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Research article

Numerical study of a nonlinear fractional chaotic Chua's circuit

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Abstract: As an exponentially growing sensitivity to modest perturbations, chaos is pervasive in nature. Chaos is expected to provide a variety of functional purposes in both technological and biological systems. This work applies the time-fractional Caputo and Caputo-Fabrizio fractional derivatives to the Chua type nonlinear chaotic systems. A numerical analysis of the mathematical models is used to compare the chaotic behavior of systems with differential operators of integer order versus systems with fractional differential operators. Even though the chaotic behavior of the classical Chua's circuit has been extensively investigated, our generalization can highlight new aspects of system behavior and the effects of memory on the evolution of the chaotic generalized circuit.

Keywords: Chaos; nonlinear chaotic systems; Caputo and Caputo-Fabrizio fractional derivatives; chaotic generalized circuit
Mathematics Subject Classification: 35R11, 35K23

1. Introduction

Architects, astronomers, and statisticians all find nonlinear models to be incredibly intriguing since so many real-world physical structures are fundamentally nonlinear. Nonlinear equations provide access to intriguing phenomena like Chaos while being challenging to investigate. In fact, even simple nonlinear dynamical frameworks can exhibit peculiar behavior, the so-called deterministic Chaos. Because chaos may also exist inside irrelevant arrangements, the chaos theory has been so astounding. We must now acknowledge that “Chaos” isn't really described. The most common definition of disordered elements is those elements that are begun by regular dynamical circumstances but have comparative or undefinable orientations from certain stochastic measures [1,2].

In the past 30 years, there has been a significant increase in the interesting nonlinear phenomenon known as chaos. It is useful or has extraordinary promise in a wide range of domains, including biomedical design, secure correspondence, information encryption, and stream components [Chen and Yu, 2003]. Complex dynamical practices in a disordered framework have unusual features, such as an extreme affectivity to minute variations in starting circumstances or constrained phase space directions. Despite this fact, synchronization and management of disorganized frameworks have attracted a wide range of research throughout time [3–5].

Notably, Chaos cannot occur continuously inside all-out orders of three or less. This attestation is predicated on conventional notions of order, such as a few states in a framework or the full spectrum of various separations or reconciliations in the framework. In fact, three differential conditions incorporating the non-numerical derivative may be used to classify the framework configuration. To comprehend this fact, we might investigate the dynamical model of partial order in the framework. The incomplete order was offered by Hartley et al. [6]. Partial frameworks were first introduced in [7] together with Chua's framework, the fragmentary order cell brain system, and the order turbulence frameworks were shown in several other studies (for example [8–16]).

Each of these instances saw the Chaos being presented in a framework with fewer than three full queries. This idea gave rise to Chua's framework, which created a further jumbled request. Additionally, the phrase “framework request” should be mentioned. If we consider the PDEs, the framework order is not equal to the number of differential equations. The framework order equates to a most elevated offshoot of the numerical model's fragmented differential equation.

Then again, fractional calculus, as speculation of integral order integration and differentiation to its non-integer (fragmentary) order partner, has ended up being an important device in the demonstrating of numerous physical marvels [17,18]. This numerical phenomenon permits to depiction of a genuine article more precisely than the old-style whole number techniques. Fractional derivatives give an incredible instrument to portraying frameworks with long haul memory [19–22], nonlocal spatial [23], and fractal properties [24]. The focal points of the fragmentary request frameworks are given degrees of opportunity in the model, and a “memory” is remembered for the model [25].

Consideration of fragmented order frameworks has recently emerged as an active research area. Fractional order frameworks' perplexing components have recently attracted a lot of attention. As hypotheses of several significant frameworks, it has been shown that incomplete request frameworks may also function loudly [26,27]. It has also been mentioned that some fragmentary request frameworks can offer confused attractors with an order under 3. Furthermore, other studies demonstrate that chaotic fragmented order frameworks may also be synced [28–30].

Due to its successful usage in several scientific disciplines, including statistics, applied mathematics, dynamics, mathematical biology, control theory, optimization, and chaos theory, fractional analysis has recently gained appeal as a topic of study. New derivative and integral operators are being defined at a rapid rate in fractional analysis. Kao et al. [31] studied Mittag-Leffler synchronization of delayed fractional memristor neural networks via adaptive control while Li et al. [32] investigated Mittag-Leffler stability of fractional-order nonlinear differential systems with state-dependent delays. Using the ideas of fractional derivatives, several new findings have been put out by scholars in many domains [33–37].

The scope of this paper is to apply the time-fractional Caputo and Caputo-Fabrizio fractional derivatives to the nonlinear chaotic systems of the Chua type. A comparison between the chaotic behavior of systems with differential operators of integer order and systems with fractional differential operators is carried out by using a numerical study of the mathematical models. Even if the classical Chua's circuit with chaotic behavior is largely studied, our generalization can highlight new aspects of system behavior and the effects of memory on the evolution of the chaotic generalized circuit.

2. A brief summary of Caputo and Caputo-Fabrizio derivatives

Fractional calculus generalizes the integration and differentiation operator to non-integer order operators.

2.1. Caputo-derivative

Definition 1. Function $k:[0,\ \infty]\ \times[0,\ 1]\ \to\ R$,

$$k(t,\alpha)=\begin{cases}\dfrac{t^{-\alpha}}{\Gamma(1-\alpha)}; & 0\le\alpha<1,\\ \delta(t); & \alpha=1,\end{cases} \tag{1}$$

where $\Gamma(\cdot)$ is Euler's integral of the second kind and $\delta(\cdot)$ is Dirac's distribution is called Caputo kernel. It is easy to see that the function from the LT $k(t,\alpha)$ is given by

$$\left(L\left\{k(t,\alpha)\right\}\right)(s)=\overline{k}(s,\alpha)=\int_0^\infty k(t,\alpha)e^{-st}dt=s^{\alpha-1};\ \alpha\in[0,1], \tag{2}$$

where LT= Laplace transform.

Definition 2. For a differentiable function, $f(t),\ t\in[0,T],\ T>0,$ the Caputo derivative of order α is defined by the following operator

$$\left({}^{C}D_t^{\alpha}f\right)(t)=k(t,\alpha)*\dot{f}(t)=\int_0^t k(t-\tau,\alpha)\dot{f}(\tau)d\tau, \tag{3}$$

where $\dot{f}(t)=\dfrac{df(t)}{dt}.$

Using the properties of LT and Eqs (2) and (3), the LT of Cd is

$$L\left\{\left({}^{C}D_t^{\alpha} f\right)(t)\right\} = \left(L\{k(t,\alpha)\}\right)(s)\left(L\left\{\dot{f}(t)\right\}\right)(s) = s^{\alpha-1}[s\overline{f}(s) - f(0)] = s^{\alpha}\overline{f}(s) - s^{\alpha-1}f(0). \quad (4)$$

Definition 3. Function $p:[0,\infty]\times[0,1]\to R,$

$$p(t,\alpha) = \begin{cases} \delta(t); & \alpha = 0, \\ \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}; & 0 < \alpha \le 1, \end{cases}$$

is called Riemann-Lioville kernel.

