)\le\frac{1}{p(b-a)}\int_a^{qb+(1-p)a}\bar{\Xi}(x)\ {}_a\mathrm{d}_{p,q}x\le\frac{q\bar{\Xi}(a)+p\bar{\Xi}(b)}{[2]_{p,q}}, \tag{2.8}$$

where $0<q<p\le 1$.

In [17], Vivas-Cortez et al. proved the following Hermite-Hadamard type inequalities for convex functions via $(p,q)^b$ integral:

Theorem 2.4. *For a convex mapping $\bar{\Xi}:[a,b]\to\mathbb{R}$ which is differentiable on $[a,b]$, the following inequalities hold for $(p,q)^b$ integral:*

$$\bar{\Xi}\left(\frac{pa+qb}{[2]_{p,q}}\right)\le\frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b}\bar{\Xi}(x)\ {}^b\mathrm{d}_{p,q}x\le\frac{p\bar{\Xi}(a)+q\bar{\Xi}(b)}{[2]_{p,q}}, \tag{2.9}$$

where $0<q<p\le 1$.

2.3. $\bar{T}_q$-integrals and related inequalities

In this subsection, we present definitions and properties given by using trapezoids.

Alp and Sarikaya, by using the area of trapezoids, introduced the following generalized quantum integral which is called ${}_a\bar{T}_q$-integral.

Definition 2.7 ([1, 2, 10]). *Let $\bar{\Xi} : [a,b] \to \mathbb{R}$ is continuous function. for $x \in [a,b]$*

$$\int_a^b \bar{\Xi}(\bar{\varrho})\ {}_a\mathsf{d}_q^{\bar{T}}\bar{\varrho} = \frac{(1-q)(b-a)}{2q}\left[(1+q)\sum_{\tilde{\varpi}=0}^{\infty} q^{\tilde{\varpi}}\bar{\Xi}\left(q^{\tilde{\varpi}}b + \left(1-q^{\tilde{\varpi}}\right)a\right) - \bar{\Xi}(b)\right].$$

Theorem 2.5 (${}_a\bar{T}_q$-Hermite-Hadamard). *[1, 2] Let $\bar{\Xi} : [a,b] \to \mathbb{R}$ be a convex continuous function on $[a,b]$. Then we have*

$$\bar{\Xi}\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \bar{\Xi}(\bar{\varrho})\ {}_a\mathsf{d}_q^{\bar{T}}\bar{\varrho} \leq \frac{\bar{\Xi}(a)+\bar{\Xi}(b)}{2}.$$

In [10], Kara et al. introduced the following generalized quantum integral which is called ${}^b\bar{T}_q$-integral.

Definition 2.8 ([10]). *Let $\bar{\Xi} : [a,b] \to \mathbb{R}$ is continuous function. for $x \in [a,b]$,*

$$\int_a^b \bar{\Xi}(\bar{\varrho})\ {}^b\mathsf{d}_q^{\bar{T}}\bar{\varrho} = \frac{(1-q)(b-a)}{2q}\left[(1+q)\sum_{\tilde{\varpi}=0}^{\infty} q^{\tilde{\varpi}}\bar{\Xi}\left(q^{\tilde{\varpi}}a + \left(1-q^{\tilde{\varpi}}\right)b\right) - \bar{\Xi}(a)\right].$$

Theorem 2.6 (${}^b\bar{T}_q$-Hermite-Hadamard). *[10] Let $\bar{\Xi} : [a,b] \to \mathbb{R}$ be a convex continuous function on $[a,b]$. Then we have*

$$\bar{\Xi}\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \bar{\Xi}(\bar{\varrho})\ {}^b\mathsf{d}_q^{\bar{T}}\bar{\varrho} \leq \frac{\bar{\Xi}(a)+\bar{\Xi}(b)}{2}.$$

3. New definition and some basic properties

First we define ${}_a\bar{T}_{p,q}$-integral.

Definition 3.1. *Let $\bar{\Xi} : I \to \mathbb{R}$ be continuous functions, then for $0 < q < p \leq 1$ we have*

$$\int_a^x \bar{\Xi}(\bar{\varrho})\ {}_a\mathsf{d}_{p,q}^{\bar{T}}\bar{\varrho}$$
$$= \frac{(p-q)(x-a)}{2q}\left[(p+q)\sum_{\tilde{\varpi}=0}^{\infty}\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}\bar{\Xi}\left(\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}x + \left(1-\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}\right)a\right) - \bar{\Xi}\left(\frac{x+(p-1)a}{p}\right)\right],$$

where $x \in [a, qb + (1-p)a]$.

Remark 3.1. *If we take $p = 1$, then we obtain ${}_a\bar{T}_q$ integrals.*

We now discuss some basic properties of ${}_a\bar{T}_{p,q}$-integral.

Theorem 3.1. *For $\alpha \in \mathbb{R} \setminus \{-1\}$, then*

$$\int_a^x (\bar{\varrho} - a)^\alpha \; {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho} = \frac{(p^\alpha + q^\alpha)(x-a)^{\alpha+1}}{2p^\alpha \, [\alpha+1]_{p,q}}.$$

Proof. From the definition of ${}_a\bar{T}_{p,q}$-integrals, we have

$$\begin{aligned}
\int_a^x (\bar{\varrho} - a)^\alpha \; {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho} &= \frac{(p-q)(x-a)}{2q}\Bigg[(p+q)\sum_{\tilde{\varpi}=0}^{\infty} \frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}\left(\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}x + \left(1 - \frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}\right)a - a\right)^\alpha \\
&\quad - \left(\frac{x+(p-1)a}{p} - a\right)^\alpha\Bigg] \\
&= \frac{(p-q)(x-a)}{2q}\left[(p+q)\sum_{\tilde{\varpi}=0}^{\infty}\left(\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}\right)^{\alpha+1}(x-a)^\alpha - \left(\frac{x-a}{p}\right)^\alpha\right] \\
&= \frac{(p-q)(x-a)^{\alpha+1}}{2q}\left[\frac{(p+q)}{p^{\alpha+1}}\sum_{\tilde{\varpi}=0}^{\infty}\left(\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}}}\right)^{\alpha+1} - \left(\frac{1}{p}\right)^\alpha\right] \\
&= \frac{(p-q)(p^\alpha + q^\alpha)(x-a)^{\alpha+1}}{2p^\alpha(p^{\alpha+1} - q^{\alpha+1})}.
\end{aligned}$$

This completes the proof. □

Note that if we take $p = 1$, we have the following result.

Corollary 3.1. *For $\alpha \in \mathbb{R} \setminus \{-1\}$, then*

$$\int_a^x (\bar{\varrho} - a)^\alpha \; {}_a\mathsf{d}^{\bar{\Xi}}_{q}\bar{\varrho} = \frac{(1 + q^\alpha)(x-a)^{\alpha+1}}{2\,[\alpha+1]_q}.$$

Lemma 3.1. *Let $p > 1$ and $(a - \bar{\varrho})^{\tilde{\varpi}}_{p,q}$ is (p,q)-binomial, then*

$$\int_0^x (a - \bar{\varrho})^{\tilde{\varpi}}_{p,q} \; {}_0\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho} = \frac{a^{\tilde{\varpi}+1}(p+q)}{2p[\tilde{\varpi}+1]_{p,q}} - \left(\frac{q\left(a - \frac{px}{q}\right)^{\tilde{\varpi}+1} + p(a-x)^{\tilde{\varpi}+1}}{2p[\tilde{\varpi}+1]_{p,q}}\right).$$

Proof. By using (p,q)-binomial formula and Gauss binomial formula

$$\begin{aligned}
&\int_0^x (a - \bar{\varrho})^{\tilde{\varpi}}_{p,q} \; {}_0\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho} \\
&= \int_0^x \sum_{k=0}^{\tilde{\varpi}} (-1)^k \binom{\tilde{\varpi}}{k}_{p,q} p^{\frac{(\tilde{\varpi}-k)(\tilde{\varpi}-k-1)}{2}} q^{\frac{k(k-1)}{2}} \bar{\varrho}^k a^{\tilde{\varpi}-k} \; {}_0\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho} \\
&= \int_0^x \sum_{k=0}^{\tilde{\varpi}} (-1)^k \frac{[\tilde{\varpi}]_{p,q}!}{[k]_{p,q}![\tilde{\varpi}-k]_{p,q}!}_{p,q} p^{\frac{(\tilde{\varpi}-k)(\tilde{\varpi}-k-1)}{2}} q^{\frac{k(k-1)}{2}} \bar{\varrho}^k a^{\tilde{\varpi}-k} \; {}_0\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho} \\
&= \sum_{k=0}^{\tilde{\varpi}} (-1)^k \frac{[\tilde{\varpi}]_{p,q}!}{[k]_{p,q}![\tilde{\varpi}-k]_{p,q}!}_{p,q} p^{\frac{(\tilde{\varpi}-k)(\tilde{\varpi}-k-1)}{2}} q^{\frac{k(k-1)}{2}} a^{\tilde{\varpi}-k} \int_0^x \bar{\varrho}^k \; {}_0\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}
\end{aligned}$$

$$= \sum_{k=0}^{\tilde{\varpi}} (-1)^k \frac{[\tilde{\varpi}]_{p,q}!}{[k]_{p,q}![\tilde{\varpi}-k]_{p,q}!} p^{\frac{(\tilde{\varpi}-k)(\tilde{\varpi}-k-1)}{2}} q^{\frac{k(k-1)}{2}} a^{\tilde{\varpi}-k} \frac{(p^k+q^k)x^{k+1}}{2[k+1]_{p,q}}$$

$$= \sum_{k=0}^{\tilde{\varpi}} (-1)^k \frac{[\tilde{\varpi}]_{p,q}!}{[k+1]_{p,q}![\tilde{\varpi}-k]_{p,q}!} p^{\frac{(\tilde{\varpi}-k)(\tilde{\varpi}-k-1)}{2}} q^{\frac{k(k-1)}{2}} a^{\tilde{\varpi}-k} \frac{(p^k+q^k)x^{k+1}}{2}.$$

