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Research article

A hybrid analytical technique for solving multi-dimensional time-fractional Navier-Stokes system

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Abstract: In this research, a hybrid method, entitled the Laplace Residual Power Series technique, is adapted to find series solutions to a time-fractional model of Navier-Stokes equations in the sense of Caputo derivative. We employ the proposed method to construct analytical solutions to the target problem using the idea of the Laplace transform and the residual function with the concept of limit at infinity. A simple modification of the suggested method is presented to deal easily with the nonlinear terms constructed on the properties of the power series. Three interesting examples are solved and compared with the exact solutions to test the reliability, simplicity, and capacity of the presented method of solving systems of fractional partial differential equations. The results indicate that the used technique is a simple approach for solving nonlinear fractional differential equations since it depends only on the residual functions and the concept of the limit at infinity without needing differentiation or other complex computations.

Keywords: Caputo-fractional derivative; Laplace transform; series solutions; Navier-Stokes system
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1. Introduction

In the past decade, many researchers have studied many fractional partial differential equations (FPDEs) types. Since the beginning of fractional calculus history in 1695, when L'Hospital raised the question: what is the meaning of $\frac{d^n y}{dx^n}$ if $n = \frac{1}{2}$? That is, what if n is fractional? [1] Even for new researchers, the fractional derivatives were complicated-although it appears in many parts of sciences such as physics, engineering, bioengineering, COVID-19 studies, and many other branches of sciences [2–9]. In addition, many definitions of fractional derivatives have been given [10–11]. Fractional order derivatives of a given function involve the entire function history where the following state of a fractional order system is dependent on its current state and all its historical states [4–11].

There are several analytical and numerical techniques for handling fractional problems, B-spline functions, Bernoulli polynomials, Adomian decomposition, variational iteration, Homotopy analysis, and many others [12–17]. On top of that, some applicable analytical methods are developed to address nonlinear problems with fractional derivatives. One of these approaches is the Laplace residual power series method (LRPSM) [18–29], which shows its efficiency and applicability in solving nonlinear problems.

The LRPSM is a very modern technique, and it is a hybrid method of two approaches, the Laplace transform (LT) and the idea of the residual power series method (RPSM) [30–33]. In 2020, the authors in the article [18] were able to adapt the LT to solve nonlinear neutral fractional pantograph equations using the residual function and the RPSM idea. The LT, usually, is implemented to solve linear equations only, but the LRPSM can overcome this disadvantage and thus adapts it to solve nonlinear equations of different types. The LRPSM presents an approximate analytical solution with a series form using the concept of the Laurent series and the power series [18–25]. What distinguishes LRPSM from RPSM is the use of the idea of limit at infinity in getting the coefficients of a series solution rather than the concept of a fractional derivative as in RPSM. Many articles used the proposed method to treat several types of differential equations of fractional orders. In 2021, El-Ajou adapted LRPSM to establish solitary solutions of nonlinear dispersive FPDEs [19] and to present series solutions for systems of Caputo FPDEs with variable coefficients [20]. Newly, the LRPSM is used for solving Fuzzy Quadratic Riccati Differential Equations [21], time-fractional nonlinear water wave PDE [22], fractional Lane-Emden equations [23], Fisher's equation and logistic system model [24], and nonlinear fractional reaction-diffusion for bacteria growth model [25].

Claude Louis Navier and Gabriel Stokes have created the so-called Navier-Stokes equations (NSEs). A French mechanical engineer Claude was affiliated in continuum mechanics with a physicist specializing and the French government, whose main contribution was the Navier-Stokes equations (1822). This famous equation made his name among the several names incised on the Eiffel Tower. Moreover, Newton's second law for fluid substance, which is central to fluid mechanics has been used in describing many physical phenomena in many applied sciences [34–37]. For example, the study of airflow around a wing and water flow in pipes and used as one of the continuity equations needed to build microscopic models in 1985 and also as a special case considered to establish the relationship between external and pressure forces on the fluid to the responses of fluid flow [38].

The motivation of this work is to adapt the LRPSM to provide analytical solutions for a multi-dimensional time-fractional Navier-Stokes (M-DT-FNS) system which takes the following form [39]:

$$D_t^{\alpha} u + (u \cdot \nabla) u = \upsilon \nabla^2 u - \frac{1}{\rho} \nabla p, 0 < \alpha \leq 1, \tag{1.1}$$

where D_t^{α} is the Caputo fractional-derivative operator of order α, $p = p(\chi, \varsigma, \zeta, t)$ is the pressure, ρ is the density, u is a vector field that represents the flow velocity vector, $\upsilon = \frac{\mu}{\rho}$ is the kinematic viscosity (μ is the dynamic viscosity), and ∇ & ∇^2 are the gradient and Laplacian operators, respectively, subject to the initial conditions (ICs) at the initial velocity:

$$u = \varphi. \quad (1.2)$$

If the density is constant throughout the fluid domain, then the vector Eq (1.1) is an incompressible NSEs.

The vector Eqs (1.1) and (1.2) can be separated in a system form as follows [39,40]:

$$D_t^\alpha u_1 + u_1 \frac{\partial u_1}{\partial \chi} + u_2 \frac{\partial u_1}{\partial \varsigma} + u_3 \frac{\partial u_1}{\partial \zeta} = \upsilon \left(\frac{\partial^2 u_1}{\partial \chi^2} + \frac{\partial^2 u_1}{\partial \varsigma^2} + \frac{\partial^2 u_1}{\partial \zeta^2} \right) - \frac{1}{\rho} \frac{\partial p}{\partial \chi},$$
$$D_t^\alpha u_2 + u_1 \frac{\partial u_2}{\partial \chi} + u_2 \frac{\partial u_2}{\partial \varsigma} + u_3 \frac{\partial u_2}{\partial \zeta} = \upsilon \left(\frac{\partial^2 u_2}{\partial \chi^2} + \frac{\partial^2 u_2}{\partial \varsigma^2} + \frac{\partial^2 u_2}{\partial \zeta^2} \right) - \frac{1}{\rho} \frac{\partial p}{\partial \varsigma},$$
$$D_t^\alpha u_3 + u_1 \frac{\partial u_3}{\partial \chi} + u_2 \frac{\partial u_3}{\partial \varsigma} + u_3 \frac{\partial u_3}{\partial \zeta} = \upsilon \left(\frac{\partial^2 u_3}{\partial \chi^2} + \frac{\partial^2 u_3}{\partial \varsigma^2} + \frac{\partial^2 u_3}{\partial \zeta^2} \right) - \frac{1}{\rho} \frac{\partial p}{\partial \zeta}, \quad (1.3)$$

subject to the initial conditions (ICs):

$$u_1(\chi, \varsigma, \zeta, 0) = f(\chi, \varsigma, \zeta) = f,$$
$$u_2(\chi, \varsigma, \zeta, 0) = h(\chi, \varsigma, \zeta) = h,$$
$$u_3(\chi, \varsigma, \zeta, 0) = g(\chi, \varsigma, \zeta) = g, \quad (1.4)$$

where $u = \langle u_1, u_2, u_3 \rangle$ and $\varphi = \langle f, h, g \rangle$ such that u_1,u_2,u_3, and p are analytical functions of four variables χ, ς, ζ & t.

In this equation, the solution represents the fluid velocity and pressure. It is commonly used to describe the motion of fluids in models relevant to weather, ocean currents, water flow in pipes, etc.

The novelty of this study is obvious in the proposed method when dealing with the Navier-Stokes problem, we show the simplicity and the applicability of the method, and we mention also that the method needs no differentiation, linearization, or discretization, the only mathematical step we need after taking the LT and defining the residual functions, is taking the limit at infinity which is much easier compared to other analytical techniques. Moreover, in this research, we obtain a general formula of the solution that neither researcher has, allowing us to compute as many possible terms of the series solution directly.

This study is prepared as follows: After the introduction section, a few fundamental principles and theories are reviewed for constructing an analytic series solution to the M-DT-FNS system using LRPSM. In Section 3, we constructed a Laplace residual power series (LRPS) solution to the goal problem. Three interesting examples are presented to explain the technique's simplicity and accuracy, which are displayed in Section 4. Finally, some conclusions are made about the features of the method used and its applicability in solving other types of problems.

2. Basic concepts

This part presents fundamental definitions and properties of fractional operators and power series.

Definition 2.1. [1] The time Caputo fractional-derivative of order α of the multivariable function $u(\chi, \varsigma, \zeta, t)$, is defined by

$$D_t^{\alpha}[u(\chi,\varsigma,\zeta,t)] = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{\frac{\partial^n}{\partial\xi^n}u(\chi,\varsigma,\zeta,\xi)}{(t-\xi)^{\alpha+1-n}}d\xi, & n-1<\alpha<n, \\ \frac{\partial^n}{\partial t^n}u(\chi,\varsigma,\zeta,t), & \alpha = n. \end{cases} \tag{2.1}$$

Lemma 2.1. [1] The operator D_t^{α}, $t \geq 0$, $n-1<\alpha\leq n$ satisfies the following properties:

1) $D_t^{\alpha}c = 0, c \in \mathbb{R}$.
2) $D_t^{\alpha}t^q = \begin{cases} 0 & , q<\alpha, \ q \text{ is integer} \\ \frac{\Gamma(q+1)}{\Gamma(q-\alpha+1)}t^{q-\alpha} & , \text{otherwise} \end{cases}$, for $q > -1$.

Definition 2.2. [18] The time-LT for the multivariable function $u(\chi,\varsigma,\zeta,t)$ is defined by:

$$U(\chi,\varsigma,\zeta,s) = \mathcal{L}[u(\chi,\varsigma,\zeta,t)] = \int_0^{\infty} e^{-st}u(\chi,\varsigma,\zeta,t)dt, \ s>\delta. \tag{2.2}$$

We denote the inverse LT of the function $U(\chi,v,\zeta,s)$ and define it as

$$u(\chi,\varsigma,\zeta,t) = \mathcal{L}^{-1}[U(\chi,\varsigma,\zeta,s)] = \int_{z-i\infty}^{z+i\infty} e^{st}\,U(\chi,\varsigma,\zeta,s)ds; \ z = Re(s) > z_0. \tag{2.3}$$

The most popular properties of the LT are mentioned below.

Lemma 2.2. [18,19] Assume that $U(\chi,\varsigma,\zeta,s) = \mathcal{L}[u(\chi,\varsigma,\zeta,t)]$ and $V(\chi,\varsigma,\zeta,s) = \mathcal{L}[v(\chi,\varsigma,\zeta,t)]$. Then

1) $\lim_{s\to\infty} s\,U(\chi,\varsigma,\zeta,s) = u(\chi,\varsigma,\zeta,0)$.
2) $\mathcal{L}\left[a\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}\right] = \frac{a}{s^{i\alpha+1}}, a\in\mathbb{R}, i = 0,1,\cdots$.
3) $\mathcal{L}[D_t^{m\alpha}u(\chi,\varsigma,\zeta,t)] = s^{m\alpha}U(\chi,\varsigma,\zeta,s) - \sum_{n=0}^{m-1} s^{(m-n)\alpha-1}D_t^{n\alpha}u(\chi,\varsigma,\zeta,0), 0<\alpha\leq 1$, where $D_t^{m\alpha} = D_t^{\alpha}D_t^{\alpha}\ldots D_t^{\alpha}$ (m-times).

Theorem 2.1. [19] The power series expansion of the multivariable function $u(\chi,\varsigma,\zeta,t)$ can be expressed as

$$u(\chi,\varsigma,\zeta,t) = \sum_{i=0}^{\infty}\frac{f_i(\chi,\varsigma,\zeta)t^{i\alpha}}{\Gamma(i\alpha+1)}, \tag{2.4}$$

where $f_i(\chi,\varsigma,\zeta) = D_t^{i\alpha}u(\chi,\varsigma,\zeta,0)$, $i = 0,1,\cdots$.

Thus, the LT of Eq (2.4) is a Laurent expansion in the Laplace space of the following form:

$$u(\chi,\varsigma,\zeta,t) = \sum_{i=0}^{\infty}\frac{f_i(\chi,\varsigma,\zeta)t^{i\alpha}}{\Gamma(i\alpha+1)}. \tag{2.5}$$

Theorem 2.2. [19] Assume that

$$\left|s\mathcal{L}\left[D_t^{(m+1)\alpha}u(\chi,\varsigma,\zeta,t)\right]\right| \leq M,$$

on $0\leq s\leq q$, $0<\alpha\leq 1$ and $M = M(\chi,\varsigma,\zeta)$ for some χ, ς, and $\zeta\in I$. Then the remainder $R_m(\chi,\varsigma,\zeta,s)$ of the new fractional Laurent series (2.5) satisfies the following inequality

$$|R_m(\chi,\varsigma,\zeta,s)| \leq \frac{M}{s^{(m+1)\alpha+1}}.$$

It is known that the LT cannot be distributed in the case of multiplication. Therefore, the following Lemma is introduced to simplify the calculations at the application of LRPSM, based on the characteristics of the powers of the power series.

