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Research article

On generalizations of trapezoid and Bullen type inequalities based on generalized fractional integrals

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Abstract: In this paper, we establish an integral identity involving differentiable functions and generalized fractional integrals. Then, using the newly established identity, we prove some new general versions of Bullen and trapezoidal type inequalities for differentiable convex functions. The main benefit of the newly established inequalities is that they can be converted into similar inequalities for classical integrals, Riemann-Liouville fractional integrals, k-Riemann-Liouville fractional integrals, Hadamard fractional integrals, etc. Moreover, the inequalities presented in the paper are extensions of several existing inequalities in the literature.

Keywords: Hermite-Hadamard inequality; fractional integral operators; convex function
Mathematics Subject Classification: 26D07, 26D10, 26D15

1. Introduction

In literature, the theory of inequality plays an important role in mathematics. There are many studies on the known Hermite-Hadamard inequality and related inequalities such as trapezoid, midpoint, Simpson's inequality, and Bullen's inequality.

Over the years, many articles have focused on finding trapezoid and midpoint inequalities that give boundaries to the right and left side of Hermite-Hadamard inequality, respectively. For example, Dragomir and Agarwal first established trapezoid inequalities in convex activities in [8], while Kirmacı first, found the midpoint of convex activity in [22]. In addition to [28], Qaisar and Hussain introduced several generalized inequalities of midpoint type. Sarikaya et al. and Iqbal et al. prove fractional trapezoid inequality and midpoint inequality for convex functions in [17, 32], respectively. In [4, 5], researchers established some generalized midpoint type inequalities for Riemann-Liouville fractional integrals.

Many mathematicians have focused the results of Simpson-type for convex functions. More precisely, some inequalities of Simpson's type for s-convex functions are proved by using differentiable functions [1]. In the papers [33, 34], it is investigated the new variants of Simpson's type inequalities based on the differentiable convex mapping. For more information about Simpson type inequalities for various convex classes, we refer the reader to Refs. [9, 12, 16, 24, 27, 29, 30] and the references therein.

In [6], Bullen established the well-known Bullen inequalities in the literature in 1978. In [35], Sarikaya et al. proved generalized Bullen inequality for generalized convex function. Erden and Sarikaya established the generalized Bullen-type inequalities involving local fractional integrals on fractal sets in [11]. Du et al. used the generalized fractional integrals to obtain Bullen-type inequalities in [10]. In [7], Çakmak proved some Bullen type inequalities for conformable fractional integrals.

On the other hand recently, Sarikaya and Ertugral [36] have defined a new class of fractional integrals, called generalized fractional and they used these integrals to prove general version of Hermite-Hadamard type inequalities for convex functions. In [39], the authors used generalized fractional integrals and proved some trapezoidal type inequalities for harmonic convex functions. Budak et al. [3] proved several variants of Ostrowski's and Simpson's type for differentiable convex functions via generalized fractional integrals. For more inequalities via fractional integrals, one can consult [2, 18–20, 37, 38, 40] and references therein.

Inspired by the ongoing studies, we prove some new inequalities of Bullen type inequalities for differentiable convex functions using the generalized fractional integrals. The main benefit of the inequalities and operators used to obtain them is that these inequalities can be turned into some existing results for Riemann integrals and new results for Riemann-Liouville fractional integral inequalities and k-fractional integrals.

2. Fractional integrals and related inequalities

In this section, we recall some basic notations and notions of the fractional integrals. We also recall some inequalities via different fractional integrals.

Definition 2.1. *[15, 21] Let $\mathcal{F} \in L_1[\theta, \vartheta]$. The Riemann-Liouville fractional integrals (RLFIs) $J_{\theta+}^{\alpha}\mathcal{F}$ and $J_{\vartheta-}^{\alpha}\mathcal{F}$ of order $\alpha > 0$ are defined as follows:*

$$J_{\theta+}^{\alpha}\mathcal{F}(\varkappa) = \frac{1}{\Gamma(\alpha)} \int_{\theta}^{\varkappa} (\varkappa - \lambda)^{\alpha-1} \mathcal{F}(\lambda)\, d\lambda,\ \varkappa > \theta$$

and

$$J_{\vartheta-}^{\alpha}\mathcal{F}(\varkappa) = \frac{1}{\Gamma(\alpha)} \int_{\varkappa}^{\vartheta} (\lambda - \varkappa)^{\alpha-1} \mathcal{F}(\lambda)\, d\lambda,\ \varkappa < \vartheta,$$

respectively, where Γ is the well-known Gamma function and its described as follows:

$$\Gamma(\alpha) = \int_0^{\infty} e^{-u} u^{\alpha-1} du.$$

Definition 2.2. *[26] Let $\mathcal{F} \in L_1[\theta, \vartheta]$. The k-fractional integrals (KFIs) $J_{\theta+}^{\alpha,k}\mathcal{F}$ and $J_{\vartheta-,k}^{\alpha,k}\mathcal{F}$ of order $\alpha, k > 0$ are defined as follows:*

$$J_{\theta+}^{\alpha,k}\mathcal{F}(\varkappa) = \frac{1}{k\Gamma_k(\alpha)} \int_\theta^\varkappa (\varkappa - \lambda)^{\frac{\alpha}{k}-1} \mathcal{F}(\lambda)\, d\lambda,\ \varkappa > \theta$$

and

$$J_{\vartheta-}^{\alpha,k}\mathcal{F}(\varkappa) = \frac{1}{k\Gamma_k(\alpha)} \int_\varkappa^\vartheta (\lambda - \varkappa)^{\frac{\alpha}{k}-1} \mathcal{F}(\lambda)\, d\lambda,\ \varkappa < \vartheta,$$

respectively, where Γ_k is the well-known k-Gamma function and its described as follows:

$$\Gamma_k(\alpha) = \int_0^\infty e^{-\frac{u^k}{k}} u^{\alpha-1} du.$$

Definition 2.3. *[36] Let $\mathcal{F} \in L_1[\theta, \vartheta]$. The generalized fractional integrals (GFIs) ${}_{\theta+}I_\varphi\mathcal{F}$ and ${}_{\vartheta-}I_\varphi\mathcal{F}$ with $\theta \geq 0$ are defined as follows:*

$${}_{\theta+}I_\varphi\mathcal{F}(\varkappa) = \int_\theta^\varkappa \frac{\varphi(\varkappa - \lambda)}{\varkappa - \lambda} \mathcal{F}(\lambda)\, d\lambda,\ \varkappa > \theta$$

and

$${}_{\vartheta-}I_\varphi\mathcal{F}(\varkappa) = \int_\varkappa^\vartheta \frac{\varphi(\lambda - \varkappa)}{\lambda - \varkappa} \mathcal{F}(\lambda)\, d\lambda,\ \varkappa < \vartheta,$$

respectively, where $\varphi : [0, \infty) \to [0, \infty)$ is a function. For more properties of the the functions φ, one can consult [36].

Remark 2.4. *The importance of the GFIs is that these can be turned into classical Riemann integrals, RLFIs and KFIs for $\varphi(\lambda) = \lambda$, $\varphi(\lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)}$ and $\varphi(\lambda) = \frac{\lambda^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, respectively.*

Theorem 2.5. *Let $\mathcal{F} : I \to \mathbb{R}$ be a convex function on I with $\theta, \vartheta \in I$ such that $\theta < \vartheta$. If $\mathcal{F} \in L_1[\theta, \vartheta]$, the following inequality holds:*

$$\mathcal{F}\left(\frac{\theta + \vartheta}{2}\right) \leq \frac{1}{2\Lambda(1)} \left[{}_{\theta+}I_\varphi\mathcal{F}(\vartheta) + {}_{\vartheta-}I_\varphi\mathcal{F}(\theta)\right] \leq \frac{\mathcal{F}(\theta) + \mathcal{F}(\vartheta)}{2},$$

where $\Lambda(1) = \int_0^1 \frac{\varphi((\vartheta-\theta)\lambda)}{\lambda} d\lambda$.

