Mathematics

Research article

Statistical property analysis for a stochastic chemostat model with degenerate diffusion

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Abstract: By considering the fact that the growth of microorganisms in a chemostat is subject to white noise, we construct a stochastic chemostat model with degenerate diffusion by using a discrete Markov chain. By solving the corresponding Fokker-Planck equation, we derive the explicit expression of the stationary joint probability density, which peaks near the deterministic equilibrium. Next, we simulate the the marginal probability density functions for different noise intensities and further discuss the relationship of the marginal probability density function and noise intensities. For the statistical properties of the stochastic model, we mainly investigate the effect of white noise on the variance and skewness of the concentration of microorganisms.

1. Introduction

Chemostat refers to a laboratory apparatus used for growing microorganisms in a cultured environment. The simplest chemostat apparatus consists of three interconnected tanks called feed bottle, culture vessel and collection vessel. The growth limiting substrate is pumped from the feed bottle to the culture vessel, where the interaction between the microorganisms and the substrate takes place, and another flow is pumped from the culture vessel to the collection vessel such that the volume of the culture vessel remains constant [1].

Chemostat tools play a key role in population dynamics, the chemical field, industrial treatment, biochemical development and other fields [2, 3]. Chemostat is also used in wastewater treatment, with which wastewater can be converted into treated water and reused in daily activities [4–6]. The classical mathematical model describing the continuous culture of the microorganisms in the chemostat takes

the following form:

$$\begin{cases} \frac{dS}{dt} = \frac{Q}{V}\left(S^0 - S\right) - \gamma^{-1}\mu(S)x, \\ \frac{dx}{dt} = \mu(S)x - \frac{Q}{V}x, \end{cases} \tag{1.1}$$

where $S(t)$ and $x(t)$ denote the concentrations of the growth limiting substrate and microorganisms at time t, respectively. S^0 is the input concentration, Q is the input flow, V is the volume of liquid media inside the culture vessel and the ratio Q/V is usually denoted by D which is also called the dilution rate. $\mu(S)$ is known as the response function representing the substrate uptake rate, and the constant γ measures the growth yield of the microorganisms. Taking the Holling type-II response function (i.e., $\mu(S) = mS/(a + S)$) and introducing the variable transformation $x_{new} = \gamma^{-1}x_{old}$ such that the yield coefficient γ can be scaled out in model (1.1), then it can be written as

$$\begin{cases} \frac{dS}{dt} = D\left(S^0 - S\right) - \frac{mSx}{a+S}, \\ \frac{dx}{dt} = \left(\frac{mS}{a+S} - D\right)x. \end{cases} \tag{1.2}$$

The dynamics of model (1.2) is completely determined by the break-even concentration λ which satisfies $m\lambda/(a + \lambda) = D$ [7, 8]. If $\lambda \geq S^0$, the washout equilibrium $E^0 = (S^0, 0)$ is globally stable and the microorganisms become extinct. If $\lambda < S^0$, the unique positive equilibrium $E^* = (S^*, x^*)$ is globally stable and the microorganisms persist.

All along, the research on various chemostat models has not been interrupted. For instance, Chen et al. [9, 10] analyzed the extinction and permanence of chemostat models with pulsed input. Shi et al. [11] studied the coexistence of competing species in a reaction-diffusion chemostat model. Alzahrani et al. [12] researched the global dynamics of a cell quota-based model in a chemostat. Baratti et al. [13] analytically characterized the nonlinear stochastic dynamics of a class of two-state bioreactors with isotonic or nonisotonic kinetics using Fokker-Planck theory. Lu et al. [14] proposed a (state, input)-disturbed continuous stirred tank reactors (CSTRs) model that considers unknown but bounded fluctuations in kinetics, flow rates, and heat exchange, they developed a feedback control law that stabilizes the CSTRs system to reach noise-to-state exponential stability. One can see Refs. [15–21] and the references cited therein for more studies on the chemostat models.

