AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(1): 1889–1900.
DOI:10.3934/math.2023097
Received: 25 June 2022
Revised: 28 September 2022
Accepted: 13 October 2022
Published: 26 October 2022

Research article

Some properties of a new subclass of tilted star-like functions with respect to symmetric conjugate points

Daud Mohamad, Nur Hazwani Aqilah Abdul Wahid* and Nurfatin Nabilah Md Fauzi

Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA, 40450 Shah Alam, Selangor, Malaysia

* **Correspondence:** Email: hazwaniaqilah@tmsk.uitm.edu.my.

Abstract: In this paper, we introduced a new subclass $S^*_{SC}(\alpha, \delta, A, B)$ of tilted star-like functions with respect to symmetric conjugate points in an open unit disk and obtained some of its basic properties. The estimation of the Taylor-Maclaurin coefficients, the Hankel determinant, Fekete-Szegö inequality, and distortion and growth bounds for functions in this new subclass were investigated. A number of new or known results were presented to follow upon specializing in the parameters involved in our main results.

Keywords: univalent functions; star-like functions with respect to symmetric conjugate points; subordination; coefficient estimates; Hankel determinant; Fekete-Szegö inequality; distortion bound; growth bound
Mathematics Subject Classification: 30C45, 30C50

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f with the normalized condition $f(0) = f'(0) - 1 = 0$ in an open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$. The function $f \in \mathcal{A}$ has the Taylor-Maclaurin series expansion given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \; z \in E. \tag{1.1}$$

Further, we denote S the subclass of $\mathcal{A}$ consisting of univalent functions in E.

Let H denote the class of Schwarz functions ω which are analytic in E such that $\omega(0) = 0$ and $|\omega(z)| < 1$. The function $\omega \in H$ has the series expansion given by

$$\omega(z) = \sum_{k=1}^{\infty} b_k z^k, \; z \in E. \tag{1.2}$$

We say that the analytic function f is subordinate to another analytic function g in E, expressed as $f \prec g$, if and only if there exists a Schwarz function $\omega \in H$ such that $f(z) = g(\omega(z))$ for all $z \in E$. In particular, if g is univalent in E, then $f \prec g \Leftrightarrow f(0) = g(0)$ and $f(E) = g(E)$.

A function p analytic in E with $p(0) = 1$ is said to be in the class of Janowski if it satisfies

$$p(z) \prec \frac{1 + Az}{1 + Bz}, \ -1 \leqslant B < A \leqslant 1, \ z \in E, \tag{1.3}$$

where

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \tag{1.4}$$

This class was introduced by Janowski [1] in 1973 and is denoted by $P(A, B)$. We note that $P(1, -1) \equiv P$, the well-known class of functions with positive real part consists of functions p with $\operatorname{Re} p(z) > 0$ and $p(0) = 1$.

A function $f \in \mathcal{A}$ is called star-like with respect to symmetric conjugate points in E if it satisfies

$$\operatorname{Re}\left\{\frac{2zf'(z)}{f(z) - \overline{f(-\overline{z})}}\right\} > 0, \ z \in E. \tag{1.5}$$

This class was introduced by El-Ashwah and Thomas [2] in 1987 and is denoted by S^*_{SC}. In 1991, Halim [3] defined the class $S^*_{SC}(\delta)$ consisting of functions $f \in \mathcal{A}$ that satisfy

$$\operatorname{Re}\left\{\frac{2zf'(z)}{f(z) - \overline{f(-\overline{z})}}\right\} > \delta, \ 0 \leqslant \delta < 1, \ z \in E. \tag{1.6}$$

In terms of subordination, in 2011, Ping and Janteng [4] introduced the class $S^*_{SC}(A, B)$ consisting of functions $f \in \mathcal{A}$ that satisfy

$$\frac{2zf'(z)}{f(z) - \overline{f(-\overline{z})}} \prec \frac{1 + Az}{1 + Bz}, \ -1 \leqslant B < A \leqslant 1, \ z \in E. \tag{1.7}$$