Definition 4. For an inferable function $f(t),\ t\in[0,T],\ T>0,$ the Riemann Liouville fractional integral operator of order α is defined as

$$\left(J_t^{\alpha} f\right)(t) = p(t,\alpha) * f(t) = \int_0^t p(t-\tau,\alpha) f(\tau)d\tau. \quad (5)$$

Hence, LT of (5) is

$$L\left\{\left(J_t^{\alpha} f\right)(t)\right\} = s^{-\alpha}\overline{f}(s). \quad (6)$$

The following properties are useful.

Properties:

(1) $\left({}^{C}D_t^{0} f\right)(t) = k(t,0) * \dot{f}(t) = \int_0^t \dot{f}(\tau)d\tau = f(t) - f(0),$ (7)

(2) $\left({}^{C}D_t^{1} f\right)(t) = k(t,1) * \dot{f}(t) = \delta(t) * \dot{f}(t) = \dfrac{df(t)}{dt},$ (8)

(3) $\left(J_t^{0} f\right)(t) = p(t,0) * f(t) = \delta(t) * f(t) = f(t),$ (9)

(4) $\left(J_t^{1} f\right)(t) = p(t,1) * f(t) = \int_0^t f(\tau)d\tau,$ (10)

(5) $\left({}^{C}D_t^{\alpha} J_t^{\alpha} f\right)(t) = f(t);\ \left(J_t^{\alpha}\,{}^{C}D_t^{\alpha} f\right)(t) = f(t) - f(0).$ (11)

Proof:

$$L\left\{\left({}^{C}D_t^{\alpha} J_t^{\alpha} f\right)(t)\right\} = s^{\alpha}L\left\{\left(J_t^{\alpha} f\right)(t)\right\} - s^{\alpha-1}\left(J_t^{\alpha} f\right)(0) = s^{\alpha}L\left\{\left(J_t^{\alpha} f\right)(t)\right\} = s^{\alpha}s^{-\alpha}\overline{f}(s) = L\{f(t)\}.$$

Applying the inverse operator L^{-1}, we obtain $\left({}^{C}D_t^{\alpha} J_t^{\alpha} f\right)(t) = f(t).$

In the same way, the second relation (11) is obtained.

2.2. Caputo-Fabrizio derivative

Definition 5. Function $w:[0,\infty]\times[0,1]\to R,$

$$w(t,\alpha)=\begin{cases}\frac{1}{1-\alpha}e^{-\frac{\alpha}{1-\alpha}}; & 0\le\alpha<1,\\ \delta(t); & \alpha=1,\end{cases} \tag{12}$$

is called Caputo-Fabrizio kernel.

The LT of function $w(t,\alpha)$ is

$$L\{w(t,\alpha)\}=\bar{w}(s,\alpha)=\frac{1}{(1-\alpha)s+\alpha},\ \alpha\in[0,1]. \tag{13}$$

Definition 6. For a differentiable function $f(t),\ t\in[0,T],\ T>0,$ Caputo-Fabrizio derivative of order α is given by

$$\left({}^{CF}D_t^{\alpha}f\right)(t)=w(t,\alpha)*\dot{f}(t)=\int_0^t w(t-\tau,\alpha)\dot{f}(\tau)d\tau, \tag{14}$$

from (13) and (14) it is obtained the LT of Caputo-Fabrizio derivative

$$L\left\{\left({}^{CF}D_t^{\alpha}f\right)(t)\right\}=\left(L\{w(t,\alpha)\}\right)(s)\left(L\left\{\dot{f}(t)\right\}\right)(s)=\frac{s\bar{f}(s)-f(0)}{(1-\alpha)s+\alpha}. \tag{15}$$

The fractional integral operator of order $\alpha\in[0,1]$ associated with the CFD (14) is

$$\left(I_t^{\alpha}f\right)(t)=(1-\alpha)f(t)+\alpha\int_0^t f(\tau)d\tau. \tag{16}$$

The LT of the operator (16) is

$$L\left\{\left(I_t^{\alpha}f\right)(t)\right\}=\frac{(1-\alpha)s+\alpha}{s}\bar{f}(s). \tag{17}$$

Operators defined by (14) and (16) have the listed features:

(1) $\left({}^{CF}D_t^{0}f\right)(t)=1*\dot{f}(t)=f(t)-f(0),$ (18)

(2) $\left({}^{CF}D_t^{1}f\right)(t)=w(t,1)*\dot{f}(t)=\delta(t)*\dot{f}(t)=\frac{df(t)}{dt},$ (19)

(3) $\left(I_t^0 f\right)(t) = f(t);\ \left(I_t^1 f\right)(t) = \int\limits_0^t f(\tau)d\tau,$ (20)

(4) $\left(I_t^{\alpha}\,{}^{CF}D_t^{\alpha} f\right)(t) = f(t) - f(0).$ (21)

Proof:

$$L\left\{\left(I_t^{\alpha}\,{}^{CF}D_t^{\alpha} f\right)(t)\right\} = \frac{(1-\alpha)s+\alpha}{s} L\left\{\left({}^{CF}D_t^{\alpha} f\right)(t)\right\} = \frac{(1-\alpha)s+\alpha}{s}\frac{s\overline{f}(s) - f(0)}{(1-\alpha)s+\alpha} = \overline{f}(s) - \frac{1}{s}f(0) = L\left\{f(t) - f(0)\right\}.$$

Applying the inverse Laplace transform, we have (21).

2.3. Numerical approximations of fractional operators

For this section, we give the approximate formulas for the operators defined in previous sections. For the domain of time, $[0,T],\ T>0,$ we consider a uniform discretization by points $t_n = nh,\quad n = 0,1,...,N,$ here $h = \dfrac{T}{N}$ is the step-size of time discretization. For $\tau \in [t_j, t_{j+1}],$ $j = 0,1,...,N-1$, the following approximations will be accepted.

$$f(\tau) = \frac{1}{2}\left[f(t_{j+1}) + f(t_j)\right],\ \dot{f}(\tau) = \frac{1}{h}\left[f(t_{j+1}) - f(t_j)\right]. \tag{22}$$

a) Fractional Caputo derivative

$$\begin{aligned}\left({}^{C}D_t^{\alpha} f\right)(t_n) &= \frac{1}{\Gamma(1-\alpha)}\int\limits_0^{t_n}(t_n-\tau)^{-\alpha}\dot{f}(\tau)d\tau = \sum_{j=0}^{n-1}\int\limits_{t_j}^{t_{j+1}}\frac{(t_n-\tau)^{-\alpha}\dot{f}(\tau)}{\Gamma(1-\alpha)}d\tau = \\ &= \sum_{j=0}^{n-1}\frac{\left[f(t_{j+1}) - f(t_j)\right]}{h\Gamma(1-\alpha)}\int\limits_{t_j}^{t_{j+1}}(t_n-\tau)^{-\alpha}d\tau = \sum_{j=0}^{n-1}a_{nj}\left[f_{j+1} - f_j\right],\end{aligned} \tag{23}$$

where

$$f_j = f(t_j),\ a_{nj} = \frac{h^{-\alpha}}{\Gamma(2-\alpha)}\left[(n-j)^{1-\alpha} - (n-j-)^{1-\alpha}\right],\ {}_{\substack{n=1,2,...,N,\\ j=0,1,...,n-1,}}\ \alpha \in (0,1). \tag{24}$$