Replacing k by $k-1$

$$= \sum_{k=1}^{\tilde{\varpi}+1} (-1)^{k-1} \frac{[\tilde{\varpi}+1]_{p,q}!}{[k]_{p,q}![\tilde{\varpi}+1-k]_{p,q}!} p^{\frac{(\tilde{\varpi}-k+1)(\tilde{\varpi}-k)}{2}} q^{\frac{(k-1)(k-2)}{2}} a^{\tilde{\varpi}+1-k} \frac{(p^k+q^k)x^k}{2}$$

$$= \frac{-1}{2[\tilde{\varpi}+1]_{p,q}} \left[\frac{q}{p} \sum_{k=1}^{\tilde{\varpi}+1} (-1)^{k-1} \frac{[\tilde{\varpi}]_{p,q}!}{[k]_{p,q}![\tilde{\varpi}+1-k]_{p,q}!} p^{\frac{(\tilde{\varpi}-k+1)(\tilde{\varpi}-k)}{2}} q^{\frac{k(k-1)}{2}} a^{\tilde{\varpi}+1-k} \frac{x^k}{q^k 2} \right.$$

$$\left. + \sum_{k=1}^{\tilde{\varpi}+1} (-1)^{k-1} \frac{[\tilde{\varpi}+1]_{p,q}!}{[k]_{p,q}![\tilde{\varpi}+1-k]_{p,q}!} p^{\frac{(\tilde{\varpi}-k+1)(\tilde{\varpi}-k)}{2}} q^{\frac{k(k-1)}{2}} a^{\tilde{\varpi}+1-k} x^k \right]$$

$$= \frac{a^{\tilde{\varpi}+1}(p+q)}{2p[\tilde{\varpi}+1]_{p,q}} - \left(\frac{q\left(a-\frac{px}{q}\right)^{\tilde{\varpi}+1} + p(a-x)^{\tilde{\varpi}+1}}{2p[\tilde{\varpi}+1]_{p,q}} \right).$$

□

Example 3.1. *Let $p>1$ and $(1-\bar{\varrho})^{\tilde{\varpi}}_{p,q}$ is (p,q)-binomial, then*

$$\int_0^1 (1-\bar{\varrho})^{\tilde{\varpi}}_{p,q}\ {}_0\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho} = \frac{(p+q)}{2p[\tilde{\varpi}+1]_{p,q}}.$$

Theorem 3.2. *Let $\bar{\Xi}_1, \bar{\Xi}_2 : I \to \mathbb{R}$ be continuous functions, then*

(1) $\int_a^x [\bar{\Xi}_1(\bar{\varrho}) + \bar{\Xi}_2(\bar{\varrho})]\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho} = \int_a^x \bar{\Xi}_1(\bar{\varrho})\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho} + \int_a^x \bar{\Xi}_2(\bar{\varrho})\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}$;

(2) $\int_a^x \alpha\bar{\Xi}(\bar{\varrho})\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho} = \alpha \int_a^x \bar{\Xi}(\bar{\varrho})\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}$;

(3) ${}_aD_{p,q} \int_a^x \bar{\Xi}(\bar{\varrho})\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho} = \frac{\bar{\Xi}(x)+\bar{\Xi}\left(\frac{(qx+(p-q)a)}{p}\right)}{2}$.

Proof. We omit the details of the proof of (1) and (2). For (3), from the definition of ${}_a\bar{T}_{p,q}$-integrals, we have

$$\int_a^x \bar{\Xi}(\bar{\varrho})\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho} = \frac{(p-q)(x-a)}{2q} \left[(p+q) \sum_{\tilde{\varpi}=0}^{\infty} \frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}} \bar{\Xi}\left(\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}} x + \left(1-\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}\right) a \right) \right.$$

$$\left. - \bar{\Xi}\left(\frac{x+(p-1)a}{p} \right) \right].$$

Taking (p,q)-derivative , we have

$$
\begin{aligned}
&{}_aD_{p,q}\int_a^x \bar{\Xi}(\bar{\varrho})\ {}_a\mathsf{d}_{p,q}^{\bar{T}}\bar{\varrho}\\
&=\frac{p}{2q}\left[(p+q)\sum_{\tilde{\varpi}=0}^{\infty}\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}\bar{\Xi}\left(\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}(px+(1-p)a)+\left(1-\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}\right)a\right)-\bar{\Xi}(x)\right]\\
&-\frac{q}{2q}\left[(p+q)\sum_{\tilde{\varpi}=0}^{\infty}\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}\bar{\Xi}\left(\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}(qx+(1-q)a)+\left(1-\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}\right)a\right)\right.\\
&\left.-\bar{\Xi}\left(\frac{qx+(p-q)a}{p}\right)\right]\\
&=\frac{1}{2q}\left[(p+q)\sum_{\tilde{\varpi}=0}^{\infty}\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}}}\bar{\Xi}\left(\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}}}x+\left(1-\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}}}\right)a\right)-p\bar{\Xi}(x)\right.\\
&\left.-(p+q)\sum_{\tilde{\varpi}=0}^{\infty}\frac{q^{\tilde{\varpi}+1}}{p^{\tilde{\varpi}+1}}\bar{\Xi}\left(\frac{q^{\tilde{\varpi}+1}}{p^{\tilde{\varpi}+1}}x+\left(1-\frac{q^{\tilde{\varpi}+1}}{p^{\tilde{\varpi}+1}}\right)a\right)+q\bar{\Xi}\left(\frac{qx+(p-q)a}{p}\right)\right]\\
&=\frac{1}{2q}\left[(p+q)\sum_{\tilde{\varpi}=0}^{\infty}\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}}}\bar{\Xi}\left(\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}}}x+\left(1-\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}}}\right)a\right)-p\bar{\Xi}(x)\right.\\
&\left.-(p+q)\sum_{\tilde{\varpi}=1}^{\infty}\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}}}\bar{\Xi}\left(\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}}}x+\left(1-\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}}}\right)a\right)+q\bar{\Xi}\left(\frac{qx+(p-q)a}{p}\right)\right]\\
&=\frac{1}{2q}\left[(p+q)\sum_{\tilde{\varpi}=0}^{\infty}\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}}}\bar{\Xi}\left(\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}}}x+\left(1-\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}}}\right)a\right)-p\bar{\Xi}(x)\right.\\
&\left.-(p+q)\sum_{\tilde{\varpi}=0}^{\infty}\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}}}\bar{\Xi}\left(\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}}}x+\left(1-\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}}}\right)a\right)+(p+q)\bar{\Xi}(x)+q\bar{\Xi}\left(\frac{qx+(p-q)a}{p}\right)\right]\\
&=\frac{1}{2q}\left[(p+q)\bar{\Xi}(x)-p\bar{\Xi}(x)+q\bar{\Xi}\left(\frac{qx+(p-q)a}{p}\right)\right]\\
&=\frac{\bar{\Xi}(x)+\bar{\Xi}\left(\frac{(qx+(p-q)a)}{p}\right)}{2}.
\end{aligned}
$$

This completes the proof. □

Theorem 3.3 (Change of variable property)**.** *Let* $\bar{\Xi}: I\to\mathbb{R}$ *be a function with* $0<q<p\leq 1$ *, then*

$$
\int_0^p \bar{\Xi}(\bar{\varrho}b+(1-\bar{\varrho})a)\ {}_0\mathsf{d}_{p,q}^{\bar{T}}\bar{\varrho}=\frac{1}{b-a}\int_a^{qb+(1-p)a}\bar{\Xi}(\bar{\varrho})\ {}_a\mathsf{d}_{p,q}^{\bar{T}}\bar{\varrho}.
$$

Proof. From the definition of ${}_a\bar{T}_{p,q}$-integral, we have

$$
\begin{aligned}
&\int_0^p \bar{\Xi}(\bar{\varrho}b+(1-\bar{\varrho})a)\ {}_0\mathsf{d}_{p,q}^{\bar{T}}\bar{\varrho}\\
&=\frac{(p-q)p}{2q}\left[(p+q)\sum_{\tilde{\varpi}=0}^{\infty}\bar{\Xi}\left(\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}}}b+\left(1-\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}}}\right)a\right)-\bar{\Xi}\left(b\right)\right]
\end{aligned}
$$

$$= \frac{(p-q)p}{2q}\left[(p+q)\sum_{\tilde{\varpi}=0}^{\infty}\bar{\Xi}\left(\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}(qb+(1-p)a)+\left(1-\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}\right)a\right)-\bar{\Xi}(b)\right]$$

$$= \frac{1}{b-a}\int_a^{qb+(1-p)a}\bar{\Xi}(\bar{\varrho})\ {}_a\mathrm{d}_{p,q}^{\bar{T}}\bar{\varrho}.$$

This completes the proof. □

Theorem 3.4. *Let $\bar{\Xi}: I \to \mathbb{R}$ be a continuous function with $c \in (a, x)$, then*

$$\int_c^x {}_aD_{p,q}\bar{\Xi}(\bar{\varrho})\ {}_a\mathrm{d}_{p,q}^{\bar{T}}\bar{\varrho} = \frac{q\bar{\Xi}(x)-q\bar{\Xi}(c)+p\bar{\Xi}\left(\frac{qx+(p-q)a}{p}\right)-p\bar{\Xi}\left(\frac{qc+(p-q)a}{p}\right)}{2q}.$$