It follows from (1.7) that $f \in S^*_{SC}(A, B)$ if and only if

$$\frac{2zf'(z)}{f(z) - \overline{f(-\overline{z})}} = \frac{1 + A\omega(z)}{1 + B\omega(z)} = p(z), \ \omega \in H. \tag{1.8}$$

Motivated by the work mentioned above, for functions $f \in \mathcal{A}$, we now introduce the subclass of the tilted star-like functions with respect to symmetric conjugate points as follows:

Definition 1.1. *Let $S^*_{SC}(\alpha, \delta, A, B)$ be the class of functions defined by*

$$\left(e^{i\alpha}\frac{zf'(z)}{h(z)} - \delta - i\sin\alpha\right)\frac{1}{\tau_{\alpha\delta}} \prec \frac{1 + Az}{1 + Bz}, \ z \in E, \tag{1.9}$$

where $h(z) = \frac{f(z) - \overline{f(-\overline{z})}}{2}$, $\tau_{\alpha\delta} = \cos\alpha - \delta$, $0 \leqslant \delta < 1$, $|\alpha| < \frac{\pi}{2}$, and $-1 \leqslant B < A \leqslant 1$.

By definition of subordination, it follows from (1.9) that there exists a Schwarz function ω which satisfies $\omega(0) = 0$ and $|\omega(z)| < 1$, and

$$\left(e^{i\alpha}\frac{zf'(z)}{h(z)} - \delta - i\sin\alpha\right)\frac{1}{\tau_{\alpha\delta}} = \frac{1 + A\omega(z)}{1 + B\omega(z)} = p(z), \ \omega \in H. \tag{1.10}$$

We observe that for particular values of the parameters $\alpha,\ \delta,\ A$, and B, the class $S^*_{SC}(\alpha, \delta, A, B)$ reduces to the following existing classes:

(a) For $\alpha = \delta = 0, A = 1$ and $B = -1$, the class $S^*_{SC}(0, 0, 1, -1) = S^*_{SC}$ introduced by El-Ashwah and Thomas [2].

(b) For $\alpha = 0, A = 1$ and $B = -1$, the class $S^*_{SC}(0, \delta, 1, -1) = S^*_{SC}(\delta)$ introduced by Halim [3].

(c) For $\alpha = \delta = 0$, the class $S^*_{SC}(0, 0, A, B) = S^*_{SC}(A, B)$ introduced by Ping and Janteng [4].

It is obvious that $S^*_{SC}(0) \equiv S^*_{SC}$, $S^*_{SC}(1, -1) \equiv S^*_{SC}$, and $S^*_{SC}(0, 0, 1, -1) \equiv S^*_{SC}$.

Aside from that, in recent years, several authors obtained many interesting results for various subclasses of star-like functions with respect to other points, i.e., symmetric points and conjugate points. This includes, but is not limited to, these properties: coefficient estimates, Hankel and Toeplitz determinants, Fekete-Szegö inequality, growth and distortion bounds, and logarithmic coefficients. We may point interested readers to recent advances in these subclasses as well as their geometric properties, which point in a different direction than the current study, for example [5–21].

In this paper, we obtained some interesting properties for the class $S^*_{SC}(\alpha, \delta, A, B)$ given in Definition 1.1. The paper is organized as follows: the authors presented some preliminary results in Section 2 and obtained the estimate for the general Taylor-Maclaurin coefficients $|a_n|$, $n \geqslant 2$, the upper bounds of the second Hankel determinant $|H_2(2)| = \left|a_2a_4 - a_3{}^2\right|$, the Fekete-Szegö inequality $\left|a_3 - \mu a_2{}^2\right|$ with complex parameter μ, and the distortion and growth bounds for functions belonging to the class $S^*_{SC}(\alpha, \delta, A, B)$ as the main results in Section 3. In addition, some consequences of the main results from Section 3 were presented in Section 4. Finally, the authors offer a conclusion and some suggestions for future study in Section 5.

2. Preliminary results

We need the following lemmas to derive our main results.