Remark 2.6. *In Theorem 2.5, we have*

(i) If we set $\varphi(\lambda) = \lambda$, then we have the following classical Hermite-Hadamard inequality (see, [31, p. 137]):

$$\mathcal{F}\left(\frac{\theta + \vartheta}{2}\right) \leq \frac{1}{\vartheta - \theta} \int_\theta^\vartheta \mathcal{F}(\varkappa)\, d\varkappa \leq \frac{\mathcal{F}(\theta) + \mathcal{F}(\vartheta)}{2}.$$

(ii) If we set $\varphi(\lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)}$, then we have the following RLFIs Hermite-Hadamard inequality (see, [32]):

$$\mathcal{F}\left(\frac{\theta + \vartheta}{2}\right) \leq \frac{\Gamma(\alpha + 1)}{2(\vartheta - \theta)^\alpha} \left[J_{\theta+}^\alpha\mathcal{F}(\vartheta) + J_{\vartheta-}^\alpha\mathcal{F}(\theta)\right] \leq \frac{\mathcal{F}(\theta) + \mathcal{F}(\vartheta)}{2}.$$

(iii) If we set $\varphi(\lambda) = \frac{\lambda^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then we have the following KFIs Hermite-Hadamard inequality (see, [14]):

$$\mathcal{F}\left(\frac{\theta + \vartheta}{2}\right) \leq \frac{\Gamma_k(\alpha + k)}{2(\vartheta - \theta)^{\frac{\alpha}{k}}} \left[J_{\theta+}^{\alpha,k}\mathcal{F}(\vartheta) + J_{\vartheta-}^{\alpha,k}\mathcal{F}(\theta)\right] \leq \frac{\mathcal{F}(\theta) + \mathcal{F}(\vartheta)}{2}.$$

3. Main results

In this section, firstly we need to give a lemma for differentiable functions which will help us to prove our main theorems. Then, we present some midpoint type inequalities which are the generalization of those given in earlier works.

Throughout this study, for brevity, we define

$$\Lambda_1(\lambda) = \int_0^\lambda \frac{\varphi\left(\frac{\varkappa-\theta}{2}u\right)}{u}du \quad \Lambda_2(\lambda) = \int_0^\lambda \frac{\varphi\left(\frac{\vartheta-\varkappa}{2}u\right)}{u}du.$$

Lemma 3.1. *Let $\mathcal{F} : [\theta, \vartheta] \to \mathbb{R}$ be differentiable function on (θ, ϑ) with $\theta < \vartheta$. If $\mathcal{F}' \in L[\theta, \vartheta]$, then we have the following identity for GFIs:*

$$\begin{aligned} &\frac{(\vartheta-\theta)\mathcal{F}(\varkappa)+(\varkappa-\theta)\mathcal{F}(\theta)+(\vartheta-\varkappa)\mathcal{F}(\vartheta)}{2} \\ &-\frac{\varkappa-\theta}{2\Lambda_1(1)}\left[{}_{\varkappa^-}I_\varphi\mathcal{F}\left(\frac{\varkappa+\theta}{2}\right)+{}_{\theta^+}I_\varphi\mathcal{F}\left(\frac{\varkappa+\theta}{2}\right)\right]-\frac{\vartheta-\varkappa}{2\Lambda_2(1)}\left[{}_{\varkappa^+}I_\varphi\mathcal{F}\left(\frac{\varkappa+\vartheta}{2}\right)+{}_{\vartheta^-}I_\varphi\mathcal{F}\left(\frac{\varkappa+\vartheta}{2}\right)\right] \\ =\ &\frac{(\varkappa-\theta)^2}{4\Lambda_1(1)}\int_0^1\Lambda_1(\lambda)\mathcal{F}'\left(\frac{1+\lambda}{2}\varkappa+\frac{1-\lambda}{2}\theta\right)d\lambda-\frac{(\varkappa-\theta)^2}{4\Lambda_1(1)}\int_0^1\Lambda_1(\lambda)\mathcal{F}'\left(\frac{1-\lambda}{2}\varkappa+\frac{1+\lambda}{2}\theta\right)d\lambda \\ &-\frac{(\vartheta-\varkappa)^2}{4\Lambda_2(1)}\int_0^1\Lambda_2(\lambda)\mathcal{F}'\left(\frac{1+\lambda}{2}\varkappa+\frac{1-\lambda}{2}\vartheta\right)d\lambda+\frac{(\vartheta-\varkappa)^2}{4\Lambda_2(1)}\int_0^1\Lambda_2(\lambda)\mathcal{F}'\left(\frac{1-\lambda}{2}\varkappa+\frac{1+\lambda}{2}\vartheta\right)d\lambda. \end{aligned} \tag{3.1}$$

Proof. First, we consider

$$\begin{aligned} &\frac{(\varkappa-\theta)^2}{4\Lambda_1(1)}\int_0^1\Lambda_1(\lambda)\mathcal{F}'\left(\frac{1+\lambda}{2}\varkappa+\frac{1-\lambda}{2}\theta\right)d\lambda-\frac{(\varkappa-\theta)^2}{4\Lambda_1(1)}\int_0^1\Lambda_1(\lambda)\mathcal{F}'\left(\frac{1-\lambda}{2}\varkappa+\frac{1+\lambda}{2}\theta\right)d\lambda \\ &-\frac{(\vartheta-\varkappa)^2}{4\Lambda_2(1)}\int_0^1\Lambda_2(\lambda)\mathcal{F}'\left(\frac{1+\lambda}{2}\varkappa+\frac{1-\lambda}{2}\vartheta\right)d\lambda+\frac{(\vartheta-\varkappa)^2}{4\Lambda_2(1)}\int_0^1\Lambda_2(\lambda)\mathcal{F}'\left(\frac{1-\lambda}{2}\varkappa+\frac{1+\lambda}{2}\vartheta\right)d\lambda \\ =\ &\frac{(\varkappa-\theta)^2}{4\Lambda_1(1)}I_1-\frac{(\varkappa-\theta)^2}{4\Lambda_1(1)}I_2-\frac{(\vartheta-\varkappa)^2}{4\Lambda_2(1)}I_3+\frac{(\vartheta-\varkappa)^2}{4\Lambda_2(1)}I_4. \end{aligned} \tag{3.2}$$