Proof. Applying (p,q)-derivative, change of variable and ${}_a\bar{T}_{p,q}$-integrals, we have

$$\begin{aligned}
&\int_c^x {}_aD_{p,q}\bar{\Xi}(\bar{\varrho})\ {}_a\mathrm{d}_{p,q}^{\bar{T}}\bar{\varrho}\\
&= \int_a^x {}_aD_{p,q}\bar{\Xi}(\bar{\varrho})\ {}_a\mathrm{d}_{p,q}^{\bar{T}}\bar{\varrho} - \int_a^c {}_aD_{pq}\bar{\Xi}(\bar{\varrho})\ {}_a\mathrm{d}_{p,q}^{\bar{T}}\bar{\varrho}\\
&= \int_a^x \frac{\bar{\Xi}(p\bar{\varrho}+(1-p)a)-\bar{\Xi}(q\bar{\varrho}+(1-q)a)}{(p-q)(\bar{\varrho}-a)}\ {}_a\mathrm{d}_{p,q}^{\bar{T}}\bar{\varrho}\\
&\quad - \int_a^c \frac{\bar{\Xi}(p\bar{\varrho}+(1-p)a)-\bar{\Xi}(q\bar{\varrho}+(1-q)a)}{(p-q)(\bar{\varrho}-a)}{}_a\mathrm{d}_{\bar{p},\bar{q}}\bar{\varrho}\\
&= \int_a^x \frac{\bar{\Xi}(p\bar{\varrho}+(1-p)a)}{(p-q)(\bar{\varrho}-a)}\ {}_a\mathrm{d}_{p,q}^{\bar{T}}\bar{\varrho} - \int_a^x \frac{\bar{\Xi}(q\bar{\varrho}+(1-q)a)}{(p-q)(\bar{\varrho}-a)}\ {}_a\mathrm{d}_{p,q}^{\bar{T}}\bar{\varrho}\\
&\quad - \int_a^c \frac{\bar{\Xi}(p\bar{\varrho}+(1-p)a)}{(p-q)(\bar{\varrho}-a)}\ {}_a\mathrm{d}_{p,q}^{\bar{T}}\bar{\varrho} + \int_a^c \frac{\bar{\Xi}(q\bar{\varrho}+(1-q)a)}{(p-q)(\bar{\varrho}-a)}\ {}_a\mathrm{d}_{p,q}^{\bar{T}}\bar{\varrho}\\
&= \int_a^{px+(1-p)a} \frac{\bar{\Xi}(\bar{\varrho})}{(p-q)(\bar{\varrho}-a)}\ {}_a\mathrm{d}_{p,q}^{\bar{T}}\bar{\varrho} - \int_a^{qx+(1-q)a} \frac{\bar{\Xi}(\bar{\varrho})}{(p-q)(\bar{\varrho}-a)}\ {}_a\mathrm{d}_{p,q}^{\bar{T}}\bar{\varrho}\\
&\quad - \int_a^{pc+(1-p)a} \frac{\bar{\Xi}(\bar{\varrho})}{(p-q)(\bar{\varrho}-a)}\ {}_a\mathrm{d}_{p,q}^{\bar{T}}\bar{\varrho} + \int_a^{qc+(1-q)a} \frac{\bar{\Xi}(\bar{\varrho})}{(p-q)(\bar{\varrho}-a)}\ {}_a\mathrm{d}_{p,q}^{\bar{T}}\bar{\varrho}\\
&= I_1 - I_2 - I_3 + I_4,
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
I_1 &= \int_a^{px+(1-p)a} \frac{\bar{\Xi}(\bar{\varrho})}{(p-q)(\bar{\varrho}-a)}\ {}_a\mathrm{d}_{p,q}^{\bar{T}}\bar{\varrho}\\
&= \frac{1}{2q}\left[(p+q)\sum_{\tilde{\varpi}=0}^{\infty}\bar{\Xi}\left(\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}}}x+\left(1-\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}}}\right)a\right)-p\bar{\Xi}(x)\right],\\
I_2 &= \int_a^{qx+(1-q)a} \frac{\bar{\Xi}(\bar{\varrho})}{(p-q)(\bar{\varrho}-a)}\ {}_a\mathrm{d}_{p,q}^{\bar{T}}\bar{\varrho}\\
&= \frac{1}{2q}\left[(p+q)\sum_{\tilde{\varpi}=0}^{\infty}\bar{\Xi}\left(\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}}}x+\left(1-\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}}}\right)a\right)-(p+q)\bar{\Xi}(x)-p\bar{\Xi}\left(\frac{qx+(p-q)a}{p}\right)\right],
\end{aligned}$$

$$I_3 = \int_a^{pc+(1-p)a} \frac{\bar{\Xi}(\bar{\varrho})}{(p-q)(\bar{\varrho}-a)} \ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}$$
$$= \frac{1}{2q}\left[(p+q)\sum_{\tilde{\varpi}=0}^{\infty}\bar{\Xi}\left(\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}}}c+\left(1-\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}}}\right)a\right)-p\bar{\Xi}(c)\right],$$
$$I_4 = \int_a^{qc+(1-q)a} \frac{\bar{\Xi}(\bar{\varrho})}{(p-q)(\bar{\varrho}-a)} \ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}$$
$$= \frac{1}{2q}\left[(p+q)\sum_{\tilde{\varpi}=0}^{\infty}\bar{\Xi}\left(\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}}}c+\left(1-\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}}}\right)a\right)-(p+q)\bar{\Xi}(c)-p\bar{\Xi}\left(\frac{qc+(p-q)a}{p}\right)\right].$$

Substituting the values of I_1, I_2, I_3 and I_4 in (3.1), we obtain the required result. □

Theorem 3.5. *Let $\bar{\Xi}_1, \bar{\Xi}_2 : I \to \mathbb{R}$ be continuous functions with $c \in (a, x)$, then*

$$\int_c^x \bar{\Xi}_1(p\bar{\varrho}+(1-p)a)_a D_{p,q}\bar{\varrho}\bar{\Xi}_2(\bar{\varrho}) \ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}$$
$$= \left.\frac{q\bar{\Xi}_1(\bar{\varrho})\bar{\Xi}_2(\bar{\varrho})+p\bar{\Xi}_1\left(\frac{q\bar{\varrho}+(p-q)a}{p}\right)\bar{\Xi}_2\left(\frac{q\bar{\varrho}+(p-q)a}{p}\right)}{2q}\right|_c^x$$
$$-\int_c^x \bar{\Xi}_2(q\bar{\varrho}+(1-q)a)_a D_{p,q}\bar{\varrho}\bar{\Xi}_1(\bar{\varrho}) \ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}. \tag{3.2}$$

Proof. Using the definition of (p, q)-derivative, we have

$${}_aD_{p,q}\bar{\varrho}\bar{\Xi}_1(\bar{\varrho})\bar{\Xi}_2(\bar{\varrho})$$
$$= \frac{\bar{\Xi}_1(p\bar{\varrho}+(1-p)a)\bar{\Xi}_2(p\bar{\varrho}+(1-p)a)-\bar{\Xi}_1(q\bar{\varrho}+(1-q)a)\bar{\Xi}_1(q\bar{\varrho}+(1-q)a)}{(p-q)(\bar{\varrho}-a)}$$
$$= \bar{\Xi}_1(p\bar{\varrho}+(1-p)a)_a D_{p,q}\bar{\varrho}\bar{\Xi}_2(\bar{\varrho})+\bar{\Xi}_2(q\bar{\varrho}+(1-q)a)_a D_{p,q}\bar{\varrho}\bar{\Xi}_1(\bar{\varrho}).$$

Now by taking ${}_a\bar{T}_{p,q}$-integrals, we have

$$\int_c^x {}_aD_{p,q}\bar{\varrho}\bar{\Xi}_1(\bar{\varrho})\bar{\Xi}_2(\bar{\varrho}) \ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}$$
$$= \int_c^x \bar{\Xi}_1(p\bar{\varrho}+(1-p)a)_a D_{p,q}\bar{\varrho}\bar{\Xi}_2(\bar{\varrho}) \ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}+\int_c^x \bar{\Xi}_2(q\bar{\varrho}+(1-q)a)_a D_{p,q}\bar{\varrho}\bar{\Xi}_1(\bar{\varrho}) \ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}.$$

Now by making use of Theorem 3.4, we have

$$\int_c^x \bar{\Xi}_1(p\bar{\varrho}+(1-p)a)_a D_{p,q}\bar{\varrho}\bar{\Xi}_2(\bar{\varrho}) \ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}$$
$$= \left.\frac{q\bar{\Xi}_1(\bar{\varrho})\bar{\Xi}_2(\bar{\varrho})+p\bar{\Xi}_1\left(\frac{q\bar{\varrho}+(p-q)a}{p}\right)\bar{\Xi}_2\left(\frac{q\bar{\varrho}+(p-q)a}{p}\right)}{2q}\right|_c^x$$
$$-\int_c^x \bar{\Xi}_2(q\bar{\varrho}+(1-q)a)_a D_{p,q}\bar{\varrho}\bar{\Xi}_1(\bar{\varrho}) \ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}.$$

This completes the proof. □

Theorem 3.6. *Let $\bar{\Xi}_1, \bar{\Xi}_2 : I \to \mathbb{R}$ be continuous functions such that $\bar{\Xi}_1(\bar{\varrho}) \leq \bar{\Xi}_2(\bar{\varrho})$ for all $\bar{\varrho} \in [a, b]$, then we have*

$$\int_a^x \bar{\Xi}_1(\bar{\varrho})\ {}_a\mathsf{d}_{p,q}^{\bar{T}}\bar{\varrho} \leq \int_a^x \bar{\Xi}_2(\bar{\varrho})\ {}_a\mathsf{d}_{p,q}^{\bar{T}}\bar{\varrho},$$

for $x \in [a, qb + (1-p)a]$.

Proof. By definition ${}_a\bar{T}_{p,q}$ integral, we have

$$\begin{aligned}
&\int_a^x \bar{\Xi}_1(\bar{\varrho})\ {}_a\mathsf{d}_{p,q}^{\bar{T}}\bar{\varrho} \\
&= \frac{(p-q)(x-a)}{2q}\left[(p+q)\sum_{\tilde{\varpi}=0}^{\infty}\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}\bar{\Xi}_1\left(\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}x + \left(1 - \frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}\right)a\right) - \bar{\Xi}_1\left(\frac{x+(p-1)a}{p}\right)\right] \\
&= \frac{(p-q)(x-a)(p+q)}{2q}\sum_{\tilde{\varpi}=1}^{\infty}\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}\bar{\Xi}_1\left(\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}x + \left(1 - \frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}\right)a\right) \\
&\leq \frac{(p-q)(x-a)(p+q)}{2q}\sum_{\tilde{\varpi}=1}^{\infty}\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}\bar{\Xi}_2\left(\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}x + \left(1 - \frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}\right)a\right) \\
&= \frac{(p-q)(x-a)}{2q}\left[(p+q)\sum_{\tilde{\varpi}=0}^{\infty}\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}\bar{\Xi}_2\left(\frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}x + \left(1 - \frac{q^{\tilde{\varpi}}}{p^{\tilde{\varpi}+1}}\right)a\right) - \bar{\Xi}_2\left(\frac{x+(p-1)a}{p}\right)\right] \\
&= \int_a^x \bar{\Xi}_2(\bar{\varrho})\ {}_a\mathsf{d}_{p,q}^{\bar{T}}\bar{\varrho}.
\end{aligned}$$

This completes the proof. □

4. ${}_a\bar{T}_{p,q}$-analogues of certain inequalities

In this section, we derive some new ${}_a\bar{T}_{p,q}$-analogues of certain classical inequalities.