Lemma 2.1. *[22] If the function p of the form $p(z) = 1 + \sum\limits_{n=1}^{\infty} p_n z^n$ is analytic in E and $p(z) \prec \frac{1+Az}{1+Bz}$, then*

$$|p_n| \leqslant A - B, \ -1 \leqslant B < A \leqslant 1, \ n \in \mathbb{N}.$$

Lemma 2.2. *[23] Let $p \in P$ of the form $p(z) = 1 + \sum\limits_{n=1}^{\infty} p_n z^n$ and $\mu \in \mathbb{C}$. Then*

$$|p_n - \mu p_k p_{n-k}| \leqslant 2max\{1, |2\mu - 1|\}, \ 1 \leqslant k \leqslant n - 1.$$

If $|2\mu - 1| \geqslant 1$, then the inequality is sharp for the function $p(z) = \frac{1+z}{1-z}$ or its rotations. If $|2\mu - 1| < 1$, then the inequality is sharp for the function $p(z) = \frac{1+z^n}{1-z^n}$ or its rotations.

Lemma 2.3. *[24] For a function $p \in P$ of the form $p(z) = 1 + \sum\limits_{n=1}^{\infty} p_n z^n$, the sharp inequality $|p_n| \leqslant 2$ holds for each $n \geqslant 1$. Equality holds for the function $p(z) = \frac{1+z}{1-z}$.*

3. Main results

This section is devoted to our main results. We begin by finding the upper bound of the Taylor-Maclaurin coefficients $|a_n|$, $n \geqslant 2$ for functions belonging to $S^*_{SC}(\alpha,\delta,A,B)$. Further, we estimate the upper bounds of the second Hankel determinant $|H_2(2)|$ and the Fekete-Szegö inequality $\left|a_3-\mu a_2{}^2\right|$ with complex parameter μ, and the distortion and growth bounds for functions in the class $S^*_{SC}(\alpha,\delta,A,B)$.

3.1. Coefficient estimates

Theorem 3.1. *Let $f \in S^*_{SC}(\alpha,\delta,A,B)$. Then for $n \geqslant 1$,*

$$|a_{2n}| \leqslant \frac{\psi}{n!2^n}\prod_{j=1}^{n-1}(\psi+2j) \tag{3.1}$$

and

$$|a_{2n+1}| \leqslant \frac{\psi}{n!2^n}\prod_{j=1}^{n-1}(\psi+2j), \tag{3.2}$$

where $\psi=(A-B)\tau_{\alpha\delta}$ and $\tau_{\alpha\delta}=\cos\alpha-\delta$.

Proof. In view of (1.10), we have

$$e^{i\alpha}zf'(z)=h(z)\left(\tau_{\alpha\delta}p(z)+\delta+i\sin\alpha\right). \tag{3.3}$$

Using (1.1) and (1.4), (3.3) yields

$$\begin{aligned}&e^{i\alpha}\left(z+2a_2z^2+3a_3z^3+4a_4z^4+5a_5z^5+\ldots\right)\\&=e^{i\alpha}\left(z+a_3z^3+a_5z^5+\ldots\right)+\tau_{\alpha\delta}\left[p_1z^2+p_2z^3+(p_3+a_3p_1)z^4+(p_4+a_3p_2)z^5+\ldots\right].\end{aligned} \tag{3.4}$$

Comparing the coefficients of the like powers of z^n, $n \geqslant 1$ on both sides of the series expansions of (3.4), we obtain

$$2a_2=\tau_{\alpha\delta}e^{-i\alpha}p_1, \tag{3.5}$$

$$2a_3=\tau_{\alpha\delta}e^{-i\alpha}p_2, \tag{3.6}$$

$$4a_4=\tau_{\alpha\delta}e^{-i\alpha}(p_3+a_3p_1), \tag{3.7}$$

$$4a_5=\tau_{\alpha\delta}e^{-i\alpha}(p_4+a_3p_2), \tag{3.8}$$

$$2na_{2n}=\tau_{\alpha\delta}e^{-i\alpha}\left(p_{2n-1}+a_3p_{2n-3}+a_5p_{2n-5}+\ldots+a_{2n-1}p_1\right), \tag{3.9}$$

and

$$2na_{2n+1}=\tau_{\alpha\delta}e^{-i\alpha}\left(p_{2n}+a_3p_{2n-2}+a_5p_{2n-4}+\ldots+a_{2n-1}p_2\right). \tag{3.10}$$