By integration by parts, we have

$$\begin{aligned} I_1 &= \int_0^1\Lambda_1(\lambda)\mathcal{F}'\left(\frac{1+\lambda}{2}\varkappa+\frac{1-\lambda}{2}\theta\right)d\lambda \\ &= \frac{2\Lambda_1(\lambda)}{\varkappa-\theta}\mathcal{F}\left(\frac{1+\lambda}{2}\varkappa+\frac{1-\lambda}{2}\theta\right)\Bigg|_0^1-\frac{2}{\varkappa-\theta}\int_0^1\frac{\varphi\left(\frac{\varkappa-\theta}{2}\lambda\right)}{\lambda}\mathcal{F}\left(\frac{1+\lambda}{2}\varkappa+\frac{1-\lambda}{2}\theta\right)d\lambda \\ &= \frac{2\Lambda_1(1)}{\varkappa-\theta}\mathcal{F}(\varkappa)-\frac{2}{\varkappa-\theta}\int_{\frac{\theta+\varkappa}{2}}^{\varkappa}\frac{\varphi\left(y-\frac{\varkappa+\theta}{2}\right)}{y-\frac{\varkappa+\theta}{2}}\mathcal{F}(y)\,dy \\ &= \frac{2\Lambda_1(1)}{\varkappa-\theta}\mathcal{F}(\varkappa)-\frac{2}{\varkappa-\theta}\,{}_{\varkappa^-}I_\varphi\mathcal{F}\left(\frac{\varkappa+\theta}{2}\right) \end{aligned} \tag{3.3}$$

and similarly

$$I_2 = \int_0^1\Lambda_1(\lambda)\mathcal{F}'\left(\frac{1-\lambda}{2}\varkappa+\frac{1+\lambda}{2}\theta\right)d\lambda = \frac{-2\Lambda_1(1)}{\varkappa-\theta}\mathcal{F}(\theta)+\frac{2}{\varkappa-\theta}\,{}_{\theta^+}I_\varphi\mathcal{F}\left(\frac{\varkappa+\theta}{2}\right) \tag{3.4}$$

$$I_3 = \int_0^1 \Lambda_2(\lambda)\mathcal{F}'\left(\frac{1+\lambda}{2}\varkappa + \frac{1-\lambda}{2}\vartheta\right)d\lambda = -\frac{2\Lambda_2(1)}{\vartheta-\varkappa}\mathcal{F}(\varkappa) + \frac{2}{\vartheta-\varkappa}\,{}_{\varkappa^+}I_\varphi\mathcal{F}\left(\frac{\varkappa+\vartheta}{2}\right) \tag{3.5}$$

$$I_4 = \int_0^1 \Lambda_2(\lambda)\mathcal{F}'\left(\frac{1-\lambda}{2}\varkappa + \frac{1+\lambda}{2}\vartheta\right)d\lambda = \frac{2\Lambda_2(1)}{\vartheta-\varkappa}\mathcal{F}(\varkappa) - \frac{2}{\vartheta-\varkappa}\,{}_{\vartheta^-}I_\varphi\mathcal{F}\left(\frac{\varkappa+\vartheta}{2}\right). \tag{3.6}$$

By substituting the equalities (3.3)–(3.6) in (3.2), then we obtain the desired result. □

Remark 3.2. *If we choose $\varphi(\lambda) = \lambda$ for all $\lambda \in [\theta, \vartheta]$ in Lemma 3.1, then Lemma 3.1 reduces to [23, Lemma 1].*

Theorem 3.3. *Let $\mathcal{F} : [\theta, \vartheta] \to \mathbb{R}$ be differentiable function on (θ, ϑ). If $|\mathcal{F}'|$ is convex function, then we have the following inequality for GFIs:*

$$\begin{aligned}&\Bigg|\frac{(\vartheta-\theta)\mathcal{F}(\varkappa) + (\varkappa-\theta)\mathcal{F}(\theta) + (\vartheta-\varkappa)\mathcal{F}(\vartheta)}{2} \\ &-\frac{\varkappa-\theta}{2\Lambda_1(1)}\left[{}_{\varkappa^-}I_\varphi\mathcal{F}\left(\frac{\varkappa+\theta}{2}\right) + {}_{\theta^+}I_\varphi\mathcal{F}\left(\frac{\varkappa+\theta}{2}\right)\right] - \frac{\vartheta-\varkappa}{2\Lambda_2(1)}\left[{}_{\varkappa^+}I_\varphi\mathcal{F}\left(\frac{\varkappa+\vartheta}{2}\right) + {}_{\vartheta^-}I_\varphi\mathcal{F}\left(\frac{\varkappa+\vartheta}{2}\right)\right]\Bigg| \\ \le\;& \frac{(\varkappa-\theta)^2}{4\Lambda_1(1)}\left(\int_0^1 |\Lambda_1(\lambda)|\,d\lambda\right)\left(|\mathcal{F}'(\varkappa)| + |\mathcal{F}'(\theta)|\right) + \frac{(\vartheta-\varkappa)^2}{4\Lambda_2(1)}\left(\int_0^1 |\Lambda_2(\lambda)|\,d\lambda\right)\left(|\mathcal{F}'(\varkappa)| + |\mathcal{F}'(\vartheta)|\right).\end{aligned} \tag{3.7}$$

Proof. By taking modulus in Lemma 3.1, we have

$$\begin{aligned}&\Bigg|\frac{(\vartheta-\theta)\mathcal{F}(\varkappa) + (\varkappa-\theta)\mathcal{F}(\theta) + (\vartheta-\varkappa)\mathcal{F}(\vartheta)}{2} \\ &-\frac{\varkappa-\theta}{2\Lambda_1(1)}\left[{}_{\varkappa^-}I_\varphi\mathcal{F}\left(\frac{\varkappa+\theta}{2}\right) + {}_{\theta^+}I_\varphi\mathcal{F}\left(\frac{\varkappa+\theta}{2}\right)\right] - \frac{\vartheta-\varkappa}{2\Lambda_2(1)}\left[{}_{\varkappa^+}I_\varphi\mathcal{F}\left(\frac{\varkappa+\vartheta}{2}\right) + {}_{\vartheta^-}I_\varphi\mathcal{F}\left(\frac{\varkappa+\vartheta}{2}\right)\right]\Bigg| \\ \le\;& \frac{(\varkappa-\theta)^2}{4\Lambda_1(1)}\int_0^1 |\Lambda_1(\lambda)|\left|\mathcal{F}'\left(\frac{1+\lambda}{2}\varkappa + \frac{1-\lambda}{2}\theta\right)\right|d\lambda + \frac{(\varkappa-\theta)^2}{4\Lambda_1(1)}\int_0^1 |\Lambda_1(\lambda)|\left|\mathcal{F}'\left(\frac{1-\lambda}{2}\varkappa + \frac{1+\lambda}{2}\theta\right)\right|d\lambda \\ &+\frac{(\vartheta-\varkappa)^2}{4\Lambda_2(1)}\int_0^1 |\Lambda_2(\lambda)|\left|\mathcal{F}'\left(\frac{1+\lambda}{2}\varkappa + \frac{1-\lambda}{2}\vartheta\right)\right|d\lambda + \frac{(\vartheta-\varkappa)^2}{4\Lambda_2(1)}\int_0^1 |\Lambda_2(\lambda)|\left|\mathcal{F}'\left(\frac{1-\lambda}{2}\varkappa + \frac{1+\lambda}{2}\vartheta\right)\right|d\lambda.\end{aligned} \tag{3.8}$$