4.1. ${}_a\bar{T}_{p,q}$-Young's inequality

Theorem 4.1. *Let $a, b > 0$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ with $\alpha > 1$, then*

$$a.b \leq \frac{1}{2}\left[\frac{(p^{\alpha-1} + q^{\alpha-1})}{[\alpha]_{p,q}}a^{\alpha} + \frac{(p^{\beta-1} + q^{\beta-1})}{[\beta]_{p,q}}b^{\beta}\right].$$

Proof. Choose $g = x^{\alpha-1}$ functions for $\alpha > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. We draw the graph of $g = x^{\alpha-1}$.

$$c_1 = \int_0^a x^{\alpha-1}\ {}_0\mathsf{d}_{p,q}^{\bar{T}}x = \frac{(p^{\alpha-1} + q^{\alpha-1})}{2[\alpha]_{p,q}}a^{\alpha}.$$

and

$$c_2 = \int_0^b g^{\frac{1}{\alpha-1}}\ {}_0\mathsf{d}_{p,q}^{\bar{T}}g = \frac{(p^{\beta-1} + q^{\beta-1})}{2[\beta]_{p,q}}b^{\beta}.$$

According to Figure 1,

$$
\begin{aligned}
a.b &\leq c_1 + c_2 \\
&\leq \frac{1}{2}\left[\frac{(p^{\alpha-1}+q^{\alpha-1})}{[\alpha]_{p,q}}a^{\alpha} + \frac{(p^{\beta-1}+q^{\beta-1})}{[\beta]_{p,q}}b^{\beta}\right].
\end{aligned}
$$

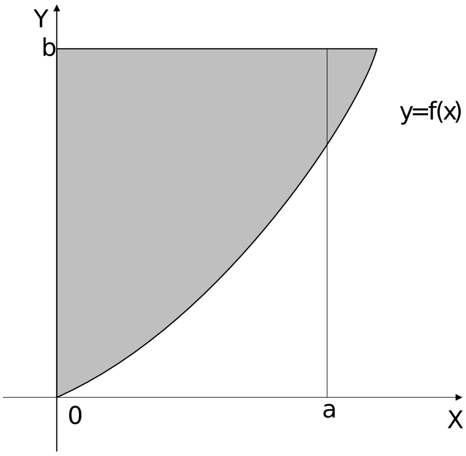

Figure 1. $g = x^{\alpha-1}$.

This completes the proof. □

4.2. ${}_a\bar{T}_{p,q}$-Hölder's inequality

Theorem 4.2. *Let* $\frac{1}{\alpha}+\frac{1}{\beta}=1$ *with* $\alpha > 1$, *then we have*

$$
\int_a^b \bar{\Xi}_1(\bar{\varrho})\bar{\Xi}_2(\bar{\varrho})\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho} = \left[\frac{(p^{\alpha-1}+q^{\alpha-1})}{2[\alpha]_{p,q}} + \frac{(p^{\beta-1}+q^{\beta-1})}{2[\beta]_{p,q}}\right]\|\bar{\Xi}_1\|_{\alpha}\|\bar{\Xi}_2\|_{\beta},
$$

where $\|\bar{\Xi}_1\|_r = \left(\int_a^b |\bar{\Xi}_1(\bar{\varrho})|^r\,{}_a d_{\bar{p},\bar{q}}\bar{\varrho}\right)^{\frac{1}{r}}$.

Proof. By taking $a = \frac{|\bar{\Xi}_1(\bar{\varrho})|}{\|\bar{\Xi}_1\|_{\alpha}}$, $b = \frac{|\bar{\Xi}_2(\bar{\varrho})|}{\|\bar{\Xi}_2\|_{\beta}}$ and using ${}_a\bar{T}_{p,q}$-Young Inequality,

$$
\frac{|\bar{\Xi}_1(\bar{\varrho})|}{\|\bar{\Xi}_1\|_{\alpha}}\frac{|\bar{\Xi}_2(\bar{\varrho})|}{\|\bar{\Xi}_2\|_{\beta}} \leq \frac{(p^{\alpha-1}+q^{\alpha-1})}{2[\alpha]_{p,q}}\frac{|\bar{\Xi}_1(\bar{\varrho})|^{\alpha}}{\|\bar{\Xi}_1\|_{\alpha}^{\alpha}} + \frac{(p^{\beta-1}+q^{\beta-1})}{2[\beta]_{p,q}}\frac{|\bar{\Xi}_2(\bar{\varrho})|^{\beta}}{\|\bar{\Xi}_2\|_{\beta}^{\beta}}.
$$

Now by taking ${}_a\bar{T}_{p,q}$-integral of the above inequality,

$$
\begin{aligned}
&\int_a^b \frac{|\bar{\Xi}_1(\bar{\varrho})||\bar{\Xi}_2(\bar{\varrho})|\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}}{\|\bar{\Xi}_1\|_{\alpha}\|\bar{\Xi}_2\|_{\beta}} \leq \int_a^b \frac{(p^{\alpha-1}+q^{\alpha-1})}{2[\alpha]_{p,q}}\frac{|\bar{\Xi}_1(\bar{\varrho})|^{\alpha}}{\|ft\|_{\alpha}^{\alpha}}\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho} + \frac{(p^{\beta-1}+q^{\beta-1})}{2[\beta]_{p,q}}\int_a^b \frac{|\bar{\Xi}_2(\bar{\varrho})|^{\beta}}{\|\bar{\Xi}_2\|_{\beta}^{\beta}}\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho} \\
&\leq \frac{(p^{\alpha-1}+q^{\alpha-1})}{2[\alpha]_{p,q}} + \frac{(p^{\beta-1}+q^{\beta-1})}{2[\beta]_{p,q}}.
\end{aligned}
$$

This completes the proof. □

4.3. ${}_{a}\bar{T}_{p,q}$-Minkowski inequality

Theorem 4.3. *Let $\frac{1}{\alpha}+\frac{1}{\beta}=1$ with $\alpha>1$, then we have*

$$\|\bar{\Xi}_1+\bar{\Xi}_2\|_\alpha=\left[\frac{(p^{\alpha-1}+q^{\alpha-1})}{2[\alpha]_{p,q}}+\frac{(p^{\beta-1}+q^{\beta-1})}{2[\beta]_{p,q}}\right][\|\bar{\Xi}_1\|_\alpha+\|\bar{\Xi}_2\|_\alpha],$$

where $\|\bar{\Xi}_1\|_r=\left(\int_a^b|\bar{\Xi}_1(\varrho)|^r\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}\right)^{\frac{1}{r}}$.

Proof. From ${}_{a}\bar{T}_{p,q}$ Hölder's inequality, we get

$$\begin{aligned}
&\|\bar{\Xi}_1+\bar{\Xi}_2\|_\alpha^\alpha\\
&\leq\int_a^b|\bar{\Xi}_1(\varrho)||\bar{\Xi}_1(\varrho)+\bar{\Xi}_2(\varrho)|^{\alpha-1}\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}+\int_a^b|\bar{\Xi}_2(\varrho)||\bar{\Xi}_1(\varrho)+\bar{\Xi}_2(\varrho)|^{\alpha-1}\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}\\
&\leq\left[\frac{(p^{\alpha-1}+q^{\alpha-1})}{2[\alpha]_{p,q}}+\frac{(p^{\beta-1}+q^{\beta-1})}{2[\beta]_{p,q}}\right]\left(\int_a^b|\bar{\Xi}_1(\varrho)+\bar{\Xi}_2(\varrho)|^{\beta(\alpha-1)}\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}\right)^{\frac{1}{\beta}}\\
&\times\left(\left(\int_a^b|\bar{\Xi}_1(\varrho)|^\alpha\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}\right)^{\frac{1}{\alpha}}+\left(\int_a^b|\bar{\Xi}_1(\varrho)|^\alpha\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}\right)^{\frac{1}{\alpha}}\right)\\
&\leq\|\bar{\Xi}_1+\bar{\Xi}_2\|_\alpha^{\alpha-1}\left[\frac{(p^{\alpha-1}+q^{\alpha-1})}{2[\alpha]_{p,q}}+\frac{(p^{\beta-1}+q^{\beta-1})}{2[\beta]_{p,q}}\right][\|\bar{\Xi}_1\|_\alpha+\|\bar{\Xi}_2\|_\alpha].
\end{aligned}$$

Dividing both sides by $\|\bar{\Xi}_1+\bar{\Xi}_2\|_\alpha^{\alpha-1}$,we obtain our required result. □

4.4. ${}_{a}\bar{T}_{p,q}$-Ostrowski inequality

Lemma 4.1. *Let $\bar{\Xi}:[a,b]\to\mathbb{R}$ be (p,q)-differentiable function (a,b), then for $0<q<p\leq1$, then*

$$\begin{aligned}
&\frac{1}{2q}\left[q\bar{\Xi}(x)+p\bar{\Xi}\left(\frac{qx+(p-q)a}{p}\right)-(p-q)\bar{\Xi}\left(qb+(p-q)a\right)\right]\\
&-\frac{1}{p\,(b-a)}\int_a^{qb+(1-p)a}\bar{\Xi}(q\bar{\varrho}+(1-q)a)\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}\\
&=\frac{1}{p\,(b-a)}\left[\int_a^x p(\bar{\varrho}-a)_aD_{p,q}\bar{\Xi}(\bar{\varrho})\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}+\int_x^{qb+(1-p)a}p(\bar{\varrho}-b)_aD_{p,q}\bar{\Xi}(\bar{\varrho})\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}\right].
\end{aligned}$$