We prove (3.1) and (3.2) using mathematical induction. Using Lemma 2.1, from (3.5)−(3.8), respectively, we obtain

$$|a_2| \leqslant \frac{(A-B)\tau_{\alpha\delta}}{2}, \tag{3.11}$$

$$|a_3| \leqslant \frac{(A-B)\,\tau_{\alpha\delta}}{2}, \tag{3.12}$$

$$|a_4| \leqslant \frac{(A-B)\,\tau_{\alpha\delta}\,((A-B)\,\tau_{\alpha\delta}+2)}{8}, \tag{3.13}$$

and

$$|a_5| \leqslant \frac{(A-B)\,\tau_{\alpha\delta}\,((A-B)\,\tau_{\alpha\delta}+2)}{8}. \tag{3.14}$$

It follows that (3.1) and (3.2) hold for $n = 1, 2$.
For simplicity, we denote $\psi = (A-B)\,\tau_{\alpha\delta}$ and from (3.1) in conjunction with Lemma 2.1 yields

$$|a_{2n}| \leqslant \frac{\psi}{2n}\left[1 + \sum_{k=1}^{n-1} |a_{2k+1}|\right]. \tag{3.15}$$

Next, we assume that (3.1) holds for $k = 3, 4, 5, \ldots, (n-1)$.
From (3.15), we get

$$|a_{2n}| \leqslant \frac{\psi}{2n}\left[1 + \sum_{k=1}^{n-1} \frac{\psi}{k!2^k} \prod_{j=1}^{k-1} (\psi + 2j)\right]. \tag{3.16}$$

To complete the proof, it is sufficient to show that

$$\frac{\psi}{2m}\left[1 + \sum_{k=1}^{m-1} \frac{\psi}{k!2^k} \prod_{j=1}^{k-1} (\psi + 2j)\right] = \frac{\psi}{m!2^m} \prod_{j=1}^{m-1} (\psi + 2j), \tag{3.17}$$

for $m = 3, 4, 5, \ldots, n$. It is easy to see that (3.17) is valid for $m = 3$.
Now, suppose that (3.17) is true for $m = 4, \ldots, (n-1)$. Then, from (3.16) we get

$$\begin{aligned}
&\frac{\psi}{2n}\left[1 + \sum_{k=1}^{n-1} \frac{\psi}{k!2^k} \prod_{j=1}^{k-1} (\psi + 2j)\right] \\
&= \frac{n-1}{n}\left[\frac{\psi}{2\,(n-1)}\left(1 + \sum_{k=1}^{n-2} \frac{\psi}{k!2^k} \prod_{j=1}^{k-1} (\psi + 2j)\right)\right] + \frac{\psi}{2n}\frac{\psi}{(n-1)!2^{n-1}} \prod_{j=1}^{n-2} (\psi + 2j) \\
&= \frac{(n-1)}{n}\frac{\psi}{(n-1)!2^{n-1}} \prod_{j=1}^{n-2} (\psi + 2j) + \frac{\psi}{2n}\frac{\psi}{(n-1)!2^{n-1}} \prod_{j=1}^{n-2} (\psi + 2j) \\
&= \frac{\psi}{(n-1)!2^{n-1}}\left(\frac{\psi + 2\,(n-1)}{2n}\right) \prod_{j=1}^{n-2} (\psi + 2j) \\
&= \frac{\psi}{n!2^n} \prod_{j=1}^{n-1} (\psi + 2j).
\end{aligned}$$

Thus, (3.17) holds for $m = n$ and hence (3.1) follows. Similarly, we can prove (3.2) and is omitted. This completes the proof of Theorem 3.1. □

3.2. Hankel determinant

Theorem 3.2. *If $f \in S^{*}_{SC}(\alpha, \delta, A, B)$, then*

$$|H_2(2)| \leqslant \frac{\psi^2}{16}\left[2\left|2\Upsilon^+ + 1\right| + \left|\psi e^{-i\alpha} - 4\right| + \Upsilon^+\left|\psi e^{-i\alpha} + 2\Upsilon^+\right|\right], \tag{3.18}$$

where $\psi = (A - B)\,\tau_{\alpha\delta}$, $\tau_{\alpha\delta} = \cos\alpha - \delta$, and $\Upsilon^+ = 1 + B$.