By using the convexity of $|\mathcal{F}'|$ we get

$$\begin{aligned}&\Bigg|\frac{(\vartheta-\theta)\mathcal{F}(\varkappa) + (\varkappa-\theta)\mathcal{F}(\theta) + (\vartheta-\varkappa)\mathcal{F}(\vartheta)}{2} \\ &-\frac{(\varkappa-\theta)}{2\Lambda_1(1)}\left[{}_{\varkappa^-}I_\varphi\mathcal{F}\left(\frac{\varkappa+\theta}{2}\right) + {}_{\theta^+}I_\varphi\mathcal{F}\left(\frac{\varkappa+\theta}{2}\right)\right] - \frac{(\vartheta-\varkappa)}{2\Lambda_2(1)}\left[{}_{\varkappa^+}I_\varphi\mathcal{F}\left(\frac{\varkappa+\vartheta}{2}\right) + {}_{\vartheta^-}I_\varphi\mathcal{F}\left(\frac{\varkappa+\vartheta}{2}\right)\right]\Bigg| \\ \le\;& \frac{(\varkappa-\theta)^2}{4\Lambda_1(1)}\left[\frac{|\mathcal{F}'(\varkappa)|}{2}\int_0^1 |\Lambda_1(\lambda)|(1+\lambda)\,d\lambda + \frac{|\mathcal{F}'(\theta)|}{2}\int_0^1 |\Lambda_1(\lambda)|(1-\lambda)\,d\lambda\right] \\ &+\frac{(\varkappa-\theta)^2}{4\Lambda_1(1)}\left[\frac{|\mathcal{F}'(\varkappa)|}{2}\int_0^1 |\Lambda_1(\lambda)|(1-\lambda)\,d\lambda + \frac{|\mathcal{F}'(\theta)|}{2}\int_0^1 |\Lambda_1(\lambda)|(1+\lambda)\,d\lambda\right]\end{aligned}$$

$$+\frac{(\vartheta-\varkappa)^2}{4\Lambda_2(1)}\left[\frac{|\mathcal{F}'(\varkappa)|}{2}\int_0^1|\Lambda_2(\lambda)|(1+\lambda)\,d\lambda+\frac{|\mathcal{F}'(\vartheta)|}{2}\int_0^1|\Lambda_2(\lambda)|(1-\lambda)\,d\lambda\right]$$
$$+\frac{(\vartheta-\varkappa)^2}{4\Lambda_2(1)}\left[\frac{|\mathcal{F}'(\varkappa)|}{2}\int_0^1|\Lambda_2(\lambda)|(1-\lambda)\,d\lambda+|\mathcal{F}'(\vartheta)|\int_0^1|\Lambda_2(\lambda)|(1+\lambda)\,d\lambda\right]$$
$$\leq \frac{(\varkappa-\theta)^2}{4\Lambda_1(1)}\left[|\mathcal{F}'(\varkappa)|+|\mathcal{F}'(\theta)|\right]\left(\int_0^1|\Lambda_1(\lambda)|\,d\lambda\right)+\frac{(\vartheta-\varkappa)^2}{4\Lambda_2(1)}\left[|\mathcal{F}'(\varkappa)|+|\mathcal{F}'(\vartheta)|\right]\left(\int_0^1|\Lambda_2(\lambda)|\,d\lambda\right).$$

This completes the proof. □

Remark 3.4. *If we choose $\varphi(\lambda)=\lambda$ for all $\lambda\in[\theta,\vartheta]$ in Theorem 3.3, then Theorem 3.3 reduces to [23, Theorem 1].*

Remark 3.5. *If we choose $\varphi(\lambda)=\frac{\lambda^\alpha}{\Gamma(\alpha)}$, $\alpha>1$, for all $\lambda\in[\theta,\vartheta]$ in Theorem 3.3, then Theorem 3.3 reduces to [25, Theorem 1].*

Remark 3.6. *If we take $\varkappa=\theta$ (or $\varkappa=\vartheta$) in Theorem 3.3, then we have the following Trapezoid type inequality which is proved by Ertuğral et al. in [13];*

$$\left|\frac{\mathcal{F}(\theta)+\mathcal{F}(\vartheta)}{2}-\frac{1}{2\Lambda_2(1)}\left[{}_{\theta^+}I_\varphi\mathcal{F}\left(\frac{\theta+\vartheta}{2}\right)+{}_{\vartheta^-}I_\varphi\mathcal{F}\left(\frac{\theta+\vartheta}{2}\right)\right]\right|$$
$$\leq \frac{\vartheta-\theta}{4\Delta(1)}\left(\int_0^1|\Delta(\lambda)|\,d\lambda\right)\left(|\mathcal{F}'(\theta)|+|\mathcal{F}'(\vartheta)|\right),$$

where

$$\Delta(\lambda)=\int_0^\lambda\frac{\varphi\left(\frac{\vartheta-\theta}{2}u\right)}{u}du.$$

Corollary 3.7. *Under assumption of Theorem 3.3, if we take $\varkappa=\frac{\theta+\vartheta}{2}$ in Theorem 3.3 then we have the following Bullen type inequality*

$$\left|\frac{1}{2}\left[\frac{\mathcal{F}(\theta)+\mathcal{F}(\vartheta)}{2}+\mathcal{F}\left(\frac{\theta+\vartheta}{2}\right)\right]\right.$$
$$\left.-\frac{1}{4\Psi(1)}\left[{}_{\frac{\theta+\vartheta}{2}^-}I_\varphi\mathcal{F}\left(\frac{3\theta+\vartheta}{4}\right)+{}_{\theta^+}I_\varphi\mathcal{F}\left(\frac{3\theta+\vartheta}{4}\right)\right]-\frac{1}{4\Psi(1)}\left[{}_{\frac{\theta+\vartheta}{2}^+}I_\varphi\mathcal{F}\left(\frac{\theta+3\vartheta}{4}\right)+{}_{\vartheta^-}I_\varphi\mathcal{F}\left(\frac{\theta+3\vartheta}{4}\right)\right]\right|$$
$$\leq \frac{\vartheta-\theta}{8\Psi(1)}\left(\int_0^1|\Psi(\lambda)|\,d\lambda\right)\left(\left|\mathcal{F}'\left(\frac{\theta+\vartheta}{2}\right)\right|+\frac{|\mathcal{F}'(\theta)|+|\mathcal{F}'(\vartheta)|}{2}\right)$$
$$\leq \frac{\vartheta-\theta}{8\Psi(1)}\left(\int_0^1|\Psi(\lambda)|\,d\lambda\right)\left(|\mathcal{F}'(\theta)|+|\mathcal{F}'(\vartheta)|\right),$$

where

$$\Psi(\lambda)=\int_0^\lambda\frac{\varphi\left(\frac{\vartheta-\theta}{4}u\right)}{u}du.$$

Corollary 3.8. *If we choose $\varphi(\lambda) = \lambda$ for all $\lambda \in [\theta, \vartheta]$ in Corollary 3.7, then Corollary 3.7 reduces to [23, Corollary 1].*

Corollary 3.9. *If we choose $\varphi(\lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)}$, $\alpha > 1$, for all $\lambda \in [\theta, \vartheta]$ in Corollary 3.7, then we have the following Bullen type inequality for RLFIs*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{\mathcal{F}(\theta)+\mathcal{F}(\vartheta)}{2}+\mathcal{F}\left(\frac{\theta+\vartheta}{2}\right)\right]-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(\vartheta-\theta)^{\alpha-1}}\left[J^{\alpha}_{\frac{\theta+\vartheta}{2}^-}\mathcal{F}\left(\frac{3\theta+\vartheta}{4}\right)+J^{\alpha}_{\theta^+}\mathcal{F}\left(\frac{3\theta+\vartheta}{4}\right)\right]\right.\\
&\left.-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(\vartheta-\varkappa)^{\alpha-1}}\left[J^{\alpha}_{\frac{\theta+\vartheta}{2}^+}\mathcal{F}\left(\frac{\theta+3\vartheta}{4}\right)+J^{\alpha}_{\vartheta^-}\mathcal{F}\left(\frac{\theta+3\vartheta}{4}\right)\right]\right|\\
\leq\ & \frac{\vartheta-\theta}{8(\alpha+1)}\left(\left|\mathcal{F}'\left(\frac{\theta+\vartheta}{2}\right)\right|+\frac{|\mathcal{F}'(\theta)|+|\mathcal{F}'(\vartheta)|}{2}\right)\\
\leq\ & \frac{\vartheta-\theta}{8(\alpha+1)}\left(|\mathcal{F}'(\theta)|+|\mathcal{F}'(\vartheta)|\right).
\end{aligned}$$