Proof. Consider the right hand side and making use of Theorem 3.5, we have

$$\begin{aligned}
&\frac{1}{b-a}\left[\int_a^x p(\bar{\varrho}-a)_aD_{p,q}\bar{\Xi}(\bar{\varrho})\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}+\int_x^b(p\bar{\varrho}+(1-p)a-b)_aD_{p,q}\bar{\Xi}(\bar{\varrho})\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}\right]\\
&=\frac{1}{b-a}[I_3+I_4],
\end{aligned}\tag{4.1}$$

where

$$
\begin{aligned}
I_3 &= \int_a^x p(\bar{\varrho}-a)_a D_{p,q}\bar{\Xi}(\bar{\varrho})\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho} \\
&= \left.\frac{q(\bar{\varrho}-a)\bar{\Xi}(\bar{\varrho}) + (q\bar{\varrho}+(p-q)a-pa)\bar{\Xi}\left(\frac{q\bar{\varrho}+(p-q)a}{p}\right)}{2q}\right|_a^x \\
&\quad - \int_a^x \bar{\Xi}(q\bar{\varrho}+(1-q)a)_a D_{p,q}(\bar{\varrho}-a)\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho} \\
&= \frac{q(x-a)\bar{\Xi}(x) + q(x-a)\bar{\Xi}\left(\frac{qx+(p-q)a}{p}\right)}{2q} \\
&\quad - \int_a^x \bar{\Xi}(q\bar{\varrho}+(1-q)a)\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
I_4 &= \int_x^{qb+(1-p)a} p(\bar{\varrho}-b)_a D_{p,q}\bar{\Xi}(\bar{\varrho})\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho} \\
&= \int_x^{qb+(1-p)a} (p\bar{\varrho}+(1-p)a-(qb+(1-p)a))_a D_{p,q}\bar{\Xi}(\bar{\varrho})\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho} \\
&= \frac{q(qb+(1-p)a-x)\bar{\Xi}(x) - p\,(p-q)\,(b-a)\bar{\Xi}\,(qb+(1-q)a) - \left[q(x-a)-p^2(b-a)\right]\bar{\Xi}\left(\frac{qx+(p-q)a}{p}\right)}{2q} \\
&\quad - \int_x^{qb+(1-p)a} \bar{\Xi}(q\bar{\varrho}+(1-q)a)\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}.
\end{aligned}
$$

By substituting the values of I_1 and I_2 in (4.1), we obtain our required result. □

Lemma 4.2. *Let $p > 1$ and $(a-\bar{\varrho})^{\tilde{\varpi}}_{p,q}$ is (p,q)-binomial, then*

$$
\int_0^x (a-\bar{\varrho})^{\tilde{\varpi}}_{p,q}\ {}_0\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho} = \frac{a^{\tilde{\varpi}+1}(p+q)}{2p[\tilde{\varpi}+1]_{p,q}} - \left(\frac{q\left(a-\frac{px}{q}\right)^{\tilde{\varpi}+1} + p(a-x)^{\tilde{\varpi}+1}}{2p[\tilde{\varpi}+1]_{p,q}}\right).
$$

Proof. By using (p,q)-binomial formula and Gauss binomial formula

$$
\begin{aligned}
&\int_0^x (a-\bar{\varrho})^{\tilde{\varpi}}_{p,q}\ {}_0\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho} \\
&= \int_0^x \sum_{k=0}^{\tilde{\varpi}} (-1)^k \binom{\tilde{\varpi}}{k}_{p,q} p^{\frac{(\tilde{\varpi}-k)(\tilde{\varpi}-k-1)}{2}} q^{\frac{k(k-1)}{2}} \bar{\varrho}^k a^{\tilde{\varpi}-k}\ {}_0\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho} \\
&= \int_0^x \sum_{k=0}^{\tilde{\varpi}} (-1)^k \frac{[\tilde{\varpi}]_{p,q}!}{[k]_{p,q}![\tilde{\varpi}-k]_{p,q}!}_{p,q} p^{\frac{(\tilde{\varpi}-k)(\tilde{\varpi}-k-1)}{2}} q^{\frac{k(k-1)}{2}} \bar{\varrho}^k a^{\tilde{\varpi}-k}\ {}_0\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho} \\
&= \sum_{k=0}^{\tilde{\varpi}} (-1)^k \frac{[\tilde{\varpi}]_{p,q}!}{[k]_{p,q}![\tilde{\varpi}-k]_{p,q}!}_{p,q} p^{\frac{(\tilde{\varpi}-k)(\tilde{\varpi}-k-1)}{2}} q^{\frac{k(k-1)}{2}} a^{\tilde{\varpi}-k} \int_0^x \bar{\varrho}^k\ {}_0\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho} \\
&= \sum_{k=0}^{\tilde{\varpi}} (-1)^k \frac{[\tilde{\varpi}]_{p,q}!}{[k]_{p,q}![\tilde{\varpi}-k]_{p,q}!} p^{\frac{(\tilde{\varpi}-k)(\tilde{\varpi}-k-1)}{2}} q^{\frac{k(k-1)}{2}} a^{\tilde{\varpi}-k} \frac{(p^k+q^k)x^{k+1}}{2[k+1]_{p,q}}
\end{aligned}
$$

$$= \sum_{k=0}^{\tilde{\varpi}} (-1)^k \frac{[\tilde{\varpi}]_{p,q}!}{[k+1]_{p,q}![\tilde{\varpi}-k]_{p,q}!} p^{\frac{(\tilde{\varpi}-k)(\tilde{\varpi}-k-1)}{2}} q^{\frac{k(k-1)}{2}} a^{\tilde{\varpi}-k} \frac{(p^k+q^k)x^{k+1}}{2}.$$

Replacing k by $k-1$

$$= \sum_{k=1}^{\tilde{\varpi}+1} (-1)^{k-1} \frac{[\tilde{\varpi}+1]_{p,q}!}{[k]_{p,q}![\tilde{\varpi}+1-k]_{p,q}!} p^{\frac{(\tilde{\varpi}-k+1)(\tilde{\varpi}-k)}{2}} q^{\frac{(k-1)(k-2)}{2}} a^{\tilde{\varpi}+1-k} \frac{(p^k+q^k)x^k}{2}$$

$$= \frac{-1}{2[\tilde{\varpi}+1]_{p,q}} \left[\frac{q}{p} \sum_{k=1}^{\tilde{\varpi}+1} (-1)^{k-1} \frac{[\tilde{\varpi}]_{p,q}!}{[k]_{p,q}![\tilde{\varpi}+1-k]_{p,q}!} p^{\frac{(\tilde{\varpi}-k+1)(\tilde{\varpi}-k)}{2}} q^{\frac{k(k-1)}{2}} a^{\tilde{\varpi}+1-k} \frac{x^k}{q^k 2} \right.$$

$$\left. + \sum_{k=1}^{\tilde{\varpi}+1} (-1)^{k-1} \frac{[\tilde{\varpi}+1]_{p,q}!}{[k]_{p,q}![\tilde{\varpi}+1-k]_{p,q}!} p^{\frac{(\tilde{\varpi}-k+1)(\tilde{\varpi}-k)}{2}} q^{\frac{k(k-1)}{2}} a^{\tilde{\varpi}+1-k} x^k \right]$$

$$= \frac{a^{\tilde{\varpi}+1}(p+q)}{2p[\tilde{\varpi}+1]_{p,q}} - \left(\frac{q\left(a-\frac{px}{q}\right)^{\tilde{\varpi}+1} + p(a-x)^{\tilde{\varpi}+1}}{2p[\tilde{\varpi}+1]_{p,q}} \right).$$

□

Example 4.1. *Let $p > 1$ and $(1-\varrho)_{p,q}^{\tilde{\varpi}}$ is (p,q)-binomial, then*

$$\int_0^1 (1-\varrho)_{p,q}^{\tilde{\varpi}} \, {}_0\mathsf{d}_{p,q}^{\bar{T}}\varrho = \frac{(p+q)}{2p[\tilde{\varpi}+1]_{p,q}}.$$

Theorem 4.4. *Let $\bar{\Xi} : I = [a,b] \to \mathbb{R}$ be a (p,q)-differentiable convex function on (a,b) and $\left|{}_aD_{p,q}\bar{\Xi}(x)\right| \le \mathcal{M}$ for all $x \in [a,b]$, then we have*

$$\left| \frac{1}{2q} \left[q\bar{\Xi}(x) + p\bar{\Xi}\left(\frac{qx+(p-q)a}{p}\right) - (p-q)\bar{\Xi}\left(qb+(p-q)a\right) \right] \right.$$
$$\left. - \frac{1}{p(b-a)} \int_a^{qb+(1-p)a} \bar{\Xi}(q\varrho+(1-q)a) \, {}_a\mathsf{d}_{p,q}^{\bar{T}}\varrho \right|$$
$$\le \frac{\mathcal{M}}{p(b-a)} \left((x-a)^2 + \frac{p^2(b-a)^2}{2} - p(x-a)(b-a) \right).$$

Proof. Using Lemma 4.1, modulus property and $\left|{}_aD_{p,q}\bar{\Xi}(\varrho)\right| \le \mathcal{M}$

$$\left| \frac{1}{2q} \left[q\bar{\Xi}(x) + p\bar{\Xi}\left(\frac{qx+(p-q)a}{p}\right) - (p-q)\bar{\Xi}\left(qb+(p-q)a\right) \right] \right.$$
$$\left. - \frac{1}{p(b-a)} \int_a^{qb+(1-p)a} \bar{\Xi}(q\varrho+(1-q)a) \, {}_a\mathsf{d}_{p,q}^{\bar{T}}\varrho \right|$$
$$\le \frac{1}{p(b-a)} \left[\int_a^x p|(\varrho-a)||{}_aD_{p,q}\bar{\Xi}(\varrho)| \, {}_a\mathsf{d}_{p,q}^{\bar{T}}\varrho + \int_x^{qb+(1-p)a} |p(\varrho-b)||{}_aD_{p,q}\bar{\Xi}(\varrho)| \, {}_a\mathsf{d}_{p,q}^{\bar{T}}\varrho \right]$$
$$\le \frac{\mathcal{M}}{p(b-a)} \left[\int_a^x p(\varrho-a) \, {}_a\mathsf{d}_{p,q}^{\bar{T}}\varrho + \int_x^{qb+(1-p)a} p(b-\varrho) \, {}_a\mathsf{d}_{p,q}^{\bar{T}}\varrho \right]$$

$$\leq \frac{\mathcal{M}}{p(b-a)}\left(\frac{(x-a)^2}{2}+\frac{p^2(b-a)^2}{2}-p(x-a)(b-a)+\frac{(x-a)^2}{2}\right)$$
$$\leq \frac{\mathcal{M}}{p(b-a)}\left((x-a)^2+\frac{p^2(b-a)^2}{2}-p(x-a)(b-a)\right).$$

