AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(1): 1871–1888.
DOI: 10.3934/math.2023096
Received: 05 July 2022
Revised: 29 September 2022
Accepted: 10 October 2022
Published: 26 October 2022

*Research article*

# On employing linear algebra approach to hybrid Sheffer polynomials

**Mdi Begum Jeelani***

Department of Mathematics, Imam Mohammad Ibn Saud Islamic University(IMSIU), Riyadh, Kingdom of Saudi Arabia

* **Correspondence:** Email: mbshaikh@imamu.edu.sa, write2mohammadi@gmail.com.

**Abstract:** By employing practical and effective matrix algebra, this article aims to investigate specific properties of truncated exponential-Sheffer polynomials. This method provides a valuable tool for researching multivariable special polynomial properties. The properties and association between the Pascal functional and Wronskian matrices are used to build the recursive equations and differential equation for these polynomials, as well as for several members of the truncated exponential-Sheffer family. The corresponding results for the truncated exponential-associated Sheffer and truncated exponential-Appell families is specified, as well as some examples are given. Finally a conclusion with a truncated exponential-Sheffer polynomial identity is provided.

**Keywords:** truncated exponential-Sheffer polynomials; Pascal functional matrix; Wronskian matrix; recursive formulae; differential equation
**Mathematics Subject Classification:** 15A15, 15A24, 33E30, 65QXX

## 1. Introduction and preliminaries

Special polynomials with two variables are critical from the standpoint of applications. These polynomials make it simple to derive a variety of useful identities and aid in the introduction of new families of special polynomials. Bretti et al. [5], established the general classes of two-variable Appell polynomials in utilising the features of an iterated isomorphism linked to Laguerre-type exponentials. Several writers investigated the two variable forms of the Hermite, Laguerre, and truncated exponential polynomials, as well as their generalisations (see [3, 6–8].

The truncated exponential polynomials (TEP) $e_m(u)$ possess the series [1]:

$$\sum_{i=0}^{m} \frac{u^i}{i!} = e_m(u). \tag{1.1}$$

Many issues in optics and quantum physics use these polynomials. The features of these polynomials aren't well understood. The literatures of Dattoli et al. [7] provided the first detailed research of some properties of these polynomials.

The Succeeding series formulation can be used to deduce the most notable features of these polynomials, and the integral illustration for $e_m(u)$ is given as:

$$\int_0^{+\infty} e^{-\xi}(u+\xi)^m d\xi = m!\, e_m(u) \tag{1.2}$$

which being a significant outcome of relation [1]:

$$\int_0^{+\infty} e^{-\xi}\xi^m d\xi = m!. \tag{1.3}$$

Consequently, $e_m(u)$ are defined [7, p. 596 (4)]:

$$\sum_{m=0}^{\infty} e_m(u)t^m = \frac{e^{ut}}{1-t}. \tag{1.4}$$

Dattoli et al. [7] developed a 2-variable extension of the truncated exponential polynomials(TEP), which is being useful in the evaluation of integrals involving products of special functions and in a variety of optics and quantum mechanics situations.

We recall that the following generating relation [7] for 2-variable TEP is given by

$$\sum_{m=0}^{\infty} {}_{[2]}e_m(u,v)t^m = \frac{e^{ut}}{1-vt^2} \tag{1.5}$$

and the succeeding series relation

$$\sum_{i=0}^{[\frac{m}{2}]} \frac{v^k\, u^{m-2i}}{(m-2i)!} = {}_{[2]}e_m(u,v). \tag{1.6}$$

Recalling the generating relation of higher order 2-variable TEP [7, p. 599 (31)]

$$\sum_{m=0}^{\infty} {}_{[r]}e_m(u,v)t^m = \frac{e^{ut}}{1-vt^r} \tag{1.7}$$

and the succeeding series relation:

$$\sum_{i=0}^{[\frac{m}{r}]} \frac{v^i\, u^{m-ri}}{(m-ri)!} = {}_{[r]}e_m(u,v). \tag{1.8}$$

In light of the expressions (1.7), (1.5) and (1.4), we find

$$\begin{aligned} {}_{[2]}e_m(u,v) &= e_m^{(2)}(u,v), \\ e_m(u) &= e_m^{(1)}(u,1). \end{aligned} \tag{1.9}$$

Note down that

$$U_m(v) = {}_{[2]}e_m(0, v), \tag{1.10}$$

where $U_m(v)$ denotes the second-order Chebyshev polynomials given by the generating relation [1]

$$\sum_{m=0}^{\infty} U_m(v)t^m = \frac{1}{(1 - 2ut + t^2)}, \qquad |t| < 1; \quad u \leq 1. \tag{1.11}$$

We know that the class of Sheffer sequences [14] is an important class which appears in a variety of problems in applied mathematics, theoretical physics, approximation theory, and other disciplines of mathematics.

$$\sum_{m=0}^{\infty} f_m \frac{t^m}{m!} = f(t), \;\; f_0 = 0, \;\; f_1 \neq 0 \tag{1.12}$$

and

$$\sum_{m=0}^{\infty} \mathsf{g}_m \frac{t^m}{m!} = \mathsf{g}(t), \;\; \mathsf{g}_0 \neq 0. \tag{1.13}$$

$s_m(u)$ is determined uniquely in Roman [12] by two formal power series and it is given by exponential generating relation for $s_m(u)$ as

$$\sum_{m=0}^{\infty} s_m(u) \frac{t^m}{m!} = \frac{e^{uf^{-1}(t)}}{\mathsf{g}(f^{-1}(t))}, \qquad u \in \mathbb{C}, \tag{1.14}$$

where the compositional inverse of $f(t)$ is $f^{-1}(t)$ .

It should be noted that for $\mathsf{g}(t) = 1$, the $s_m(u)$ reduces to $\mathfrak{s}_m(u)$ called associated-Sheffer sequence and for $f(t) = t$, it reduces to $A_m(u)$ known as the Appell sequence [2].

These sequences are given by the generating relations

$$\sum_{m=0}^{\infty} \mathfrak{s}_m(u) \frac{t^m}{m!} = e^{uf^{-1}(t)} \tag{1.15}$$

and

$$\sum_{m=0}^{\infty} A_m(u) \frac{t^m}{m!} = \frac{e^{ut}}{\mathsf{g}(t)}, \tag{1.16}$$

respectively.

In the table below, chosen members of the Sheffer, associated Sheffer, and Appell polynomial families are listed:

**Table 1.** Certain members of the Sheffer-Appell and associated Sheffer families.

| **Sheffer family** | | | |
|---|---|---|---|
| S. No. | $g(t); f(t); f^{-1}(t)$ | Generating function | Polynomials |
| I. | $e^{\frac{t^2}{4}}; \frac{t}{2}; 2t$ | $e^{2ut-t^2} = \sum\limits_{m=0}^{\infty} H_m(u)\frac{t^m}{m!}$ | Hermite polynomials $H_m(u)$ [1] |
| II. | $(1-t)^{-\gamma}; \ln(1-t); 1-e^t$ | $\exp(\gamma t + u(1-e^t)) = \sum\limits_{m=0}^{\infty} a_m^{(\gamma)}(u)\frac{t^m}{m!}$ | Actuarial polynomials $a_m^{(\gamma)}(u)$ [12] |
| III. | $(1-t)^{-(p+1)}$ | $\frac{1}{(1-t)^{p+1}}e^{ut} = \sum\limits_{m=0}^{\infty} G_m^{(p)}(u)\frac{t^m}{m!}$ | MIller-Lee polynomials $G_m^{(p)}(u)$ [1] |
| IV. | $\frac{e^t+1}{2}$ | $\frac{2}{e^t+1}e^{ut} = \sum\limits_{m=0}^{\infty} E_m(u)\frac{t^m}{m!}$ | Euler polynomials $E_m(u)$ [11] |
| **Associated Sheffer family** | | | |
| S. No. | $f(t); f^{-1}(t)$ | Generating function | Polynomials |
| I. | $\ln(1+t); e^t-1$ | $\exp(u(e^t-1)) = \sum\limits_{m=0}^{\infty} \phi_m(u)\frac{t^m}{m!}$ | Exponential polynomials $\phi_n(x)$ [4] |
| II. | $\frac{e^t-1}{e^t+1}$ ; $\log\left(\frac{1+t}{1-t}\right)$ | $\left(\frac{1+t}{1-t}\right)^u = \sum\limits_{m=0}^{\infty} M_m(u)\frac{t^m}{m!}$ | Mittag-Leffler polynomials $M_m(u)$ [12] |

The truncated exponential-Sheffer polynomials are introduced and studied by Khan et al. [9] in 2014. Recalling that the truncated exponential-Sheffer polynomials (TESP) ${}_{e^{(r)}}s_m(u,v)$ are defined by the following generating function:

$$\frac{1}{g(f^{-1}(t))(1-v(f^{-1}(t))^r)}\exp\left(xf^{-1}(t)\right) = \sum_{m=0}^{\infty} {}_{e^{(r)}}s_m(u,v)\frac{t^m}{m!}, \quad u,v\in\mathbb{C}, \tag{1.17}$$

where $f^{-1}(t)$ is the compositional inverse of $f(t)$.