Theorem 3.10. *Let $\mathcal{F} : [\theta, \vartheta] \to \mathbb{R}$ be differentiable function on (θ, ϑ). If $|\mathcal{F}'|^q$, $q > 1$, is convex function, then we have the following inequality for GFIs:*

$$\begin{aligned}
&\left|\frac{(\vartheta-\theta)\mathcal{F}(\varkappa)+(\varkappa-\theta)\mathcal{F}(\theta)+(\vartheta-\varkappa)\mathcal{F}(\vartheta)}{2}\right.\\
&\left.-\frac{\varkappa-\theta}{2\Lambda_1(1)}\left[{}_{\varkappa^-}I_{\varphi}\mathcal{F}\left(\frac{\varkappa+\theta}{2}\right)+{}_{\theta^+}I_{\varphi}\mathcal{F}\left(\frac{\varkappa+\theta}{2}\right)\right]-\frac{\vartheta-\varkappa}{2\Lambda_2(1)}\left[{}_{\varkappa^+}I_{\varphi}\mathcal{F}\left(\frac{\varkappa+\vartheta}{2}\right)+{}_{\vartheta^-}I_{\varphi}\mathcal{F}\left(\frac{\varkappa+\vartheta}{2}\right)\right]\right|\\
\leq\ & \frac{(\varkappa-\theta)^2}{4\Lambda_1(1)}\left(\int_0^1|\Lambda_1(\lambda)|^p d\lambda\right)^{\frac{1}{p}}\left[\left(\frac{3|\mathcal{F}'(\varkappa)|^q+|\mathcal{F}'(\theta)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{F}'(\varkappa)|^q+3|\mathcal{F}'(\theta)|^q}{4}\right)^{\frac{1}{q}}\right]\\
&+\frac{(\vartheta-\varkappa)^2}{4\Lambda_2(1)}\left(\int_0^1|\Lambda_2(\lambda)|^p d\lambda\right)^{\frac{1}{p}}\left[\left(\frac{3|\mathcal{F}'(\varkappa)|^q+|\mathcal{F}'(\vartheta)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{F}'(\varkappa)|^q+3|\mathcal{F}'(\vartheta)|^q}{4}\right)^{\frac{1}{q}}\right]\\
\leq\ & \frac{(\varkappa-\theta)^2}{2^{\frac{2}{q}}\Lambda_1(1)}\left(\int_0^1|\Lambda_1(\lambda)|^p d\lambda\right)^{\frac{1}{p}}\left[|\mathcal{F}'(\varkappa)|+|\mathcal{F}'(\theta)|\right]+\frac{(\vartheta-\varkappa)^2}{2^{\frac{2}{q}}\Lambda_2(1)}\left(\int_0^1|\Lambda_2(\lambda)|^p d\lambda\right)^{\frac{1}{p}}\left[|\mathcal{F}'(\varkappa)|+|\mathcal{F}'(\vartheta)|\right],
\end{aligned} \tag{3.9}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using the well-known Hölder inequality in (3.8), we obtain

$$\begin{aligned}
&\left|\frac{(\vartheta-\theta)\mathcal{F}(\varkappa)+(\varkappa-\theta)\mathcal{F}(\theta)+(\vartheta-\varkappa)\mathcal{F}(\vartheta)}{2}\right.\\
&\left.-\frac{\varkappa-\theta}{2\Lambda_1(1)}\left[{}_{\varkappa^-}I_{\varphi}\mathcal{F}\left(\frac{\varkappa+\theta}{2}\right)+{}_{\theta^+}I_{\varphi}\mathcal{F}\left(\frac{\varkappa+\theta}{2}\right)\right]-\frac{\vartheta-\varkappa}{2\Lambda_2(1)}\left[{}_{\varkappa^+}I_{\varphi}\mathcal{F}\left(\frac{\varkappa+\vartheta}{2}\right)+{}_{\vartheta^-}I_{\varphi}\mathcal{F}\left(\frac{\varkappa+\vartheta}{2}\right)\right]\right|\\
\leq\ & \frac{(\varkappa-\theta)^2}{4\Lambda_1(1)}\left(\int_0^1|\Lambda_1(\lambda)|^p d\lambda\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{F}'\left(\frac{1+\lambda}{2}\varkappa+\frac{1-\lambda}{2}\theta\right)\right|^q d\lambda\right)^{\frac{1}{q}}
\end{aligned} \tag{3.10}$$

$$+\frac{(\varkappa-\theta)^2}{4\Lambda_1(1)}\left(\int_0^1|\Lambda_1(\lambda)|^p\right)\left(\int_0^1\left|\mathcal{F}'\left(\frac{1-\lambda}{2}\varkappa+\frac{1+\lambda}{2}\theta\right)\right|^q d\lambda\right)^{\frac{1}{q}}$$
$$+\frac{(\vartheta-\varkappa)^2}{4\Lambda_2(1)}\left(\int_0^1|\Lambda_2(\lambda)|^p\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{F}'\left(\frac{1+\lambda}{2}\varkappa+\frac{1-\lambda}{2}\vartheta\right)\right|^q d\lambda\right)^{\frac{1}{q}}$$
$$+\frac{(\vartheta-\varkappa)^2}{4\Lambda_2(1)}\left(\int_0^1|\Lambda_2(\lambda)|^p\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{F}'\left(\frac{1-\lambda}{2}\varkappa+\frac{1+\lambda}{2}\vartheta\right)\right|^q d\lambda\right)^{\frac{1}{q}}.$$

Since $|\mathcal{F}'|^q$ is convex, we have

$$\begin{aligned}\int_0^1\left|\mathcal{F}'\left(\frac{1+\lambda}{2}\varkappa+\frac{1-\lambda}{2}\theta\right)\right|^q d\lambda &\leq \int_0^1\left[\frac{1+\lambda}{2}|\mathcal{F}'(\varkappa)|^q+\frac{1-\lambda}{2}|\mathcal{F}'(\theta)|^q\right]d\lambda \\ &= \frac{3|\mathcal{F}'(\varkappa)|^q+|\mathcal{F}'(\theta)|^q}{4}\end{aligned} \tag{3.11}$$

and smililarly

$$\begin{aligned}\int_0^1\left|\mathcal{F}'\left(\frac{1-\lambda}{2}\varkappa+\frac{1+\lambda}{2}\theta\right)\right|^q d\lambda &\leq \frac{|\mathcal{F}'(\varkappa)|^q+3|\mathcal{F}'(\theta)|^q}{4} \\ \int_0^1\left|\mathcal{F}'\left(\frac{1+\lambda}{2}\varkappa+\frac{1-\lambda}{2}\vartheta\right)\right|^q d\lambda &\leq \frac{3|\mathcal{F}'(\varkappa)|^q+|\mathcal{F}'(\vartheta)|^q}{4} \\ \int_0^1\left|\mathcal{F}'\left(\frac{1-\lambda}{2}\varkappa+\frac{1+\lambda}{2}\vartheta\right)\right|^q d\lambda &\leq \frac{|\mathcal{F}'(\varkappa)|^q+3|\mathcal{F}'(\vartheta)|^q}{4}.\end{aligned} \tag{3.12}$$

By substituting inequalities (3.11) and (3.12) into (3.10), we obtain the first inequalty in (3.9).