Similarly, we obtain

b) Fractional integral Riemann-Liouville operator

$$\left(J_t^{\alpha} f\right)(t_n) \simeq \sum_{j=0}^{n-1} b_{nj}\left[f_{j+1} + f_j\right], \tag{25}$$

where

$$b_{nj} = \frac{h^{\alpha}}{2\Gamma(1+\alpha)}\left[(n-j)^{\alpha} - (n-j-1)^{\alpha}\right],\ {}_{j=0,1,\dots,n-1,}^{n=1,2,\dots,N,}\ \alpha \in (0,1).$$

c) Fractional Caputo- Fabrizio derivative

$$\left(J_t^{\alpha} f\right)(t_n) \simeq \sum_{j=0}^{n-1} c_{nj}\left[f_{j+1} - f_j\right], \tag{26}$$

where

$$c_{nj} = \frac{1}{\alpha h}\left[\exp\left(\frac{\alpha h}{\alpha - 1}(n-j-1)\right) - \exp\left(\frac{\alpha h}{\alpha - 1}(n-j)\right)\right],\ {}_{j=0,1,\dots,n-1,}^{n=1,2,\dots,N,}\ \alpha \in (0,1).$$

d) Fractional integral operator associated with Caputo-Fabrizio derivative

$$\left(I_t^{\alpha} f\right)(t_n) \simeq (1-\alpha) f_0 + \sum_{j=0}^{n-1}\left[\frac{2(1-\alpha)+\alpha h}{2} f_{j+1} - \frac{2(1-\alpha)-\alpha h}{2} f_j\right];\ n = 1,2,\dots,N_1,\ \alpha \in [0,1]. \tag{27}$$

3. Generalized Chua's systems

Classical Chua's oscillator is an electronic circuit that can exhibit nonlinear dynamical phenomena such Chaos. Such a circuit is presented in Figure 1, where C_1, C_2 are the capacitors L_1 is the introduced coil and (NR) is the nonlinear resistor [38]
The equations give the mathematical model

$$\left.\begin{aligned}
\frac{dV_1(t_1)}{dt_1} &= \frac{1}{C_1}\left[GV_2(t_1) - GV_1(t_1) - \psi(V_1(t_1))\right],\\
\frac{dV_2(t_1)}{dt_1} &= \frac{1}{C_2}\left[GV_1(t_1) - GV_2(t_1) - I(t_1)\right],\\
\frac{dI(t_1)}{dt_1} &= \frac{1}{L_1}\left[-V_2(t_1) - R_L I(t_1)\right],
\end{aligned}\right\} \tag{28}$$

where $G = \frac{1}{R_2}$,
$I(t_1)$ = current
$V_1(t_1),\ V_2(t_1)$ = voltages
capacitors $= C_1$ and C_2,
and $\psi(V_1(t_1))$ = nonlinear resistor as shown in Figure 2,
and also described as

$$\psi(V_1(t_1)) = G_b V_1(t_1) + \frac{1}{2}(G_a - G_b)\left(\left|V_1(t_1) + B_p\right| - \left|V_1(t_1) - B_p\right|\right), \tag{29}$$

where $G_a < 0, G_b < 0$ are appropriate constants and B_p is the breakpoint voltage of the diode.

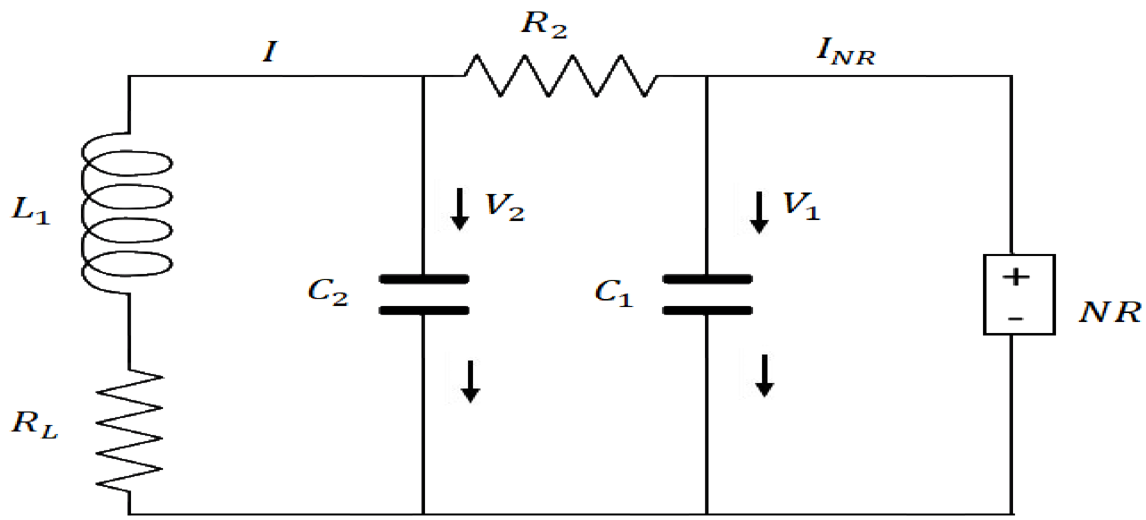


Figure 1. Graphically abstract.

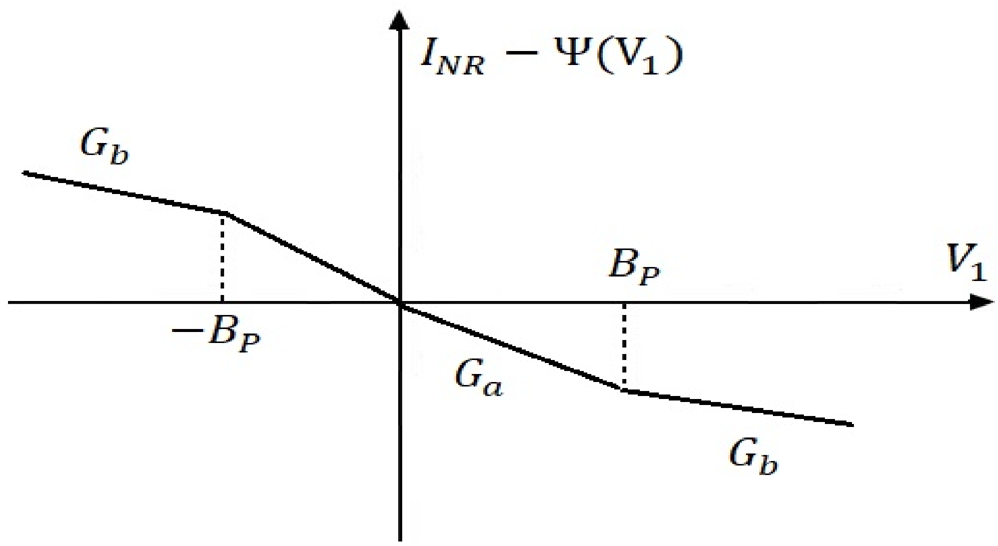


Figure 2. Graphically abstract.