Lemma 2.3. Assume that $U(\chi,\varsigma,\zeta,s)=\mathcal{L}[u(\chi,\varsigma,\zeta,t)](s)$ and $V(\chi,\varsigma,\zeta,s)=\mathcal{L}[v(\chi,\varsigma,\zeta,t)](s)$. Assume that the functions $U(\chi,\varsigma,\zeta,s)$ and $V(\chi,\varsigma,\zeta,s)$ have Laurent expansions as:

$$U(\chi,v,\zeta,s)=\sum_{i=0}^{\infty}\frac{f_i(\chi,v,\zeta)}{s^{i\alpha+1}},$$

$$V(\chi,\varsigma,\zeta,s)=\sum_{j=0}^{\infty}\frac{g_j(\chi,\varsigma,\zeta)}{s^{j\alpha+1}}. \tag{2.6}$$

Then, $\mathcal{L}[u(\chi,\varsigma,\zeta,t)v(\chi,\varsigma,\zeta,t)](s)$ can be expanded in a Laurent series form, as follows

$$\mathcal{L}[u(\chi,\varsigma,\zeta,t)v(\chi,\varsigma,\zeta,t)](s)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{f_i(\chi,\varsigma,\zeta)g_j(\chi,\varsigma,\zeta)\Gamma((i+j)\alpha+1)}{\Gamma(i\alpha+1)\Gamma(j\alpha+1)s^{(i+j)\alpha+1}}. \tag{2.7}$$

Proof. Directly, we can prove the Lemma as follows:

$$\begin{aligned}\mathcal{L}[u(\chi,\varsigma,\zeta,t)v(\chi,\varsigma,\zeta,t)](s) &= \mathcal{L}\big[\mathcal{L}^{-1}[U(\chi,\varsigma,\zeta,s)]\mathcal{L}^{-1}[V(\chi,\varsigma,\zeta,s)]\big](s)\\ &= \mathcal{L}\left[\mathcal{L}^{-1}\left[\sum_{i=0}^{\infty}\frac{f_i(\chi,\varsigma,\zeta)}{s^{i\alpha+1}}\right]\mathcal{L}^{-1}\left[\sum_{j=0}^{\infty}\frac{g_i(\chi,\varsigma,\zeta)}{s^{j\alpha+1}}\right]\right](s)\\ &= \mathcal{L}\left[\sum_{i=0}^{\infty}\frac{f_i(\chi,\varsigma,\zeta)t^{i\alpha}}{\Gamma(i\alpha+1)}\sum_{j=0}^{\infty}\frac{g_i(\chi,\varsigma,\zeta)t^{j\alpha}}{\Gamma(j\alpha+1)}\right](s)\\ &= \mathcal{L}\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{f_i(\chi,\varsigma,\zeta)g_j(\chi,\varsigma,\zeta)t^{(i+j)\alpha}}{\Gamma(i\alpha+1)\Gamma(j\alpha+1)}\right](s)\\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{f_i(\chi,\varsigma,\zeta)g_j(\chi,\varsigma,\zeta)\Gamma((i+j)\alpha+1)}{\Gamma(i\alpha+1)\Gamma(j\alpha+1)s^{(i+j)\alpha+1}}.\end{aligned}$$

It is noteworthy, that we can express the kth-truncated series of the Laurent series (2.7) as follows:

$$\mathcal{L}[u_k(\chi,\varsigma,\zeta,t)v_k(\chi,\varsigma,\zeta,t)](s)=\sum_{i=0}^{k}\sum_{j=0}^{k}\frac{f_i(\chi,\varsigma,\zeta)g_j(\chi,\varsigma,\zeta)\Gamma((i+j)\alpha+1)}{\Gamma(i\alpha+1)\Gamma(j\alpha+1)s^{(i+j)\alpha+1}}, \tag{2.8}$$

which we will use extensively throughout our work on the next pages.

3. The Laplace residual power series method

We employ the LRPSM to establish a series solution for the M-DT-FNS system (1.1) in this part of this article. This technique is mainly based on applying the LT on the target equations, assuming solutions of the generated equations have Laurent expansions, and then using the idea of the limit at infinity with the residual functions to get the unknown coefficients in expansions. Finally, we run the inverse LT to obtain the solution of the given equations in the original space.

To get the LRPS solution of the system (1.1), we first apply the LT to each equation in the system (1.1) and use the third part of Lemma 2.2 with the ICs (1.2). Then, after some simplification, we get the following algebraic system in Laplace:

$$U_1 - \frac{f}{s} + \frac{1}{s^\alpha}\mathcal{L}\left[u_1\frac{\partial u_1}{\partial \chi}\right] + \frac{1}{s^\alpha}\mathcal{L}\left[u_2\frac{\partial u_1}{\partial \varsigma}\right] + \frac{1}{s^\alpha}\mathcal{L}\left[u_3\frac{\partial u_1}{\partial \zeta}\right] - \frac{v}{s^\alpha}\left(\frac{\partial^2 U_1}{\partial \chi^2} + \frac{\partial^2 U_1}{\partial \varsigma^2} + \frac{\partial^2 U_1}{\partial \zeta^2}\right) + \frac{1}{\rho s^\alpha}\frac{\partial P}{\partial \chi} = 0,$$

$$U_2 - \frac{h}{s} + \frac{1}{s^\alpha}\mathcal{L}\left[u_1\frac{\partial u_2}{\partial \chi}\right] + \frac{1}{s^\alpha}\mathcal{L}\left[u_2\frac{\partial u_2}{\partial \varsigma}\right] + \frac{1}{s^\alpha}\mathcal{L}\left[u_3\frac{\partial u_2}{\partial \zeta}\right] - \frac{v}{s^\alpha}\left(\frac{\partial^2 U_2}{\partial \chi^2} + \frac{\partial^2 U_2}{\partial \varsigma^2} + \frac{\partial^2 U_2}{\partial \zeta^2}\right) + \frac{1}{\rho s^\alpha}\frac{\partial P}{\partial \varsigma} = 0,$$

$$U_3 - \frac{g}{s} + \frac{1}{s^\alpha}\mathcal{L}\left[u_1\frac{\partial u_3}{\partial \chi}\right] + \frac{1}{s^\alpha}\mathcal{L}\left[u_2\frac{\partial u_3}{\partial \varsigma}\right] + \frac{1}{s^\alpha}\mathcal{L}\left[u_3\frac{\partial u_3}{\partial \zeta}\right] - \frac{v}{s^\alpha}\left(\frac{\partial^2 U_3}{\partial \chi^2} + \frac{\partial^2 U_3}{\partial \varsigma^2} + \frac{\partial^2 U_3}{\partial \zeta^2}\right) + \frac{1}{\rho s^\alpha}\frac{\partial P}{\partial \zeta} = 0, \quad (3.1)$$

where $U_1 = \mathcal{L}[u_1]$, $U_2 = \mathcal{L}[u_2]$, $U_3 = \mathcal{L}[u_3]$ and $P = \mathcal{L}[p]$.

Suppose that the solution of the system (3.1), U_1, U_2, and U_3, has the following fractional Laurent expansions as follows:

$$U_1 = \sum_{i=0}^{\infty} \frac{f_i(\chi,\varsigma,\zeta)}{s^{i\alpha+1}},$$

$$U_2 = \sum_{j=0}^{\infty} \frac{h_j(\chi,\varsigma,\zeta)}{s^{j\alpha+1}},$$

$$U_3 = \sum_{r=0}^{\infty} \frac{g_r(\chi,\varsigma,\zeta)}{s^{r\alpha+1}}. \quad (3.2)$$

Assume that also P has the following Laurent expansion:

$$P = \frac{\varphi}{s} + \sum_{i=1}^{\infty} \frac{\varphi_i}{s^{i\alpha+1}}. \quad (3.3)$$

According to Theorem 2.1 and using the ICs (1.2), the expansions in Eq (3.2) can be rewritten as follows:

$$U_1 = \frac{f}{s} + \sum_{i=1}^{\infty} \frac{f_i(\chi,\varsigma,\zeta)}{s^{i\alpha+1}},$$

$$U_2 = \frac{h}{s} + \sum_{i=1}^{\infty} \frac{h_i(\chi,\varsigma,\zeta)}{s^{i\alpha+1}},$$

$$U_3 = \frac{g}{s} + \sum_{r=1}^{\infty} \frac{g_r(\chi,\varsigma,\zeta)}{s^{r\alpha+1}}. \quad (3.4)$$

The kth-truncated series of the expansions in Eqs (3.3) and (3.4) are given by:

$$U_{1,k} = \frac{f}{s} + \sum_{i=1}^{k} \frac{f_i}{s^{i\alpha+1}},$$

$$U_{2,k} = \frac{h}{s} + \sum_{j=1}^{k} \frac{h_j}{s^{j\alpha+1}},$$

$$U_{3,k} = \frac{g}{s} + \sum_{r=1}^{k} \frac{g_r}{s^{r\alpha+1}},$$

$$P_k = \frac{\varphi}{s} + \sum_{i=1}^{k} \frac{\varphi_i}{s^{i\alpha+1}}. \quad (3.5)$$

To find the coefficients in the series expansions of Eq (3.4), we establish the Laplace residual functions ($\mathcal{L}$RF) of the equations in the system (3.1) as follows:

$$\mathcal{L}Res_1(s)=U_1-\frac{f}{s}+\frac{1}{s^\alpha}\mathcal{L}\left[u_1\frac{\partial u_1}{\partial \chi}\right]+\frac{1}{s^\alpha}\mathcal{L}\left[u_2\frac{\partial u_1}{\partial \varsigma}\right]+\frac{1}{s^\alpha}\mathcal{L}\left[u_3\frac{\partial u_1}{\partial \zeta}\right]-\frac{\upsilon}{s^\alpha}\left(\frac{\partial^2 U_1}{\partial \chi}+\frac{\partial^2 U_1}{\partial \varsigma^2}+\frac{\partial^2 U_1}{\partial \zeta^2}\right)+\frac{1}{\rho s^\alpha}\frac{\partial P}{\partial \chi},$$

$$\mathcal{L}Res_2(s)=U_2-\frac{h}{s}+\frac{1}{s^\alpha}\mathcal{L}\left[u_1\frac{\partial u_2}{\partial \chi}\right]+\frac{1}{s^\alpha}\mathcal{L}\left[u_2\frac{\partial u_2}{\partial \varsigma}\right]+\frac{1}{s^\alpha}\mathcal{L}\left[u_3\frac{\partial u_2}{\partial \zeta}\right]-\frac{\upsilon}{s^\alpha}\left(\frac{\partial^2 U_2}{\partial \chi^2}+\frac{\partial^2 U_2}{\partial \varsigma^2}+\frac{\partial^2 U_2}{\partial \zeta^2}\right)+\frac{1}{\rho s^\alpha}\frac{\partial P}{\partial \varsigma},$$

$$\mathcal{L}Res_3(s)=U_3-\frac{g}{s}+\frac{1}{s^\alpha}\mathcal{L}\left[u_1\frac{\partial u_3}{\partial \chi}\right]+\frac{1}{s^\alpha}\mathcal{L}\left[u_2\frac{\partial u_3}{\partial \varsigma}\right]+\frac{1}{s^\alpha}\mathcal{L}\left[u_3\frac{\partial u_3}{\partial \zeta}\right]-\frac{\upsilon}{s^\alpha}\left(\frac{\partial^2 U_3}{\partial \chi^2}+\frac{\partial^2 U_3}{\partial \varsigma^2}+\frac{\partial^2 U_3}{\partial \zeta^2}\right)+\frac{1}{\rho s^\alpha}\frac{\partial P}{\partial \zeta}, \tag{3.6}$$

and the kth-$\boldsymbol{\mathcal{L}}$RF for $k=1,2,\dots$, as follows:

$$\begin{aligned}
\mathcal{L}Res_{1,k}(s)&=U_{1,k}-\frac{f}{s}+\frac{1}{s^\alpha}\mathcal{L}\left[u_{1,k}\frac{\partial u_{1,k}}{\partial \chi}\right]+\frac{1}{s^\alpha}\mathcal{L}\left[u_{2,k}\frac{\partial u_{1,k}}{\partial \varsigma}\right]\\
&+\frac{1}{s^\alpha}\mathcal{L}\left[u_{3,k}\frac{\partial u_{1,k}}{\partial \zeta}\right]-\frac{\upsilon}{s^\alpha}\left(\frac{\partial^2 U_{1,k}}{\partial \chi^2}+\frac{\partial^2 U_{1,k}}{\partial \varsigma^2}+\frac{\partial^2 U_{1,k}}{\partial \zeta^2}\right)+\frac{1}{\rho s^\alpha}\frac{\partial P_k}{\partial \chi},\\
\mathcal{L}Res_{2,k}(s)&=U_{2,k}-\frac{h}{s}+\frac{1}{s^\alpha}\mathcal{L}\left[u_{1,k}\frac{\partial u_{2,k}}{\partial \chi}\right]+\frac{1}{s^\alpha}\mathcal{L}\left[u_{2,k}\frac{\partial u_{2,k}}{\partial \varsigma}\right]\\
&+\frac{1}{s^\alpha}\mathcal{L}\left[u_{3,k}\frac{\partial u_{2,k}}{\partial \zeta}\right]-\frac{\upsilon}{s^\alpha}\left(\frac{\partial^2 U_{2,k}}{\partial \chi^2}+\frac{\partial^2 U_{2,k}}{\partial \varsigma^2}+\frac{\partial^2 U_{2,k}}{\partial \zeta^2}\right)+\frac{1}{\rho s^\alpha}\frac{\partial P_k}{\partial \varsigma},\\
\mathcal{L}Res_{3,k}(s)&=U_{3,k}-\frac{g}{s}+\frac{1}{s^\alpha}\mathcal{L}\left[u_{1,k}\frac{\partial u_{3,k}}{\partial \chi}\right]+\frac{1}{s^\alpha}\mathcal{L}\left[u_{2,k}\frac{\partial u_{3,k}}{\partial \varsigma}\right]\\
&+\frac{1}{s^\alpha}\mathcal{L}\left[u_{3,k}\frac{\partial u_{3,k}}{\partial \zeta}\right]-\frac{\upsilon}{s^\alpha}\left(\frac{\partial^2 U_{3,k}}{\partial \chi^2}+\frac{\partial^2 U_{3,k}}{\partial \varsigma^2}+\frac{\partial^2 U_{3,k}}{\partial \zeta^2}\right)+\frac{1}{\rho s^\alpha}\frac{\partial P_k}{\partial \zeta}.
\end{aligned} \tag{3.7}$$

Using Lemma 2.3 and substituting the expansions in Eq (3.5) into Eq (3.7) gives the Laurent series form to the kth-$\mathcal{L}$RFs as follows:

$$\begin{aligned}
\mathcal{L}Res_{1,\mathrm{k}}(s)&=\sum_{i=1}^{k}\frac{f_i}{s^{i\alpha+1}}+\sum_{i=0}^{k}\sum_{j=0}^{k}\frac{\Gamma((i+j)\alpha+1)f_i\frac{\partial f_j}{\partial \chi}}{\Gamma(i\alpha+1)\Gamma(j\alpha+1)s^{(i+j+1)\alpha+1}}\\
&+\sum_{i=0}^{k}\sum_{j=0}^{k}\frac{\Gamma((i+j)\alpha+1)h_i\frac{\partial f_j}{\partial \varsigma}}{\Gamma(i\alpha+1)\Gamma(j\alpha+1)s^{(i+j+1)\alpha+1}}\\
&+\sum_{i=0}^{k}\sum_{j=0}^{k}\frac{\Gamma((i+j)\alpha+1)g_i\frac{\partial f_j}{\partial \zeta}}{\Gamma(i\alpha+1)\Gamma(j\alpha+1)s^{(i+j+1)\alpha+1}}-\frac{\upsilon\frac{\partial^2 f}{\partial \chi^2}}{s^{\alpha+1}}-\sum_{i=1}^{k}\frac{\upsilon\frac{\partial^2 f_i}{\partial \chi^2}}{s^{(i+1)\alpha+1}}\\
&-\frac{\upsilon\frac{\partial^2 f}{\partial \varsigma^2}}{s^{\alpha+1}}-\sum_{i=1}^{k}\frac{\upsilon\frac{\partial^2 f_i}{\partial \varsigma^2}}{s^{(i+1)\alpha+1}}-\frac{\upsilon\frac{\partial^2 f}{\partial \zeta^2}}{s^{\alpha+1}}-\sum_{i=1}^{k}\frac{\upsilon\frac{\partial^2 f_i}{\partial \zeta^2}}{s^{(i+1)\alpha+1}}+\frac{\frac{\partial \varphi}{\partial \chi}}{\rho s^{\alpha+1}}\\
&+\sum_{i=1}^{k}\frac{\frac{\partial \varphi_i}{\partial \chi}}{\rho s^{(i+1)\alpha+1}},\\
\mathcal{L}Res_{2,\mathrm{k}}(s)&=\sum_{i=1}^{k}\frac{h_i}{s^{i\alpha+1}}+\sum_{i=0}^{k}\sum_{j=0}^{k}\frac{\Gamma((i+j)\alpha+1)f_i\frac{\partial h_j}{\partial \chi}}{\Gamma(i\alpha+1)\Gamma(j\alpha+1)s^{(i+j+1)\alpha+1}}\\
&+\sum_{i=0}^{k}\sum_{j=0}^{k}\frac{\Gamma((i+j)\alpha+1)h_i\frac{\partial h_j}{\partial \varsigma}}{\Gamma(i\alpha+1)\Gamma(j\alpha+1)s^{(i+j+1)\alpha+1}}
\end{aligned}$$

$$+\sum_{i=0}^{k}\sum_{j=0}^{k}\frac{\Gamma((i+j)\alpha+1)g_i\frac{\partial h_j}{\partial \zeta}}{\Gamma(i\alpha+1)\Gamma(j\alpha+1)s^{(i+j+1)\alpha+1}}-\frac{\upsilon\frac{\partial^2 h}{\partial \chi^2}}{s^{\alpha+1}}-\sum_{i=1}^{k}\frac{\upsilon\frac{\partial^2 h_i}{\partial \chi^2}}{s^{(i+1)\alpha+1}}$$

$$-\frac{\upsilon\frac{\partial^2 h}{\partial \varsigma^2}}{s^{\alpha+1}}-\sum_{i=1}^{k}\frac{\upsilon\frac{\partial^2 h_i}{\partial \varsigma^2}}{s^{(i+1)\alpha+1}}-\frac{\upsilon\frac{\partial^2 h}{\partial \zeta^2}}{s^{\alpha+1}}-\sum_{i=1}^{k}\frac{\upsilon\frac{\partial^2 h_i}{\partial \zeta^2}}{s^{(i+1)\alpha+1}}+\frac{\frac{\partial}{\partial \varsigma}\varphi}{\rho s^{\alpha+1}}$$

$$+\sum_{i=1}^{k}\frac{\frac{\partial \varphi_i}{\partial \varsigma}}{\rho s^{(i+1)\alpha+1}},$$

$$\mathcal{L}Res_{3,\mathrm{k}}(s)=\sum_{i=1}^{k}\frac{g_i}{s^{i\alpha+1}}+\sum_{i=0}^{k}\sum_{j=0}^{k}\frac{\Gamma((i+j)\alpha+1)f_i\frac{\partial g_j}{\partial \chi}}{\Gamma(i\alpha+1)\Gamma(j\alpha+1)s^{(i+j+1)\alpha+1}}$$

$$+\sum_{i=0}^{k}\sum_{j=0}^{k}\frac{\Gamma((i+j)\alpha+1)h_i\frac{\partial g_j}{\partial \varsigma}}{\Gamma(i\alpha+1)\Gamma(j\alpha+1)s^{(i+j+1)\alpha+1}}$$

$$+\sum_{i=0}^{k}\sum_{j=0}^{k}\frac{\Gamma((i+j)\alpha+1)g_i\frac{\partial g_j}{\partial \zeta}}{\Gamma(i\alpha+1)\Gamma(j\alpha+1)s^{(i+j+1)\alpha+1}}-\frac{\upsilon\frac{\partial^2 h}{\partial \chi^2}}{s^{\alpha+1}}-\sum_{i=1}^{k}\frac{\upsilon\frac{\partial^2 g_i}{\partial \chi^2}}{s^{(i+1)\alpha+1}}$$

$$-\frac{\upsilon\frac{\partial^2 g}{\partial \varsigma^2}}{s^{\alpha+1}}-\sum_{i=1}^{k}\frac{\upsilon\frac{\partial^2 g_i}{\partial \varsigma^2}}{s^{(i+1)\alpha+1}}-\frac{\upsilon\frac{\partial^2 g}{\partial \zeta^2}}{s^{\alpha+1}}-\sum_{i=1}^{k}\frac{\upsilon\frac{\partial^2 g_i}{\partial \zeta^2}}{s^{(i+1)\alpha+1}}+\frac{\frac{\partial \varphi}{\partial \zeta}}{\rho s^{\alpha+1}}$$

$$+\sum_{i=1}^{k}\frac{\frac{\partial \varphi_i}{\partial \zeta}}{\rho s^{(i+1)\alpha+1}}. \quad (3.8)$$

El-Ajou in [19] proved the validity of the following formula, which is considered the essential tool of LRPSM:

$$\lim_{s\to\infty} s^{k\alpha+1}\,\mathcal{L}Res(s)=\lim_{s\to\infty} s^{k\alpha+1}\,\mathcal{L}Res_{\mathrm{k}}(s)=0, k=1,2,3,\ldots. \quad (3.9)$$

To find the first coefficients in the expansions in Eq (3.4), consider the first truncated series of the Laurent expansions of Eq (3.5) as follows:

$$U_{1,1}=\frac{f}{s}+\frac{f_1}{s^{\alpha+1}},$$

$$U_{2,1}=\frac{h}{s}+\frac{h_1}{s^{\alpha+1}},$$

$$U_{3,1}=\frac{g}{s}+\frac{g_1}{s^{\alpha+1}},$$

$$P_1=\frac{\varphi}{s}+\frac{\varphi_1}{s^{\alpha+1}}. \quad (3.10)$$

Substituting Eq (3.10) into the first-$\mathcal{L}$Res, multiplying each equation by $s^{\alpha+1}$, and taking the limit as $s\to\infty$, we have truth in Eq (3.9):

$$\lim_{s\to\infty} s^{\alpha+1}\,\mathcal{L}Res_{1,1}(s) = f_1 + f\frac{\partial f}{\partial \chi} + h\frac{\partial f}{\partial \varsigma} + g\frac{\partial f}{\partial \zeta} - \upsilon\frac{\partial^2 f}{\partial \chi^2} - \upsilon\frac{\partial^2 f}{\partial \varsigma^2} - \upsilon\frac{\partial^2 f}{\partial \zeta^2} + \frac{1}{\rho}\frac{\partial \varphi}{\partial \chi} = 0,$$

$$\lim_{s\to\infty} s^{\alpha+1}\,\mathcal{L}Res_{2,1}(s) = h_1 + f\frac{\partial h}{\partial \chi} + h\frac{\partial h}{\partial \varsigma} + g\frac{\partial h}{\partial \zeta} - \upsilon\frac{\partial^2 h}{\partial \chi^2} - \upsilon\frac{\partial^2 h}{\partial \varsigma^2} - \upsilon\frac{\partial^2 h}{\partial \zeta^2} + \frac{1}{\rho}\frac{\partial \varphi}{\partial \varsigma} = 0,$$

$$\lim_{s\to\infty} s^{\alpha+1}\,\mathcal{L}Res_{3,1}(\mathrm{s}) = g_1 + f\frac{\partial g}{\partial \chi} + h\frac{\partial g}{\partial \varsigma} + g\frac{\partial g}{\partial \zeta} - \rho\frac{\partial^2 g}{\partial \chi^2} - \rho\frac{\partial^2 g}{\partial \varsigma^2} - \rho\frac{\partial^2 g}{\partial \zeta^2} + \frac{1}{\rho}\frac{\partial \varphi}{\partial \zeta} = 0. \quad (3.11)$$