With the accumulation of experimental data and the deepening of research, scholars have found that the growth of organisms has random fluctuation, and that influence of random environmental noise on biological populations is everywhere. Even under extremely precise experimental conditions, the growth process of microorganisms will be affected by random factors such as temperature, humidity, and light change, which will inevitably lead to the fluctuation of parameters in the established model, such as the maximum growth rate and dilution rate [22]. Therefore, it is more realistic to describe and analyze biological systems with stochastic mathematical models. With the establishment of Itô's stochastic integral, the research on stochastic differential equations and stochastic mathematical models has developed rapidly. There are many different ways of modeling stochasticity or randomness in the chemostat. For example, based on deterministic model (1.2), the authors replaced the dilution rate D by $D + \alpha\dot{B}(t)$ in Ref. [23] and replaced the maximum growth rate m by $m + \sigma\dot{B}(t)$ in Refs. [24, 25], where $\dot{B}(t)$ represents the white noise and α and σ are the noise intensities. In Refs. [26–29], the authors assumed that stochastic perturbations are the white noise type which are directly proportional to the variables S and x. Except for the white noise, telegraph noise, Markovian switching and the Ornstein-Uhlenbeck process have also been taken into consideration in the chemostat model; see Refs. [30–36].

In the aforementioned works, the authors mainly investigated the asymptotic orbital properties of the stochastic chemostat models.

In this paper, we mainly devote our attention to the investigation of the statistical properties of a stochastic chemostat model. The rest of this article is as follows. In Section 2, the construction procedure of the stochastic chemostat model is presented. In Section 3, we first derive the stationary joint probability density function of (S, x) and the marginal probability density function of x, and then we discuss the effects of noise intensity on the variance and skewness of the concentration of microorganisms. A simple discussion is given in Section 4.

2. Construction of the stochastic chemostat model

We first consider a discrete time Markov chain. For a fixed time increment $\Delta t > 0$, we define a process $X^{(\Delta t)}(t) = (S^{(\Delta t)}(t), x^{(\Delta t)}(t))^T$ for $t = 0, \Delta t, 2\Delta t, \cdots$, where $S^{(\Delta t)}(t)$ denotes the concentration of the growth limiting substrate and $x^{(\Delta t)}(t)$ is the concentration of microorganisms. Let the initial value $X^{(\Delta t)}(0) = X_0 \in \mathbb{R}_+^2$ which is deterministic. A sequence of random variables is denoted by $\left\{ R^{(\Delta t)}(k) \right\}_{k=0}^{\infty}$. Suppose that these variables are jointly independent and that within each sequence the variables are identically distributed such that

$$\mathrm{E} R^{(\Delta t)}(k) = 0, \quad \mathrm{E}\left[R^{(\Delta t)}(k)\right]^2 = \sigma^2 \Delta t, \quad \mathrm{E}\left[R^{(\Delta t)}(k)\right]^4 = o(\Delta t) \tag{2.1}$$

for $k = 0, 1, \cdots$, where $\sigma \geq 0$ is a constant that reflects the size of the stochastic effect.

The variables $R^{(\Delta t)}(k)$ ($i = 0, 1, \cdots, n$) are supposed to capture the effect of random influences on the concentration of microorganisms during the period $[k\Delta t, (k + 1)\Delta t)$. And we assume that $S^{(\Delta t)}$ and $x^{(\Delta t)}$ change within that time period according to the deterministic Eq (1.2). In addition, the random effects on the concentration of microorganisms are modelled by $R^{(\Delta t)}(k)$. Specifically, for $k = 0, 1, \cdots$, we get

$$S^{(\Delta t)}((k + 1)\Delta t) = S^{(\Delta t)}(k\Delta t) + \left\{ D\left(S^0 - S^{(\Delta t)}(k\Delta t)\right) - \frac{m S^{(\Delta t)}(k\Delta t) x^{(\Delta t)}(k\Delta t)}{a + S^{(\Delta t)}(k\Delta t)} \right\} \Delta t$$

and

$$x^{(\Delta t)}((k + 1)\Delta t) = x^{(\Delta t)}(k\Delta t) + R^{(\Delta t)}(k) x^{(\Delta t)}(k\Delta t) + \left\{ \frac{m S^{(\Delta t)}(k\Delta t) x^{(\Delta t)}(k\Delta t)}{a + S^{(\Delta t)}(k\Delta t)} - D x^{(\Delta t)}(k\Delta t) \right\} \Delta t.$$

We will show that $X^{(\Delta t)}(t)$ converges to a diffusion process as $\Delta t \to 0$. We must determine the drift coefficient and diffusion coefficient of the diffusion process. Let $\mathrm{P}^{(\Delta t)}(y, \mathrm{d}z)$ denote the transition probabilities of the homogeneous Markov chain

$$\left\{ X^{(\Delta t)}(k\Delta t) \right\}_{k=0}^{\infty},$$

that is,

$$\mathrm{P}^{(\Delta t)}(y, Z) = \mathrm{Prob}\left\{ X^{(\Delta t)}((k + 1)\Delta t) \in Z \mid X^{(\Delta t)}(k\Delta t) = y \right\}$$

for all $y = (S, x) \in \mathbb{R}_+^2$ and all Borel sets $Z \subset \mathbb{R}_+^2$.