Introducing the non-dimensional variables and functions

$$X_1 = \frac{V_1}{B_p},\ X_2 = \frac{V_2}{B_p},\ X_3 = \frac{I}{B_p G},\ t = \frac{Gt_1}{C_2},\ m_0 = \frac{G_a}{G},\ m_1 = \frac{G_b}{G},$$
$$\alpha = \frac{C_2}{C_1},\ \beta = \frac{C_2}{L_1 G^2}\ \gamma = \frac{C_2 R_L}{L_1 G}. \tag{30}$$

Equstions (28) and (29) become

$$\left.\begin{aligned}\frac{dX_1(t)}{dt}&=-X_1(t)+\alpha X_2(t)-\alpha\varphi(X_1(t)),\\ \frac{dX_2(t)}{dt}&=X_1(t)-X_2(t)+X_3(t),\\ \frac{dX_3(t)}{dt}&=-\beta X_2(t)-\gamma X_3(t),\end{aligned}\right\}\tag{31}$$

where

$$\frac{\psi(V_1(t))}{B_pG}=\varphi(X_1(t))=m_1X_1(t)+\frac{1}{2}(m_0-m_1)\left(\left|X_1(t)+1\right|-\left|X_1(t)-1\right|\right).\tag{32}$$

3.1. Fractional-order Chua's system with power-law kernel of memory

This section considers Chua's generalized system highlighted by the fractional differential equations with time-fractional Caputo derivative.

Such type systems are given by

$${}^CD_t^{\varepsilon}X_1(t)=-\alpha X_1(t)+\alpha X_2(t)-\alpha\varphi(X_1(t)),\tag{33}$$

$${}^CD_t^{\varepsilon}X_2(t)=X_1(t)-X_2(t)+X_3(t),\tag{34}$$

$${}^CD_t^{\varepsilon}X_3(t)=-\beta X_2(t)-\gamma X_3(t),\quad 0<\varepsilon\le 1.\tag{35}$$

To determine numerical solutions of the system (33)–(35) with the original conditions

$$X_1(0)=X_1^0,\quad X_2(0)=X_2^0,\ X_3(0)=X_3^0,\tag{36}$$

we will use numerical approximations given in the previous sections.

Multiplying Eq (34) by γ and adding by (35), we get

$$\gamma\,{}^CD_t^{\varepsilon}X_2(t)+{}^CD_t^{\varepsilon}X_3(t)=\gamma X_1(t)-(\beta+\gamma)X_2(t).\tag{37}$$

Applying the fractional integral operator $J_t^{\varepsilon}(\cdot)$ to Eq (37) and using the property (11), we obtain

$$X_3(t)=\gamma X_2^0+X_3^0-\gamma X_2(t)+J_t^{\varepsilon}X_1(t)-(\beta+\gamma)J_t^{\varepsilon}X_2(t).\tag{38}$$

Replacing Eq (38) into (34), we obtain the following system for the unknown functions $X_1(t)$ and $X_2(t)$:

$$ {}^{C}D_t^{\varepsilon} X_1(t) = -\alpha X_1(t) + \alpha X_2(t) - \alpha \varphi(X_1(t)), \tag{39} $$

$$ {}^{C}D_t^{\varepsilon} X_2(t) = X_1(t) - X_2(t) + \gamma X_2^0 + X_3^0 - \gamma X_2(t) + \gamma J_t^{\varepsilon} X_1(t) - (\beta+\gamma) J_t^{\varepsilon} X_2(t). \tag{40} $$

Let T be a positive number and $t \in [0,T]$. We consider a uniform discretization of the interval $[0,T]$ with the points $t_n = nh,\ n = 0,1,2,...,N,\ h = \frac{T}{N}.$

For a function $\phi(t),$ values into discretization points t_n will be denoted by $\phi(t_n) = \phi^n.$

Applying the operator $J_t^{\varepsilon}(\cdot)$ to Eq (39), we get

$$ X_1(t) - X_1^0 = -\alpha J_t^{\varepsilon} X_1(t) + \alpha J_t^{\varepsilon} X_2(t) - \alpha J_t^{\varepsilon} \varphi(X_1(t)). \tag{41} $$

For the last term, we have

$$ \begin{aligned} J_t^{\varepsilon}\varphi(X_1(t))\Big|_{t=t_n} &= \frac{1}{\Gamma(\varepsilon)}\int_0^{t_n}(t_n-\tau)^{\varepsilon-1}\varphi(X_1(\tau))d\tau = \frac{1}{\Gamma(\varepsilon)}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}(t-\tau)^{\varepsilon-1}\varphi(X_1(\tau))d\tau \\ &\simeq \sum_{j=0}^{n-1}\frac{\varphi(X_1(t_j))}{\Gamma(\varepsilon)}\int_{t_j}^{t_{j+1}}(t-\tau)^{\varepsilon-1}d\tau = \sum_{j=0}^{n-1}2b_{nj}\varphi(X_1^j). \end{aligned} \tag{42} $$

Using the approximate formulas (23), (25) and (42), Eqs (40) and (41) become

$$ X_1^n - X_1^0 = -\alpha\sum_{j=0}^{n-1}b_{nj}\left(X_1^{j+1}+X_1^j\right) + \alpha\sum_{j=0}^{n-1}b_{nj}\left(X_2^{j+1}+X_2^j\right) - \alpha\sum_{j=0}^{n-1}2b_{nj}\varphi(X_1^j), \tag{43} $$

$$ \begin{aligned} \sum_{j=0}^{n-1}a_{nj}\left(X_2^{j+1}+X_2^j\right) &= X_1^n - X_2^n + \gamma X_2^0 + X_3^0 - \gamma X_2^n + \gamma\sum_{j=0}^{n-1}b_{nj}\left(X_1^{j+1}+X_1^j\right) \\ &\quad -(\beta+\gamma)\sum_{j=0}^{n-1}b_{nj}\left(X_2^{j+1}+X_2^j\right),\ n = 1,2,...,N-1, \end{aligned} \tag{44} $$

where

$$ a_{nj} = \frac{h^{-\varepsilon}}{\Gamma(2-\varepsilon)}\left[(n-j)^{1-\varepsilon} - (n-j-1)^{1-\varepsilon}\right];\ b_{nj} = \frac{h^{\varepsilon}}{2\Gamma(1+\varepsilon)}\left[(n-j)^{\varepsilon} - (n-j-1)^{\varepsilon}\right]. \tag{45} $$

Making $n=1$ and $j=0$ into Eqs (43) and (44), we obtain the system

$$ (1+\alpha b_{10})X_1^1 - \alpha b_{10}X_2^1 = (1+\alpha b_{10})X_1^0 - \alpha b_{10}X_2^0 - 2\alpha b_{10}\varphi(X_1^0) = P_1, $$

$$ (1+\gamma b_{10})X_1^1 - \left(1+\gamma+a_{10}+(\beta+\gamma)b_{10}\right)X_2^1 = -a_{10}X_2^0 - \gamma X_2^0 - X_3^0 - \gamma b_{10}X_1^0 + (\beta+\gamma)b_{10}X_2^0 = Q_1, $$

with the solution

$$X_1(t_1) = X_1^1 = \frac{-\alpha b_{10} Q_1 + P_1[1+\gamma + a_{10} + (\beta+\gamma) b_{10}]}{a_{10}(1+\alpha b_{10}) + (\beta + \gamma - \alpha\gamma + \alpha\beta b_{10}) b_{10} - 1 - \gamma}, \tag{46}$$

$$X_2(t_1) = X_2^1 = \frac{-(1+\alpha b_{10}) Q_1 + (1+\gamma b_{10}) P_1}{a_{10}(1+\alpha b_{10}) + (\beta + \gamma + \alpha\gamma + \alpha\beta b_{10}) b_{10} + 1 + \gamma}. \tag{47}$$