Therefore, the first coefficients of the expansions of Eq (3.4) become known and take the following forms:

$$f_1 = \upsilon\left(\frac{\partial f}{\partial \chi^2} + \frac{\partial f}{\partial \varsigma^2} + \frac{\partial f}{\partial \zeta^2}\right) - \frac{1}{\rho}\frac{\partial \varphi}{\partial \chi} - \left(f\frac{\partial f}{\partial \chi} + h\frac{\partial f}{\partial \varsigma} + g\frac{\partial f}{\partial \zeta}\right),$$

$$h_1 = \upsilon\left(\frac{\partial h}{\partial \chi^2} + \frac{\partial h}{\partial \varsigma^2} + \frac{\partial h}{\partial \zeta^2}\right) - \frac{1}{\rho}\frac{\partial \varphi}{\partial \varsigma} - \left(f\frac{\partial h}{\partial \chi} + h\frac{\partial h}{\partial \varsigma} + g\frac{\partial h}{\partial \zeta}\right),$$

$$g_1 = \upsilon\left(\frac{\partial g}{\partial \chi^2} + \frac{\partial g}{\partial \varsigma^2} + \frac{\partial g}{\partial \zeta^2}\right) - \frac{1}{\rho}\frac{\partial \varphi}{\partial \zeta} - \left(f\frac{\partial g}{\partial \chi} + h\frac{\partial g}{\partial \varsigma} + g\frac{\partial g}{\partial \zeta}\right). \quad (3.12)$$

Now, to obtain the second coefficients of the series expansions in Eq (3.4), we substitute the following second truncated series:

$$U_{1,2} = \frac{f}{s} + \frac{f_1}{s^{\alpha+1}} + \frac{f_2}{s^{2\alpha+1}},$$

$$U_{2,2} = \frac{h}{s} + \frac{h_1}{s^{\alpha+1}} + \frac{h_2}{s^{2\alpha+1}},$$

$$U_{3,2} = \frac{g}{s} + \frac{g_1}{s^{\alpha+1}} + \frac{g_2}{s^{2\alpha+1}},$$

$$P_2 = \frac{\varphi}{s} + \frac{\varphi_1}{s^{\alpha+1}} + \frac{\varphi_2}{s^{2\alpha+1}}, \quad (3.13)$$

into the 2nd-$\mathcal{L}$RF, multiply each equation by $s^{2\alpha+1}$, and use Eq (3.9), we obtain the forms of required coefficients as follows:

$$f_2 = \upsilon\left(\frac{\partial^2 f_1}{\partial \chi^2} + \frac{\partial^2 f_1}{\partial \varsigma^2} + \frac{\partial^2 f_1}{\partial \zeta^2}\right) - \frac{1}{\rho}\frac{\partial \varphi_1}{\partial \chi} - \left(f_1\frac{\partial f}{\partial \chi} + f\frac{\partial f_1}{\partial \chi} + h_1\frac{\partial f}{\partial \varsigma} + h\frac{\partial f_1}{\partial \varsigma} + g_1\frac{\partial f}{\partial \zeta} + g\frac{\partial f_1}{\partial \zeta}\right),$$

$$h_2 = \upsilon\left(\frac{\partial^2 h_1}{\partial \chi^2} + \frac{\partial^2 h_1}{\partial \varsigma^2} + \frac{\partial^2 h_1}{\partial \zeta^2}\right) - \frac{1}{\rho}\frac{\partial \varphi_1}{\partial \varsigma} - \left(f_1\frac{\partial h}{\partial \chi} + f\frac{\partial h_1}{\partial \chi} + h_1\frac{\partial h}{\partial \varsigma} + h\frac{\partial h_1}{\partial \varsigma} + g_1\frac{\partial h}{\partial \zeta} + g\frac{\partial h_1}{\partial \zeta}\right),$$

$$g_2 = \upsilon\left(\frac{\partial^2 g_1}{\partial \chi^2} + \frac{\partial^2 g_1}{\partial \varsigma^2} + \frac{\partial^2 g_1}{\partial \zeta^2}\right) - \frac{1}{\rho}\frac{\partial \varphi_1}{\partial \zeta} - \left(f_1\frac{\partial g}{\partial \chi} + f\frac{\partial g_1}{\partial \chi} + h_1\frac{\partial g}{\partial \varsigma} + h\frac{\partial g_1}{\partial \varsigma} + g_1\frac{\partial g}{\partial \zeta} + g\frac{\partial g_1}{\partial \zeta}\right). \quad (3.14)$$

We can repeat the previous steps to find the third and the fourth coefficients of the expansions (3.4) as:

$$f_3 = \upsilon\left(\frac{\partial^2 f_2}{\partial \chi^2} + \frac{\partial^2 f_2}{\partial \varsigma^2} + \frac{\partial^2 f_2}{\partial \zeta^2}\right) + \frac{1}{\rho}\frac{\partial \varphi_2}{\partial \chi}$$
$$- \left(f_2\frac{\partial f}{\partial \chi} + f\frac{\partial f_2}{\partial \chi} + h_2\frac{\partial f}{\partial \varsigma} + h\frac{\partial f_2}{\partial \varsigma} + g_2\frac{\partial f}{\partial \zeta} + g\frac{\partial f_2}{\partial \zeta}\right)$$
$$- \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\left(f_1\frac{\partial f_1}{\partial \chi} + h_1\frac{\partial f_1}{\partial \varsigma} + g_1\frac{\partial f_1}{\partial \zeta}\right),$$

$$h_3 = \upsilon\left(\frac{\partial^2 h_2}{\partial\chi^2} + \frac{\partial^2 h_2}{\partial\varsigma^2} + \frac{\partial^2 h_2}{\partial\zeta^2}\right) - \frac{1}{\rho}\frac{\partial\varphi_2}{\partial\varsigma}$$
$$-\left(f_2\frac{\partial h}{\partial\chi} + f\frac{\partial h_2}{\partial\chi} + h_2\frac{\partial h}{\partial\varsigma} + h\frac{\partial h_2}{\partial\varsigma} + g_2\frac{\partial h}{\partial\zeta} + g\frac{\partial h_2}{\partial\zeta}\right)$$
$$-\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\left(f_1\frac{\partial h_1}{\partial\chi} + h_1\frac{\partial h_1}{\partial\varsigma} + g_1\frac{\partial h_1}{\partial\zeta}\right),$$
$$g_3 = \upsilon\left(\frac{\partial^2 g_2}{\partial\chi^2} + \frac{\partial^2 g_2}{\partial\varsigma^2} + \frac{\partial^2 g_2}{\partial\zeta^2}\right) - \frac{1}{\rho}\frac{\partial\varphi_2}{\partial\zeta}$$
$$-\left(f_2\frac{\partial g}{\partial\chi} + f\frac{\partial g_2}{\partial\chi} + h_2\frac{\partial g}{\partial\varsigma} + h\frac{\partial g_2}{\partial\varsigma} + g_2\frac{\partial g}{\partial\zeta} + g\frac{\partial g_2}{\partial\zeta}\right)$$
$$-\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\left(f_1\frac{\partial g_1}{\partial\chi} + h_1\frac{\partial g_1}{\partial\varsigma} + g_1\frac{\partial g_1}{\partial\zeta}\right). \quad (3.15)$$

Also, one can get

$$f_4 = \upsilon\left(\frac{\partial^2 f_3}{\partial\chi^2} + \frac{\partial^2 f_3}{\partial\varsigma^2} + \frac{\partial^2 f_3}{\partial\zeta^2}\right) - \frac{1}{\rho}\frac{\partial\varphi_3}{\partial\chi}$$
$$-\left(f_3\frac{\partial f}{\partial\chi} + f\frac{\partial f_3}{\partial\chi} + h_3\frac{\partial f}{\partial\varsigma} + h\frac{\partial f_3}{\partial\varsigma} + g_3\frac{\partial f}{\partial\zeta} + g\frac{\partial f_3}{\partial\zeta}\right)$$
$$-\frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)}\left(f_2\frac{\partial f_1}{\partial\chi} + f_1\frac{\partial f_2}{\partial\chi} + h_2\frac{\partial f_1}{\partial\varsigma} + h_1\frac{\partial f_2}{\partial\varsigma} + g_2\frac{\partial f_1}{\partial\zeta} + g_1\frac{\partial f_2}{\partial\zeta}\right),$$
$$h_4 = \upsilon\left(\frac{\partial^2 h_3}{\partial\chi^2} + \frac{\partial^2 h_3}{\partial\varsigma^2} + \frac{\partial^2 h_3}{\partial\zeta^2}\right) - \frac{1}{\rho}\frac{\partial\varphi_3}{\partial\varsigma}$$
$$-\left(f_3\frac{\partial h}{\partial\chi} + f\frac{\partial h_3}{\partial\chi} + h_3\frac{\partial h}{\partial\varsigma} + h\frac{\partial h_3}{\partial\varsigma} + g_3\frac{\partial h}{\partial\zeta} + g\frac{\partial h_3}{\partial\zeta}\right)$$
$$-\frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)}\left(f_2\frac{\partial h_1}{\partial\chi} + f_1\frac{\partial h_2}{\partial\chi} + h_2\frac{\partial h_1}{\partial\varsigma} + h_1\frac{\partial h_2}{\partial\varsigma} + g_2\frac{\partial h_1}{\partial\zeta} + g_1\frac{\partial h_2}{\partial\zeta}\right),$$
$$g_4 = \upsilon\left(\frac{\partial^2 g_3}{\partial\chi^2} + \frac{\partial^2 g_3}{\partial\varsigma^2} + \frac{\partial^2 g_3}{\partial\zeta^2}\right) - \frac{1}{\rho}\frac{\partial\varphi_3}{\partial\zeta}$$
$$-\left(f_3\frac{\partial g}{\partial\chi} + f\frac{\partial g_3}{\partial\chi} + h_3\frac{\partial g}{\partial\varsigma} + h\frac{\partial g_3}{\partial\varsigma} + g_3\frac{\partial g}{\partial\zeta} + g\frac{\partial g_3}{\partial\zeta}\right)$$
$$-\frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)}\left(f_2\frac{\partial g_1}{\partial\chi} + f_1\frac{\partial g_2}{\partial\chi} + h_2\frac{\partial g_1}{\partial\varsigma} + h_1\frac{\partial g_2}{\partial\varsigma} + g_2\frac{\partial g_1}{\partial\zeta} + g_1\frac{\partial g_2}{\partial\zeta}\right) \quad (3.16)$$

Now, we can get the form of the unknown kth-coefficients in the expansions of Eq (3.4) as follows:

$$f_k = \upsilon\left(\frac{\partial^2 f_{k-1}}{\partial\chi^2} + \frac{\partial^2 f_{k-1}}{\partial\varsigma^2} + \frac{\partial^2 f_{k-1}}{\partial\zeta^2}\right) - \frac{1}{\rho}\frac{\partial\varphi_{k-1}}{\partial\chi}$$
$$-\sum_{i=0}^{k-1}\frac{\Gamma((k-1)\alpha+1)}{\Gamma(i\alpha+1)\Gamma((k-i-1)\alpha+1)}\left(f_i\frac{\partial f_{k-i-1}}{\partial\chi} + h_i\frac{\partial f_{k-i-1}}{\partial\varsigma} + g_i\frac{\partial f_{k-i-1}}{\partial\zeta}\right), k = 1,2,\cdots,$$
$$h_k = \upsilon\left(\frac{\partial^2 h_{k-1}}{\partial\chi^2} + \frac{\partial^2 h_{k-1}}{\partial\varsigma^2} + \frac{\partial^2 h_{k-1}}{\partial\zeta^2}\right) - \frac{1}{\rho}\frac{\partial\varphi_{k-1}}{\partial\varsigma}$$
$$-\sum_{i=0}^{k-1}\frac{\Gamma((k-1)\alpha+1)}{\Gamma(i\alpha+1)\Gamma((k-i-1)\alpha+1)}\left(f_i\frac{\partial h_{k-i-1}}{\partial\chi} + h_i\frac{\partial h_{k-i-1}}{\partial\varsigma} + g_i\frac{\partial h_{k-i-1}}{\partial\zeta}\right), k = 1,2,\cdots,$$

$$g_k = \upsilon\left(\frac{\partial^2 g_{k-1}}{\partial\chi^2}+\frac{\partial^2 g_{k-1}}{\partial\varsigma^2}+\frac{\partial^2 g_{k-1}}{\partial\zeta^2}\right)-\frac{1}{\rho}\frac{\partial\varphi_{k-1}}{\partial\zeta}$$
$$-\sum_{i=0}^{k-1}\frac{\Gamma((k-1)\alpha+1)}{\Gamma(i\alpha+1)\Gamma((k-i-1)\alpha+1)}\left(f_i\frac{\partial g_{k-i-1}}{\partial\chi}+h_i\frac{\partial g_{k-i-1}}{\partial\varsigma}+g_i\frac{\partial g_{k-i-1}}{\partial\zeta}\right), k=1,2,\cdots. \quad (3.17)$$