**Note.** It should be noted that for $g(t) = 1$, the truncated exponential-Sheffer sequence ${}_{e^{(r)}}s_m(u,v)$ becomes the truncated exponential-associated Sheffer sequence ${}_{e^{(r)}}\mathfrak{s}_m(u,v)$ and for $f(t) = t$, it becomes the truncated exponential-Appell sequence ${}_{e^{(r)}}A_m(u,v)$.

We review some definitions and concepts related to the Pascal and Wronskian matrices which will be used for derivation of the results in Sections 2, 3 and 5.

Let $\{j(u) = \sum\limits_{k=0}^{\infty} a_k\frac{t^i}{i!} = \mathcal{F} \mid a_i \in \mathbb{C}\}$ be power series in the $\mathbb{C}$-algebra. The generalized Pascal functional matrix [15] of an analytic function $g(t)$ for $g(t)\in\mathcal{F}$ is a square matrix of order $(m+1)$ denoted by $\mathbb{P}_m[g(t)]$ and defined as:

$$(\mathbb{P}_m[g(t)])_{j,k} = \begin{cases} \binom{j}{k}g^{(j-k)}(t), & if \;\; j\geq k, \;\; j,k=0,1,2,\cdots,n \\ 0, & otherwise. \end{cases} \tag{1.18}$$

It should be noted that $g^{(i)}$ represents the $i^{th}$ order derivative of g and $g^i$ denotes the $i^{th}$ power of g throughout the article.

Also, Wronskian matrix of the $n^{th}$ order of an analytic functions $g_1(t), g_2(t), g_3(t), \cdots, g_m(t)$ is an $(m+1)\times m$ matrix defined by:

$$\mathcal{W}_m[g_1(t), g_2(t), g_3(t), \cdots, g_m(t)] = \begin{bmatrix} g_1(t) & g_2(t) & g_3(t) & \cdots & g_m(t) \\ g_1'(t) & g_2'(t) & g_3'(t) & \cdots & g_m'(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_1^{(n)}(t) & g_2^{(n)}(t) & g_3^{(n)}(t) & \cdots & g_m^{(n)}(t) \end{bmatrix}. \tag{1.19}$$

It is important to note that in the expressions $\mathbb{P}_m[g(x,t)]_{t=0}$ and $\mathcal{W}_m[g(x,t)]_{t=0}$, we consider $t$ as working variable and $x$ is as a parameter.

We recall certain important properties and relationships between the Pascal functional and Wronskian matrices [16].

For any $a, b \in \mathbb{C}$ and any analytic functions $g(t), g(t) \in \mathcal{F}$, the following properties hold true:

$$\mathbb{P}_m[ag(t) + bg(t)] = a\mathbb{P}_m[g(t)] + b\mathbb{P}_m[f(t)]. \tag{1.20}$$

$$\mathcal{W}_m[ag(t) + bg(t)] = a\mathcal{W}_m[g(t)] + b\mathcal{W}_m[f(t)]. \tag{1.21}$$

$$\mathbb{P}_m[g(t)]\mathbb{P}_m[f(t)] = \mathbb{P}_m[f(t)]\mathbb{P}_m[g(t)] = \mathbb{P}_m[g(t)f(t)]. \tag{1.22}$$

$$\mathbb{P}_m[g(t)]\mathcal{W}_m[f(t)] = \mathbb{P}_m[f(t)]\mathcal{W}_m[g(t)] = \mathcal{W}_m[(gf)(t)]. \tag{1.23}$$

$$\mathcal{W}_m[f(g(t))]_{t=0} = \mathcal{W}_m[1, g(t), g^2(t), g^3(t), \cdots, g^n(t)]_{t=0}\, \Lambda_n^{-1}\mathcal{W}_m[f(t)]_{t=0}, \tag{1.24}$$

where $\Lambda_m = \text{diag}[0!, 1!, 2!, \cdots, m!]$; $g(0) = 0$ and $g'(0) \neq 0$.

Further, for any analytic functions $g_1(t), g_2(t), \cdots, g_m(t)$ and $f(t)$, the following property holds true:

$$\mathbb{P}_m[f(t)]\mathcal{W}_m[g_1(t), g_2(t), \cdots, g_m(t)] = \mathcal{W}_m[(gg_1)(t), (gg_2)(t), \cdots, (gg_m)(t)]. \tag{1.25}$$

The significance of the two variable forms of special polynomials in applications, as well as recent work on Sheffer sequences using a matrix approach [10, 16], provides an incentive to develop recursive formulas and differential equations for truncated exponential-Sheffer polynomials using the generalised Pascal functional matrix of an analytic function and the Wronskian matrix of several analytic functions. Section 2 establishes several recursive formulas for truncated exponential-Sheffer polynomials ${}_{e^{(r)}}s_m(u,v)$. The differential equations for these polynomials are derived in Section 3. In Section 4, we look at few cases that provide solutions for some hybrid special polynomials. The identity for the truncated exponential-Sheffer polynomial sequences is determined in the final section.

## 2. Recursive formulas

To utilize the Wronskian matrices, the vector form of the TESP is required.

The TES vector denoted by $\overline{{}_{\mathbf{e^{(r)}}}\mathbf{s_m}}(u,v)$ is defined as:

$$\overline{{}_{\mathbf{e^{(r)}}}\mathbf{s_m}}(u,v) = [\ {}_{e^{(r)}}s_0(u,v)\ {}_{e^{(r)}}s_1(u,v)\ {}_{e^{(r)}}s_2(u,v)\ \ldots\ {}_{e^{(r)}}s_m(u,v)]^T, \tag{2.1}$$

where $\{{}_{e^{(r)}}s_m(u,v)\}$ is the TESP sequence defined by Eq (1.17).

Since $\frac{\exp(uf^{-1}(t))}{\mathrm{g}(f^{-1}(t))(1-v(f^{-1}(t))^r)}$ is analytic, therefore by Taylor's expansion, it follows that

$$ {}_{e^{(r)}}s_k(u,v) = \left(\frac{d}{dt}\right)^k \left.\frac{\exp(uf^{-1}(t))}{\mathrm{g}(f^{-1}(t))(1-v(f^{-1}(t))^r)}\right|_{t=0}, \quad k \geq 0. \tag{2.2}$$

In view of Eq (2.2), the truncated exponential-Sheffer vector (2.1) can be expressed as:

$$\begin{aligned}\overline{\mathbf{{}_{e^{(r)}}s_m}}(u,v) &= [\ {}_{e^{(r)}}s_0(u,v)\ {}_{e^{(r)}}s_1(u,v)\ {}_{e^{(r)}}s_2(u,v)\ ...\ {}_{e^{(r)}}s_m(u,v)]^T \\ &= \mathcal{W}_m \left[\frac{\exp(uf^{-1}(t))}{\mathrm{g}(f^{-1}(t))(1-v(f^{-1}(t))^r)}\right]_{t=0}.\end{aligned} \tag{2.3}$$

The following Lemma must be proved before proceeding with the formulation of the recursive formulas for the truncated exponential-Sheffer polynomial sequence.

**Lemma 2.1.** *The following property holds for the truncated exponential-Sheffer polynomial sequence* ${}_{e^{(r)}}s_m(u,v)$*:*

$$\begin{aligned}&\mathcal{W}_m[\ {}_{e^{(r)}}s_0(u,v),\ {}_{e^{(r)}}s_1(u,v),\ {}_{e^{(r)}}s_2(u,v),\ ...,\ {}_{e^{(r)}}s_m(u,v)]^T\Lambda_n^{-1} \\ &= \mathcal{W}_m\left[1, f^{-1}(t), (f^{-1}(t))^2, \cdots, (f^{-1}(t))^n\right]_{t=0}\Lambda_n^{-1}\mathbb{P}_m\left[\frac{1}{\mathrm{g}(t)(1-vt^r)}\right]_{t=0}\mathbb{P}_m[e^{xt}]_{t=0}.\end{aligned} \tag{2.4}$$

*Proof.* It follows from using property (1.24) in the r.h.s. of Eq (2.3) that

$$\begin{aligned}&[\ {}_{e^{(r)}}s_0(u,v)\ {}_{e^{(r)}}s_1(u,v)\ {}_{e^{(r)}}s_2(u,v)\ ...\ {}_{e^{(r)}}s_m(u,v)]^T \\ &= \mathcal{W}_m[1, f^{-1}(t), (f^{-1}(t))^2, \cdots, (f^{-1}(t))^n]_{t=0}\Lambda_n^{-1} \times \mathcal{W}_m\left[\frac{e^{xt}}{\mathrm{g}(t)(1-vt^r)}\right]_{t=0}.\end{aligned} \tag{2.5}$$