Proof. By definition of subordination, there exists a Schwarz function ω which satisfies $\omega(0) = 0$ and $|\omega(z)| < 1$, and from (1.10), we have

$$\left(e^{i\alpha}\frac{zf'(z)}{h(z)} - \delta - i\sin\alpha\right)\frac{1}{\tau_{\alpha\delta}} = \frac{1 + A\omega(z)}{1 + B\omega(z)}. \tag{3.19}$$

Let the function

$$p(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + \sum_{n=1}^{\infty} p_n z^n.$$

Then, we have

$$\omega(z) = \frac{p(z) - 1}{p(z) + 1}. \tag{3.20}$$

Substituting (3.20) into (3.19) yields

$$e^{i\alpha} zf'(z)\left(\Upsilon^- + p(z)\Upsilon^+\right) = g(z)\left[\left(e^{i\alpha}\Upsilon^- - \psi\right) + p(z)\left(e^{i\alpha}\Upsilon^+ + \psi\right)\right], \tag{3.21}$$

where $\Upsilon^- = 1 - B$ and $\Upsilon^+ = 1 + B$.
Using the series expansions of $f(z)$, $h(z)$, and $p(z)$, (3.21) becomes

$$\begin{aligned}
&\Upsilon^- e^{i\alpha}\left(z + 2a_2z^2 + 3a_3z^3 + 4a_4z^4 + 5a_5z^5 + ...\right) + \Upsilon^+ e^{i\alpha}\left[z + (k_1 + 2a_2)\,z^2 + (k_2 + 2a_2k_1 + 3a_3)\,z^3\right.\\
&\left.+ (k_3 + 2a_2k_2 + 3a_3k_1 + 4a_4)\,z^4 + \ldots\right]\\
&= \left(e^{i\alpha}\Upsilon^- - \psi\right)\left(z + a_3z^3 + a_5z^5 + ...\right) + \left(e^{i\alpha}\Upsilon^+ + \psi\right)\left[z + k_1z^2 + (k_2 + a_3)\,z^3 + (k_3 + a_3k_1)\,z^4 + \ldots\right].
\end{aligned}$$

On comparing coefficients of z^2, z^3, and z^4, respectively, we get

$$a_2 = \frac{p_1\psi e^{-i\alpha}}{4}, \tag{3.22}$$

$$a_3 = \frac{\psi e^{-i\alpha}\left(2p_2 - p_1^2\Upsilon^+\right)}{8}, \tag{3.23}$$

and

$$a_4 = \frac{\psi e^{-2i\alpha}\left[8p_3e^{i\alpha} + 2p_1p_2\left(\psi - 4\Upsilon^+e^{i\alpha}\right) + {p_1}^3\Upsilon^+\left(-\psi + 2\Upsilon^+e^{i\alpha}\right)\right]}{64}. \tag{3.24}$$

From (3.22)–(3.24), we obtain

$$\begin{aligned}
H_2(2) &= a_2a_4 - {a_3}^2\\
&= \frac{\psi^2 e^{-2i\alpha}}{256}\left[8p_1p_3 + 2{p_1}^2p_2\left(\psi e^{-i\alpha} + 4\Upsilon^+\right) - 16{p_2}^2 + {p_1}^4\Upsilon^+\left(-\psi e^{-i\alpha} - 2\Upsilon^+\right)\right].
\end{aligned} \tag{3.25}$$