This completes the proof. □

Theorem 4.5. *Let $\bar{\Xi}: I=[a,b]\to\mathbb{R}$ be a (p,q)-differentiable on (a,b) and $\left|{}_aD_{p,q}\bar{\Xi}(x)\right|$ is convex for all $x\in[a,b]$, then we have*

$$\left|\frac{1}{2q}\left[q\bar{\Xi}(x)+p\bar{\Xi}\left(\frac{qx+(p-q)a}{p}\right)-(p-q)\bar{\Xi}\left(qb+(p-q)a\right)\right]\right.$$
$$\left.-\frac{1}{p(b-a)}\int_a^{qb+(1-p)a}\bar{\Xi}(q\varrho+(1-q)a)\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\varrho\right|$$
$$\leq \frac{1}{(b-a)^2}\left[\left(\frac{p^2+q^2}{p^2[3]_{p,q}}(x-a)^3+\frac{p^2q}{2[3]_{p,q}}(b-a)^3-\frac{(b-a)(x-a)^2}{2p}\right)|{}_aD_{p,q}\bar{\Xi}(b)|\right.$$
$$\left.+\left(\frac{p^2+q^2}{2[3]_{p,q}}(b-a)^3-\frac{(b-a)(x-a)^2}{2p}-\frac{p^2+q^2}{p^2[3]_{p,q}}(x-a)^3-(b-a)^2(x-a)\right)|{}_aD_{p,q}\bar{\Xi}(a)|\right].$$

Proof. Using Lemma 4.1, modulus property and convexity of $\left|{}_aD_{p,q}\bar{\Xi}(\varrho)\right|$, then we have

$$\left|\frac{1}{2q}\left[q\bar{\Xi}(x)+p\bar{\Xi}\left(\frac{qx+(p-q)a}{p}\right)-(p-q)\bar{\Xi}\left(qb+(p-q)a\right)\right]\right.$$
$$\left.-\frac{1}{p(b-a)}\int_a^{qb+(1-p)a}\bar{\Xi}(q\varrho+(1-q)a)\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\varrho\right|$$
$$\leq \frac{1}{p(b-a)}\left[\int_a^x p|(\varrho-a)||{}_aD_{p,q}\bar{\Xi}(\varrho)|\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\varrho+\int_x^{qb+(1-p)a}|p(\varrho-b)||{}_aD_{p,q}\bar{\Xi}(\varrho)|\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\varrho\right]$$
$$\leq \frac{1}{p(b-a)}\left[\int_a^x p(\varrho-a)\left[\frac{\varrho-a}{b-a}|{}_aD_{p,q}\bar{\Xi}(b)|+\frac{b-\varrho}{b-a}|{}_aD_{p,q}\bar{\Xi}(a)|\right]\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\varrho\right.$$
$$\left.\int_x^{qb+(1-p)a} p(b-\varrho)\left[\frac{\varrho-a}{b-a}|{}_aD_{p,q}\bar{\Xi}(b)|+\frac{b-\varrho}{b-a}|{}_aD_{p,q}\bar{\Xi}(a)|\right]\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\varrho\right]$$
$$=\frac{1}{(b-a)^2}\left[|{}_aD_{p,q}\bar{\Xi}(b)|\left(\int_a^x(\varrho-a)^2\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\varrho+\int_x^{qb+(1-p)a}(b-\varrho)(\varrho-a)\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\varrho\right)+\right.$$
$$\left.+|{}_aD_{p,q}\bar{\Xi}(a)|\left(\int_a^x(\varrho-a)(b-\varrho)\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\varrho+\int_x^{qb+(1-p)a}(b-\varrho)^2\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\varrho\right)\right]$$
$$=\frac{1}{(b-a)^2}\left[|{}_aD_{p,q}\bar{\Xi}(b)|\left(\frac{p^2+q^2}{p^2[3]_{p,q}}(x-a)^3+\frac{p^2q}{2[3]_{p,q}}(b-a)^3-\frac{(b-a)(x-a)^2}{2p}\right)\right.$$
$$\left.+|{}_aD_{p,q}\bar{\Xi}(a)|\left(\frac{p^2+q^2}{2[3]_{p,q}}(b-a)^3-\frac{(b-a)(x-a)^2}{2p}-\frac{p^2+q^2}{p^2[3]_{p,q}}(x-a)^3-(b-a)^2(x-a)\right)\right],$$

which proves the required result. □

Theorem 4.6. *Let $\bar{\Xi} : I = [a,b] \to \mathbb{R}$ be a (p,q)-differentiable on (a,b) and $\left|{}_aD_{p,q}\bar{\Xi}(x)\right|^{\beta}$ is convex for all $x \in [a,b]$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ then we have*

$$\begin{aligned}
&\left|\frac{1}{2q}\left[q\bar{\Xi}(x) + p\bar{\Xi}\left(\frac{qx+(p-q)a}{p}\right) - (p-q)\bar{\Xi}\left(qb+(p-q)a\right)\right]\right. \\
&\left.-\frac{1}{p\,(b-a)}\int_a^{qb+(1-p)a} \bar{\Xi}(q\bar{\varrho}+(1-q)a)\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}\right| \\
&\leq \frac{1}{(b-a)^2}\left[\frac{(p^{\alpha-1}+q^{\alpha-1})}{2[\alpha]_{p,q}} + \frac{(p^{\beta-1}+q^{\beta-1})}{2[\beta]_{p,q}}\right] \\
&\times\left[(x-a)^2\left(\frac{(p^{\alpha}+q^{\alpha})}{2p^{\alpha}\,[\alpha+1]_{p,q}}\right)^{\frac{1}{\alpha}}\right. \\
&\times\left(\frac{(x-a)\,|{}_aD_{p,q}\bar{\Xi}(b)|^{\beta} + (2p(b-a)-(x-a))\,|{}_aD_{p,q}\bar{\Xi}(a)|^{\beta}}{2p}\right)^{\frac{1}{\beta}} \\
&+\left(\int_x^{qb+(1-p)a}(b-\bar{\varrho})^{\alpha}\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}\right)^{\frac{1}{\alpha}} \\
&\left.\times\left(\left(\frac{\left(p^2(b-a)^2-(x-a)^2\right)|{}_aD_{p,q}\bar{\Xi}(b)|^{\beta} + (p(b-a)-(x-a))^2\,|{}_aD_{p,q}\bar{\Xi}(a)|^{\beta}}{2p}\right)\right)^{\frac{1}{\beta}}\right].
\end{aligned}$$

Proof. Using Lemma 4.1, modulus property, Hölder's inequality and convexity of $\left|{}_aD_{p,q}\bar{\Xi}(\bar{\varrho})\right|^{\beta}$, then

$$\begin{aligned}
&\left|\frac{1}{2q}\left[q\bar{\Xi}(x) + p\bar{\Xi}\left(\frac{qx+(p-q)a}{p}\right) - (p-q)\bar{\Xi}\left(qb+(p-q)a\right)\right]\right. \\
&\left.-\frac{1}{p\,(b-a)}\int_a^{qb+(1-p)a} \bar{\Xi}(q\bar{\varrho}+(1-q)a)\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}\right| \\
&\leq \frac{1}{p\,(b-a)}\left[\int_a^x p|(\bar{\varrho}-a)||{}_aD_{p,q}\bar{\Xi}(\bar{\varrho})|\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho} + \int_x^{qb+(1-p)a}|p(\bar{\varrho}-b)||{}_aD_{p,q}\bar{\Xi}(\bar{\varrho})|\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}\right] \\
&\leq \frac{1}{b-a}\left[\frac{(p^{\alpha-1}+q^{\alpha-1})}{2[\alpha]_{p,q}} + \frac{(p^{\beta-1}+q^{\beta-1})}{2[\beta]_{p,q}}\right] \\
&\times\left[\left(\int_a^x(\bar{\varrho}-a)^{\alpha}\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}\right)^{\frac{1}{\alpha}}\left(\int_a^x|{}_aD_{p,q}\bar{\Xi}(\bar{\varrho})|^{\beta}\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}\right)^{\frac{1}{\beta}}\right. \\
&\left.+\left(\int_x^{qb+(1-p)a}(b-\bar{\varrho})^{\alpha}\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}\right)^{\frac{1}{\alpha}}\left(\int_x^{qb+(1-p)a}|{}_aD_{p,q}\bar{\Xi}(\bar{\varrho})|^{\beta}\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}\right)^{\frac{1}{\beta}}\right] \\
&\leq \frac{1}{b-a}\left[\frac{(p^{\alpha-1}+q^{\alpha-1})}{2[\alpha]_{p,q}} + \frac{(p^{\beta-1}+q^{\beta-1})}{2[\beta]_{p,q}}\right] \\
&\times\left[\left(\frac{(p^{\alpha}+q^{\alpha})(x-a)^{\alpha+1}}{2p^{\alpha}\,[\alpha+1]_{p,q}}\right)^{\frac{1}{\alpha}}\left(\int_a^x\left[\frac{\bar{\varrho}-a}{b-a}|{}_aD_{p,q}\bar{\Xi}(b)|^{\beta} + \frac{b-\bar{\varrho}}{b-a}|{}_aD_{p,q}\bar{\Xi}(a)|^{\beta}\right]\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}\right)^{\frac{1}{\beta}}\right. \\
&\left.+\left(\int_x^{qb+(1-p)a}(b-\bar{\varrho})^{\alpha}\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}\right)^{\frac{1}{\alpha}}\left(\int_x^{qb+(1-p)a}\left[\frac{\bar{\varrho}-a}{b-a}|{}_aD_{p,q}\bar{\Xi}(b)|^{\beta} + \frac{b-\bar{\varrho}}{b-a}|{}_aD_{p,q}\bar{\Xi}(a)|^{\beta}\right]\ {}_a\mathsf{d}^{\bar{T}}_{p,q}\bar{\varrho}\right)^{\frac{1}{\beta}}\right] \\
&= \frac{1}{(b-a)^2}\left[\frac{(p^{\alpha-1}+q^{\alpha-1})}{2[\alpha]_{p,q}} + \frac{(p^{\beta-1}+q^{\beta-1})}{2[\beta]_{p,q}}\right]
\end{aligned}$$

$$\times\left[\left(\frac{(p^{\alpha}+q^{\alpha})(x-a)^{\alpha+1}}{2p^{\alpha}\,[\alpha+1]_{p,q}}\right)^{\frac{1}{\alpha}}\right.$$
$$\times\left(\frac{(x-a)^2}{2p}|{}_aD_{p,q}\bar{\Xi}(b)|^{\beta}+\left((b-a)(x-a)-\frac{(x-a)^2}{2p}\right)|{}_aD_{p,q}\bar{\Xi}(a)|^{\beta}\right)^{\frac{1}{\beta}}$$
$$+\left(\int_x^{qb+(1-p)a}(b-\bar{\varrho})^{\alpha}\ {}_a\mathrm{d}^{\bar{T}}_{p,q}\bar{\varrho}\right)^{\frac{1}{\alpha}}$$
$$\left.\times\left(\left(\frac{p^2(b-a)^2-(x-a)^2}{2p}\right)|{}_aD_{p,q}\bar{\Xi}(b)|^{\beta}+\left(\frac{p(b-a)^2}{2}-(b-a)(x-a)+\frac{(x-a)^2}{2p}\right)|{}_aD_{p,q}\bar{\Xi}(a)|^{\beta}\right)^{\frac{1}{\beta}}\right].$$

This completes the proof. □

4.5. ${}_a\bar{T}_{p,q}$-Hermite-Hadamard's inequality.