For the proof of second inequality, let $\theta_1=|\mathcal{F}'(\theta)|^q$, $\vartheta_1=3|\mathcal{F}'(\varkappa)|^q$, $\theta_2=3|\mathcal{F}'(\varkappa)|^q$ and $\vartheta_2=|\mathcal{F}'(\vartheta)|^q$. Using the fact that

$$\sum_{k=1}^n(\theta_k+\vartheta_k)^s\leq\sum_{k=1}^n\theta_k^s+\sum_{k=1}^n\vartheta_k^s,\ 0\leq s<1$$

and $1+3^{\frac{1}{q}}\leq 4$ then the desired result can be obtained straightforwardly. □

Remark 3.11. *If we choose $\varphi(\lambda)=\lambda$ for all $\lambda\in[\theta,\vartheta]$ in Theorem 3.10, then Theorem 3.10 reduces to [23, Theorem 2].*

Remark 3.12. *If we choose $\varphi(\lambda)=\frac{\lambda^\alpha}{\Gamma(\alpha)}$, $\alpha>1$, for all $\lambda\in[\theta,\vartheta]$ in Theorem 3.10, then we have the following inequality for RLFIs*

$$\begin{aligned}&\left|\frac{(\vartheta-\theta)\mathcal{F}(\varkappa)+(\varkappa-\theta)\mathcal{F}(\theta)+(\vartheta-\varkappa)\mathcal{F}(\vartheta)}{2}\right.\\ &\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varkappa-\theta)^{\alpha-1}}\left[J^\alpha_{\varkappa^-}\mathcal{F}\left(\frac{\varkappa+\theta}{2}\right)+J^\alpha_{\theta^+}\mathcal{F}\left(\frac{\varkappa+\theta}{2}\right)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\vartheta-\varkappa)^{\alpha-1}}\left[J^\alpha_{\varkappa^+}\mathcal{F}\left(\frac{\varkappa+\vartheta}{2}\right)+J^\alpha_{\vartheta^-}\mathcal{F}\left(\frac{\varkappa+\vartheta}{2}\right)\right]\right|\\ \leq\ &\frac{(\varkappa-\theta)^2}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{3|\mathcal{F}'(\varkappa)|^q+|\mathcal{F}'(\theta)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{F}'(\varkappa)|^q+3|\mathcal{F}'(\theta)|^q}{4}\right)^{\frac{1}{q}}\right]\end{aligned}$$

$$+\frac{(\vartheta-\varkappa)^2}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{3\left|\mathcal{F}'(\varkappa)\right|^q+\left|\mathcal{F}'(\vartheta)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{F}'(\varkappa)\right|^q+3\left|\mathcal{F}'(\vartheta)\right|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{(\varkappa-\theta)^2}{2^{\frac{2}{q}}}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left|\mathcal{F}'(\varkappa)\right|+\left|\mathcal{F}'(\theta)\right|\right]+\frac{(\vartheta-\varkappa)^2}{2^{\frac{2}{q}}}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left|\mathcal{F}'(\varkappa)\right|+\left|\mathcal{F}'(\vartheta)\right|\right],$$

which is the same Theorem 2 of [25].

Remark 3.13. *If we take $\varkappa=\theta$ (or $\varkappa=\vartheta$) in Theorem 3.10, then we have the following Trapezoid type inequality*

$$\left|\frac{\mathcal{F}(\theta)+\mathcal{F}(\vartheta)}{2}-\frac{1}{2\Lambda_2(1)}\left[{}_{\theta^+}I_{\varphi}\mathcal{F}\left(\frac{\theta+\vartheta}{2}\right)+{}_{\vartheta^-}I_{\varphi}\mathcal{F}\left(\frac{\theta+\vartheta}{2}\right)\right]\right|$$

$$\leq \frac{\vartheta-\theta}{4\Delta(1)}\left(\int_0^1\left|\Delta(\lambda)\right|^p d\lambda\right)^{\frac{1}{p}}\left[\left(\frac{3\left|\mathcal{F}'(\theta)\right|^q+\left|\mathcal{F}'(\vartheta)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{F}'(\theta)\right|^q+3\left|\mathcal{F}'(\vartheta)\right|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{\vartheta-\theta}{2^{\frac{2}{q}}\Delta(1)}\left(\int_0^1\left|\Delta(\lambda)\right|^p d\lambda\right)^{\frac{1}{p}}\left[\left|\mathcal{F}'(\theta)\right|+\left|\mathcal{F}'(\vartheta)\right|\right],$$

which is proved by Ertuğral et al. [13].

Corollary 3.14. *Under assumption of Theorem 3.10, if we take $\varkappa=\frac{\theta+\vartheta}{2}$ in Theorem 3.10 then we have the following Bullen type inequalities*

$$\left|\frac{1}{2}\left[\frac{\mathcal{F}(\theta)+\mathcal{F}(\vartheta)}{2}+\mathcal{F}\left(\frac{\theta+\vartheta}{2}\right)\right]-\frac{1}{4\Psi(1)}\left[{}_{\frac{\theta+\vartheta}{2}^-}I_{\varphi}\mathcal{F}\left(\frac{3\theta+\vartheta}{4}\right)+{}_{\theta^+}I_{\varphi}\mathcal{F}\left(\frac{3\theta+\vartheta}{4}\right)\right]\right.$$

$$\left.-\frac{1}{4\Psi(1)}\left[{}_{\frac{\theta+\vartheta}{2}^+}I_{\varphi}\mathcal{F}\left(\frac{\theta+3\vartheta}{4}\right)+{}_{\vartheta^-}I_{\varphi}\mathcal{F}\left(\frac{\theta+3\vartheta}{4}\right)\right]\right|$$

$$\leq \frac{\vartheta-\theta}{16\Psi(1)}\left(\int_0^1\left|\Psi(\lambda)\right|^p d\lambda\right)^{\frac{1}{p}}\left\{\left[\left(\frac{3\left|\mathcal{F}'\left(\frac{\theta+\vartheta}{2}\right)\right|^q+\left|\mathcal{F}'(\theta)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{F}'\left(\frac{\theta+\vartheta}{2}\right)\right|^q+3\left|\mathcal{F}'(\theta)\right|^q}{4}\right)^{\frac{1}{q}}\right]\right.$$

$$\left.+\left[\left(\frac{3\left|\mathcal{F}'\left(\frac{\theta+\vartheta}{2}\right)\right|^q+\left|\mathcal{F}'(\vartheta)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{F}'\left(\frac{\theta+\vartheta}{2}\right)\right|^q+3\left|\mathcal{F}'(\vartheta)\right|^q}{4}\right)^{\frac{1}{q}}\right]\right\}$$

$$\leq \frac{\vartheta-\theta}{2^{\frac{2}{q}+1}\Psi(1)}\left(\int_0^1\left|\Psi(\lambda)\right|^p d\lambda\right)^{\frac{1}{p}}\left(\left|\mathcal{F}'\left(\frac{\theta+\vartheta}{2}\right)\right|+\frac{\left|\mathcal{F}'(\theta)\right|+\left|\mathcal{F}'(\vartheta)\right|}{2}\right).$$