For, $\mathcal{W}_m[e^{xt}]_{t=0} = [1\ x\ x^2\ \cdots\ x^n]^T$ and using relation (1.23), the above expression can be written as

$$\begin{aligned}&[\ {}_{e^{(r)}}s_0(u,v)\ {}_{e^{(r)}}s_1(u,v)\ {}_{e^{(r)}}s_2(u,v)\ ...\ {}_{e^{(r)}}s_m(u,v)]^T \\ &= \mathcal{W}_m[1, f^{-1}(t), (f^{-1}(t))^2, \cdots, (f^{-1}(t))^n]_{t=0}\Lambda_n^{-1} \times \mathbb{P}_m\left[\frac{1}{\mathrm{g}(t)(1-vt^r)}\right]_{t=0}[1\ x\ x^2\ \cdots\ x^n]^T.\end{aligned} \tag{2.6}$$

Division by $k!$ and differentiation of (2.6) $k$ times with regard to $x$ results in

$$\begin{aligned}&\frac{1}{k!}\left[\ {}_{e^{(r)}}s_0^{(k)}(u,v)\ {}_{e^{(r)}}s_1^{(k)}(u,v)\ {}_{e^{(r)}}s_2^{(k)}(u,v)\ ...\ {}_{e^{(r)}}s_n^{(k)}(u,v)\right]^T \\ &= \mathcal{W}_m\left[1, f^{-1}(t), (f^{-1}(t))^2, \cdots, (f^{-1}(t))^n\right]_{t=0}\Lambda_n^{-1} \\ &\quad \times \mathbb{P}_m\left[\frac{1}{\mathrm{g}(t)(1-vt^r)}\right]_{t=0}\left[0\cdots 0\ 1\ \binom{k+1}{k}x\ \binom{k+2}{k}x^2\cdots\binom{n}{k}x^{n-k}\right]^T.\end{aligned} \tag{2.7}$$

The left part of (2.7) is the $k^{th}$ column of

$$\mathcal{W}_m[\ {}_{e^{(r)}}s_0(u,v)\ {}_{e^{(r)}}s_1(u,v)\ {}_{e^{(r)}}s_2(u,v)\ ...\ {}_{e^{(r)}}s_m(u,v)]^T\Lambda_n^{-1}$$

and the right part of (2.7) is the $k^{th}$ column of

$$\mathcal{W}_m[1, f^{-1}(t), (f^{-1}(t))^2, \cdots, (f^{-1}(t))^n]_{t=0}\Lambda_n^{-1}\mathbb{P}_m\left[\frac{1}{\mathrm{g}(t)(1-vt^r)}\right]_{t=0}\mathbb{P}_m[e^{xt}]_{t=0}.$$

Thus (2.4) is proved. □

Next, for ${}_{e^{(r)}}s_m(u,v)$ certain recursive formulas are established.

First, we expresses ${}_{e^{(r)}}s_{m+1}(u,v)$ in terms of ${}_{e^{(r)}}s_m(u,v)$ by deriving a recursive formula and its derivatives in the following manner.

**Theorem 2.2.** *For ${}_{e^{(r)}}s_m(u,v)$, the succeeding recursive formula holds:*

$$\sum_{k=0}^{m}(x\delta_k+ry\mu_k+\eta_k)\frac{{}_{e^{(r)}}s_n^{(k)}(u,v)}{k!}={}_{e^{(r)}}s_{m+1}(u,v),\qquad n\geq 0,\tag{2.8}$$

*where*

$$\frac{1}{g(0)}={}_{e^{(r)}}s_0(u,v);\quad \left(\frac{1}{f'(t)}\right)^{(k)}\Big|_{t=0}=\delta_k;\quad \left(\frac{t^{r-1}}{(1-vt^r)f'(t)}\right)^{(k)}\Big|_{t=0}=\mu_k$$

*and*

$$\left(-\frac{g'(t)}{g(t)f'(t)}\right)^{(k)}\Big|_{t=0}=\eta_k.$$

*Proof.* In light of (1.19) and (2.2), we find

$$[{}_{e^{(r)}}s_1(u,v)\ {}_{e^{(r)}}s_2(u,v)\ {}_{e^{(r)}}s_3(u,v)\ \ldots\ {}_{e^{(r)}}s_{m+1}(u,v)]^T=\mathcal{W}_m\left[\frac{d}{dt}\left(\frac{\exp(uf^{-1}(t))}{g(f^{-1}(t))(1-v(f^{-1}(t))^r)}\right)\right]_{t=0}.\tag{2.9}$$

Using expressions (1.22)–(1.24) in a suitable manner and differentiating $\mathcal{W}_m\left[\frac{d}{dt}\left(\frac{\exp(uf^{-1}(t))}{g(f^{-1}(t))(1-v(f^{-1}(t))^r)}\right)\right]_{t=0}$, we find

$$\begin{aligned}&\mathcal{W}_m\left[\frac{d}{dt}\left(\frac{\exp(uf^{-1}(t))}{g(f^{-1}(t))(1-v(f^{-1}(t))^r)}\right)\right]_{t=0}\\&=\mathcal{W}_m\left[\left(x+\frac{ry(f^{-1}(t))^{r-1}}{1-v(f^{-1}(t))^r}-\frac{g'(f^{-1}(t))}{g(f^{-1}(t))}\right)\left(\frac{1}{f'(f^{-1}(t))}\right)\left(\frac{\exp(uf^{-1}(t))}{g(f^{-1}(t))(1-v(f^{-1}(t))^r)}\right)\right]_{t=0}\\&=\mathcal{W}_m[1,f^{-1}(t),(f^{-1}(t))^2,\cdots,(f^{-1}(t))^n]_{t=0}\Lambda_n^{-1}\\&\times\mathbb{P}_m\left[\frac{1}{g(t)(1-vt^r)}\right]_{t=0}\mathbb{P}_m[\exp(xt)]_{t=0}\,\mathcal{W}_m\left[\left(x+\frac{ryt^{r-1}}{1-vt^r}-\frac{g'(t)}{g(t)}\right)\frac{1}{f'(t)}\right]_{t=0},\end{aligned}\tag{2.10}$$

which in light of Lemma 2.1 becomes

$$\begin{aligned}&\mathcal{W}_m\left[\frac{d}{dt}\left(\frac{\exp(uf^{-1}(t))}{g(f^{-1}(t))(1-v(f^{-1}(t))^r)}\right)\right]_{t=0}\\&=\mathcal{W}_m[{}_{e^{(r)}}s_0(u,v),\ {}_{e^{(r)}}s_1(u,v),\ {}_{e^{(r)}}s_2(u,v),\ \cdots,\ {}_{e^{(r)}}s_m(u,v)]^T\Lambda_n^{-1}\\&\times\mathcal{W}_m\left[\left(\frac{x}{f'(t)}+\frac{ryt^{r-1}}{(1-vt^r)f'(t)}-\frac{g'(t)}{g(t)f'(t)}\right)\right]_{t=0}\end{aligned}$$

$$=\begin{bmatrix}{}_{e^{(r)}}s_0(u,v)&0&0&\cdots&0\\{}_{e^{(r)}}s_1(u,v)&\frac{{}_{e^{(r)}}s_1'(u,v)}{1!}&0&\cdots&0\\{}_{e^{(r)}}s_2(u,v)&\frac{{}_{e^{(r)}}s_2'(u,v)}{1!}&\frac{{}_{e^{(r)}}s_2''(u,v)}{2!}&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\{}_{e^{(r)}}s_m(u,v)&\frac{{}_{e^{(r)}}s_m'(u,v)}{1!}&\frac{{}_{e^{(r)}}s_m''x,y)}{2!}&\cdots&\frac{{}_{e^{(r)}}s_m^{(n)}(u,v)}{m!}\end{bmatrix}\begin{bmatrix}x\delta_0+ry\mu_0+\eta_0\\x\delta_1+ry\mu_1+\eta_1\\\vdots\\x\delta_m+ry\mu_n+\eta_m\end{bmatrix}.\tag{2.11}$$

Assertion (2.8) is obtained by comparing the last rows of (2.9) and (2.11). □

**Remark 2.3.** *For $\mathsf{g}(t) = 1 \implies \eta_k = 0$ $(k \geq 0)$, thus for $\mathsf{g}(t) = 1$ and as a result of Theorem 2.1, we arrive at the following conclusion.*

**Corollary 2.4.** *For ${}_{e^{(r)}}\mathfrak{s}_n(u, v)$, the succeeding recursive relation holds*

$$\sum_{k=0}^{m}(x\delta_k + ry\mu_k)\frac{{}_{e^{(r)}}\mathfrak{s}_n^{(k)}(u, v)}{k!} = {}_{e^{(r)}}\mathfrak{s}_{m+1}(u, v), \quad n \geq 0, \tag{2.12}$$