Thus, the solution of the algebraic system (3.1) can be expressed as:

$$U_1(\chi,\varsigma,\zeta,s)=\frac{f}{s}+\frac{f_1}{s^{\alpha+1}}+\frac{f_2}{s^{2\alpha+1}}+\cdots,$$
$$U_1(\chi,\varsigma,\zeta,s)=\frac{h}{s}+\frac{h_1}{s^{\alpha+1}}+\frac{h_2}{s^{2\alpha+1}}+\cdots,$$
$$U_3(\chi,\varsigma,\zeta,s)=\frac{g}{s}+\frac{g_1}{s^{\alpha+1}}+\frac{g_2}{s^{2\alpha+1}}+\cdots. \quad (3.18)$$

Finally, to get the LRPS solution of the M-DT-FNS systems (1.1) and (1.2) in the original space, we apply the inverse LT on the solution in Eq (3.18), to get

$$u_1(\chi,\varsigma,\zeta,t)=f+\frac{f_1t^\alpha}{\Gamma(\alpha+1)}+\frac{f_2t^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots,$$
$$u_2(\chi,\varsigma,\zeta,t)=h+\frac{h_1t^\alpha}{\Gamma(\alpha+1)}+\frac{h_2t^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots,$$
$$u_3(\chi,\varsigma,\zeta,t)=g+\frac{g_1t^\alpha}{\Gamma(\alpha+1)}+\frac{g_2t^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots. \quad (3.19)$$

4. Numerical application

In this section, we present some numerical examples that explain the working mechanism of the LRPSM. Comparisons and graphical illustrations are made to demonstrate the accuracy and efficiency of the technique.

Example 4.1. [35,36] Consider the following two-DT-FNSEs:

$$D_t^\alpha u_1+u_1\frac{\partial u_1}{\partial\chi}+u_2\frac{\partial u_1}{\partial\varsigma}=\upsilon\left(\frac{\partial^2u_1}{\partial\chi^2}+\frac{\partial^2u_1}{\partial\varsigma^2}\right),$$
$$D_t^\alpha u_2+u_1\frac{\partial u_2}{\partial\chi}+u_2\frac{\partial u_2}{\partial\varsigma}=\upsilon\left(\frac{\partial^2u_2}{\partial\chi^2}+\frac{\partial^2u_2}{\partial\varsigma^2}\right), \quad (4.1)$$

with the ICs:

$$u_1(\chi,\varsigma,0)=-\sin(\chi+\varsigma),$$
$$u_2(\chi,\varsigma,0)=\sin(\chi+\varsigma), \quad (4.2)$$

where $\upsilon\in\mathbb{R}$, and u_1 and u_2 are two functions of three variables χ,ς, and t.

Note that, when $\alpha=1$ the exact solution of the systems (4.1) and (4.2) is

$$u_1=-e^{-2\upsilon t}\sin(\chi+\varsigma),$$
$$u_2=e^{-2\upsilon t}\sin(\chi+\varsigma). \quad (4.3)$$

Based on the algorithm of the solution obtained in Section 3 and the result in Eq (3.19), we can obtain the LRPS solution of the systems (4.1) and (4.2) as follows:

$$u_1(\chi,\varsigma,t) = f + \frac{f_1 t^{\alpha}}{\Gamma(\alpha+1)} + \frac{f_2 t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots,$$

$$u_2(\chi,\varsigma,t) = h + \frac{h_1 t^{\alpha}}{\Gamma(\alpha+1)} + \frac{h_2 t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots, \tag{4.4}$$

where the coefficients f_k, h_k, $k = 1,2,...$, are given by the general formula obtained in Eq (3.17) which in the two dimensions will be as follows:

$$f_k = \upsilon\left(\frac{\partial^2 f_{k-1}}{\partial\chi^2} + \frac{\partial^2 f_{k-1}}{\partial\varsigma}\right) - \sum_{i=0}^{k-1} \frac{\Gamma((k-1)\alpha+1)\left(f_i\frac{\partial f_{k-i-1}}{\partial\chi} + h_i\frac{\partial f_{k-i-1}}{\partial\varsigma}\right)}{\Gamma(i\alpha+1)\Gamma((k-i-1)\alpha+1)},$$

$$h_k = \upsilon\left(\frac{\partial^2 h_{k-1}}{\partial\chi^2} + \frac{\partial^2 h_{k-1}}{\partial\varsigma^2}\right) - \sum_{i=0}^{k-1} \frac{\Gamma((k-1)\alpha+1)\left(f_i\frac{\partial h_{k-i-1}}{\partial\chi} + h_i\frac{\partial f h_{k-i-1}}{\partial\varsigma}\right)}{\Gamma(i\alpha+1)\Gamma((k-i-1)\alpha+1)}. \tag{4.5}$$

Thus, we conclude that

$$\begin{aligned}
&f_1 = 2\upsilon\sin(\chi+\varsigma), && h_1 = -2\upsilon\sin(\chi+\varsigma)\\
&f_2 = -(2\upsilon)^2\sin(\chi+\varsigma), && h_2 = (2\upsilon)^2\sin(\chi+\varsigma)\\
&f_3 = (2\upsilon)^3\sin(\chi+\varsigma), && h_3 = -(2\upsilon)^3\sin(\chi+\varsigma)\\
&f_4 = -(2\upsilon)^4\sin(\chi+\varsigma), && h_4 = (2\upsilon)^4\sin(\chi+\varsigma)\\
&\quad\vdots && \quad\vdots\\
&f_k = (-1)^{k+1}(2\upsilon)^k\sin(\chi+\varsigma),\ && h_k = (-2\upsilon)^k\sin(\chi+\varsigma), \qquad k = 1,2,\cdots.
\end{aligned}$$

Hence, we can express the LRPS solution of Eqs (4.1) and (4.2) as:

$$u_1(\chi,\varsigma,t) = -\sin(\chi+\varsigma)\sum_{n=0}^{\infty}\frac{(-2\upsilon)^n t^{n\alpha}}{\Gamma(n\alpha+1)},$$

$$u_2(\chi,\varsigma,t) = \sin(\chi+\varsigma)\sum_{n=0}^{\infty}\frac{(-2\upsilon)^n t^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{4.6}$$

Table 1 shows the stability of the results obtained, where the solution is stable with various values of t.

Table 1. Numerical results for Example 1 at $\alpha = 1, \chi = 0.1, \varsigma = 0.5$.

t	Numerical results of $u_1(\chi,\varsigma,t)$				
	$u_{1,6}(\chi,\varsigma,t)$	$u_{1,8}(\chi,\varsigma,t)$	$u_{1,10}(\chi,\varsigma,t)$	$u_1(\chi,\varsigma,t)$	$\lvert u_1(\chi,\varsigma,t) - u_{1,10}(\chi,\varsigma,t)\rvert$
0.16	−0.481157	−0.481157	−0.481157	−0.481157	5.551115×10^{-17}
0.32	−0.410015	−0.410015	−0.410015	−0.410015	4.957146×10^{-14}
0.48	−0.349392	−0.349391	−0.349391	−0.349391	4.238276×10^{-12}
0.64	−0.297736	−0.297732	−0.297731	−0.297732	9.907064×10^{-11}
0.80	−0.253732	−0.253710	−0.253710	−0.253710	1.138797×10^{-9}
0.96	−0.216273	−0.216199	−0.216198	−0.216198	8.355848×10^{-9}
t	Numerical results of $u_2(\chi,\varsigma,t)$				
	$u_{2,6}(\chi,\varsigma,t)$	$u_{2,8}(\chi,\varsigma,t)$	$u_{2,10}(\chi,\varsigma,t)$	$u_2(\chi,\varsigma,t)$	$\lvert u_2(\chi,\varsigma,t) - u_{2,10}(\chi,\varsigma,t)\rvert$
0.16	0.481157	0.481157	0.481157	0.481157	5.551115×10^{-17}
0.32	0.410015	0.410015	0.410015	0.410015	4.957146×10^{-14}
0.48	0.349392	0.349391	0.349391	0.349391	4.238276×10^{-12}
0.64	0.297736	0.297732	0.297731	0.297732	9.907064×10^{-11}
0.80	0.253732	0.253710	0.253710	0.253710	1.138797×10^{-9}
0.96	0.216273	0.216199	0.216198	0.216198	8.355848×10^{-9}

This solution is the same as that obtained by the Laplace decomposition method [36] and the variational iteration transform method [35]. In a special case, taking $\alpha = 1$ gives the exact solution in terms of elementary functions as follows:

$$u_1(\chi, \varsigma, t) = -e^{-2\upsilon t} \sin(\chi + \varsigma),$$
$$u_2(\chi, \varsigma, t) = e^{-2\upsilon t} \sin(\chi + \varsigma). \tag{4.7}$$

The behavior of the velocity field of the two-DT-FNSEs (4.1) and (4.2) is depicted in Figure 1 for various values of α at $t = 0.5$ and $\upsilon = 0.5$. The 10th-truncated series of Eq (4.6) is plotted in Figure 1(a-c) for $\alpha = 0.6$, $\alpha = 0.8$, and $\alpha = 1$, respectively, whereas, the exact solution at $\alpha = 1$ is plotted in Figure 1(d). The graphics indicate consistency in the behavior of the solution at various values of α, as well as the convention of the exact solution with the obtained solution in Figure 1(c,d).

Figure 2 shows the action of the 10th approximate analytical solution of the initial value problems (IVP) (4.1) and (4.2) along the line $\varsigma = \chi$ and in the region $D = \{(\chi, t): -3 \le \chi \le 3, 0 \le t < 1\}$ for distinct values of α, and at $\upsilon = 0.5$. The 10th approximate solution is plotted in Figure 2(a-c) for $\alpha = 0.6$, $\alpha = 0.8$, and $\alpha = 1$, respectively, whereas, the exact solution at $\alpha = 1$ is plotted in (d). Also, the graphics indicate consistency in the action of the solution at distinct values of α, the accord nation of the exact solution with the approximate solution in Figure 2(c,d) as well as the region of convergence of the series solution.

Example 4.2. [35,36] Consider the following two-DT-FNSEs:

$$D_t^\alpha u_1 + u_1 \frac{\partial u_1}{\partial \chi} + u_2 \frac{\partial u_1}{\partial \varsigma} = \upsilon \left(\frac{\partial^2 u_1}{\partial \chi^2} + \frac{\partial^2 u_1}{\partial \varsigma^2} \right),$$
$$D_t^\alpha u_2 + u_1 \frac{\partial u_2}{\partial \chi} + u_2 \frac{\partial u_2}{\partial \varsigma} = \upsilon \left(\frac{\partial^2 u_2}{\partial \chi^2} + \frac{\partial^2 u_2}{\partial \varsigma^2} \right), \tag{4.8}$$

with the ICs:

$$u_1(\chi, \varsigma, 0) = -e^{\chi+\varsigma},$$
$$u_2(\chi, \varsigma, 0) = e^{\chi+\varsigma}, \tag{4.9}$$

where $\upsilon \in \mathbb{R}$ and u_1 and u_2 are two functions of three variables $\chi, \varsigma,$ and t.