By suitably rearranging the terms in (3.25) and using the triangle inequality, we get

$$|H_2(2)| \leqslant \frac{\psi^2}{256}\left[8|p_1||p_3-\eta_1 p_1 p_2| + |-16p_2|\left|p_2-\eta_2 {p_1}^2\right| + |p_1|^4\left|\Upsilon^+\right|\left|-\psi e^{-i\alpha}-2\Upsilon^+\right|\right], \tag{3.26}$$

where $\eta_1 = -\Upsilon^+$ and $\eta_2 = \frac{\psi e^{-i\alpha}}{8}$.
Further, by applying Lemma 2.2 and Lemma 2.3, we have

$$8|p_1||p_3-\eta_1 p_1 p_2| \leqslant 32\left|2\Upsilon^+ + 1\right|,$$

$$|-16p_2|\left|p_2-\eta_2 {p_1}^2\right| \leqslant 16\left|\psi e^{-i\alpha}-4\right|,$$

and

$$|p_1|^4\left|\Upsilon^+\right|\left|-\psi e^{-i\alpha}-2\Upsilon^+\right| \leqslant 16\Upsilon^+\left|\psi e^{-i\alpha}+2\Upsilon^+\right|.$$

Thus, from (3.26), we obtain the required inequality (3.18). This completes the proof of Theorem 3.2. □

3.3. Fekete-Szegö inequality

Theorem 3.3. *If $f \in S^*_{SC}(\alpha,\delta,A,B)$, then for any complex number μ, we have*

$$\left|a_3-\mu {a_2}^2\right| \leqslant \frac{\psi}{2} max\left\{1, \frac{\left|2B+\mu\psi e^{-i\alpha}\right|}{2}\right\}, \tag{3.27}$$

where $\psi = (A-B)\tau_{\alpha\delta}$ and $\tau_{\alpha\delta} = \cos\alpha - \delta$.

Proof. In view of (3.22) and (3.23), we have

$$\left|a_3-\mu {a_2}^2\right| = \left|\frac{\psi e^{-i\alpha}}{4}\left(p_2-\chi p_1^2\right)\right|, \tag{3.28}$$

where $\chi = \frac{2\Upsilon^+ + \mu\psi e^{-i\alpha}}{4}$.
By the application of Lemma 2.2, our result follows. □

3.4. Distortion and growth bounds

Theorem 3.4. *If $f \in S^*_{SC}(\alpha,\delta,A,B)$, then for $|z| = r$, $0 < r < 1$, we have*

$$\frac{1}{1+r^2}\left[\frac{(1-Ar)\tau_{\alpha\delta}}{1-Br} + \sqrt{\sin^2\alpha+\delta^2}\right] \leqslant |f'(z)| \leqslant \frac{1}{1-r^2}\left[\frac{(1+Ar)\tau_{\alpha\delta}}{1+Br} + \sqrt{\sin^2\alpha+\delta^2}\right], \tag{3.29}$$

where $\tau_{\alpha\delta} = \cos\alpha - \delta$. The bound is sharp.

Proof. In view of (1.10), we have

$$\left|e^{i\alpha}\frac{zf'(z)}{h(z)} - (\delta + i\sin\alpha)\right| = |\tau_{\alpha\delta}|\left|\frac{1+A\omega(z)}{1+B\omega(z)}\right|. \tag{3.30}$$

Since h is an odd star-like function, it follows that [24]

$$\frac{r}{1+r^2} \leqslant |h(z)| \leqslant \frac{r}{1-r^2}. \tag{3.31}$$

Furthermore, for $\omega \in H$, it can be easily established that [1]

$$\frac{1-Ar}{1-Br} \leqslant \left|\frac{1+A\omega(z)}{1+B\omega(z)}\right| \leqslant \frac{1+Ar}{1+Br}. \tag{3.32}$$