The systems (43) and (44) can be written in the equivalent form

$$(1+\alpha b_{nn-1}) X_1^n - \alpha b_{nn-1} X_2^n = P_n, \; n = 2,3,\ldots,N, \tag{48}$$

$$\left(1+\gamma b_{nn-1}\right) X_1^n - \left(1+\gamma + a_{nn-1} + (\beta+\gamma) b_{nn-1}\right) X_2^n = Q_n, \; n = 2,3,\ldots,N, \tag{49}$$

where

$$P_n = \alpha \sum_{j=0}^{n-2} b_{nj} \left[\left(X_2^{j+1} + X_2^j \right) - \left(X_1^{j+1} + X_1^j \right) \right] + \alpha b_{nn-1} \left(X_2^{n-1} - X_1^{n-1} \right) - 2\alpha \sum_{j=0}^{n-1} 2 b_{nj} \varphi(X_1^j), \tag{50}$$

$$\begin{aligned} Q_n = {} & \sum_{j=0}^{n-2} a_{nj} \left(X_2^{j+1} - X_2^j \right) + \left[(\beta+\gamma) b_{nn-1} - a_{nn-1} \right] X_2^{n-1} - \gamma \sum_{j=0}^{n-2} b_{nj} \left(X_1^{j+1} + X_1^j \right) - \gamma b_{nn-1} X_1^{n-1} \\ & + (\beta+\gamma) \sum_{j=0}^{n-2} b_{nj} \left(X_2^{j+1} + X_2^j \right) - \gamma X_2^0 - X_3^0. \end{aligned} \tag{51}$$

Now, we obtain

$$X_1^n = \frac{\left[1+\gamma + a_{nn-1} + (\beta+\gamma) b_{nn-1}\right] P_n - \alpha b_{nn-1} Q_n}{1+\gamma + a_{nn-1} + (\beta + \gamma + \alpha\gamma + \alpha a_{nn-1} + \alpha\beta b_{nn-1}) b_{nn-1}}, \; n = 2,3,\ldots,N, \tag{52}$$

$$X_2^n = \frac{\left(1+\gamma b_{nn-1}\right) P_n - \left(1+\alpha b_{nn-1}\right) Q_n}{1+\gamma + a_{nn-1} + (\beta + \gamma + \alpha\gamma + \alpha a_{nn-1} + \alpha\beta b_{nn-1}) b_{nn-1}}, \; n = 2,3,\ldots,N. \tag{53}$$

4. Numerical discussions and conclusion

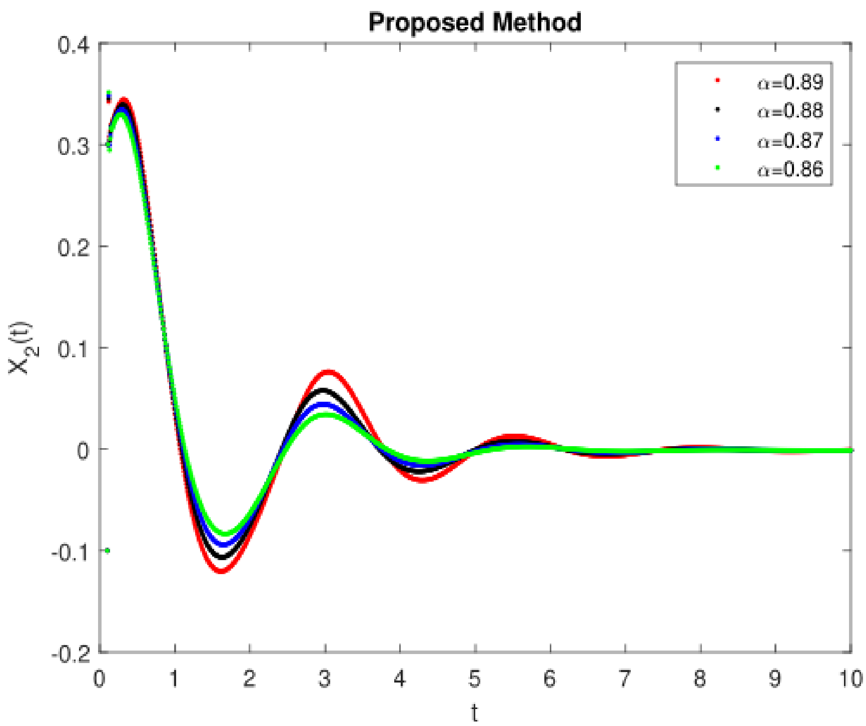
Under this part, we report the remedies of the proposed configuration generated from the nonlinear chaotic systems of the Chua type using the numerical methodology with time fractional-Caputo, Caputo-Fabrizio fractional derivatives and Laplace transform method in Section 2.

This is achieved for different fractional values of α. In this computation, we considered the sequential available parameters $a = 9.5, b = 0.15, c = -0.3, \beta = 14, \alpha = 0.98, \gamma = 0.02$ in the Caputo-Fabrizio fractional version of Chua's cubic dynamical system. Where Chua's cubic dynamical system is represented as

$$\left.\begin{aligned}\dot{x} &= a(y+bx+cx^3)\\ \dot{y} &= x-y+z\\ \dot{z} &= -\beta y+\gamma z\end{aligned}\right\} \tag{54}$$

The numerical simulations are depicted in Figures 3 for α variation versus time parameter t. It is observed that the changes in proposed models are deducting but considering the different values of fractional parameter. Initially as well as after lager time the influence are not deducted. The Chaotic behaviour of α is observed.

Figures 3–11 show numerical simulation results. The Figures 4–7, $\alpha=1$ when $a=9.5, b=0.15, c=-0.3, \beta=14, \gamma=0.02$. In this case, from figure 4, it is clear that the Caputo-Fabrizio fractional version of the equation of (54) is chaotic due to the tendency to replicate the scroll attractor or many chaotic cycles for a two-dimensional portrait for the x-axis and y-axis. In Figure 5 the two-dimensional portrait for the x-axis and z-axis is demonstrated as a diagonal chaotic behaviour with the tendency to replicate the scroll attractor. Similarly, Figure 6 shows the chaotic behaviour of the portrait with a scroll attractor of several chaotic cycles. The three-dimensional portrait for the x-axis, y-axis and z-axis in Figure 7 portrays the highly chaotic behaviour of the portrait when the Caputo-Fabrizio fractional order of Chua's cubic dynamical system is used, and the validation is seen in Figure 5. The numerical simulations of $X_1(t), X_2(t)$, $X_1(t), X_3(t)$, $X_2(t), X_3(t)$ and $X_1(t), X_2(t), X_3(t)$ are presented in Figures 8–11.