Remark 3.15. *If we choose $\varphi(\lambda)=\lambda$ for all $\lambda\in[\theta,\vartheta]$ in Corollary 3.14, then we have the following inequality*

$$\left|\frac{1}{2}\left[\frac{\mathcal{F}(\theta)+\mathcal{F}(\vartheta)}{2}+\mathcal{F}\left(\frac{\theta+\vartheta}{2}\right)\right]-\frac{1}{(\vartheta-\theta)}\int_\theta^\vartheta\mathcal{F}(\varkappa)\,d\varkappa\right| \tag{3.13}$$

$$\leq \frac{\vartheta-\theta}{16}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{3\left|\mathcal{F}'\left(\frac{\theta+\vartheta}{2}\right)\right|^q+|\mathcal{F}'(\theta)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{F}'\left(\frac{\theta+\vartheta}{2}\right)\right|^q+3|\mathcal{F}'(\theta)|^q}{4}\right)^{\frac{1}{q}}\right]$$
$$+\frac{\vartheta-\theta}{16}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{3\left|\mathcal{F}'\left(\frac{\theta+\vartheta}{2}\right)\right|^q+|\mathcal{F}'(\vartheta)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{F}'\left(\frac{\theta+\vartheta}{2}\right)\right|^q+3|\mathcal{F}'(\vartheta)|^q}{4}\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{\vartheta-\theta}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[|\mathcal{F}'(\theta)|+|\mathcal{F}'(\vartheta)|+2\left|\mathcal{F}'\left(\frac{\theta+\vartheta}{2}\right)\right|\right].$$

The first inequality of (3.13) is the same by [23, Corolalry 2].

Theorem 3.16. *Let $\mathcal{F}:[\theta,\vartheta]\to\mathbb{R}$ be differentiable function on (θ,ϑ). If $|\mathcal{F}'|^q$, $q\geq 1$, is convex function, then we have the following inequality for GFIs:*

$$\left|\frac{(\vartheta-\theta)\mathcal{F}(\varkappa)+(\varkappa-\theta)\mathcal{F}(\theta)+(\vartheta-\varkappa)\mathcal{F}(\vartheta)}{2}\right.$$
$$\left.-\frac{\varkappa-\theta}{2\Lambda_1(1)}\left[{}_{\varkappa^-}I_\varphi\mathcal{F}\left(\frac{\varkappa+\theta}{2}\right)+{}_{\theta^+}I_\varphi\mathcal{F}\left(\frac{\varkappa+\theta}{2}\right)\right]-\frac{\vartheta-\varkappa}{2\Lambda_2(1)}\left[{}_{\varkappa^+}I_\varphi\mathcal{F}\left(\frac{\varkappa+\vartheta}{2}\right)+{}_{\vartheta^-}I_\varphi\mathcal{F}\left(\frac{\varkappa+\vartheta}{2}\right)\right]\right|$$
$$\leq \frac{(\varkappa-\theta)^2}{4\Lambda_1(1)}\left(\int_0^1|\Lambda_1(\lambda)|\,d\lambda\right)^{1-\frac{1}{q}}\left\{\left(\frac{\beta_1|\mathcal{F}'(\varkappa)|^q+\beta_2|\mathcal{F}'(\theta)|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\beta_2|\mathcal{F}'(\varkappa)|^q+\beta_1|\mathcal{F}'(\theta)|^q}{2}\right)^{\frac{1}{q}}\right\}$$
$$+\frac{(\vartheta-\varkappa)^2}{4\Lambda_2(1)}\left(\int_0^1|\Lambda_2(\lambda)|\,d\lambda\right)^{1-\frac{1}{q}}\left\{\left(\frac{\beta_3|\mathcal{F}'(\varkappa)|^q+\beta_4|\mathcal{F}'(\vartheta)|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\beta_4|\mathcal{F}'(\varkappa)|^q+\beta_3|\mathcal{F}'(\vartheta)|^q}{2}\right)^{\frac{1}{q}}\right\},$$

where the numbers β_1,β_2,β_3 and β_4 are defined by

$$\begin{aligned}
\beta_1 &= \int_0^1|\Lambda_1(\lambda)|(1+\lambda)\,d\lambda,\\
\beta_2 &= \int_0^1|\Lambda_1(\lambda)|(1-\lambda)\,d\lambda,\\
\beta_3 &= \int_0^1|\Lambda_2(\lambda)|(1+\lambda)\,d\lambda,\\
\beta_4 &= \int_0^1|\Lambda_2(\lambda)|(1-\lambda)\,d\lambda.
\end{aligned}$$

Proof. By using well-known power mean inequality in (3.8), we obtain

$$\left|\frac{(\vartheta-\theta)\mathcal{F}(\varkappa)(\varkappa-\theta)\mathcal{F}(\theta)+(\vartheta-\varkappa)\mathcal{F}(\vartheta)}{2}\right. \tag{3.14}$$
$$\left.-\frac{\varkappa-\theta}{2\Lambda_1(1)}\left[{}_{\varkappa^-}I_\varphi\mathcal{F}\left(\frac{\varkappa+\theta}{2}\right)+{}_{\theta^+}I_\varphi\mathcal{F}\left(\frac{\varkappa+\theta}{2}\right)\right]-\frac{\vartheta-\varkappa}{2\Lambda_2(1)}\left[{}_{\varkappa^+}I_\varphi\mathcal{F}\left(\frac{\varkappa+\vartheta}{2}\right)+{}_{\vartheta^-}I_\varphi\mathcal{F}\left(\frac{\varkappa+\vartheta}{2}\right)\right]\right|$$

$$\leq \frac{(\varkappa-\theta)^2}{4\Lambda_1(1)}\left(\int_0^1 |\Lambda_1(\lambda)|\,d\lambda\right)^{1-\frac{1}{q}} \left\{\left(\int_0^1 |\Lambda_1(\lambda)| \left|\mathcal{F}'\left(\frac{1+\lambda}{2}\varkappa+\frac{1-\lambda}{2}\theta\right)\right|^q d\lambda\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1 |\Lambda_1(\lambda)| \left|\mathcal{F}'\left(\frac{1-\lambda}{2}\varkappa+\frac{1+\lambda}{2}\theta\right)\right|^q d\lambda\right)^{\frac{1}{q}}\right\}$$

$$+\frac{(\vartheta-\varkappa)^2}{4\Lambda_2(1)}\left(\int_0^1 |\Lambda_2(\lambda)|\,d\lambda\right)^{1-\frac{1}{q}} \left\{\left(\int_0^1 |\Lambda_2(\lambda)| \left|\mathcal{F}'\left(\frac{1+\lambda}{2}\varkappa+\frac{1-\lambda}{2}\vartheta\right)\right|^q d\lambda\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1 |\Lambda_2(\lambda)| \left|\mathcal{F}'\left(\frac{1-\lambda}{2}\varkappa+\frac{1+\lambda}{2}\vartheta\right)\right|^q d\lambda\right)^{\frac{1}{q}}\right\}.$$