The exact solution to problems (4.8) and (4.9) can be obtained, when putting $\alpha = 1$, to be $u_1 = -e^{-2\upsilon t+\chi+\varsigma}$ and $u_2 = e^{-2\upsilon t+\chi+\varsigma}$. Applying the same procedure in Example 4.1, one can obtain the following recurrence relations:

$$f_k = \upsilon \left(\frac{\partial^2 f_{k-1}}{\partial \chi^2} + \frac{\partial^2 f_{k-1}}{\partial \varsigma^2} \right) - \sum_{i=0}^{k-1} \frac{\Gamma((k-1)\alpha+1)\left(f_i \frac{\partial f_{k-i-1}}{\partial \chi} + h_i \frac{\partial f_{k-i-1}}{\partial \varsigma} \right)}{\Gamma(i\alpha+1)\Gamma((k-i-1)\alpha+1)},$$

$$h_k = \upsilon \left(\frac{\partial^2 h_{k-1}}{\partial \chi^2} + \frac{\partial^2 h_{k-1}}{\partial \varsigma^2} \right) - \sum_{i=0}^{k-1} \frac{\Gamma((k-1)\alpha+1)\left(f_i \frac{\partial h_{k-i-1}}{\partial \chi} + h_i \frac{\partial fh_{k-i-1}}{\partial \varsigma} \right)}{\Gamma(i\alpha+1)\Gamma((k-i-1)\alpha+1)}. \tag{4.10}$$

Thus, we conclude that

$$f_1 = -2\upsilon e^{\chi+\varsigma}, \quad h_1 = 2\upsilon e^{\chi+\varsigma}$$
$$f_2 = -(2\upsilon)^2 e^{\chi+\varsigma}, \quad h_2 = (2\upsilon)^2 e^{\chi+\varsigma}$$
$$f_3 = -(2\upsilon)^3 e^{\chi+\varsigma}, \quad h_3 = (2\upsilon)^3 e^{\chi+\varsigma},$$
$$f_4 = -(2\upsilon)^4 e^{\chi+\varsigma}, \quad h_3 = (2\upsilon)^3 e^{\chi+\varsigma}$$

$$\vdots \qquad\qquad\qquad \vdots$$
$$f_k = -(2\upsilon)^k e^{\chi+\varsigma}, \quad h_k = (2\upsilon)^k e^{\chi+\varsigma}, k = 1,2,\cdots.$$

Thus, the LRPS solution of the systems (4.8) and (4.9) can be expressed as follows:

$$u_1(\chi,\varsigma,t) = -e^{\chi+\varsigma}\sum_{i=0}^{\infty}\frac{(-2\upsilon)^i t^{i\alpha}}{\Gamma(i\alpha+1)},$$
$$u_2(\chi,\varsigma,t) = e^{\chi+\varsigma}\sum_{j=0}^{\infty}\frac{(-2\upsilon)^j t^{j\alpha}}{\Gamma(j\alpha+1)}. \qquad (4.11)$$

For $\alpha = 1$, the solution in Eq (4.11) has the form:

$$u_1(\chi,\varsigma,t) = -e^{-2\upsilon t+\chi+\varsigma},$$
$$u_2(\chi,\varsigma,t) = e^{-2\upsilon t+\chi+\varsigma}.$$

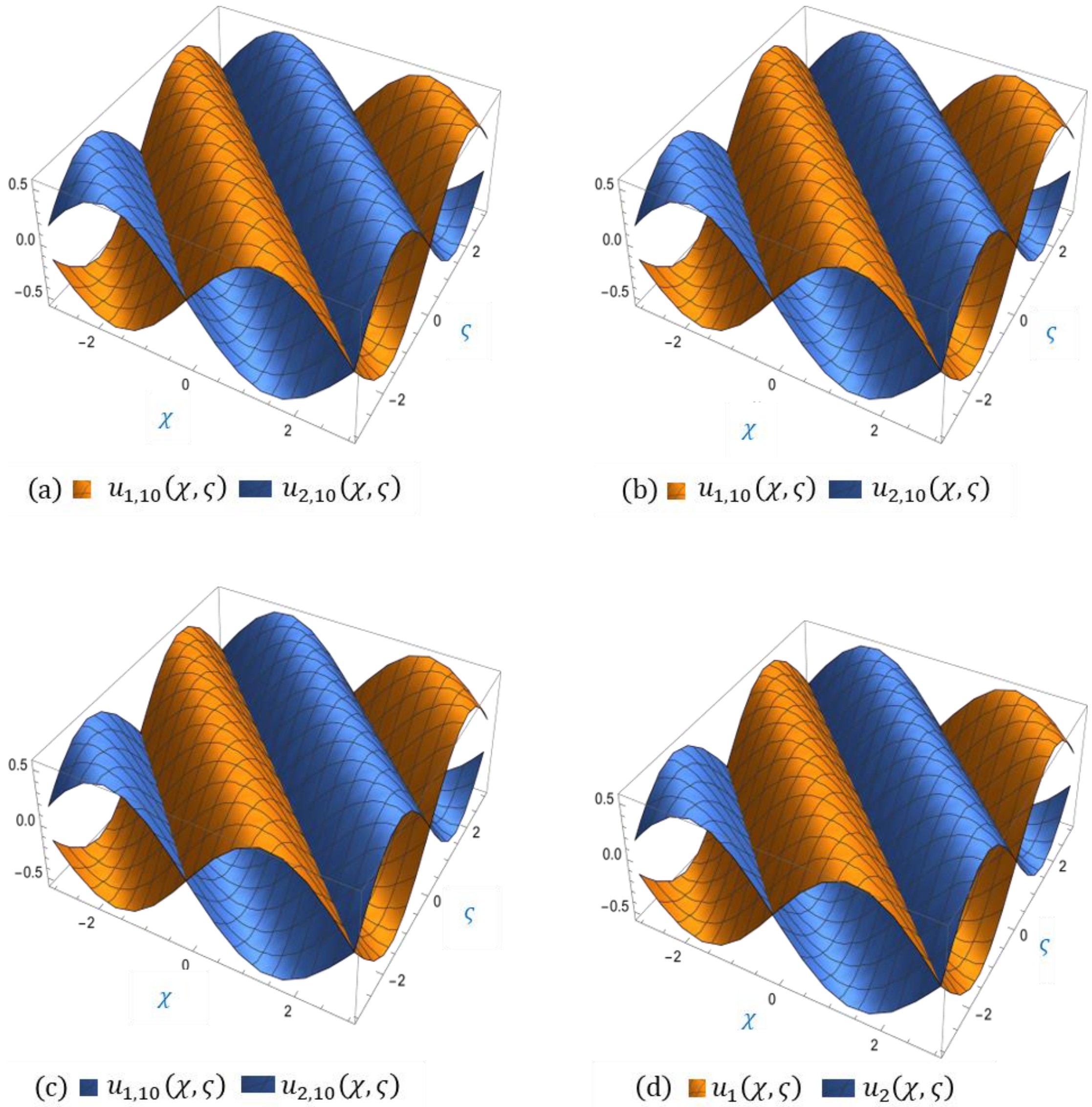

Figure 1. The 3D surface plot of the 10th approximate solutions of u_1 and u_2 at different values of α and $t = 0.5$ & $\upsilon = 0.5$ for the problem in Example 4.1. (a) $\alpha = 0.6$, (b) $\alpha = 0.8$, (c) $\alpha = 1$, (d) $\alpha = 1$ (Exact solutions).

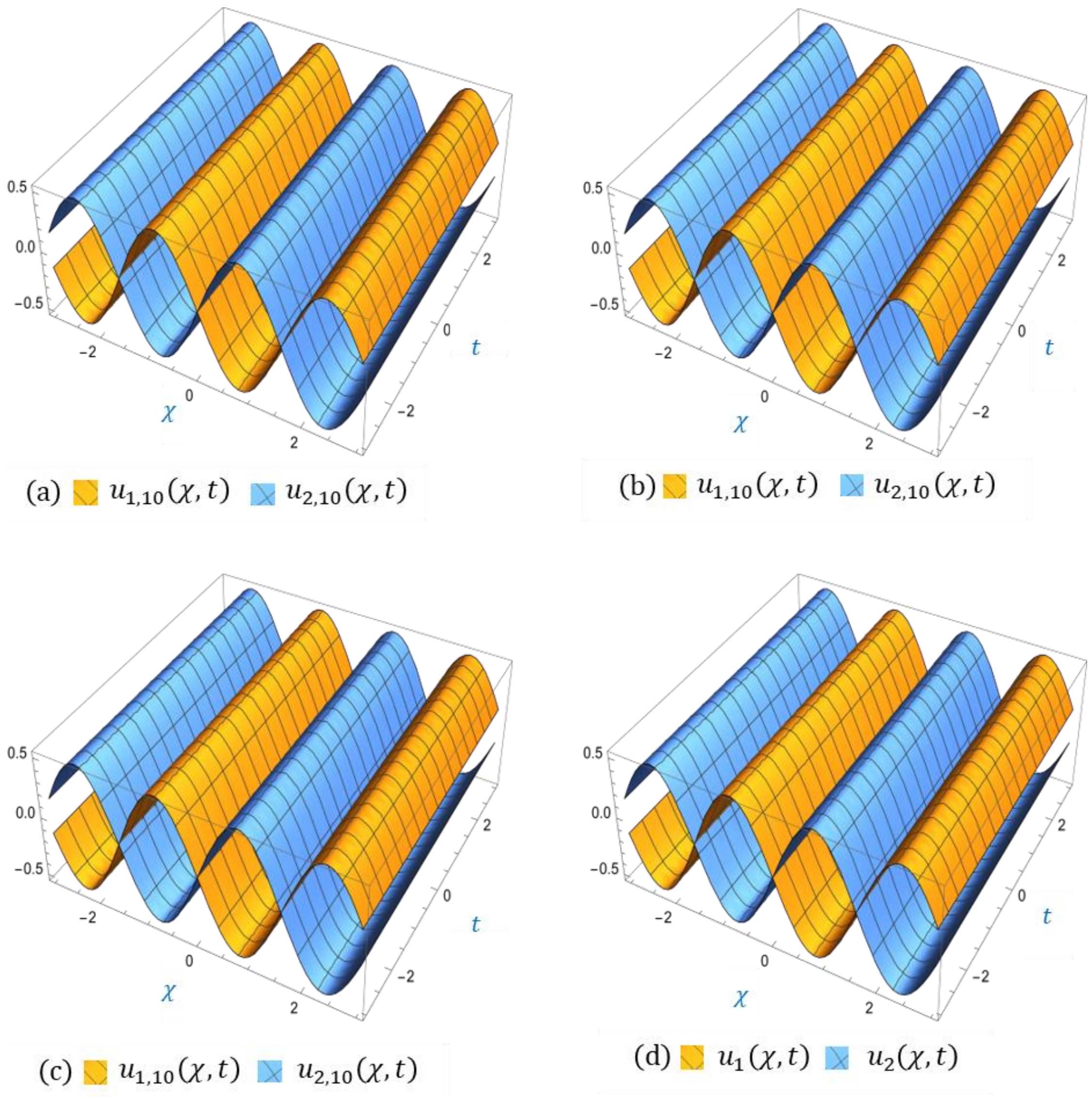

Figure 2. The graph of the 3D surface of the 10th approximate solutions of u_1 and u_2 along the line $\varsigma = \chi$ and at various values of α and $\upsilon = 0.5$ for the problem in Example 4.1. (a) $\alpha = 0.6$, (b) $\alpha = 0.8$, (c) $\alpha = 1$, (d) $\alpha = 1$ (Exact).

Figure 3 shows the velocity field behavior of the two-DT-FNSEs (4.8) and (4.9) for distinct values of α at $t = 0.5$ and $\upsilon = 0.5$. The 10th LRPS approximate analytical solution of the IVP (4.8) and (4.8) plotted in Figure 3(a-c) for $\alpha = 0.6$, $\alpha = 0.8$, and $\alpha = 1$ respectively, while the exact solution at $\alpha = 1$ is plotted in (d). The graphics indicate the consistency in the solution behavior at various values of α, as well as the exact solution agreement with the proposed analytical solution in Figure 3(c,d).