Let

$$F^{(\Delta t)}(y) = \begin{pmatrix} f_0^{(\Delta t)}(y) \\ f_1^{(\Delta t)}(y) \end{pmatrix}$$

and

$$G^{(\Delta t)}(y) \begin{pmatrix} g_{00}^{(\Delta t)}(y) & g_{01}^{(\Delta t)}(y) \\ g_{10}^{(\Delta t)}(y) & g_{11}^{(\Delta t)}(y) \end{pmatrix}$$

denote the drift coefficient and diffusion coefficient, respectively. It follows from (2.1) that

$$
\begin{aligned}
f_0^{(\Delta t)}(y) &= \frac{1}{\Delta t} \int (z_0 - S) \mathrm{P}^{(\Delta t)}(y, \mathrm{d}z) \\
&= D\left(S^0 - S\right) - \frac{mS x}{a + S},
\end{aligned}
\tag{2.2}
$$

$$
\begin{aligned}
f_1^{(\Delta t)}(y) &= \frac{1}{\Delta t} \int (z_1 - x) \mathrm{P}^{(\Delta t)}(y, \mathrm{d}z) \\
&= \left(\frac{mS}{a + S} - D\right) x + \frac{x}{\Delta t} \mathrm{E} R^{(\Delta t)}(0) \\
&= \left(\frac{mS}{a + S} - D\right) x,
\end{aligned}
\tag{2.3}
$$

and

$$
\begin{aligned}
g_{11}^{(\Delta t)}(y) &= \frac{1}{\Delta t} \int (z_1 - x)^2 \mathrm{P}^{(\Delta t)}(y, \mathrm{d}z) \\
&= \frac{1}{\Delta t} \mathrm{E}\left[\left(\frac{mS}{a + S} - D\right) x \Delta t + R^{(\Delta t)}(0)x\right]^2 \\
&= \left(\frac{mS}{a + S} - D\right)^2 x^2 \Delta t + 2\left(\frac{mS}{a + S} - D\right) x^2 \mathrm{E} R^{(\Delta t)}(0) + \frac{x^2}{\Delta t} \mathrm{E}[R^{(\Delta t)}(0)]^2 \\
&= \left(\frac{mS}{a + S} - D\right)^2 x^2 \Delta t + \sigma^2 x^2.
\end{aligned}
$$

Therefore,

$$\lim_{\Delta t \to 0^+} \sup_{\|y\| \le K} |\, g_{11}^{(\Delta t)}(y) - \sigma^2 x^2 \,| = 0 \tag{2.4}$$

for all $0 < K < \infty$. Similarly, one can verify

$$\lim_{\Delta t \to 0^+} \sup_{\|y\| \le K} |\, g_{00}^{(\Delta t)}(y) \,| = \lim_{\Delta t \to 0^+} \sup_{\|y\| \le K} |\, g_{01}^{(\Delta t)}(y) \,| = \lim_{\Delta t \to 0^+} \sup_{\|y\| \le K} |\, g_{10}^{(\Delta t)}(y) \,| = 0. \tag{2.5}$$

Besides, form (2.1), we know that, for all $0 < K < \infty$,

$$\lim_{\Delta t \to 0^+} \sup_{\|y\| \le K} \frac{1}{\Delta t} \int \|\, z - y \,\|^3 \, \mathrm{P}^{(\Delta t)}(y, \mathrm{d}z) = 0. \tag{2.6}$$

In line with Ref. [26], the definition of $X^{(\Delta t)}(t)$ can be extended to all $t \ge 0$ by setting $X^{(\Delta t)}(t) = X^{(\Delta t)}(k\Delta t)$ for $t \in [k\Delta t, (k + 1)\Delta t)$. According to Theorem 7.1 and Lemma 8.2 in [37], and (2.2)–(2.6), we can conclude that, as $\Delta t \to 0$, $X^{(\Delta t)}(t)$ converges weakly to the solution of the following stochastic differential equation:

$$
\begin{cases}
\mathrm{d}S = \left[D\left(S^0 - S\right) - \frac{mS x}{a+S}\right] \mathrm{d}t, \\
\mathrm{d}x = \left(\frac{mS}{a+S} - D\right) x \mathrm{d}t + \sigma x \mathrm{d}B,
\end{cases}
\tag{2.7}
$$

with the initial condition $X(0) = X_0$, where $B = B(t)$ is the one-dimensional standard Brownian motion, σ is the noise intensity and other symbols have the same meanings as in the deterministic model (1.2). For the uniqueness of the global positive solution to the stochastic Model (2.7), we have the following lemma.