*where*

$$1 = {}_{e^{(r)}}\mathfrak{s}_0(u, v); \quad \left(\frac{1}{f'(t)}\right)^{(k)}\Big|_{t=0} = \delta_k; \quad \left(\frac{t^{r-1}}{(1 - vt^r)f'(t)}\right)^{(k)}\Big|_{t=0} = \mu_k.$$

**Remark 2.5.** *Consequently for $f(t) = t \implies \delta_0 = 1$; $\delta_k = 0$ $(k \neq 0)$ , thus for $t = f(t)$ as a result of Theorem 2.1, we arrive at the following conclusion.*

**Corollary 2.6.** *For the ${}_{e^{(r)}}A_m(u, v)$, the succeeding recursive formula holds*

$$x_{e^{(r)}}A_m(u, v) + \sum_{k=0}^{m}(ry\vartheta_k + \sigma_k)\frac{{}_{e^{(r)}}A_m^{(k)}(u, v)}{k!} = {}_{e^{(r)}}A_{m+1}(u, v), \quad n \geq 0, \tag{2.13}$$

*where*

$$\frac{1}{\mathsf{g}(0)} = {}_{e^{(r)}}A_0(u, v); \quad \left(\frac{t^{r-1}}{(1 - vt^r)}\right)^{(k)}\Big|_{t=0} = \vartheta_k; \quad \left(-\frac{\mathsf{g}'(t)}{\mathsf{g}(t)}\right)^{(k)}\Big|_{t=0} = \sigma_k.$$

Further, a pure recursive formula for the truncated exponential-Sheffer polynomials is derived by proving the result.

**Theorem 2.7.** *For ${}_{e^{(r)}}s_m(u, v)$, the succeeding recursive formula holds.*

$$x\ {}_{e^{(r)}}s_m(u, v) + \sum_{k=0}^{m}\binom{m}{k}(\varsigma_k + \theta_k){}_{e^{(r)}}s_{n-k}(u, v) - \sum_{k=1}^{n}\binom{m}{k}\epsilon_k\ {}_{e^{(r)}}s_{m+1-k}(u, v) = \epsilon_0\ {}_{e^{(r)}}s_{m+1}(u, v), \quad n \geq 0, \tag{2.14}$$

*where*

$$\frac{1}{\mathsf{g}(0)} = {}_{e^{(r)}}s_0(u, v); \quad (f'(f^{-1}(t)))^{(k)}\Big|_{t=0} = \left(\frac{1}{(f^{-1}(t))'}\right)^{(k)}\Big|_{t=0} = \epsilon_k;$$

$$\left(\frac{(f^{-1}(t))^{r-1}}{(1 - v(f^{-1}(t))^r)f'(f^{-1}(t))}\right)^{(k)}\Big|_{t=0} = \varsigma_k; \quad \left(-\frac{\mathsf{g}'(f^{-1}(t))}{\mathsf{g}(f^{-1}(t))}\right)^{(k)}\Big|_{t=0} = \theta_k.$$

*Proof.* In light of property (1.23), the expression $\mathcal{W}_m\left[f'(f^{-1}(t))\frac{d}{dt}\left(\frac{\exp(uf^{-1}(t))}{\mathsf{g}(f^{-1}(t))(1-v(f^{-1}(t))^r)}\right)\right]_{t=0}$ takes the form:

$$\begin{aligned}
&\mathcal{W}_m\left[f'(f^{-1}(t))\frac{d}{dt}\left(\frac{\exp(uf^{-1}(t))}{\mathsf{g}(f^{-1}(t))(1 - v(f^{-1}(t))^r)}\right)\right]_{t=0} \\
&= \mathbb{P}_m\left[\frac{d}{dt}\left(\frac{\exp(uf^{-1}(t))}{\mathsf{g}(f^{-1}(t))(1 - v(f^{-1}(t))^r)}\right)\right]_{t=0}\mathcal{W}_m[f'(f^{-1}(t))]_{t=0} \\
&= \begin{bmatrix}
{}_{e^{(r)}}s_1(u,v) & 0 & 0 & \cdots & 0 \\
{}_{e^{(r)}}s_2(u,v) & {}_{e^{(r)}}s_1(u,v) & 0 & \cdots & 0 \\
{}_{e^{(r)}}s_3(u,v) & \binom{2}{1}{}_{e^{(r)}}s_2(u,v) & {}_{e^{(r)}}s_1(u,v) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
{}_{e^{(r)}}s_{m+1}(u,v) & \binom{n}{1}{}_{e^{(r)}}s_m(u,v) & \binom{n}{2}{}_{e^{(r)}}s_{n-1}(u,v) & \cdots & {}_{e^{(r)}}s_1(u,v)
\end{bmatrix}
\begin{bmatrix} \epsilon_0 \\ \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}.
\end{aligned} \tag{2.15}$$

When differentiation is performed in the same expression and properties (1.21) and (1.23) are used, it follows that

$$\begin{aligned}
&\mathcal{W}_m\left[f'(f^{-1}(t))\frac{d}{dt}\left(\frac{\exp(uf^{-1}(t))}{\mathsf{g}(f^{-1}(t))(1-v(f^{-1}(t))^r)}\right)\right]_{t=0}\\
&= x\mathcal{W}_m\left[\frac{\exp(uf^{-1}(t))}{\mathsf{g}(f^{-1}(t))(1-v(f^{-1}(t))^r)}\right]_{t=0} + \mathbb{P}_m\left[\frac{\exp(uf^{-1}(t))}{\mathsf{g}(f^{-1}(t))(1-v(f^{-1}(t))^r)}\right]_{t=0}\\
&\times\ \mathcal{W}_m\left[\frac{ry(f^{-1}(t))^{r-1}}{1-v(f^{-1}(t))^r} - \frac{\mathsf{g}'(f^{-1}(t))}{\mathsf{g}(f^{-1}(t))}\right]_{t=0} \qquad (2.16)\\
&= x\begin{bmatrix} {}_{e^{(r)}}s_0(u,v)\\ {}_{e^{(r)}}s_1(u,v)\\ {}_{e^{(r)}}s_2(u,v)\\ \vdots\\ {}_{e^{(r)}}s_m(u,v)\end{bmatrix} + \begin{bmatrix} {}_{e^{(r)}}s_0(u,v) & 0 & 0 & \cdots & 0\\ {}_{e^{(r)}}s_1(u,v) & {}_{e^{(r)}}s_0(u,v) & 0 & \cdots & 0\\ {}_{e^{(r)}}s_2(u,v) & \binom{2}{1}{}_{e^{(r)}}s_1(u,v) & {}_{e^{(r)}}s_0(u,v) & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ {}_{e^{(r)}}s_m(u,v) & \binom{n}{1}{}_{e^{(r)}}s_{n-1}(u,v) & \binom{n}{2}{}_{e^{(r)}}s_{n-2}(u,v) & \cdots & {}_{e^{(r)}}s_0(u,v)\end{bmatrix}\begin{bmatrix} ry\varsigma_0+\theta_0\\ ry\varsigma_1+\theta_1\\ ry\varsigma_2+\theta_2\\ \vdots\\ ry\varsigma_m+\theta_m\end{bmatrix}.
\end{aligned}$$

Assertion (2.14) is established by comparing the last rows of (2.15) and (2.16). □

**Remark 2.8.** *Since* $\mathsf{g}(t) = 1 \implies \theta_k = 0\ (k \geq 0)$*, thus for* $\mathsf{g}(t) = 1$ *and as a result of Theorem 2.2, we arrive at the following conclusion.*

**Corollary 2.9.** *For* ${}_{e^{(r)}}\mathfrak{s}_n(u,v)$*, the succeeding pure recursive formula holds.*

$$x\ {}_{e^{(r)}}s_m(u,v) + \sum_{k=0}^{m}\binom{m}{k}\varsigma_k\ {}_{e^{(r)}}s_{n-k}(u,v) - \sum_{k=1}^{n}\binom{m}{k}\epsilon_k\ {}_{e^{(r)}}s_{m+1-k}(u,v) = \epsilon_0\ {}_{e^{(r)}}s_{m+1}(u,v), \quad n \geq 0, \qquad (2.17)$$

*where*

$$\frac{1}{\mathsf{g}(0)} = {}_{e^{(r)}}s_0(u,v); \quad (f'(f^{-1}(t)))^{(k)}\Big|_{t=0} = \left(\frac{1}{(f^{-1}(t))'}\right)^{(k)}\Big|_{t=0} = \epsilon_k;$$

$$\left(\frac{(f^{-1}(t))^{r-1}}{(1-v(f^{-1}(t))^r)f'(f^{-1}(t))}\right)^{(k)}\Big|_{t=0} = \varsigma_k.$$

**Remark 2.10.** *Since* $f(t) = t \implies \epsilon_0 = 1;\ \epsilon_k = 0\ (k \neq 0)$*, thus for* $f(t) = t$*.*

**Corollary 2.11.** *For* ${}_{e^{(r)}}A_m(u,v)$*, the succeeding pure recursive formula holds*

$$x\ {}_{e^{(r)}}s_m(u,v) + \sum_{k=0}^{m}\binom{m}{k}(\tau_k + \upsilon_k)\ {}_{e^{(r)}}s_{n-k}(u,v) = {}_{e^{(r)}}s_{m+1}(u,v), \qquad n \geq 0, \qquad (2.18)$$

*where*

$$\frac{1}{\mathsf{g}(0)} = {}_{e^{(r)}}s_0(u,v); \ \left(\frac{t^{r-1}}{(1-vt^r)f'(t)}\right)^{(k)}\Big|_{t=0} = \tau_k; \ \left(-\frac{\mathsf{g}'(t)}{\mathsf{g}(t)}\right)^{(k)}\Big|_{t=0} = \upsilon_k.$$

Finally, by proving the following result, we derive a pure recursive formula that provides a representation of ${}_{e^{(r)}}s_{m+1}(u,v)$ in terms of ${}_{e^{(r)}}s_{n-k}(u,v)$ $(k = 0, 1, 2, cdotsn)$.