Applying (3.31) and (3.32), and for $|z| = r$, we find that, after some simplification,

$$|f'(z)| \leqslant \frac{1}{1-r^2}\left[\frac{(1+Ar)\tau_{\alpha\delta}}{1+Br} + \sqrt{\sin^2\alpha + \delta^2}\right] \tag{3.33}$$

and

$$|f'(z)| \geqslant \frac{1}{1+r^2}\left[\frac{(1-Ar)\tau_{\alpha\delta}}{1-Br} + \sqrt{\sin^2\alpha + \delta^2}\right], \tag{3.34}$$

which yields the desired result (3.29). The result is sharp due to the extremal functions corresponding to the left and right sides of (3.29), respectively,

$$f(z) = \int_0^z \frac{1}{1+t^2}\left[\frac{(1-At)\tau_{\alpha\delta}}{1-Bt} + \sqrt{\sin^2\alpha + \delta^2}\right]dt$$

and

$$f(z) = \int_0^z \frac{1}{1-t^2}\left[\frac{(1+At)\tau_{\alpha\delta}}{1+Bt} + \sqrt{\sin^2\alpha + \delta^2}\right]dt.$$

□

Theorem 3.5. *If* $f \in S^*_{SC}(\alpha, \delta, A, B)$, *then for* $|z| = r$, $0 < r < 1$, *we have*

$$\begin{aligned} &\frac{1}{1+B^2}\left[\psi \ln\left(\frac{1-Br}{\sqrt{1+r^2}}\right) + \tau_{\alpha\delta}(1+AB)\tan^{-1}r\right] + \sqrt{\sin^2\alpha+\delta^2}\tan^{-1}r \leqslant |f(z)| \\ &\leqslant \frac{1}{1-B^2}\left[\psi \ln\left(\frac{1+Br}{\sqrt{1-r^2}}\right) + \tau_{\alpha\delta}(1-AB)\ln\left(\sqrt{\frac{1+r}{1-r}}\right)\right] + \sqrt{\sin^2\alpha+\delta^2}\ln\left(\sqrt{\frac{1+r}{1-r}}\right), \end{aligned} \tag{3.35}$$

where $\psi = (A-B)\tau_{\alpha\delta}$ *and* $\tau_{\alpha\delta} = \cos\alpha - \delta$. *The bound is sharp.*

Proof. Upon elementary integration of (3.29) yields (3.35). The result is sharp due to the extremal functions corresponding to the left and right sides of (3.35), respectively,

$$f(z) = \int_0^z \frac{1}{1+t^2}\left[\frac{(1-At)\tau_{\alpha\delta}}{1-Bt} + \sqrt{\sin^2\alpha + \delta^2}\right]dt$$

and

$$f(z) = \int_0^z \frac{1}{1-t^2}\left[\frac{(1+At)\tau_{\alpha\delta}}{1+Bt} + \sqrt{\sin^2\alpha + \delta^2}\right]dt.$$

□

4. Consequences and corollaries

In this section, we apply our main results in Section 3 to deduce each of the following consequences and corollaries as shown in Tables 1–4.

Table 1. General Taylor-Maclaurin coefficients.

Corollary 4.1	Class	$\lvert a_n\rvert,\ n \geqslant 2$
(a)	$S^*_{SC}(0,0,1,-1)$	$\lvert a_{2n}\rvert \leqslant \frac{2^{2-n}}{n!}\prod_{j=1}^{n-1}(1+j),\ n \geqslant 1$
		$\lvert a_{2n+1}\rvert \leqslant \frac{2^{2-n}}{n!}\prod_{j=1}^{n-1}(1+j),\ n \geqslant 1$
(b)	$S^*_{SC}(0,\delta,1,-1)$	$\lvert a_{2n}\rvert \leqslant \frac{2^{2-n}(1-\delta)}{n!}\prod_{j=1}^{n-1}((1-\delta)+j),\ n \geqslant 1$
		$\lvert a_{2n+1}\rvert \leqslant \frac{2^{2-n}(1-\delta)}{n!}\prod_{j=1}^{n-1}((1-\delta)+j),\ n \geqslant 1$
(c)	$S^*_{SC}(0,0,A,B)$	$\lvert a_{2n}\rvert \leqslant \frac{(A-B)}{n!2^n}\prod_{j=1}^{n-1}((A-B)+2j),\ n \geqslant 1$
		$\lvert a_{2n+1}\rvert \leqslant \frac{(A-B)}{n!2^n}\prod_{j=1}^{n-1}((A-B)+2j),\ n \geqslant 1$

Remark 4.1. The results obtained in Corollary 4.1(c) coincide with the results obtained in [4].