Lemma 2.1. *For any given initial value* $(S(0), x(0)) \in \mathbb{R}^2_+$, *model (2.7) has a unique positive solution* $(S(t), x(t))$ *on* $t \geq 0$; *in addition, the solution* $(S(t), x(t))$ *will remain in* $\mathbb{R}^2_+$ *with a probability of one, namely,* $(S(t), x(t)) \in \mathbb{R}^2_+$ *for all* $t \geq 0$ *almost surely.*

Proof. Notice that the local Lipschitz continuity of the coefficients in model (2.7) for any given initial value $(S(0), x(0)) \in \mathbb{R}^2_+$; so, model (2.7) has a unique positive local solution $(S(t), x(t))$ on $t \in [0, t_e)$, where t_e is the explosion time. Next, we will show that $t_e = \infty$.

Since the solution $(S(t), x(t))$ is positive on $t \in [0, t_e)$, from (2.7) we have

$$\mathrm{d}S \leq D\left(S^0 - S\right)\mathrm{d}t, \tag{2.8}$$

and

$$\mathrm{d}x \leq (m - D)\, x\mathrm{d}t + \sigma x\mathrm{d}B. \tag{2.9}$$

Denote by $\Psi(t)$ the solution of the following equation:

$$\begin{cases} \mathrm{d}\Psi(t) = D\left(S^0 - \Psi(t)\right)\mathrm{d}t, \\ \Psi(0) = S(0), \end{cases} \tag{2.10}$$

and denote by $\Phi(t)$ the solution of the stochastic differential equation

$$\begin{cases} \mathrm{d}\Phi(t) = (m - D)\,\Phi(t)\mathrm{d}t + \sigma\Phi(t)\mathrm{d}B, \\ \Phi(0) = x(0). \end{cases} \tag{2.11}$$

It follows from the comparison theorem for stochastic equations [38] that $S(t) \leq \Psi(t)$, $x(t) \leq \Phi(t)$, and $t \in [0, t_e)$, a.s.

Similarly, from (2.7), we can also have

$$\mathrm{d}S \geq \left[D\left(S^0 - S\right) - m\Phi\right]\mathrm{d}t, \tag{2.12}$$

and

$$\mathrm{d}x \geq -x\left(D\mathrm{d}t - \sigma\mathrm{d}B\right), \tag{2.13}$$

on $t \in [0, t_e)$. Denote by $\psi(t)$ the solution of the stochastic differential equation

$$\begin{cases} \mathrm{d}\psi(t) = \left[D\left(S^0 - \psi\right) - m\Phi\right]\mathrm{d}t, \\ \psi(0) = S(0), \end{cases} \tag{2.14}$$

and denote by $\phi(t)$ the solution of the equation

$$\begin{cases} \mathrm{d}\phi = -\phi\left(D\mathrm{d}t - \sigma\mathrm{d}B\right), \\ \phi(0) = x(0). \end{cases} \tag{2.15}$$

Using the comparison theorem for stochastic equations [38] again, we have $S(t) \geq \psi(t), x(t) \geq \phi(t), \ t \in [0, t_e)$, a.s.

To sum up,

$$\begin{cases} \psi(t) \leq S(t) \leq \Psi(t), \\ \phi(t) \leq x(t) \leq \Phi(t), t \in [0, t_e), a.s. \end{cases} \tag{2.16}$$

Since Eqs (2.10), (2.11), (2.14) and (2.15), are all linear, the expressions of $\Psi(t), \Phi(t), \psi(t)$, and $\phi(t)$ can all be explicitly solved from Eqs (2.10), (2.11), (2.14) and (2.15), respectively. Obviously, these solutions are all positive and globally existent for all $t \in [0, \infty)$. Thus, we can get that the explosion time $t_e = \infty$ from (2.16). The proof is thus completed.