We now derive some new variants of classical Hermite-Hadamard's inequality essentially using ${}_a\bar{T}_{p,q}$-integrals.

Theorem 4.7. *Let $\bar{\Xi}: I=[a,b]\to\mathbb{R}$ be a continuous and convex function on (a,b) with $0<q<p\leq 1$, we have*

$$\bar{\Xi}\left(\frac{a+b}{2}\right)\leq\frac{1}{\mathfrak{p}\,(b-a)}\int_a^{pb+(1-p)a}\bar{\Xi}(x)\ {}_a\mathrm{d}^{\bar{T}}_{p,q}x\leq\frac{\bar{\Xi}(a)+\bar{\Xi}(b)}{2}. \tag{4.2}$$

Proof. Since $\bar{\Xi}$ is a differentiable function at $[a,b]$ then there exist a tangent at $\frac{a+b}{2}\in(a,b)$, tangent line (see Figure 2) can be expressed as function $\bar{\Xi}_1(x)=\bar{\Xi}\left(\frac{a+b}{2}\right)+\bar{\Xi}'\left(\frac{a+b}{2}\right)\left(x-\frac{a+b}{2}\right)$.

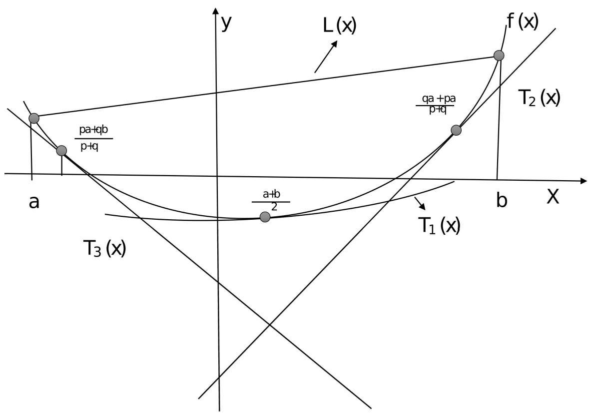

Figure 2. $\bar{\Xi}_1(x)=\bar{\Xi}\left(\frac{a+b}{2}\right)+\bar{\Xi}'\left(\frac{a+b}{2}\right)\left(x-\frac{a+b}{2}\right)$.

Since $\bar{\Xi}$ is convex function on $[a,b]$, then

$$\bar{\Xi}_1(x)=\bar{\Xi}\left(\frac{a+b}{2}\right)+\bar{\Xi}'\left(\frac{a+b}{2}\right)\left(x-\frac{a+b}{2}\right)\leq\bar{\Xi}(x),$$

for all $x\in[a,b]$. By Theorem 3.6, we have

$$\int_a^{pb+(1-p)a}\bar{\Xi}_1(x)\ {}_a\mathrm{d}^{\bar{T}}_{p,q}x\leq\int_a^{pb+(1-p)a}\bar{\Xi}(x)\ {}_a\mathrm{d}^{\bar{T}}_{p,q}x.$$

By definiton of ${}_a\bar{T}_{p,q}$-integrals, we have

$$\begin{aligned}
&\int_a^{pb+(1-p)a} \bar{\Xi}_1(x)\ {}_a\mathsf{d}_{p,q}^{\bar{T}}x \\
&= \int_a^{pb+(1-p)a} \left[\bar{\Xi}\left(\frac{a+b}{2}\right) + \bar{\Xi}'\left(\frac{a+b}{2}\right)\left(x-\frac{a+b}{2}\right)\right]\ {}_a\mathsf{d}_{p,q}^{\bar{T}}x \\
&= p(b-a)\bar{\Xi}\left(\frac{a+b}{2}\right) + \bar{\Xi}'\left(\frac{a+b}{2}\right)\int_a^{pb+(1-p)a}\left(x-a-\frac{b-a}{2}\right)\ {}_a\mathsf{d}_{p,q}^{\bar{T}}x \\
&= p(b-a)\bar{\Xi}\left(\frac{a+b}{2}\right). \qquad (4.3)
\end{aligned}$$

This completes the first ineqıality in (4.2).

Also the line connecting the points $(a, \bar{\Xi}(a))$ and $(b, \bar{\Xi}(b))$ is expressed as

$$\bar{\Xi}(x) \leq k(x) = \bar{\Xi}(a) + \frac{\bar{\Xi}(b)-\bar{\Xi}(a)}{b-a}(x-a).$$

This implies

$$\begin{aligned}
\int_a^{pb+(1-p)a} \bar{\Xi}(x)\ {}_a\mathsf{d}_{p,q}^{\bar{T}}x &\leq \int_a^{pb+(1-p)a}\left(\bar{\Xi}(a) + \frac{\bar{\Xi}(b)-\bar{\Xi}(a)}{b-a}(x-a)\right)\ {}_a\mathsf{d}_{p,q}^{\bar{T}}x \\
&\leq \bar{\Xi}(a)p(b-a) + \frac{\bar{\Xi}(b)-\bar{\Xi}(a)}{b-a}\frac{p(b-a)^2}{2} \\
&\leq p(b-a)\frac{\bar{\Xi}(a)+\bar{\Xi}(b)}{2}. \qquad (4.4)
\end{aligned}$$

This compeletes the proof. □

Theorem 4.8. *Let $\bar{\Xi} : I = [a, b] \to \mathbb{R}$ be a convex continuous function on (a, b) with $0 < q < p \leq 1$, we have*

$$\bar{\Xi}\left(\frac{qa+qb}{p+q}\right) - \frac{(b-a)(p-q)}{2(p+q)}\bar{\Xi}'\left(\frac{qa+qb}{p+q}\right) \leq \frac{1}{p(b-a)}\int_a^{pb+(1-p)a}\bar{\Xi}(x)\ {}_a\mathsf{d}_{p,q}^{\bar{T}}x \leq \frac{\bar{\Xi}(a)+\bar{\Xi}(b)}{2}. \quad (4.5)$$

Proof. Since $\bar{\Xi}$ is a differentiable function at $[a, b]$ then there exist a tangent at $\frac{a+b}{2} \in (a, b)$, tangent line can be expressed as function $\bar{\Xi}_2(x) = \bar{\Xi}\left(\frac{qa+qb}{p+q}\right) + \bar{\Xi}'\left(\frac{qa+qb}{p+q}\right)\left(x - \frac{qa+qb}{p+q}\right)$. since $\bar{\Xi}$ is convex function on $[a, b]$, then

$$\bar{\Xi}_2(x) = \bar{\Xi}\left(\frac{qa+qb}{p+q}\right) + \bar{\Xi}'\left(\frac{qa+qb}{p+q}\right)\left(x-\frac{qa+qb}{p+q}\right) \leq \bar{\Xi}(x), \quad \forall x \in [a, b].$$

Taking ${}_a\bar{T}_{p,q}$-integrals of the above inequality over $[a, b]$,

$$\begin{aligned}
&\int_a^{pb+(1-p)a} \bar{\Xi}_2(x)\ {}_a\mathsf{d}_{p,q}^{\bar{T}}x \qquad (4.6) \\
&= \int_a^{pb+(1-p)a}\left(\bar{\Xi}\left(\frac{qa+qb}{p+q}\right) + \bar{\Xi}'\left(\frac{qa+qb}{p+q}\right)\left(x-\frac{qa+qb}{p+q}\right)\right)\ {}_a\mathsf{d}_{p,q}^{\bar{T}}x
\end{aligned}$$

$$= p\,(b-a)\,\bar{\Xi}\left(\frac{qa+qb}{p+q}\right)+\bar{\Xi}'\left(\frac{qa+qb}{p+q}\right)\int_a^{pb+(1-p)a}\left(x-a-p\frac{b-a}{p+q}\right)\,{}_a\mathsf{d}^{\bar{T}}_{p,q}x$$
$$= p\,(b-a)\,\bar{\Xi}\left(\frac{qa+qb}{p+q}\right)-\bar{\Xi}'\left(\frac{qa+qb}{p+q}\right)\frac{p\,(b-a)^2\,(p-q)}{2\,(p+q)}$$
$$\leq \int_a^{pb+(1-p)a}\bar{\Xi}(x)\,{}_a\mathsf{d}^{\bar{T}}_{p,q}x.$$

Comparing (4.6) and (4.4), we obtain our required result. □

Theorem 4.9. *Let* $\bar{\Xi}: I=[a,b]\to\mathbb{R}$ *be a convex continuous function on* (a,b) *with* $0<q<p\leq 1$*, then*

$$\bar{\Xi}\left(\frac{pa+qb}{p+q}\right)+\frac{(b-a)(p-q)}{2(p+q)}\bar{\Xi}'\left(\frac{pa+qb}{p+q}\right)\leq\frac{1}{p\,(b-a)}\int_a^{pb+(1-p)a}\bar{\Xi}(x)\,{}_a\mathsf{d}^{\bar{T}}_{p,q}x\leq\frac{\bar{\Xi}(a)+\bar{\Xi}(b)}{2}. \quad (4.7)$$