Since $|\mathcal{F}'|^q$ is convex, we have

$$\begin{aligned} &\int_0^1 |\Lambda_1(\lambda)| \left|\mathcal{F}'\left(\frac{1+\lambda}{2}\varkappa+\frac{1-\lambda}{2}\theta\right)\right|^q d\lambda \\ \leq\ &\int_0^1 |\Lambda_1(\lambda)| \left[\frac{1+\lambda}{2}\left|\mathcal{F}'(\varkappa)\right|^q+\frac{1-\lambda}{2}\left|\mathcal{F}'(\theta)\right|^q\right] d\lambda \\ \leq\ &\int_0^1 |\Lambda_1(\lambda)|(1+\lambda)\frac{\left|\mathcal{F}'(\varkappa)\right|^q}{2}d\lambda+\int_0^1 |\Lambda_1(\lambda)|(1-\lambda)\frac{\left|\mathcal{F}'(\theta)\right|^q}{2}d\lambda \\ \leq\ &\frac{\left[\beta_1\left|\mathcal{F}'(\varkappa)\right|^q+\beta_2\left|\mathcal{F}'(\theta)\right|^q\right]}{2} \end{aligned} \tag{3.15}$$

and similarly

$$\int_0^1 |\Lambda_1(\lambda)| \left|\mathcal{F}'\left(\frac{1-\lambda}{2}\varkappa+\frac{1+\lambda}{2}\theta\right)\right|^q d\lambda \leq \frac{\beta_2\left|\mathcal{F}'(\varkappa)\right|^q+\beta_1\left|\mathcal{F}'(\theta)\right|^q}{2} \tag{3.16}$$

$$\int_0^1 |\Lambda_2(\lambda)| \left|\mathcal{F}'\left(\frac{1+\lambda}{2}\varkappa+\frac{1-\lambda}{2}\vartheta\right)\right|^q d\lambda \leq \frac{\beta_3\left|\mathcal{F}'(\varkappa)\right|^q+\beta_4\left|\mathcal{F}'(\vartheta)\right|^q}{2} \tag{3.17}$$

$$\int_0^1 |\Lambda_2(\lambda)| \left|\mathcal{F}'\left(\frac{1-\lambda}{2}\varkappa+\frac{1+\lambda}{2}\vartheta\right)\right|^q d\lambda \leq \frac{\beta_4\left|\mathcal{F}'(\varkappa)\right|^q+\beta_3\left|\mathcal{F}'(\vartheta)\right|^q}{2}. \tag{3.18}$$

By considering the inequalities (3.15)–(3.18) in (3.14), then we obtain the required result. □

Remark 3.17. *If we choose $\varphi(\lambda) = \lambda$ for all $\lambda \in [\theta, \vartheta]$ in Theorem 3.16, then Theorem 3.16 reduces to [23, Theorem 3].*

Remark 3.18. *If we choose $\varphi(\lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)}$, $\alpha > 1$, for all $\lambda \in [\theta, \vartheta]$ in Theorem 3.16, then Theorem 3.16 reduces to [25, Theorem 3].*

Remark 3.19. *If we take $\varkappa = \theta$ (or $\varkappa = \vartheta$) in Theorem 3.16, then we have the following Trapezoid type inequality which proved by Ertuğral et al. in [13]:*

$$\begin{aligned}&\left|\frac{\mathcal{F}(\theta)+\mathcal{F}(\vartheta)}{2}-\frac{1}{2\Lambda_2(1)}\left[{}_{\theta^+}I_\varphi\mathcal{F}\left(\frac{\theta+\vartheta}{2}\right)+{}_{\vartheta^-}I_\varphi\mathcal{F}\left(\frac{\theta+\vartheta}{2}\right)\right]\right|\\ \leq\ &\frac{(\vartheta-\theta)^2}{4\Delta(1)}\left(\int_0^1|\Delta(\lambda)|\,d\lambda\right)^{1-\frac{1}{q}}\left[\left(\frac{\beta_5\,|\mathcal{F}'(\varkappa)|^q+\beta_6\,|\mathcal{F}'(\vartheta)|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\beta_5\,|\mathcal{F}'(\varkappa)|^q+\beta_6\,|\mathcal{F}'(\vartheta)|^q}{2}\right)^{\frac{1}{q}}\right],\end{aligned}$$

where β_5 and β_6 are defined by

$$\beta_5=\int_0^1|\Delta(\lambda)|\,(1+\lambda)\,d\lambda,\qquad \beta_6=\int_0^1|\Delta_1(\lambda)|\,(1-\lambda)\,d\lambda.$$

Corollary 3.20. *Under assumption of Theorem 3.16, if we take $\varkappa = \frac{\theta+\vartheta}{2}$ then Theorem 3.16 reduces to following inequalities*

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{\mathcal{F}(\theta)+\mathcal{F}(\vartheta)}{2}+\mathcal{F}\left(\frac{\theta+\vartheta}{2}\right)\right]-\frac{\vartheta-\theta}{4\Lambda_1(1)}\left[{}_{\frac{\theta+\vartheta}{2}^-}I_\varphi\mathcal{F}(\frac{3\theta+\vartheta}{4})+{}_{\theta^+}I_\varphi\mathcal{F}(\frac{3\theta+\vartheta}{4})\right]\right.\\ &\left.-\frac{\vartheta-\theta}{4\Lambda_2(1)}\left[{}_{\frac{\theta+\vartheta}{2}^+}I_\varphi\mathcal{F}(\frac{\theta+3\vartheta}{4})+{}_{\vartheta^-}I_\varphi\mathcal{F}(\frac{\theta+3\vartheta}{4})\right]\right|\\ \leq\ &\frac{\vartheta-\theta}{16\Lambda_1(1)}\left(\int_0^1|\Lambda_1(\lambda)|\,d\lambda\right)^{1-\frac{1}{q}}\left\{\left[\beta_1\left|\mathcal{F}'\left(\frac{\theta+\vartheta}{2}\right)\right|^q+\beta_2\,|\mathcal{F}'(\theta)|^q\right]^{\frac{1}{q}}+\left[\beta_2\left|\mathcal{F}'\left(\frac{\theta+\vartheta}{2}\right)\right|^q+\beta_1\,|\mathcal{F}'(\theta)|^q\right]^{\frac{1}{q}}\right\}\\ &+\frac{\vartheta-\theta}{2^{4+\frac{1}{q}}\Lambda_2(1)}\left(\int_0^1|\Lambda_2(\lambda)|\,d\lambda\right)^{1-\frac{1}{q}}\left\{\left[\beta_3\left|\mathcal{F}'\left(\frac{\theta+\vartheta}{2}\right)\right|^q+\beta_4\,|\mathcal{F}'(\vartheta)|^q\right]^{\frac{1}{q}}+\left[\beta_4\left|\mathcal{F}'\left(\frac{\theta+\vartheta}{2}\right)\right|^q+\beta_3\,|\mathcal{F}'(\vartheta)|^q\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

Corollary 3.21. *If we choose $\varphi(\lambda) = \lambda$ Corollary 3.20, then we have the following inequality*

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{\mathcal{F}(\theta)+\mathcal{F}(\vartheta)}{2}+\mathcal{F}\left(\frac{\theta+\vartheta}{2}\right)\right]-\frac{1}{(\vartheta-\theta)}\int_\theta^\vartheta\mathcal{F}(\varkappa)\,d\varkappa\right|\\ \leq\ &\frac{\vartheta-\theta}{16}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{\left(\frac{5\left|\mathcal{F}'\left(\frac{\theta+\vartheta}{2}\right)\right|^q+|\mathcal{F}'(\theta)|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{F}'\left(\frac{\theta+\vartheta}{2}\right)\right|^q+5\,|\mathcal{F}'(\theta)|^q}{6}\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\frac{5\left|\mathcal{F}'\left(\frac{\theta+\vartheta}{2}\right)\right|^q+|\mathcal{F}'(\vartheta)|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{F}'\left(\frac{\theta+\vartheta}{2}\right)\right|^q+5\,|\mathcal{F}'(\vartheta)|^q}{6}\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

4. Conclusions

In this work, we established some new integral inequalities for differentiable convex functions via the GFIs. We also discussed many special cases of newly established inequalities and obtained several new midpoint and trapezoidal type inequalities for differentiable convex functions through different integral operators. It is an interesting and new problem that researchers can obtain similar inequalities for different kinds of convexity in their future work.

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Conflict of interest

The authors declare no conflict of interest.

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