**Theorem 2.12.** *For ${}_{e^{(r)}}s_m(u,v)$, the succeeding pure recursive formula holds:*

$$\sum_{k=0}^{m}\binom{m}{k}(x\phi_k + ry\xi_k + \psi_k)\,{}_{e^{(r)}}s_{n-k}(u,v) = {}_{e^{(r)}}s_{m+1}(u,v), \quad n \geq 0, \tag{2.19}$$

*where*

$$\frac{1}{\mathsf{g}(0)} = {}_{e^{(r)}}s_0(u,v); \quad \left(\frac{1}{f'(f^{-1}(t))}\right)^{(k)}\Big|_{t=0} = \phi_k; \quad \left(\frac{(f^{-1}(t))^{r-1}}{(1-v(f^{-1}(t))^r)f'(f^{-1}(t))}\right)\Big|_{t=0} = \xi_k;$$

$$\left(-\frac{\mathsf{g}'(f^{-1}(t))}{\mathsf{g}(f^{-1}(t))}\frac{1}{f'(f^{-1}(t))}\right)^{(k)}\Big|_{t=0} = \psi_k.$$

*Proof.* Differentiating $\mathcal{W}_m\left[\frac{d}{dt}\left(\frac{\exp(uf^{-1}(t))}{\mathsf{g}(f^{-1}(t))(1-v(f^{-1}(t))^r)}\right)\right]_{t=0}$ and then using (1.23), we have

$$\mathcal{W}_m\left[\frac{d}{dt}\left(\frac{\exp(uf^{-1}(t))}{\mathsf{g}(f^{-1}(t))(1-v(f^{-1}(t))^r)}\right)\right]_{t=0}$$

$$= \mathbb{P}_m\left[\left(x + \frac{ry(f^{-1}(t))^{r-1}}{1-v(f^{-1}(t))^r} - \frac{\mathsf{g}'(f^{-1}(t))}{\mathsf{g}(f^{-1}(t))}\right)\frac{1}{f'(f^{-1}(t))}\right]_{t=0}\mathcal{W}_m\left[\frac{\exp(uf^{-1}(t))}{\mathsf{g}(f^{-1}(t))(1-v(f^{-1}(t))^r)}\right]_{t=0} \tag{2.20}$$

$$= \begin{bmatrix} (x\phi_0 + ry\xi_0) + \psi_0 & 0 & \cdots & 0 \\ (x\phi_1 + ry\xi_1) + \psi_1 & (x\phi_0 + ry\xi_0) + \psi_0 & \cdots & 0 \\ (x\phi_2 + ry\xi_2) + \psi_2 & \binom{2}{1}((x\phi_1 + ry\xi_1) + \psi_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (x\phi_n + ry\xi_n) + \psi_n & \binom{n}{1}((x\phi_{n-1} + ry\xi_{n-1}) + \psi_{n-1}) & \cdots & (x\phi_0 + ry\xi_0) + \psi_0 \end{bmatrix}.$$

Assertion (2.19) is established by comparing the last rows of (2.9) and (2.20). □

**Remark 2.13.** *Since $\mathsf{g}(t) = 1 \implies \psi_k = 0\ (k \geq 0)$, therefore for $\mathsf{g}(t) = 1$and as a result of Theorem 2.3, we arrive at the following conclusion.*

**Corollary 2.14.** *For ${}_{L}\mathfrak{s}_n(u,v)$, the succeeding pure recursive formula holds*

$$\sum_{k=0}^{m}\binom{m}{k}(x\phi_k + ry\xi_k)\,{}_{e^{(r)}}s_{n-k}(u,v) = {}_{e^{(r)}}s_{m+1}(u,v), \quad n \geq 0, \tag{2.21}$$

*where*

$$\frac{1}{\mathsf{g}(0)} = {}_{e^{(r)}}s_0(u,v); \quad \left(\frac{1}{f'(f^{-1}(t))}\right)^{(k)}\Big|_{t=0} = \phi_k; \quad \left(\frac{(f^{-1}(t))^{r-1}}{(1-v(f^{-1}(t))^r)f'(f^{-1}(t))}\right)\Big|_{t=0} = \xi_k.$$

**Remark 2.15.** *Noting down for $f(t) = t \implies \phi_0 = 1$; $\phi_k = 0\ (k \neq 0)$. Thus, taking $f(t) = t$, in Theorem 2.3, we deduce the consequence of Theorem 2.3.*

**Corollary 2.16.** *For ${}_{e^{(r)}}A_m(u,v)$, the succeeding pure recursive formula holds*

$$x\ {}_{e^{(r)}}s_{n-k}(u,v)+\sum_{k=0}^{m}\binom{m}{k}(ry\chi_k+\omega_k)\ {}_{e^{(r)}}s_{n-k}(u,v)={}_{e^{(r)}}s_{m+1}(u,v),\quad n\geq 0,$$

*where*

$$\frac{1}{g(0)}={}_{e^{(r)}}s_0(u,v);\quad \left(\frac{t^{r-1}}{(1-yt^r)f'(t)}\right)\Big|_{t=0}=\chi_k;\quad \left(-\frac{g'(t)}{g(t)}\frac{1}{f'(t)}\right)^{(k)}\Big|_{t=0}=\omega_k.$$

The differential equation satisfied by the truncated exponential-Sheffer polynomials ${}_{e^{(r)}}s_m(u,v)$ is derived in the following section.

## 3. Differential equation

We prove the following result in order to derive the differential equation for the truncated exponential-Sheffer polynomials ${}_{e^{(r)}}s_m(u,v)$.

**Theorem 3.1.** *The following differential equation is satisfied by the truncated exponential-Sheffer polynomials ${}_{e^{(r)}}s_m(u,v)$.*

$$\sum_{k=1}^{n}(\beta_k x+ry\gamma_k+\alpha_k)\frac{\partial^k}{\partial x^k}\frac{{}_{e^{(r)}}s_m(u,v)}{k!}-n\ {}_{e^{(r)}}s_m(u,v)=0, \tag{3.1}$$

*where*

$$\left(-\frac{g'(t)}{g(t)}\frac{f(t)}{f'(t)}\right)^{(k)}\Big|_{t=0}=\alpha_k;\quad \left(\frac{f(t)}{f'(t)}\right)^{(k)}\Big|_{t=0}=\beta_k;\quad \left(\frac{t^{r-1}f(t)}{(1-vt^r)f'(t)}\right)^{(k)}\Big|_{t=0}=\gamma_k.$$

*Proof.* In light of (1.23), the relation $\mathcal{W}_n\left[t\frac{d}{dt}\left(\frac{\exp(uf^{-1}(t))}{g(f^{-1}(t))(1-v(f^{-1}(t))^r)}\right)\right]_{t=0}$ takes the form

$$\begin{aligned}
&\mathcal{W}_m\left[t\frac{d}{dt}\left(\frac{\exp(uf^{-1}(t))}{g(f^{-1}(t))(1-v(f^{-1}(t))^r)}\right)\right]_{t=0}\\
=&\mathbb{P}_m[t]_{t=0}\mathcal{W}_m\left[\frac{d}{dt}\left(\frac{\exp(uf^{-1}(t))}{g(f^{-1}(t))(1-v(f^{-1}(t))^r)}\right)\right]_{t=0}\\
=&\begin{bmatrix}0&0&0&0&\cdots&0&0&0\\1&0&0&0&\cdots&0&0&0\\0&2&0&0&\cdots&0&0&0\\0&0&3&0&\cdots&0&0&0\\\vdots&\vdots&\vdots&\ddots&\ddots&\vdots&\vdots&\vdots\\&&&&\ddots&\ddots&&\\0&0&0&0&\cdots&n-1&0&0\\0&0&0&0&\cdots&0&n&0\end{bmatrix}\begin{bmatrix}{}_{e^{(r)}}s_1(u,v)\\{}_{e^{(r)}}s_2(u,v)\\{}_{e^{(r)}}s_3(u,v)\\\vdots\\{}_{e^{(r)}}s_m(u,v)\\{}_{e^{(r)}}s_{m+1}(u,v)\end{bmatrix}.
\end{aligned} \tag{3.2}$$