3. Statistical property analysis

In this section, we derive the stationary joint probability density function of (S, x) and the marginal probability density function of x based on the stochastic model (2.7). In view of the biological significance, we consider the case that the deterministic model (1.2) has a stable positive equilibrium $E^* = (S^*, x^*)$, i.e., the microorganisms persists in the chemostat. Let $y = \ln S$ and $z = \ln x$, then, the stochastic model (2.7) can be rewritten as

$$\begin{cases} dy = \left[DS^0 e^{-y} - D - \frac{me^z}{a+e^y} \right] dt, \\ dz = \left(\frac{me^y}{a+e^y} - D \right) dt + \sigma dB, \end{cases}$$

and its linearized version near $(\ln S^*, \ln x^*)$ is as follows:

$$\begin{cases} dy = (a_{11}y + a_{12}z) \, dt, \\ dz = a_{21}y dt + \sigma dB, \end{cases} \tag{3.1}$$

where

$$a_{11} = \frac{mS^*x^*}{(a+S^*)^2} - \frac{DS^0}{S^*} = -\frac{D[(S^*)^2 + aS^0]}{(a+S^*)S^*}, \quad a_{12} = -\frac{mx^*}{a+S^*}, \quad a_{21} = \frac{maS^*}{(a+S^*)^2}. \tag{3.2}$$

By using the variable transformations $u = y$ and $v = -\frac{a_{12}}{a_{11}}z$ for Eq (3.1), we obtain

$$\begin{cases} du = (a_{11}u - a_{11}v) \, dt, \\ dv = -\frac{a_{12}}{a_{11}} a_{21} u dt - \frac{a_{12}}{a_{11}} \sigma dB. \end{cases} \tag{3.3}$$

It follows from the theory of Markov processes [39–41] that the transition probability density function $P(u, v, t | u_0, v_0, t_0)$ is governed by the following Fokker-Planck equation:

$$\frac{\partial P}{\partial t} = \frac{\partial^2}{\partial v^2} \left(\frac{a_{12}^2 \sigma^2}{2a_{11}^2} P \right) + \frac{\partial}{\partial u} \left[a_{11}(v-u)P \right] + \frac{\partial}{\partial v} \left(\frac{a_{12}a_{21}}{a_{11}} uP \right),$$

and the stationary probability density $P(u, v)$ is determined by

$$\frac{\partial^2}{\partial v^2} \left(\frac{a_{12}^2 \sigma^2}{2a_{11}^2} P \right) + \frac{\partial}{\partial u} \left[a_{11}(v-u)P \right] + \frac{\partial}{\partial v} \left(\frac{a_{12}a_{21}}{a_{11}} uP \right) = 0,$$

i.e.,

$$\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial u}\right)\left[\frac{a_{12}^2\sigma^2}{2a_{11}^2}\frac{\partial P}{\partial v} + a_{11}(u-v)P\right] + \frac{\partial}{\partial v}\left[\frac{a_{12}^2\sigma^2}{2a_{11}^2}\frac{\partial P}{\partial u} + a_{11}(v-u)P + \frac{a_{12}a_{21}}{a_{11}}uP\right] = 0. \qquad (3.4)$$

One can find that

$$P(u,v) = C_0 \exp\left\{-\frac{a_{11}^2}{a_{12}^2\sigma^2}\left[\left(\frac{a_{12}a_{21}}{a_{11}} - a_{11}\right)u^2 + 2a_{11}uv - a_{11}v^2\right]\right\}$$

$$= C_0 \exp\left\{\frac{a_{11}}{a_{12}^2\sigma^2}\left[a_{11}^2(u-v)^2 - a_{12}a_{21}u^2\right]\right\} \qquad (3.5)$$

satisfies

$$\begin{cases} \frac{a_{12}^2\sigma^2}{2a_{11}^2}\frac{\partial P}{\partial v} + a_{11}(u-v)P = 0, \\ \frac{a_{12}^2\sigma^2}{2a_{11}^2}\frac{\partial P}{\partial u} + a_{11}(v-u)P + \frac{a_{12}a_{21}}{a_{11}}uP = 0, \end{cases}$$

where C_0 is a positive constant such that $\iint_{\mathbb{R}^2} P(u,v)\mathrm{d}u\mathrm{d}v = 1$. That is to say, the function $P(u,v)$ showed in (3.5) is a solution of Eq (3.4), and it is the stationary joint probability density of (u,v) corresponding to the linear model (3.3). Then, the stationary joint probability density of (S,x) corresponding to the linearized version of the stochastic model (2.7) is as follows

$$P(S,x) = C_0 \exp\left\{\frac{a_{11}}{a_{12}^2\sigma^2}\left[a_{11}^2\left(\ln\frac{S}{S^*} + \frac{a_{12}}{a_{11}}\ln\frac{x}{x^*}\right)^2 - a_{12}a_{21}\left(\ln\frac{S}{S^*}\right)^2\right]\right\}.$$

Thus, the result obtained so far in this section can be summarized as the following theorem.