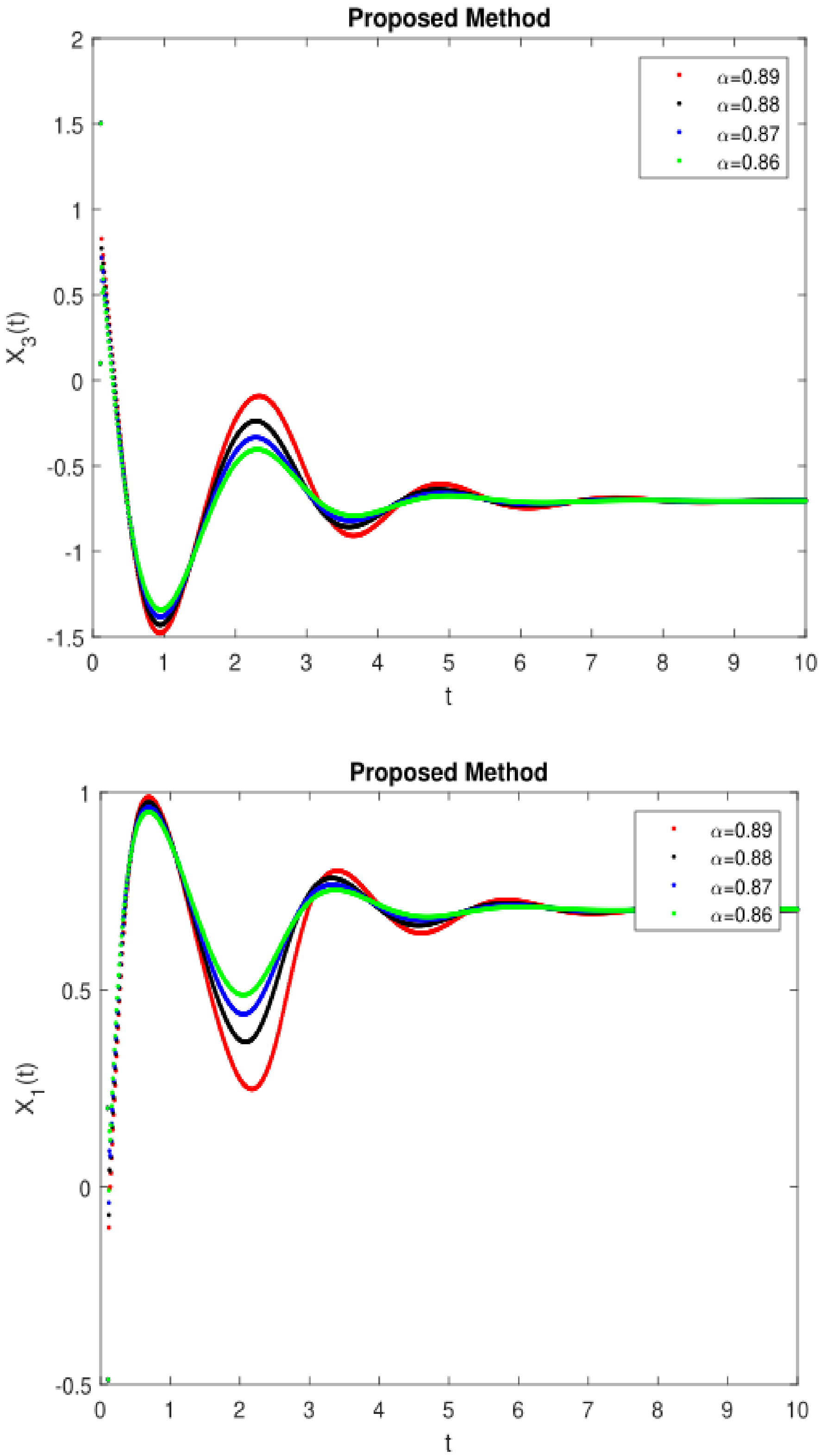


Figure 3. Numerical simulations of $X_1(t)$, $X_2(t)$, $X_3(t)$ for α variation.

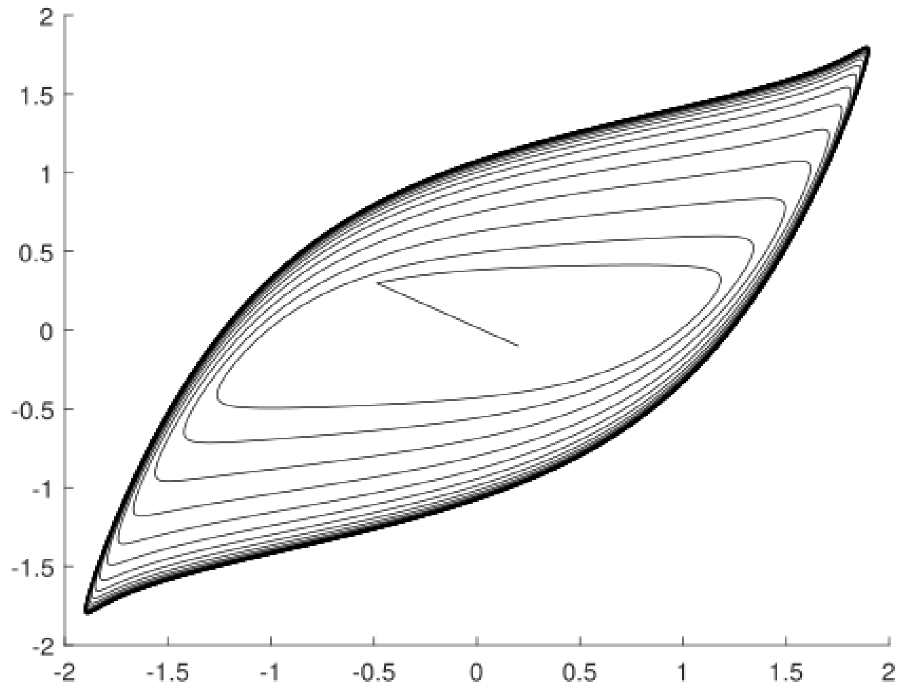


Figure 4. xy phase plane projection.