On the other hand, by performing differentiation in the same expression and appropriately applying properties (1.23)–(1.25), we have

$$\begin{aligned}
&\mathcal{W}_m\left[t\frac{d}{dt}\left(\frac{\exp(uf^{-1}(t))}{\mathsf{g}(f^{-1}(t))(1-v(f^{-1}(t))^r)}\right)\right]_{t=0}\\
&=\mathcal{W}_m[1,f^{-1}(t),(f^{-1}(t))^2,\cdots,(f^{-1}(t))^n]_{t=0}\,\Lambda_n^{-1}\,\mathbb{P}_m\left[\frac{1}{(1-vt^r)\mathsf{g}(t)}\right]_{t=0}\,\mathbb{P}_m\left[e^{xt}\right]_{t=0}\\
&\times\mathcal{W}_m\left[\left(x+\frac{ryt^{r-1}}{(1-vt^r)}\right)\frac{f(t)}{f'(t)}-\frac{\mathsf{g}'(t)}{\mathsf{g}(t)}\frac{f(t)}{f'(t)}\right]\\
&=\mathcal{W}_m[1,f^{-1}(t),(f^{-1}(t))^2,\cdots,(f^{-1}(t))^n]_{t=0}\Lambda_n^{-1}\,\mathbb{P}_m\left[\frac{1}{(1-vt^r)\mathsf{g}(t)}\right]_{t=0}\,\mathbb{P}_m\left[e^{xt}\right]_{t=0}\\
&\times\mathcal{W}_m\left[\left(x\frac{f(t)}{f'(t)}+\left(\frac{ryt^{r-1}}{1-vt^r}\right)\frac{f(t)}{f'(t)}-\frac{\mathsf{g}'(t)}{\mathsf{g}(t)}\frac{f(t)}{f'(t)}\right)\right].
\end{aligned}\tag{3.3}$$

In light of Lemma 2.1, (3.3) becomes

$$\begin{aligned}
&\mathcal{W}_m\left[t\frac{d}{dt}\left(\frac{\exp(uf^{-1}(t))}{\mathsf{g}(f^{-1}(t))(1-v(f^{-1}(t))^r)}\right)\right]_{t=0}\\
&=\left(\mathcal{W}_m[{}_{e^{(r)}}s_0(u,v),\ {}_{e^{(r)}}s_1(u,v),\ {}_{e^{(r)}}s_2(u,v),\ \cdots,\ {}_{e^{(r)}}s_m(u,v)]\right)^T\\
&\times\ \Lambda_n^{-1}\,\mathcal{W}_m\left[\left(x\frac{f(t)}{f'(t)}+\left(\frac{ryt^{r-1}}{1-vt^r}\right)\frac{f(t)}{f'(t)}-\frac{\mathsf{g}'(t)}{\mathsf{g}(t)}\frac{f(t)}{f'(t)}\right)\right]_{t=0}\\
&=\begin{bmatrix}
{}_{e^{(r)}}s_0(u,v) & 0 & 0 & \cdots & 0\\
{}_{e^{(r)}}s_1(u,v) & \frac{{}_{e^{(r)}}s_1'(u,v)}{1!} & 0 & \cdots & 0\\
{}_{e^{(r)}}s_2(u,v) & \frac{{}_{e^{(r)}}s_2'(u,v)}{1!} & \frac{{}_{e^{(r)}}ss_2''(u,v)}{2!} & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots & \\
{}_{e^{(r)}}s_m(u,v) & \frac{{}_{e^{(r)}}s_m'(u,v)}{1!} & \frac{{}_{e^{(r)}}s_m''(u,v)}{2!} & \cdots & \frac{{}_{e^{(r)}}s_m^{(n)}(u,v)}{m!}
\end{bmatrix}
\begin{bmatrix}
x\beta_0+ry\gamma_0+\alpha_0\\
x\beta_1+ry\gamma_1+\alpha_1\\
\vdots\\
x\beta_m+ry\gamma_m+\alpha_m
\end{bmatrix}.
\end{aligned}\tag{3.4}$$

Assertion (3.1) is proved by comparing the last rows of (3.2) and (3.4) and in light of the fact that

$$f(0)=0\implies\alpha_0=\beta_0=\gamma_0=0.$$

□

**Remark 3.2.** *For* $\mathsf{g}(t)=1$*, the polynomials* ${}_{e^{(r)}}s_m(u,v)$ *reduces to the polynomials* ${}_{e^{(r)}}\mathfrak{s}_n(u,v)$ *and since* $\mathsf{g}(t)=1\implies\alpha_k=0\ (k\geq 1)$*, the succeeding consequence of Theorem 3.1 is obtained.*

**Corollary 3.3.** *The polynomials* ${}_{e^{(r)}}\mathfrak{s}_n(u,v)$ *satisfy the succeeding differential equation*

$$\sum_{k=1}^{n}(x\beta_k+ry\gamma_k)\frac{\partial^k}{\partial x^k}\frac{{}_{e^{(r)}}\mathfrak{s}_n(u,v)}{k!}-n\ {}_{e^{(r)}}\mathfrak{s}_n(u,v)=0,\tag{3.5}$$

*where*

$$\left(\frac{f(t)}{f'(t)}\right)^{(k)}\Big|_{t=0}=\beta_k and\ \left(\frac{t^{r-1}\ f(t)}{(1-vt^r)f'(t)}\right)=\gamma_k.$$

**Remark 3.4.** *For $f(t) = t$, the polynomials ${}_{e^{(r)}}s_m(u, v)$ reduces to the polynomials ${}_{e^{(r)}}A_m(u, v)$ and since $f(t) = t \implies \beta_1 = 1$; $\beta_k = 0$ $(k \neq 1)$, therefore, for $f(t) = t$, the succeeding consequence of Theorem 3.1 is obtained.*

**Corollary 3.5.** *The polynomials ${}_{e^{(r)}}A_m(u, v)$ satisfy the succeeding differential equation*

$$x\frac{\partial}{\partial x}{}_{e^{(r)}}A_m(u, v) + \sum_{k=1}^{n}(ry\pi_k + A_k)\frac{\partial^k}{\partial x^k}\frac{{}_{e^{(r)}}A_m(u, v)}{k!} - n\,{}_{e^{(r)}}A_m(u, v) = 0, \tag{3.6}$$

*where*

$$\left(-\frac{tg'(t)}{g(t)}\right)^{(k)}\Big|_{t=0} = A_k;\quad \left(\frac{t^r}{1 - vt^r}\right)^{(k)}\Big|_{t=0} = \pi_k.$$

The differential equation and recursive formulas for certain members of the truncated exponential-Sheffer and truncated exponential-associated Sheffer families are derived in the following section.

## 4. Examples

We use Theorem 3.1 to derive the differential equation and Theorems 2.1–2.3 to find the recursive formulas for certain members of the truncated exponential-Sheffer family.

**Example 4.1.** *For $e^{(\frac{t}{v})^m}$, $f(t) = \frac{t}{v} = g(t)$ and $vt = f^{-1}(t)$, the Sheffer polynomials become the generalized Hermite polynomials $H_{n,m,v}(x)$ and consequently the truncated exponential-Sheffer polynomials become the truncated exponential-generalized Hermite polynomials ${}_{e^{(r)}}H_{n,m,v}(u, v)$.*

*It folllows from Theorem 3.1 that*

$$-m\frac{m!}{v^m} = \alpha_m;\ \ \alpha_k = 0\ (k \neq m),\ \ \beta_1 = 1;\ \ \beta_k = 0\ (k \neq 1),\ \ \left(\frac{t^r}{1 - vt^r}\right)^{(k)}\Big|_{t=0} = \gamma_k. \tag{4.1}$$

*Inserting (4.1) into expression (3.1), the differential equation for the truncated exponential-generalized Hermite polynomials ${}_{e^{(r)}}H_{n,m,v}(u, v)$ is obtained as follows*

$$n\,{}_{e^{(r)}}H_{n,m,v}(u, v) = (x - \frac{m}{v^m}\frac{\partial^{m-1}}{\partial x^{m-1}})\frac{\partial}{\partial x}{}_{e^{(r)}}H_{n,m,v}(u, v) + \sum_{k=1}^{n} ry\,\gamma_k\frac{\partial^k}{\partial x^k}\frac{{}_{e^{(r)}}H_{n,m,v}(u, v)}{k!}. \tag{4.2}$$