Theorem 3.1. *If $\frac{Da}{m-D} < S^0$ holds, then the distribution of (S,x) has a density function $P(S,x)$, and it has the form*

$$P(S,x) = C_0 \exp\left\{\frac{a_{11}}{a_{12}^2\sigma^2}\left[a_{11}^2\left(\ln\frac{S}{S^*} + \frac{a_{12}}{a_{11}}\ln\frac{x}{x^*}\right)^2 - a_{12}a_{21}\left(\ln\frac{S}{S^*}\right)^2\right]\right\},$$

where (S,x) is a solution to model (2.7), where any initial value $(S(0), x(0)) \in \mathbb{R}_+^2$, (S^*, x^*) is the positive equilibrium of the deterministic model (1.2) and a_{11}, a_{12} and a_{21} are defined in (3.2).

Remark 3.1. *We should point out that the explicit expression of the stationary joint probability density showed in Theorem 3.1 is obtained by solving the Fokker-Planck equation associated with the linearization of the chemostat model at the positive equilibrium. It means that the stochastic trajectories with initial values near the deterministic positive equilibrium will converge to a distribution which has the probability density $P(S,x)$.*

Next, we perform some numerical simulations to verify our results and try to simulate the influence of environmental fluctuations on the density function.

Fixing parameters $S^0 = 4$, $D = 2$, $m = 3$ and $a = 1$, we know that the deterministic model (1.2) has a stable positive equilibrium $E^* = (2,2)$. By taking the noise intensity $\sigma = 0.15$, we show the stationary joint probability density $P(S,x)$ in Figure 1, which indicates that the peak of joint probability density is near the deterministic equilibrium E^*.
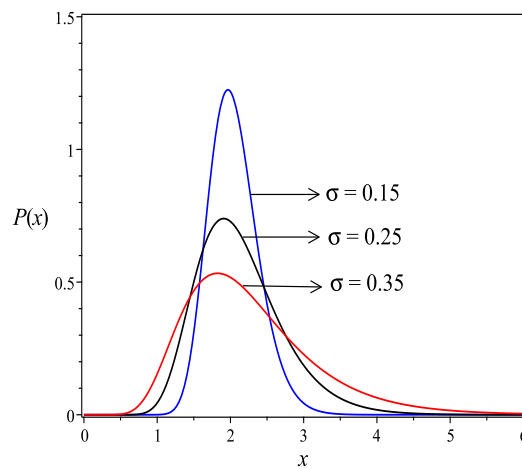
Besides, the marginal probability density functions $P(S)$ and $P(x)$, namely the probability densities of the concentrations of the substrate and microorganisms, respectively, can be represented as

$$P(S) = \int_0^\infty P(S, x)\mathrm{d}x, \quad P(x) = \int_0^\infty P(S, x)\mathrm{d}S.$$
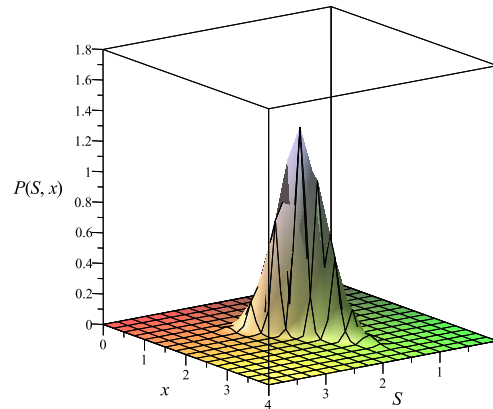
Figure 1. Joint density function $P(S, x)$ for $S^0 = 4$, $D = 2$, $m = 3$, $a = 1$ and $\sigma = 0.15$.