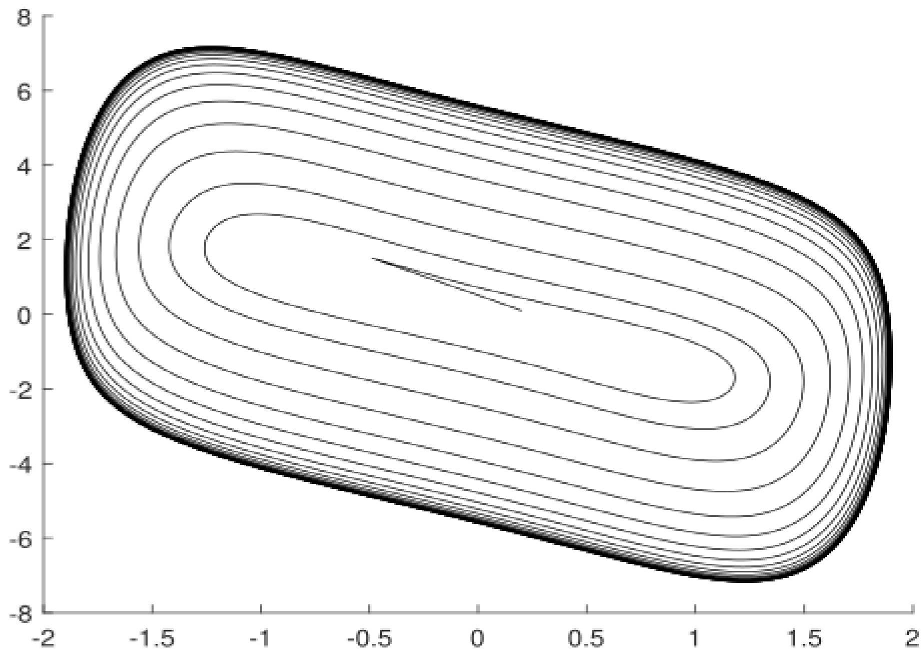


Figure 5. xz phase plane projection.

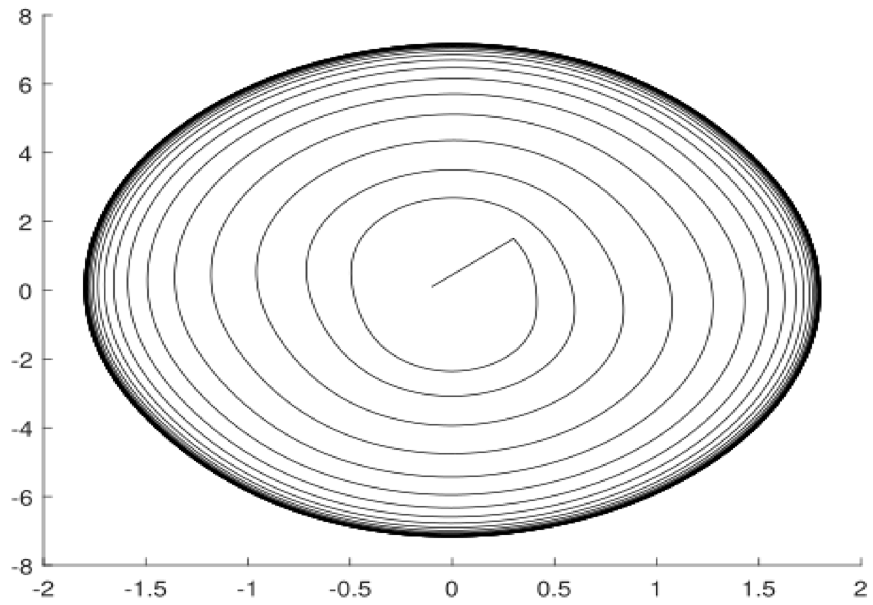


Figure 6. yz phase plane projection.

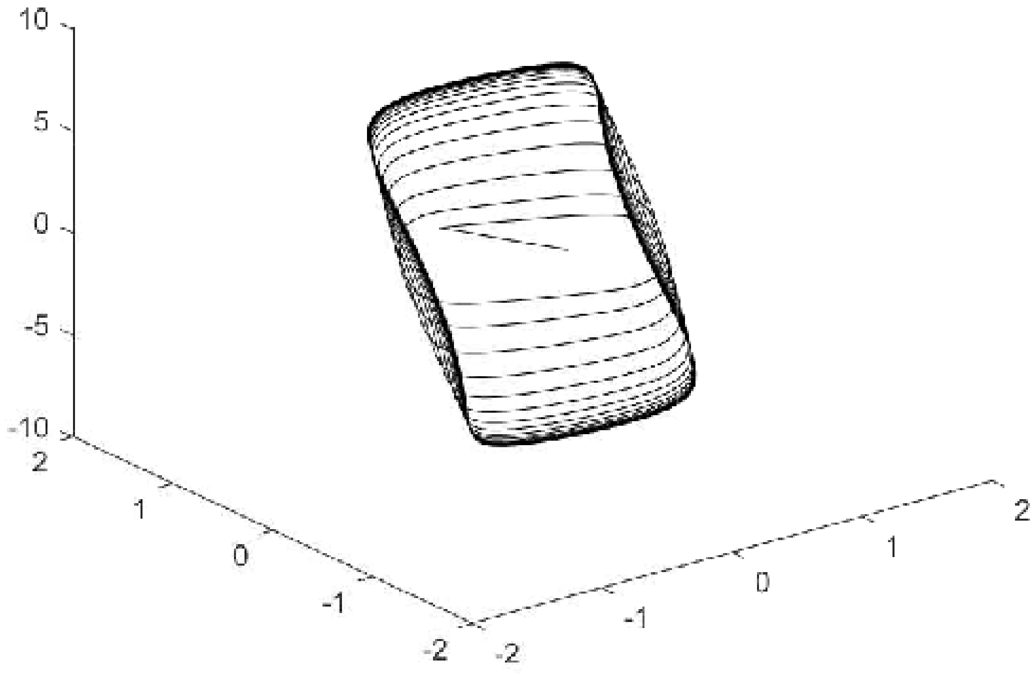


Figure 7. xyz phase space projection.

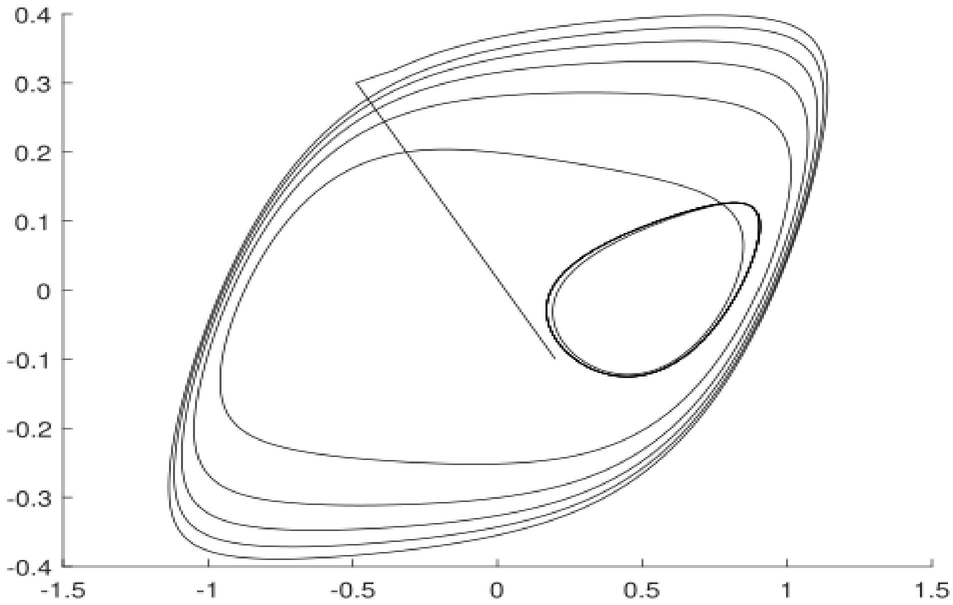


Figure 8. Numerical simulations of $X_1(t)$, $X_2(t)$.