Proof. As $\bar{\Xi}$ is differentiable function on (a,b), so there exist a tangent at $\frac{pa+qb}{p+q}\in(a,b)$, tangent line can be expressed as function $\bar{\Xi}_3(x)=\bar{\Xi}\left(\frac{pa+qb}{p+q}\right)+\bar{\Xi}'\left(\frac{pa+qb}{p+q}\right)\left(x-\frac{pa+qb}{p+q}\right)$. since $\bar{\Xi}$ is convex function on $[a,b]$, then

$$\bar{\Xi}_3(x)=\bar{\Xi}\left(\frac{pa+qb}{p+q}\right)+\bar{\Xi}'\left(\frac{pa+qb}{p+q}\right)\left(x-\frac{pa+qb}{p+q}\right)\leq\bar{\Xi}(x),\quad\forall x\in[a,b].$$

Taking ${}_a\bar{T}_{p,q}$-integrals of the above inequality over $[a,b]$, we have

$$\int_a^{pb+(1-p)a}\bar{\Xi}_3(x)\,{}_a\mathsf{d}^{\bar{T}}_{p,q}x \quad (4.8)$$
$$=\int_a^{pb+(1-p)a}\bar{\Xi}\left(\frac{pa+qb}{p+q}\right)+\bar{\Xi}'\left(\frac{pa+qb}{p+q}\right)\left(x-\frac{pa+qb}{p+q}\right)\,{}_a\mathsf{d}^{\bar{T}}_{p,q}x$$
$$=p(b-a)\bar{\Xi}\left(\frac{pa+qb}{p+q}\right)+\bar{\Xi}'\left(\frac{pa+qb}{p+q}\right)\int_a^{pb+(1-p)a}\left(x-a-q\frac{b-a}{p+q}\right)\,{}_a\mathsf{d}^{\bar{T}}_{p,q}x$$
$$=p(b-a)\bar{\Xi}\left(\frac{pa+qb}{p+q}\right)+\bar{\Xi}'\left(\frac{pa+qb}{p+q}\right)\left(\frac{p(b-a)^2}{2}-q\frac{p(b-a)^2}{p+q}\right)$$
$$=p(b-a)\bar{\Xi}\left(\frac{pa+qb}{p+q}\right)+\frac{p(b-a)^2(p-q)}{2(p+q)}\bar{\Xi}'\left(\frac{pa+qb}{p+q}\right)\leq\int_a^{pb+(1-p)a}\bar{\Xi}(x)\,{}_a\mathsf{d}^{\bar{T}}_{p,q}x.$$

Comparing (4.8) and (4.4), we obtain our required result. □

Theorem 4.10. *Let* $\bar{\Xi}: I=[a,b]\to\mathbb{R}$ *be a convex continuous function on* (a,b) *with* $0<q<p\leq 1$*, we have*

$$\max\{\mathcal{N}_1,\mathcal{N}_2,\mathcal{N}_3\}\leq\frac{1}{p\,(b-a)}\int_a^{pb+(1-p)a}\bar{\Xi}(x)\,{}_a\mathsf{d}^{\bar{T}}_{p,q}x\leq\frac{\bar{\Xi}(a)+\bar{\Xi}(b)}{2},$$

where

$$\mathcal{N}_1=\bar{\Xi}\left(\frac{a+b}{2}\right),$$

$$\mathcal{N}_2 = \bar{\Xi}\left(\frac{qa+qb}{p+q}\right) - \frac{(b-a)(p-q)}{2(p+q)}\bar{\Xi}'\left(\frac{qa+qb}{p+q}\right),$$
$$\mathcal{N}_3 = \bar{\Xi}\left(\frac{pa+qb}{p+q}\right) + \frac{(b-a)(p-q)}{2(p+q)}\bar{\Xi}'\left(\frac{pa+qb}{p+q}\right).$$

Proof. Combining (4.2), (4.5) and (4.7), we can obtain our required result easily. □

5. Conclusions

We defined new post quantum integrals using trapezoidal strips and discussed their basic properties. Utilizing these integrals, we have established new (p, q) variants of Young's, Hölder's, Minkowski's, Ostrowski's and Hermite-Hadamard's inequalities. It is worth mentioning that by using these integrals, one can also develop new analogues of Chebychev and, Hermite-Hadamard-Mercer, Gruss-like inequalities. In future, we will extend the idea for q-fractional and interval analysis to obtain some new refinements of classical inequalities.

Acknowledgements

The authors are thankful to the editor and the anonymous reviewers for their valuable comments and suggestions. Muhammad Uzair Awan is thankful to HEC Pakistan for 8081/Punjab/NRPU/R&D/HEC/2017.

This research is supported by Researchers Supporting Project number (RSP-2021/158), King Saud University, Riyadh, Saudi Arabia.

Conflict of interests

The authors declare that they have no conflicts of interests.

References

1. N. Alp, M. Z. Sarikaya, A new definition and properties of quantum integral which calls $\bar{q}$–integral, *Konuralp J. Math.*, **5** (2017), 146–159.

2. N. Alp, M. Z. Sarikaya, $\overline{q}$-Inequalities on quantum integral, *Malaya J. Matematik*, **8** (2020), 2035–2044. https://doi.org/10.26637/MJM0804/0121

3. N. Alp, M. Z. Sarikaya, M. Kunt, I. Iscan, q–Hermite–Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions, *J. King Saud. Univ. Sci.*, **30** (2018), 193–203. https://doi.org/10.1016/j.jksus.2016.09.007

4. S. Bermudo, P. Kórus, J. E. N. Valdés, On q-Hermite-Hadamard inequalities for general convex functions, *Acta Math. Hung.*, **162** (2020), 364–374.

5. R. Chakrabarti, R. Jagannathan, A (p, q)–oscillator realization of two-parameter quantum algebras, *J. Phys. A: Math. Gen.*, **24** (1991), L711.

6. Y. M. Chu, M. U. Awan, S. Talib, M. A. Noor, K. I. Noor, New post quantum analogues of Ostrowski type inequalities using new definitions of left–right (p,q)-derivatives and definite integrals, *Adv. Differ. Equ.*, **2020** (2020), 634. https://doi.org/10.1186/s13662-020-03094-x

7. T. S. Du, C. Y. Luo, B. Yu, Certain quantum estimates on the parameterized integral inequalities and their applications, *J. Math. Inequal.*, **15** (2021), 201–228. https://doi.org/10.7153/jmi-2021-15-16

8. F. H. Jackson, On a q-definite integrals, *Q. J. Pure Appl. Math.*, **41** (1910), 193–203.

9. V. Kac, P. Cheung, *Quantum calculus*, Springer, 2002.

10. H. Kara, H. Budak, N. Alp, H. Kalsoom, M. Z. Sarikaya, On new generalized quantum integrals and related Hermite-Hadamard inequalities, *J. Inequal. Appl.*, **2021** (2021), 180. https://doi.org/10.1186/s13660-021-02715-7

11. M. Kunt, I. Işcan, N. Alp, M. Z. Sarikaya, (p,q)–Hermite–Hadamard inequalities and (p,q)–estimates for midpoint type inequalities via convex and quasi–convex functions, *Rascam. Rev. R. Acad. Math.*, **112** (2018), 969–992. https://doi.org/10.1007/s13398-017-0402-y

12. M. A. Noor, K. I. Noor, M. U. Awan, Some quantum estimates for Hermite–Hadamard inequalities, *Appl. Math. Comput.*, **251** (2015), 675–679. https://doi.org/10.1016/j.amc.2014.11.090

13. W. Sudsutad, S. K. Ntouyas, J. Tariboon, Quantum integral inequalities for convex functions, *J. Math. Inequal.*, **9** (2015), 781–793. https://doi.org/10.7153/jmi-09-64

14. J. Tariboon, S. K. Ntouyas, Quantum calculus on finite intervals and applications to impulsive difference equations, *Adv. Differ. Equ.*, **2013** (2013), 282. https://doi.org/10.1186/1687-1847-2013-282

15. J. Tariboon, S. K. Ntouyas, Quantum integral inequalities on finite intervals, *J. Inequal. Appl.*, **2014** (2014), 121. https://doi.org/10.1186/1029-242X-2014-121

16. M. Tunc, E. Gov, Some integral inequalities via (p,q)-calculus on finite intervals, *Filomat*, **35** (2021), 1421–1430. https://doi.org/10.2298/FL2105421T

17. M. Vivas-Cortez, M. A. Ali, H. Budak, H. Kalsoom, P. Agarwal, Some new Hermite–Hadamard and related inequalities for convex functions via (p,q)-integral, entropy, *Entropy*, **23** (2021), 828. https://doi.org/10.3390/e23070828

18. B. Yu, C. Y. Luo, T. S. Du, On the refinements of some important inequalities via (p,q)–calculus and their applications, *J. Inequal. Appl.*, **2021** (2021), 1–26. https://doi.org/10.1186/s13660-021-02617-8

19. Y. Zhang, T. S. Du, H. Wang, Y. J. Shen, Different types of quantum integral inequalities via (α,m)-convexity, *J. Inequal. Appl.*, **2018** (2018), 1–24. https://doi.org/10.1186/s13660-018-1860-2

20. M. A. Abbas, L. Chen, A. R. Khan, G. Muhammad, B. Sun, S. Hussain, et al., Some new Anderson type h and q integral inequalities in quantum calculus, *Symmetry*, **14** (2022), 1294. https://doi.org/10.3390/sym14071294

21. Y. Deng, M. U. Awan, S. Wu, Quantum integral inequalities of Simpson-type for strongly preinvex functions, *Mathematics*, **7** (2019), 751. https://doi.org/10.3390/math7080751

22. S. Erden, S. Iftikhar, M. R. Delavar, P. Kumam, P. Thounthong, W. Kumam, On generalizations of some inequalities for convex functions via quantum integrals, *Rascam. Rev. R. Acad. Math.*, **114** (2020), 1–15. https://doi.org/10.1007/s13398-020-00841-3

23. D. F. Zhao, G. Gulshan, M. A. Ali, K. Nonlaopon, Some new midpoint and trapezoidal-type inequalities for general convex functions in q-calculus, *Mathematics*, **10** (2022), 444. https://doi.org/10.3390/math10030444

24. B. Bin-Mohsin, M. Saba, M. Z. Javed, M. U. Awan, H. Budak, K. Nonlaopon, A quantum calculus view of Hermite-Hadamard-Jensen-Mercer inequalities with applications, *Symmetry*, **14** (2022), 1246. https://doi.org/10.3390/sym14061246

AIMS Press