*Similarly, in light of Theorems 2.1–2.3, the recursive formulas for the truncated exponential-generalized Hermite polynomials ${}_{e^{(r)}}H_{n,m,v}(u, v)$ are obtained as follows*

$$1 = {}_{e^{(r)}}H_{0,m,v}(u, v), \tag{4.3}$$

$$(x - \frac{m}{v^{m-1}}\frac{\partial^{m-1}}{\partial x^{m-1}})\,{}_{e^{(r)}}H_{n,m,v}(u, v) + \sum_{k=0}^{n} ry\,\mu_k\frac{{}_{e^{(r)}}H^{(k)}_{n,m,v}(u, v)}{k!} = {}_{e^{(r)}}H_{m+1,m,v}(u, v),\quad n \geq 0, \tag{4.4}$$

$$vx\,{}_{e^{(r)}}H_{n,m,v}(u, v) + \binom{n}{m-1} - m!\,{}_{e^{(r)}}H_{n-m+1,m,v}(u, v)$$

$$+ v\sum_{k=0}^{n}\binom{m}{k}\varsigma_k\ {}_{e^{(r)}}H_{n-k,m,v}(u, v) = {}_{e^{(r)}}H_{m+1,m,v}(u, v),\quad n \geq 0, \tag{4.5}$$

*and*

$$\nu x\ {}_{e^{(r)}}H_{n,m,\nu}(u,v) - \binom{n}{m-1}\frac{m!}{\nu}\,{}_{e^{(r)}}H_{n-m+1,m,\nu}(u,v)$$

$$+\sum_{k=0}^{n}\binom{m}{k}ry\ \ \xi_{k\,e^{(r)}}H_{n-k,m,\nu}(u,v) = {}_{e^{(r)}}H_{m+1,m,\nu}(u,v), \qquad n \geq 0. \tag{4.6}$$

**Example 4.2.** *For* $(1-t)^{-\kappa-1} = g(t)$ *and* $\frac{t}{t-1} = f^{-1}(t)$*, the Sheffer polynomials become the generalized Laguerre polynomials* $L_n^{\kappa}(x)$ *and consequently the truncated exponential-Sheffer polynomials become the generalized truncated exponential-Laguerre polynomials* ${}_{e^{(r)}}L_n^{\kappa}(u,v)$*.*

*From Theorem 3.1, we find*

$$\alpha_1 = -(\kappa+1);\ \ \alpha_k = 0\ (k \neq 1),\ \ \beta_1 = 1;\ \beta_2 = -2;$$
$$\beta_k = 0\ (k \neq 1,2),\ \ \gamma_k = \left(-\frac{t^r(t-1)}{1-\nu t^r}\right)^{(k)}\Big|_{t=0}. \tag{4.7}$$

*Inserting* (4.7) *into* (3.1)*, the succeeding differential equation for* ${}_{e^{(r)}}L_n^{\kappa}(u,v)$ *is obtained*

$$(x-\kappa-1)\left(\frac{\partial}{\partial x}\right){}_{e^{(r)}}L_n^{\kappa}(u,v) - x\frac{\partial^2}{\partial x^2}{}_{e^{(r)}}L_n^{\kappa}(u,v) + \sum_{k=1}^{n} ry\ \gamma_k\frac{\partial^k}{\partial x^k}\frac{{}_{e^{(r)}}L_n^{\kappa}(u,v)}{k!} = n\ {}_{e^{(r)}}L_n^{\kappa}(u,v). \tag{4.8}$$

*Similarly, in light of Theorems 2.1–2.3, the succeeding recursive formulas for* ${}_{e^{(r)}}L_n^{\kappa}(u,v)$ *are obtained*

$$1 = {}_{e^{(r)}}L_n^{\kappa}(u,v), \tag{4.9}$$

$${}_{e^{(r)}}L_{m+1}^{\kappa}(u,v) = (-x+\kappa+1)\ {}_{e^{(r)}}L_n^{\iota}(u,v) + (2x-\kappa-1)\frac{\partial}{\partial x}{}_{e^{(r)}}L_n^{\kappa}(u,v) - x\frac{\partial^2}{\partial x^2}\ \ {}_{e^{(r)}}L_n^{\kappa}(u,v)$$

$$+\sum_{k=0}^{n} ry\ \mu_k\frac{\partial^k}{\partial x^k}\frac{{}_{e^{(r)}}L_n^{\kappa}(u,v)}{k!}, \qquad n \geq 0, \tag{4.10}$$

$${}_{e^{(r)}}L_n^{\kappa}(u,v) = (-x+\kappa+1+2n)\ {}_{e^{(r)}}L_n^{\kappa}(u,v) - (n\kappa+n+2n(n-1))_{e^{(r)}}L_{n-1}^{\kappa}(u,v)$$

$$-\sum_{k=0}^{n}\binom{m}{k}\varsigma_k\ {}_{e^{(r)}}L_{n-k}^{\kappa}(u,v), \quad n \geq 0, \tag{4.11}$$

$${}_{e^{(r)}}L_{m+1}^{\kappa}(u,v) = (-x-\kappa-1)\ {}_{e^{(r)}}L_n^{\kappa}(u,v) + \sum_{k=1}^{n}\frac{m!}{(n-k)!}\left(-x(k+1)+(\kappa+1)\right){}_{e^{(r)}}L_{n-k}^{\kappa}(u,v)$$

$$+\sum_{k=0}^{n}\binom{m}{k}\xi_k\ {}_{e^{(r)}}L_{n-k}^{\kappa}(u,v), \qquad n \geq 0. \tag{4.12}$$

*Next, The differential equation is then derived using Corollary 3.1, and the recursive formulas for certain members of the truncated exponential-associated Sheffer family are derived using Corollaries 2.1, 2.3, and 2.5.*

**Example 4.3.** *For* $\ln(1+t) = f(t)$ *and* $e^t - 1 = f^{-1}(t)$*, the associated Sheffer polynomials reduces to the exponential polynomials* $\phi_n(x)$ *and consequently the truncated exponential-associated Sheffer polynomials become the truncated exponential-exponential polynomials* ${}_{e^{(r)}}\phi_n(u,v)$*.*

*It follows from Corollary 3.1*

$$\beta_2 = 1 = \beta_1;\ \ (-1)^k(k-2)! = \beta_k\ (k \neq 1,2). \tag{4.13}$$

*Inserting* (4.13) *into* (3.5)*, the succeeding differential equation for polynomials* ${}_{e^{(r)}}\phi_n(u,v)$ *is obtained:*

$$\begin{aligned} n\ {}_{e^{(r)}}\phi_n(u,v) &= x\left(\frac{\partial}{\partial x} - \frac{1}{2!}\frac{\partial^2}{\partial x^2}\right){}_{e^{(r)}}\phi_n(u,v) \\ &\quad + \sum_{k=3}^{n}\Big((-1)^k(k-2)!x + ry\,\gamma_k\Big)\frac{\partial^k}{\partial y^k}\frac{{}_{e^{(r)}}\phi_n(u,v)}{k!}. \end{aligned} \tag{4.14}$$

*Similarly, Corollaries 2.1, 2.3, and 2.5 are used to obtain the recursive formulas for the polynomials* ${}_{e^{(r)}}\phi_n(u,v)$

$$1 = {}_{e^{(r)}}\phi_0(u,v), \tag{4.15}$$

$$x(1+\frac{\partial}{\partial x})\ {}_{e^{(r)}}\phi_n(u,v) + \sum_{k=0}^{n} ry\,\mu_k\ {}_{e^{(r)}}\phi_n(u,v) = {}_{e^{(r)}}\phi_{m+1}(u,v), \quad n \geq 0, \tag{4.16}$$

$$x\ {}_{e^{(r)}}\phi_n(u,v) + \sum_{k=1}^{n}\binom{m}{k}\varsigma_k\ {}_{e^{(r)}}\phi_{m-k}(u,v) - \sum_{k=1}^{m}\binom{m}{k}(-1)^k\ {}_{e^{(r)}}\phi_{m+1-k}(u,v) = {}_{e^{(r)}}\phi_{m+1}(u,v), \quad n \geq 0, \tag{4.17}$$

*and*

$$\sum_{k=0}^{n}\binom{m}{k}(x + ry\,\xi_k)\ {}_{e^{(r)}}\phi_{n-k}(u,v), \quad n \geq 0 = {}_{e^{(r)}}\phi_{m+1}(u,v). \tag{4.18}$$

In the following section, we look at an identity for truncated exponential-Sheffer polynomials.

## 5. Appendix

The approach used in the preceding sections can be extended to yield additional results. As an example, we prove the following result to derive an identity for the truncated exponential-Sheffer polynomial sequences.