In Figure 2, we depict the marginal probability density function $P(x)$ with different noise intensities. It shows that as the noise intensity σ increases, the peak of $P(x)$ is shifted to the left and the peak height becomes lower. That is to say, the microorganisms concentration taking the highest probability decreases with the noise intensity, and the probability of a relatively lower or higher microorganisms concentration increases with the noise intensity. Besides, it follows from the geometrical asymmetry of $P(x)$ (especially the red curve in Figure 2) that the frequency distribution has a positive skew distribution, indicating that the probability of a relatively lower microorganisms concentration is larger than the probability of a higher concentration.


Figure 2. Marginal density function $P(x)$ for $S^0 = 4$, $D = 2$, $m = 3$, $a = 1$ and different noise intensities.

In order to quantitatively investigate the statistical properties of the microorganisms concentration, with the help of the marginal probability density function $P(x)$, we can further introduce the normalized variance and skewness of the variable x as follows:

$$V_x = \frac{\langle x^2 \rangle}{\langle x \rangle^2} - 1, \ \ S_x = \frac{\langle x^3 \rangle}{\langle x \rangle^3} - 3V_x - 1,$$

where $\langle x^n \rangle = \int_0^\infty x^n P(x)\mathrm{d}x$. Figure 3 shows that the numerical calculation results of the variance V_x and skewness S_x as a function of noise intensity σ, which were obtained by taking the same parameters as those used in Figure 2. Obviously, the curves of $V_x(\sigma)$ and $S_x(\sigma)$ are monotonically increasing, and all values of S_x are positive, which are consistent with the simulation findings in Figure 3.
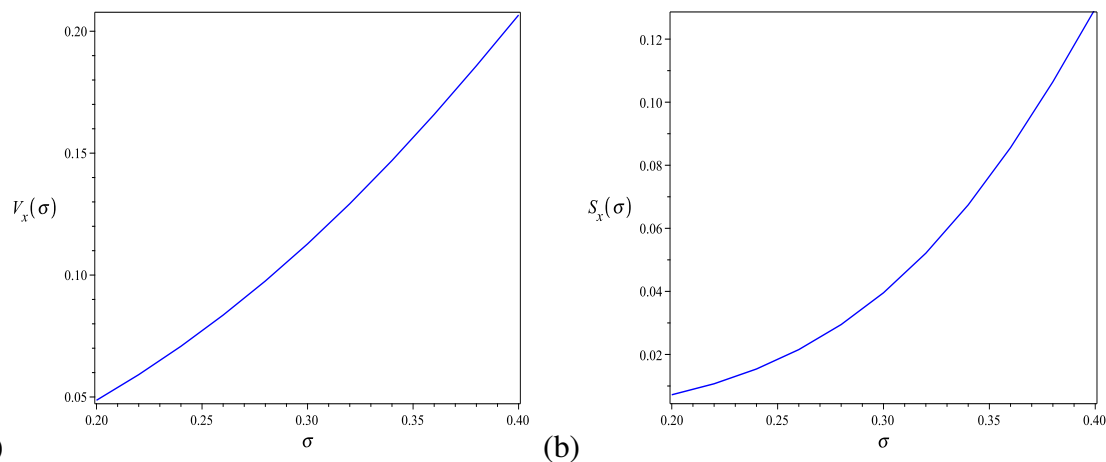
Figure 3. Normalized variance $V_x(\sigma)$ and skewness $S_x(\sigma)$ of variable x for $S^0 = 4$, $D = 2$, $m = 3$ and $a = 1$.

4. Conclusions

This paper is concerned with the effect of white noise on the continuous culture of microorganisms in the chemostat, which was evaluatedby constructing a stochastic chemostat model and investigating its statistical properties. Different from the previous methods for analyzing stochastic differential systems, we derived explicit expressions for the stationary joint probability density by solving the corresponding Fokker-Planck equation. The results obtained show that the continuous culture of microorganisms is less stable when the chemostat is subjected to environmental noise, and that the noise is adverse to the survival of microorganisms. It is consistent with the conclusion of Refs. [23, 42], in which the asymptotic orbital properties of stochastic chemostat models were investigated.

We need to point out that this paper deals with a stochastic chemostat model with degenerate diffusion. It is interesting to study the stochastic chemostat model with non-degenerate diffusion [43]. Another interesting problem is analyzing the effect of colored noise on the statistical properties of a stochastic chemostat model. These problems deserve further investigation.

Acknowledgments

Conflict of interest

All authors declare no conflicts of interest regarding this paper.

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