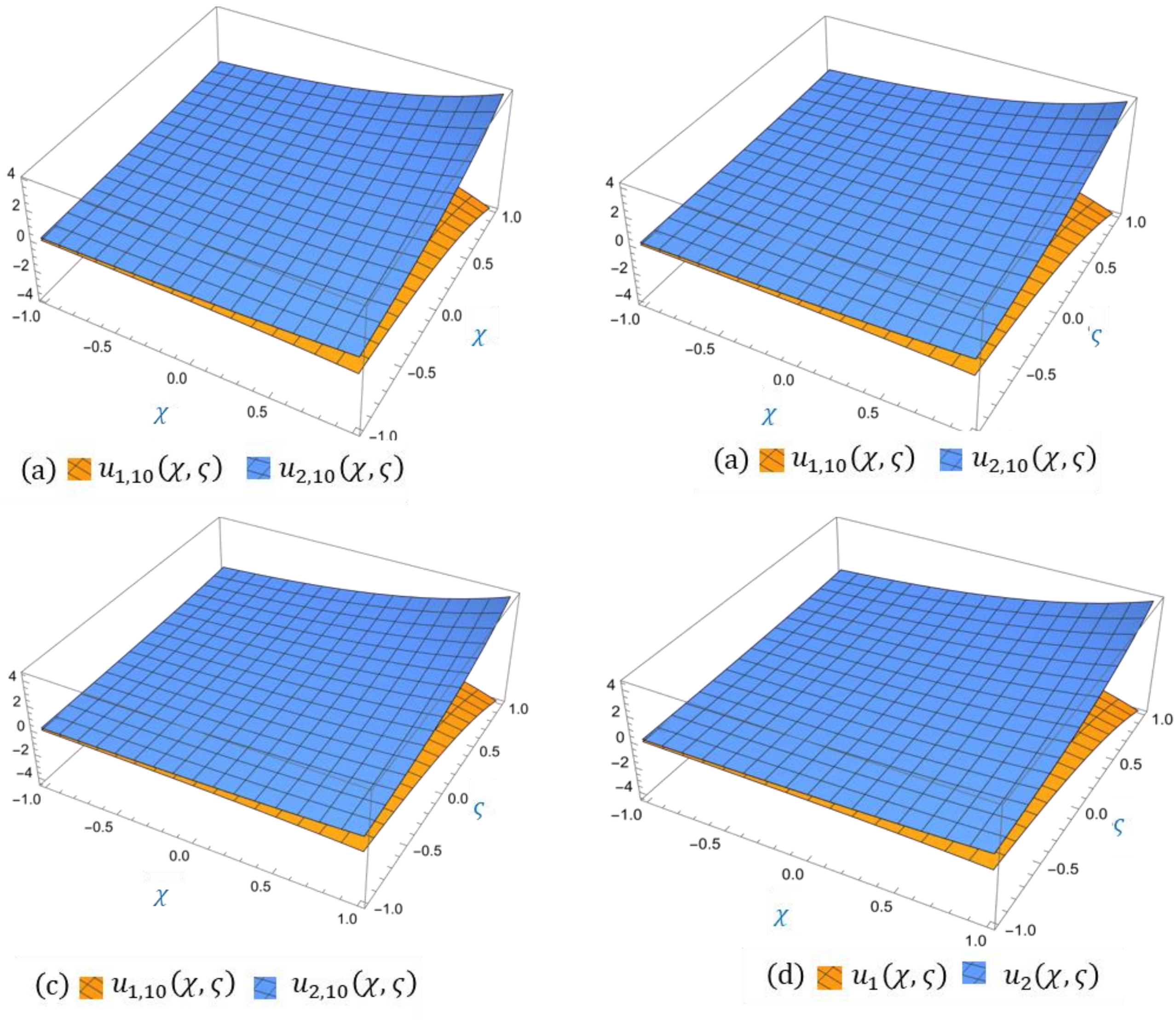

Figure 3. The 3D surface plot of the 10th approximate solutions of u_1 and u_2 at distinct values of α and $t = 0.5$ & $\upsilon = 0.5$ for the problem in Example 4.2. (a) $\alpha = 0.6$, (b) $\alpha = 0.8$, (c) $\alpha = 1$, (d) $\alpha = 1$ (Exact solutions).

Figure 4 illustrates the behavior of the 10th approximate solution of the IVP (4.8) and (4.9) along the line $\varsigma = \chi$ and in the region $D = \{(\chi, t): -1 \leq \chi \leq 1, 0 \leq t < 1\}$ for different values of α, and at $\upsilon = 0.5$. The 10th approximate solution is plotted in Figure 4(a-c) for $\alpha = 0.6$, $\alpha = 0.8$, and $\alpha = 1$, respectively, whereas, the exact solution at $\alpha = 1$ is plotted in (d). Also, the graphics indicate consistency in the action of the solution at distinct values of α, the coordination between the exact solution and the approximate analytical solution as illustrated in Figure 4(c,d) as well as the determination of the region of convergence for the series solution, is clear.

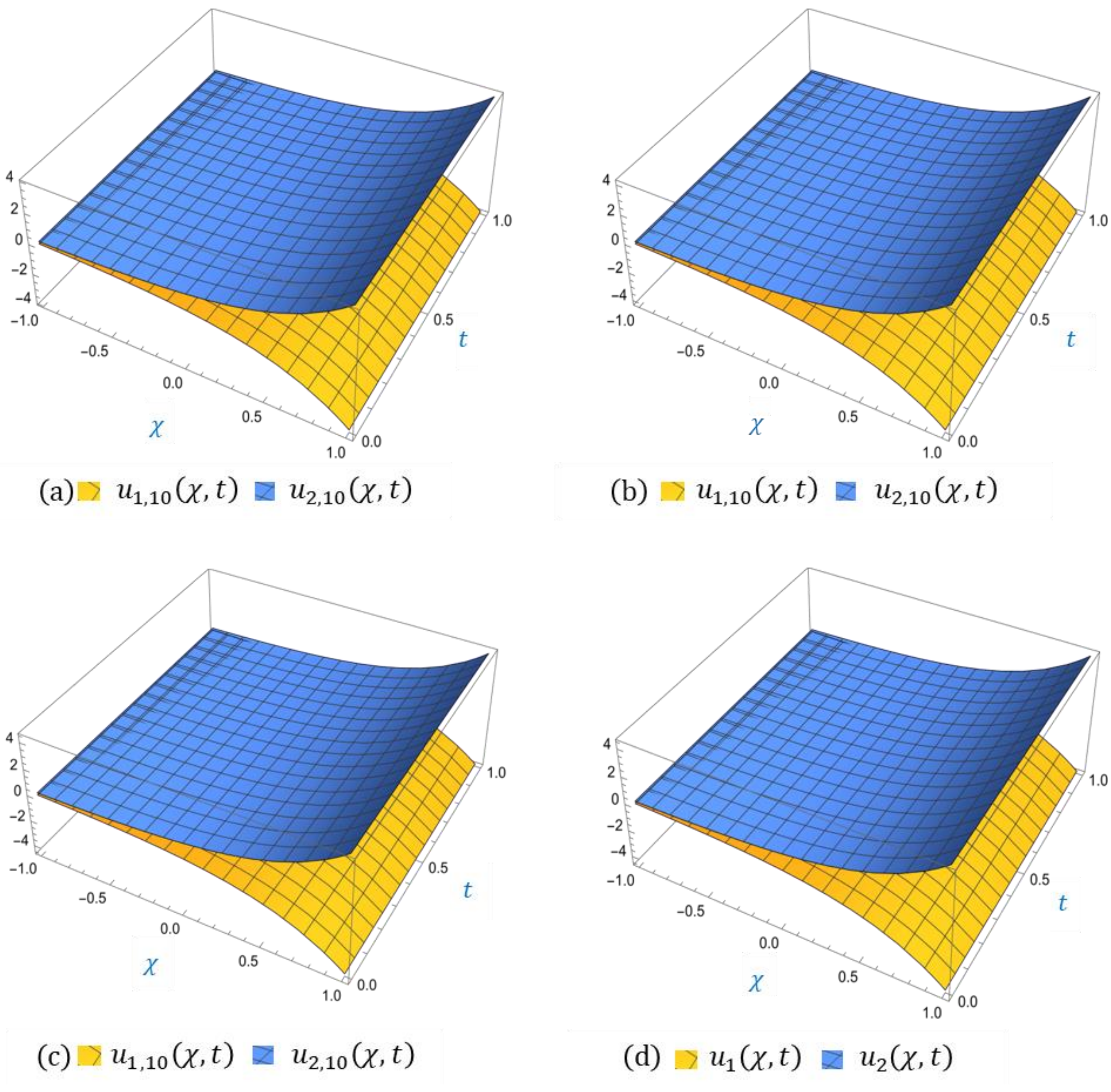

Figure 4. u_1 and u_2 along the line $\varsigma = \chi$ and at various of α and $\upsilon = 0.5$ for the problem in Example 4.2. (a) $\alpha = 0.6$, (b) $\alpha = 0.8$, (c) $\alpha = 1$, (d) $\alpha = 1$ (Exact).

Example 4.3. [36,41] Consider the following three-DT-FNSEs:

$$D_t^\alpha u_1 + u_1\frac{\partial u_1}{\partial \chi} + u_2\frac{\partial u_1}{\partial \varsigma} + u_3\frac{\partial u_1}{\partial \zeta} = \upsilon\left(\frac{\partial^2 u_1}{\partial \chi^2} + \frac{\partial^2 u_1}{\partial \varsigma^2} + \frac{\partial^2 u_1}{\partial \zeta^2}\right),$$

$$D_t^\alpha u_2 + u_1\frac{\partial u_2}{\partial \chi} + u_2\frac{\partial u_2}{\partial \varsigma} + u_3\frac{\partial u_2}{\partial \zeta} = \upsilon\left(\frac{\partial^2 u_2}{\partial \chi^2} + \frac{\partial^2 u_2}{\partial \varsigma^2} + \frac{\partial^2 u_2}{\partial \zeta^2}\right),$$

$$D_t^\alpha u_3 + u_1\frac{\partial u_3}{\partial \chi} + u_2\frac{\partial u_3}{\partial \varsigma} + u_3\frac{\partial u_3}{\partial \zeta} = \upsilon\left(\frac{\partial^2 u_3}{\partial \chi^2} + \frac{\partial^2 u_3}{\partial \varsigma^2} + \frac{\partial^2 u_3}{\partial \zeta^2}\right), \qquad (4.12)$$

where $\upsilon \in \mathbb{R}$, and with the ICs:

$$u_1(\chi,\varsigma,\zeta,0) = -0.5\chi + \varsigma + \zeta,$$
$$u_2(\chi,\varsigma,\zeta,0) = \chi - 0.5\varsigma + \zeta,$$
$$u_3(\chi,\varsigma,\zeta,0) = \chi + \varsigma - 0.5\zeta, \qquad (4.13)$$

and the exact solution, when $\alpha = 1$, is

$$u_1 = \frac{-0.5\chi + \varsigma + \zeta - 2.25\,\chi\,t}{1 - 2.25\,t^2}, u_2 = \frac{\chi - 0.5\varsigma + \zeta - 2.25\varsigma t}{1 - 2.25\,t^2}, u_3 = \frac{\chi + \varsigma - 0.5\zeta - 2.25\zeta t}{1 - 2.25\,t^2}.$$

The general formula (3.12) for the systems (4.12) and (4.13) will be as follows:

$$f_k = \upsilon\left(\frac{\partial^2 f_{k-1}}{\partial\chi^2} + \frac{\partial^2 f_{k-1}}{\partial\varsigma^2} + \frac{\partial^2 f_{k-1}}{\partial\zeta^2}\right) - \sum_{i=0}^{k-1} \frac{\Gamma((k-1)\alpha+1)\left(f_i\frac{\partial f_{k-i-1}}{\partial\chi} + h_i\frac{\partial f_{k-i-1}}{\partial\varsigma} + g_i\frac{\partial f_{k-i-1}}{\partial\zeta}\right)}{\Gamma(i\alpha+1)\Gamma\left((k-i-1)\alpha+1\right)},$$

$$h_k = \upsilon\left(\frac{\partial^2 h_{k-1}}{\partial\chi^2} + \frac{\partial^2 h_{k-1}}{\partial\varsigma^2} + \frac{\partial^2 h_{k-1}}{\partial\zeta^2}\right) - \sum_{i=0}^{k-1} \frac{\Gamma\left((k-1)\alpha+1\right)\left(f_i\frac{\partial h_{k-i-1}}{\partial\chi} + h_i\frac{\partial h_{k-i-1}}{\partial\varsigma} + g_i\frac{\partial h_{k-i-1}}{\partial\zeta}\right)}{\Gamma(i\alpha+1)\Gamma\left((k-i-1)\alpha+1\right)},$$

$$g_k = \upsilon\left(\frac{\partial^2 g_{k-1}}{\partial\chi^2} + \frac{\partial^2 g_{k-1}}{\partial\varsigma^2} + \frac{\partial^2 g_{k-1}}{\partial\zeta^2}\right) - \sum_{i=0}^{k-1} \frac{\Gamma\left((k-1)\alpha+1\right)\left(f_i\frac{\partial g_{k-i-1}}{\partial\chi} + h_i\frac{\partial g_{k-i-1}}{\partial\varsigma} + g_i\frac{\partial g_{k-i-1}}{\partial\zeta}\right)}{\Gamma(i\alpha+1)\Gamma\left((k-i-1)\alpha+1\right)}.$$