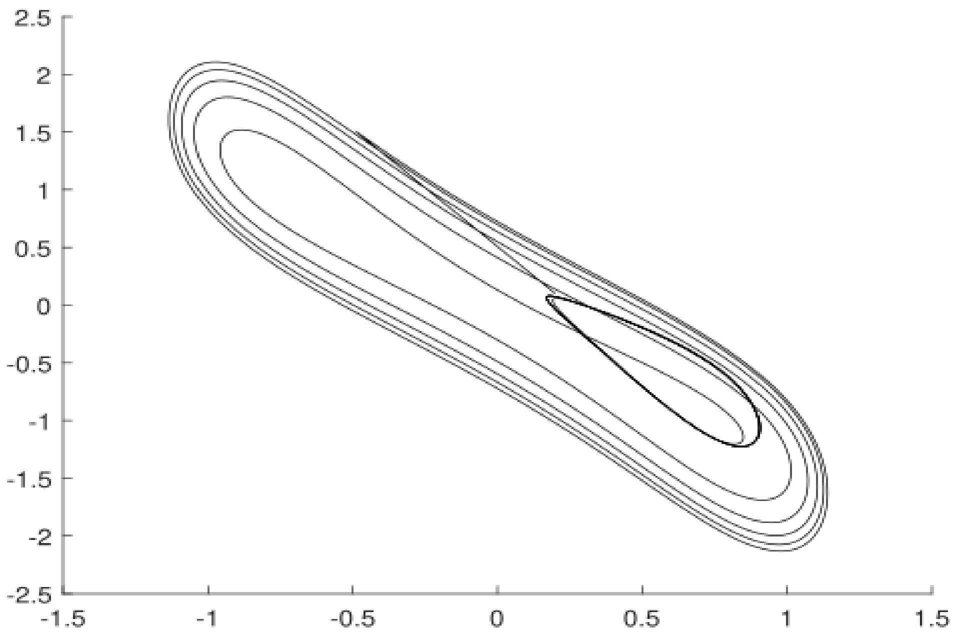


Figure 9. Numerical simulations of $X_1(t)$, $X_3(t)$.

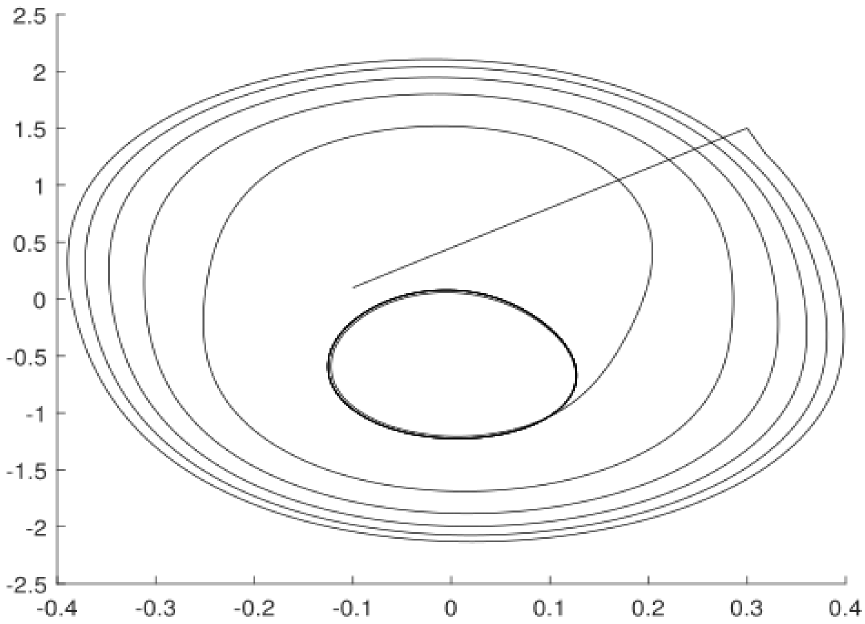


Figure 10. Numerical simulations of $X_2(t)$, $X_3(t)$.

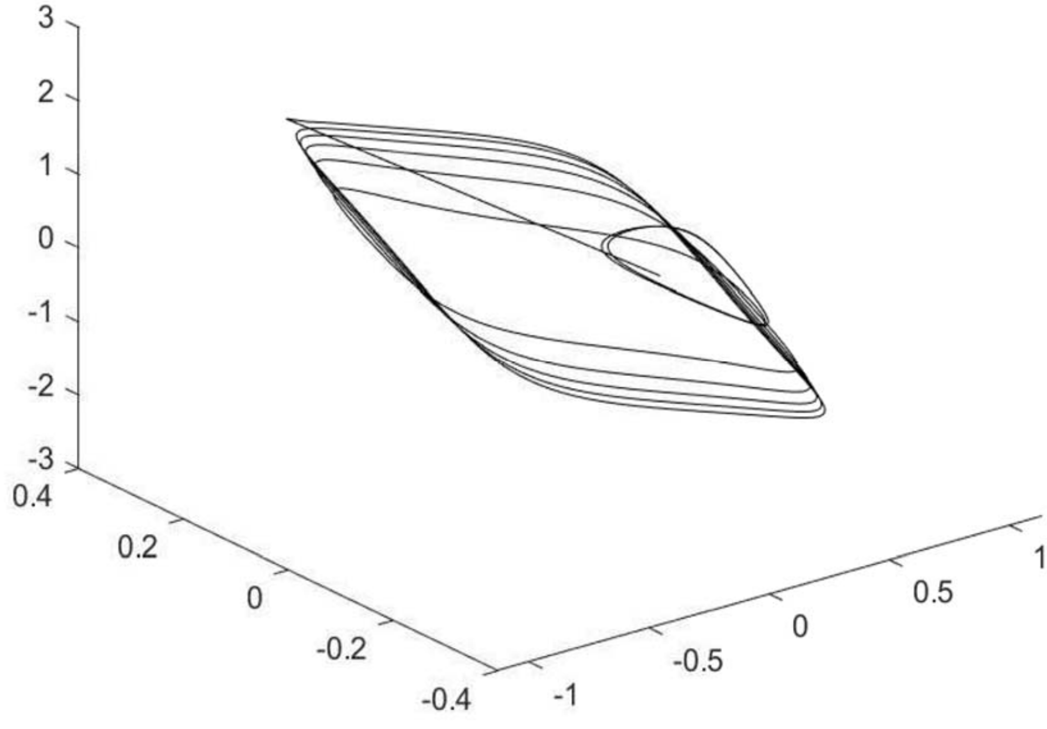


Figure 11. Numerical simulations of $X_1(t)$, $X_2(t)$, $X_3(t)$.

5. Conclusions

The objective of this work is to analyze the memory effects of time-fractional Caputo and Caputo-Fabrizio fractional derivatives nonlinear chaotic systems of the Chua type. A numerical analysis of the mathematical models is used to compare the chaotic behavior of systems with differential operators of integer order versus systems with fractional differential operators and has good influence. Even though the chaotic behavior of the classical Chua's circuit has been extensively investigated, our generalization can highlight new aspects of system behavior and the effects of memory on the evolution of the chaotic generalized circuit.

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Conflict of interest

The authors declared no conflict of interest.

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