**Theorem 5.1.** *The following identity for the truncated exponential-Sheffer polynomials* ${}_{e^{(r)}}s_m(u,v)$ *holds true*

$$\frac{1}{m+1}\sum_{k=1}^{m+1}\frac{\Lambda_k}{k!}\ {}_{e^{(r)}}s_{m+1}^{(k)}(u,v) = {}_{e^{(r)}}s_m(u,v), \quad n \geq 0, \tag{5.1}$$

*where* $f^{(k)}(0) = \Lambda_k$*.*

*Proof.* In light of property (1.23), expression $\mathcal{W}_m\left[t\left(\frac{\exp(uf^{-1}(t))}{g(f^{-1}(t))(1-v(f^{-1}(t))^r)}\right)\right]_{t=0}$ takes the the form as given under

$$\mathbb{P}_m[t]_{t=0}\ \mathcal{W}_m\left[\frac{\exp(uf^{-1}(t))}{g(f^{-1}(t))(1-v(f^{-1}(t))^r)}\right]_{t=0} = \mathcal{W}_m\left[t\left(\frac{\exp(uf^{-1}(t))}{g(f^{-1}(t))(1-v(f^{-1}(t))^r)}\right)\right]_{t=0}. \tag{5.2}$$

In contrast, using Lemma 2.1 and the properties (1.22)–(1.24), we have

$$\begin{aligned}
&\mathcal{W}_m\left[f(f^{-1}(t))\frac{\exp(uf^{-1}(t))}{g(f^{-1}(t))(1-v(f^{-1}(t))^r)}\right]_{t=0}\\
&=\mathcal{W}_m\left[1,f^{-1}(t),\cdots,(f^{-1}(t))^n\right]_{t=0}\Lambda_m^{-1}\,\mathcal{W}_m\left[f(t)\frac{\exp(ut)}{g(t)(1-vt^r)}\right]_{t=0}\\
&=\mathcal{W}_m\left[1,f^{-1}(t),\cdots,(f^{-1}(t))^n\right]_{t=0}\Lambda_m^{-1}\,\mathbb{P}_m\left[\frac{1}{g(t)(1-vt^r)}\right]_{t=0}\mathbb{P}_m\left[\exp(ut)\right]_{t=0}\,\mathcal{W}_m[f(t)]_{t=0}\\
&=\mathcal{W}_m\left[{}_{e^{(r)}}s_0(u,v),{}_{e^{(r)}}s_1(u,v),\cdots,{}_{e^{(r)}}s_m(u,v)\right]^T\Lambda_m^{-1}\,\mathcal{W}_m[f(t)]_{t=0}\\
&=\begin{bmatrix}
{}_{e^{(r)}}s_0(u,v) & 0 & 0 & \cdots & 0\\
{}_{e^{(r)}}s_1(u,v) & \frac{{}_{e^{(r)}}s_1'(u,v)}{1!} & 0 & \cdots & 0\\
{}_{e^{(r)}}s_2(u,v) & \frac{{}_{e^{(r)}}s_2'(u,v)}{1!} & \frac{{}_{e^{(r)}}s_2''(u,v)}{2!} & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots & \\
{}_{e^{(r)}}s_m(u,v) & \frac{{}_{e^{(r)}}s_m'(u,v)}{1!} & \frac{{}_{e^{(r)}}s_m''(u,v)}{2!} & \cdots & \frac{{}_{e^{(r)}}s_m^{(n)}(u,v)}{m!}
\end{bmatrix}
\begin{bmatrix}\Lambda_0\\ \Lambda_1\\ \Lambda_2\\ \vdots\\ \Lambda_m\end{bmatrix}.
\end{aligned}\tag{5.3}$$

Assertion (5.1) is proved by comparing the $n^{th}$ rows of (5.2) and (5.3) and replacing $m$ by $m+1$. □

To demonstrate the application of Theorem 5.1, consider the following examples in the form of a table.

**Table 2.** Certain special cases of truncated exponential-Sheffer, associated-Sheffer and Appell polynomial family.

| S. No. | Cases | Name of the polynomial | Identity |
|---|---|---|---|
| **I.** | $\Lambda_k = (ln(1-t))^{(k)}\Big\vert_{t=0}$ | Truncated exponential-actuarial polynomials ${}_{e^{(r)}}A_m^{(\gamma)}(u,v)$ | ${}_{e^{(r)}}A_m^{(\gamma)}(u,v) = \frac{-1}{m+1}\Bigg(\left({}_{e^{(r)}}a_{m+1}^{(\gamma)}(u,v)\right)' + \sum_{k=2}^{m+1}\frac{({}_{e^{(r)}}a_{m+1}^{(\gamma)}(u,v))^k}{k}\Bigg)$ |
| **II.** | $\Lambda_k = (ln(1+t))^{(k)}\Big\vert_{t=0}$ | Truncated exponential-exponential polynomials ${}_{e^{(r)}}\phi_m(u,v)$ | ${}_{e^{(r)}}\phi_m(u,v) = \frac{1}{m+1}\Bigg({}_{e^{(r)}}\phi_{m+1}'(u,v) + \sum_{k=2}^{m+1}(-1)^{k-1}\frac{{}_{e^{(r)}}\phi_{m+1}^{(k)}(u,v)}{k}\Bigg)$ |
| **III.** | $\Lambda_k = (t)^{(k)}\vert_{t=0}$ | Truncated exponential-Miller-Lee polynomials ${}_{e^{(r)}}G_m^{(m)}(u,v)$ | ${}_{e^{(r)}}G_m^{(m)}(u,v) = \frac{1}{m+1}\left({}_{e^{(r)}}G_{m+1}^{(m)}(u,v)\right)'$ |

The recursive formulas and differential equations are critical in the analysis of algorithms [13]. These appear in infinite impulse response (IIR) digital filters. The attempt to model population dynamics gave rise to recurrence relations. Fibonacci numbers, for example, were once used as a model for rabbit population growth.

## 6. Conclusions

Hence we provided some specific properties of truncated exponential-Sheffer polynomials and multi variable special polynomial properties. The properties and association between the Pascal functional and Wronskian matrices are used to build the recursive equations and differential equation for these polynomials, as well as for several members of the truncated exponential-Sheffer family. The corresponding results for the truncated exponential-associated Sheffer and truncated exponential-Appell families are provided with some examples.

## Conflict of interest

The author declares that there is no conflict of interest.

## References


1. L. C. Andrews, *Special functions for engineers and applied mathematicians*, New York: Macmillan Publishing Company, 1985.
2. P. Appell, Sur une classe de polynômes, *Ann. Sci. I'École Norm. Sup.*, **9** (1880), 119–144. https://doi.org/10.24033/asens.186
3. P. Appell, J. K. de Fériet, *Fonctions hypergéométriques et hypersphériques: Polynômes d'Hermite*, Paris: Gauthier-Villars, 1926.
4. E. T. Bell, Exponential polynomials, *Ann. Math.*, **35** (1934), 258–277. https://doi.org/10.2307/1968431
5. G. Bretti, C. Cesarano, P. E. Ricci, Laguerre-type exponentials and generalized Appell polynomials, *Comput. Math. Appl.*, **48** (2004), 833–839. https://doi.org/10.1016/j.camwa.2003.09.031
6. G. Dattoli, *Hermite-Bessel and Laguerre-Bessel functions: A byproduct of the monomiality principle*, Advanced Special Functions and Applications, 2000.
7. G. Dattoli, C. Cesarano, D. Sacchetti, A note on truncated polynomials, *Appl. Math. Comput.*, **134** (2003), 595–605. https://doi.org/10.1016/S0096-3003(01)00310-1
8. G. Dattoli, S. Lorenzutta, A. M. Mancho, A. Torre, Generalized polynomials and associated operational identities, *J. Comput. Appl. Math.*, **108** (1999), 209–218. https://doi.org/10.1016/S0377-0427(99)00111-9
9. S. Khan, G. Yasmeen, N. Ahmad, On a new family related to truncated exponential and Sheffer polynomials, *J. Math. Anal. Appl.*, **418** (2014), 921–937. https://doi.org/10.1016/j.jmaa.2014.04.028
10. D. S. Kim, T. Kim, A matrix approach to some identities involving Sheffer polynomial sequences, *Appl. Math. Comput.*, **253** (2015), 83–101. https://doi.org/10.1016/j.amc.2014.12.048
11. E. D. Rainville, *Special functions*, New York: The Macmillan Company, 1960.
12. S. Roman, *The umbral calculus*, New York: Dover Publications, 1984.

13. R. Sedgewick, F. Flajolet, *An introduction to the analysis of algorithms*, Pearson Education India, 2013.

14. I. M. Sheffer, Some properties of polynomial sets of type zero, *Duke Math. J.*, **5** (1939), 590–622. https://doi.org/10.1215/S0012-7094-39-00549-1

15. Y. Yang, C. Micek, Generalized Pascal functional matrix and its applications, *Linear Algebra Appl.*, **423** (2007), 230–245. https://doi.org/10.1016/j.laa.2006.12.014

16. H. Youn, Y. Yang, Differential equation and recursive formulas of Sheffer polynomial sequences, *Int. Scholarly Res. Not.*, **2011** (2011), 476462. https://doi.org/10.5402/2011/476462