Table 2. Second Hankel determinant.

Corollary 4.2	Class	$\lvert H_2(2)\rvert$
(a)	$S^*_{SC}(0,0,1,-1)$	$\lvert H_2(2)\rvert \leqslant 1$
(b)	$S^*_{SC}(0,\delta,1,-1)$	$\lvert H_2(2)\rvert \leqslant \frac{(1-\delta)^2(2+\delta)}{2}$
(c)	$S^*_{SC}(0,0,A,B)$	$\lvert H_2(2)\rvert \leqslant \frac{(A-B)^2}{16}\left[2\lvert 2\Upsilon^+ + 1\rvert + \lvert (A-B)-4\rvert + \Upsilon^+\lvert (A-B)+2\Upsilon^+\rvert\right]$

Remark 4.2. The result in Corollary 4.2(a) coincides with the findings of Singh [10].

Table 3. Fekete-Szegö inequality.

Corollary 4.3	Class	$\lvert a_3-\mu a_2{}^2\rvert$
(a)	$S^*_{SC}(0,0,1,-1)$	$\lvert a_3-\mu a_2{}^2\rvert \leqslant max\{1,\lvert -1+\mu\rvert\}$
(b)	$S^*_{SC}(0,\delta,1,-1)$	$\lvert a_3-\mu a_2{}^2\rvert \leqslant (1-\delta)\,max\{1,\lvert -1+\mu(1-\delta)\rvert\}$
(c)	$S^*_{SC}(0,0,A,B)$	$\lvert a_3-\mu a_2{}^2\rvert \leqslant \frac{(A-B)}{2}max\left\{1,\frac{\lvert 2B+\mu(A-B)\rvert}{2}\right\}$

Table 4. Distortion bound.

Corollary 4.4	Class	Distortion bound
(a)	$S^*_{SC}(0,0,1,-1)$	$\frac{1-r}{(1+r^2)(1+r)} \leqslant \lvert f'(z)\rvert \leqslant \frac{1+r}{(1-r^2)(1-r)}$
(b)	$S^*_{SC}(0,\delta,1,-1)$	$\frac{1}{1+r^2}\left[\frac{(1-r)(1-\delta)}{1+r}+\delta\right] \leqslant \lvert f'(z)\rvert \leqslant \frac{1}{1-r^2}\left[\frac{(1+r)(1-\delta)}{1-r}+\delta\right]$
(c)	$S^*_{SC}(0,0,A,B)$	$\frac{1-Ar}{(1+r^2)(1-Br)} \leqslant \lvert f'(z)\rvert \leqslant \frac{1+Ar}{(1-r^2)(1+Br)}$

Remark 4.3. The result obtained in Corollary 4.4(c) coincides with the result obtained in [4].

Remark 4.4. Setting $\alpha = 0$ and $\delta = 0$, Theorem 3.5 reduces to the result of Ping and Janteng [4].

5. Conclusions

In this paper, we considered a new subclass of tilted star-like functions with respect to symmetric conjugate points. Various interesting properties of these functions were investigated, such as coefficient bounds, the second Hankel determinant, Fekete-Szegö inequality, distortion bound, and growth bound. The results presented in this paper not only generalize some results obtained by Ping and Janteng [4] and Singh [10], but also give new results as special cases based on the various special choices of the involved parameters. Other interesting properties for functions belonging to the class $S^*_{SC}(\alpha,\delta,A,B)$ could be estimated in future work, such as the upper bounds of the Toeplitz determinant, the Hankel determinant of logarithmic coefficients, the Zalcman coefficient functional, the radius of star-likeness, partial sums, etc.

Acknowledgments

This research was funded by Universiti Teknologi MARA grant number 600-RMC/GPM LPHD 5/3 (060/2021). The authors wish to thank the referees for their valuable comments and suggestions.

Conflict of interest

The authors declare that they have no competing interests.

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