Thus, we conclude that

$f_1 = -2.25\chi,$ $\quad h_1 = -2.25\varsigma,$ $\quad g_1 = -2.25\zeta,$

$f_2 = -2.25\chi + 4.5\varsigma + 4.5\zeta,$ $\quad h_2 = 4.5\chi - 2.25\varsigma + 4.5\zeta,$ $\quad g_2 = 4.5\chi + 4.5\varsigma - 2.25\zeta,$

$$f_3 = -\chi\left(\frac{5.0625\,\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2} + 20.25\right),$$

$$h_3 = -\varsigma\left(\frac{5.0625\,\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2} + 20.25\right),$$

$$g_3 = -\zeta\left(\frac{5.0625\,\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2} + 20.25\right),$$

$$f_4 = \frac{\Gamma(2\alpha+1)(-5.0625\chi + 10.125\varsigma + 10.125\zeta)}{\Gamma(\alpha+1)^2} + \frac{\Gamma(3\alpha+1)(-10.125\chi + 20.25\varsigma + 20.25\zeta)}{\Gamma\,(\alpha+1)\,\Gamma(2\alpha+1)} -20.25\chi + 40.5\varsigma + 40.5\zeta,$$

$$h_4 = \frac{\Gamma(2\alpha+1)(10.125\chi - 5.0625\varsigma + 10.125\zeta)}{\Gamma(\alpha+1)^2} + \frac{\Gamma(3\alpha+1)(20.25\chi - 10.125\varsigma + 20.25\zeta)}{\Gamma(\alpha+1)\,\Gamma(2\alpha+1)} +40.5\chi - 20.25\varsigma + 40.5\zeta,$$

$$g_4 = \frac{\Gamma(2\alpha+1)(10.125\chi + 10.125\varsigma - 5.0625\zeta)}{\Gamma(\alpha+1)^2} + \frac{\Gamma(3\alpha+1)(20.25\chi + 20.25\varsigma - 10.125\zeta)}{\Gamma(\alpha+1)\,\Gamma(2\alpha+1)}$$

$$+40.5\chi + 40.5\varsigma - 20.25\zeta,$$

$$f_5 = \chi\begin{pmatrix} -182.25 - \dfrac{22.78125\,\Gamma(4\alpha+1)\,\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^3\,\Gamma(3\alpha+1)} - \dfrac{45.5625\,\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2} \\ + \dfrac{-\dfrac{91.125\,\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)} - \dfrac{91.125\,\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)}}{\Gamma(\alpha+1)} - \dfrac{45.5625\,\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)^2} \end{pmatrix},$$

$$h_5 = \varsigma\begin{pmatrix} -182.25 - \dfrac{22.78125\,\Gamma(4\alpha+1)\,\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^3\,\Gamma(3\alpha+1)} - \dfrac{45.5625\,\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2} \\ + \dfrac{-\dfrac{91.125\,\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)} - \dfrac{91.125\,\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)}}{\Gamma(\alpha+1)} - \dfrac{45.5625\,\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)^2} \end{pmatrix},$$

$$g_5 = \zeta\begin{pmatrix} -182.25 - \dfrac{22.78125\,\Gamma(4\alpha+1)\,\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^3\,\Gamma(3\alpha+1)} - \dfrac{45.5625\,\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2} \\ + \dfrac{-\dfrac{91.125\,\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)} - \dfrac{91.125\,\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)}}{\Gamma(\alpha+1)} - \dfrac{45.5625\,\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)^2} \end{pmatrix}.$$

So, the LRPS solution of the systems (4.12) and (4.13) has the following series form:

$$u_1(\chi,\varsigma,\zeta,t) = -0.5\chi + \varsigma + \zeta - \frac{2.25}{\Gamma(1+\alpha)}\chi t^{\alpha} + \frac{2(2.25)}{\Gamma(1+2\alpha)}(-0.5\chi + \varsigma + \zeta)t^{2\alpha}$$
$$-\frac{(2.25)^2}{\Gamma(1+3\alpha)}\left(4 + \frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}\right)\chi t^{3\alpha} + \frac{(2.25)^2}{\Gamma(1+4\alpha)}\left(8 + \frac{2\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}\right.$$
$$\left.+\frac{4\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\right)(-0.5\chi + \varsigma + \zeta)t^{4\alpha} + \cdots,$$

$$u_2(\chi,\varsigma,\zeta,t) = \chi - 0.5\varsigma + \zeta - \frac{2.25}{\Gamma(1+\alpha)}\varsigma t^{\alpha} + \frac{2(2.25)}{\Gamma(1+2\alpha)}(\chi - 0.5\varsigma + \zeta)t^{2\alpha}$$
$$-\frac{(2.25)^2}{\Gamma(1+3\alpha)}\left(4 + \frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}\right)\varsigma t^{3\alpha} + \frac{(2.25)^2}{\Gamma(1+4\alpha)}\left(8 + \frac{2\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}\right.$$
$$\left.+\frac{4\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\right)(\chi - 0.5\varsigma + \zeta)t^{4\alpha} + \cdots,$$

$$u_3(\chi,\varsigma,\zeta,t) = \chi + \varsigma - 0.5\zeta - \frac{2.25}{\Gamma(1+\alpha)}\zeta t^{\alpha} + \frac{2(2.25)}{\Gamma(1+2\alpha)}(\chi + \varsigma - 0.5\zeta)t^{2\alpha}$$
$$-\frac{(2.25)^2}{\Gamma(1+3\alpha)}\left(4 + \frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}\right)\zeta t^{3\alpha} + \frac{(2.25)^2}{\Gamma(1+4\alpha)}\left(8 + \frac{2\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}\right.$$
$$\left.+\frac{4\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\right)(\chi + \varsigma - 0.5\zeta)t^{4\alpha} + \cdots. \quad (4.14)$$

The sum of the series in Eq (4.14) at $\alpha = 1$ has the following rational forms:

$$u_1(\chi,\varsigma,\zeta,t)=\frac{-0.5\chi+\varsigma+\zeta-2.25\chi t}{1-2.25t^2},$$

$$u_2(\chi,\varsigma,\zeta,t)=\frac{\chi-0.5\varsigma+\zeta-2.25\varsigma t}{1-2.25t^2},$$

$$u_3(\chi,\varsigma,\zeta,t)=\frac{\chi+\varsigma-0.5\zeta-2.25\zeta t}{1-2.25t^2}. \tag{4.15}$$

The behavior of the velocity field of the Three-DT-FNSEs (4.12) and (4.13) is depicted in Figure 5 for various values of α at $t=0.1$ and $\zeta=3$. The 10th-truncated series of Eq (4.10) is plotted in Figure 5(a-c) for $\alpha=0.6$, $\alpha=0.8$ and $\alpha=1$, respectively, whereas, the exact solution at $\alpha=1$ is plotted in (d). The graphics indicate consistency in the behavior of the solution at various values of α, as well as the agreement of the exact solution with the approximate solution in Figure 5(c,d).

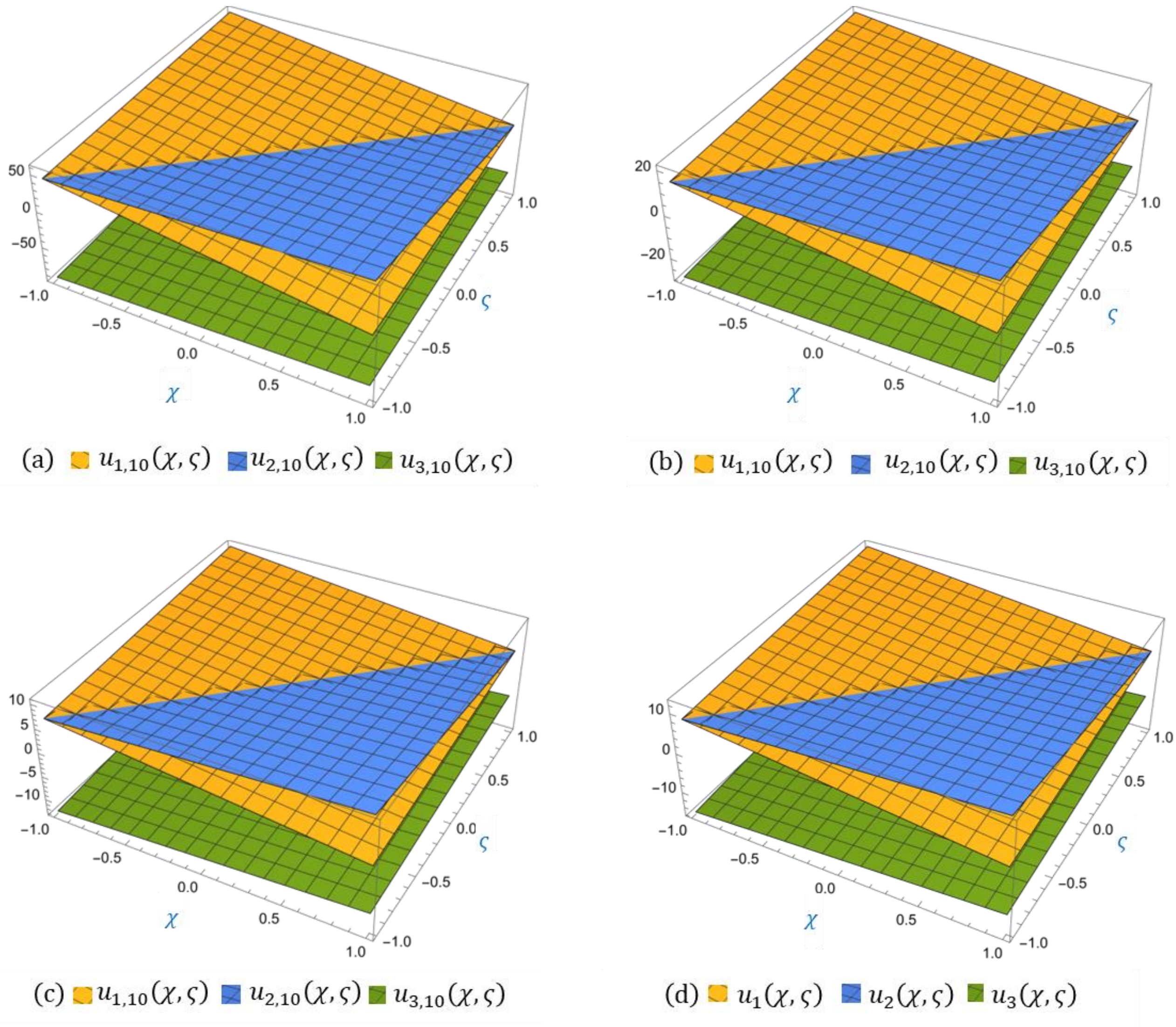

Figure 5. The 3D surfaces plot of the 10th approximate solutions of u_1, u_2, and u_3 at various values of α and $t=0.5$ & $\zeta=3$ for the problem in Example 4.3. (a) $\alpha=0.6$, (b) $\alpha=0.8$, (c) $\alpha=1$, (d) $\alpha=1$ (Exact solutions).

5. Mathematical reviews

This article presents the LRPSM in a new scheme. We proposed the method and used it to solve the M-DT-FNS system. In the following, we state the advantages of using the presented method and the disadvantages of treating the M-DT-FNS.

5.1 Advantages of the method

1) The method is simple to apply to solve linear and non-linear FPDE compared to other techniques, other power series methods are based on finding derivatives and the calculations are usually complex, but LRPSM mainly depends on computing the limit at infinity which is much easier.
2) The proposed method is applicable in finding approximate solutions for physical applications, and in finding many terms of the analytical series solutions.
3) The method is accurate and gives approximate solutions close to the exact ones.

5.2 Disadvantages of the method

LRPSM needs first to find the LT of the target equations and finally to run the inverse LT to obtain the solution in the original space. So, if we have nonhomogeneous equations, the source functions need to be piecewise continuous and of exponential order, and after the computations, the inverse LT must exist.

6. Conclusions

In this article, we have introduced the LRPSM in a new scheme and simplified the technique to present series solutions for the M-DT-FNS system in the sense of the Caputo derivative. It is worth noting here that we obtained a general formula for an analytic solution of M-DT-FN, which other researchers have not previously obtained by other methods. We tested three examples by solving them in the proposed technique and then analyzing the results. In the future, we will use LRPSM to solve more problems and make new modifications to address the flaws of the presented technique.

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Conflict of interest

The authors declare no conflicts of interest.

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