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Research article

Mittag-Leffler stabilization of anti-periodic solutions for fractional-order neural networks with time-varying delays

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Abstract: Mittag-Leffler stabilization of anti-periodic solutions for fractional-order neural networks with time-varying delays are investigated in the article. We derive the relationship between the fractional-order integrals of the state function with and without delays through the division of time interval, using the properties of fractional calculus, and initial conditions. Moreover, by constructing the sequence solution of the system function which converges to a continuous function uniformly with the Arzela-Asoli theorem, a sufficient condition is obtained to ensure the existence of an anti-periodic solution and Mittag-Leffler stabilization of the system. In the final, we verify the correctness of the conclusion by numerical simulation.

Keywords: fractional-order; time-varying delays; anti-periodic solutions; Mittag-Leffler stabilization; neural networks
Mathematics Subject Classification: 92B20, 34K20

1. Introduction

The stabilization and existence of anti-periodic solutions have major significance in dynamic behavior on nonlinear differential equations, which plays a key role in various physical phenomena, such as anti-periodic characteristics in vibration equations and so on [1–5]. As a special case of periodic solutions, many scholars have studied the existence and stabilization of anti-periodic solutions of several kinds of neural networks in recent years. The authors [6] studied the existence and stabilization of anti-periodic solutions for BAM Cohen-Grossberg neural networks. In [7] authors investigated the existence and global exponential stabilization of anti-periodic solutions for quaternion numerical cellular neural networks with impulse effect. The existence and exponential stabilization of anti-periodic solutions for BAM neural networks is studied in [8,9]. The authors [10] studied the global exponential stabilization of anti-periodic solutions for Cohen-Grossberg neural networks. All studies in [6–10] are integer-order models, however, the research on fractional-order neural networks has attracted attention and obtained important research results in recent years.

The existence and stabilization of anti-periodic solutions are of great significance in the dynamic behavior of nonlinear differential equations, such as [1–5]. From previous data, there are only discussions on the asymptotic ω-periodic solution, almost periodic solutions and s-asymptotic ω-periodic solutions for fractional-order neural networks (e.g., [11–15]), we haven't found the existence and stabilization of anti-periodic solutions yet. We focus on the problem of the existence of anti-periodic solutions and Mittag-Leffler stabilization for a class of fractional order neural networks in this paper, this is a new research topic, our characteristics mainly include three points:

1) Deriving the relationship between fractional-order integrals of state functions with and without time delay through the division of time interval and the properties of fractional-order calculus;

2) Constructing function sequence solution, and it uniformly converges to a continuous function with Arzela-Asoli theorem, then giving a sufficiency for the existence of anti-periodic solutions and Mittag-Leffler stabilization of the system, the results are new;

3) Verifying the correctness of the theorems by numerical simulation instances. It provides a new criterion for dynamic system research.

We consider fractional-order neural networks with time-varying delays:

$$\mathrm{D}_t^{\alpha} x_i(t) = -\beta_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_{ij}(t))) + I_i(t), i = 1, 2, \cdots, n. \tag{1}$$

Where $t \geq 0$, D_t^{α} is Riemann-Liouville derivative with α-order, $0 < \alpha < 1$; $x_i(t)$ is the state of the ith neuron at time t; $\beta_i > 0$; a_{ij}, b_{ij} are connection weights of neurons; $f_j(\cdot)$ is an excitation function of the jth neuron; $I_i(t)$ is an external input function of the ith neuron at time t; $\tau_{ij}(t)$ is a signal transmission delay between the ith neuron and the jth neuron, and $\tau_{ij}(t) > 0$.

Given the initial conditions of the system (1):

$$x_i(s) = \varphi_i(s),\ D_t^{\alpha} x_i(s) = \psi_i(s),\ -\tau \leq s \leq 0,\ i = 1, 2, \cdots, n. \tag{2}$$

Here $\tau = \sup_{1 \leq i, j \leq n,\, t > 0} \{\tau_{ij}(t)\}$, $\varphi_i(s), \psi_i(s)$ are bounded continuous functions.

The structure of this article is as follow. First a few preliminaries are given in Section 2. In Section 3, by the properties of fractional-order calculus, constructing function sequence solution, and the Arzela-Asoli theorem, a sufficient case is derived for the existence of anti-periodic solutions and Mittag-Leffler stabilization of the system. An illustrative example to show the effectiveness of the proposed theory in Section 4.

2. Preliminaries

Definition 1. [16] Define the q-order fractional-order integral of $f(t)$ (Riemann-Liouville integral) as

$$D_t^{-q} f(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-r)^{q-1} f(r) \mathrm{d}r,$$

where $t \ge t_0 \ge 0$, q is a positive real number, $\Gamma(\cdot)$ is a Gamma function, and $\Gamma(r) = \int_0^{+\infty} t^{r-1} e^{-t} \mathrm{d}t, r > 0.$

Definition 2. [16] Define the q-order fractional-order derivative of $f(t)$ (Riemann-Liourille derivative) as

$$D_t^q f(t) = \frac{1}{\Gamma(n-q)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{t_0}^{t} \frac{f(s)}{(t-s)^{q-n+1}} \mathrm{d}s,$$

where $t \ge t_0 \ge 0$, $n-1 \le q < n, n \in Z^+$, $\Gamma(\cdot)$ is a Gamma function.

Definition 3. [17] A Mittag-Leffler function with parameter q is defined

$$E_q(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(kq+1)},$$

where $R\mathrm{e}(q) > 0$ is the real part of q, z is plural, $\Gamma(\cdot)$ is a Gamma function.

Definition 4. Let $X^{\mathrm{T}}(t)$ and $\bar{X}^{\mathrm{T}}(t)$ are the solutions of $x_i(s) = \varphi_i(s), D_t^{\alpha} x_i(s) = \psi_i(s)$ and $\bar{x}_i(s) = \bar{\varphi}_i(s)$, $D_t^{\alpha} \bar{x}_i(s) = \bar{\psi}_i(s)$, $-\tau \le s \le 0$. If there exist $\rho_1 > 0$, $\rho_2 > 0$, $\bar{X}^{\mathrm{T}}(t)$ and $X^{\mathrm{T}}(t)$ satisfy

$$\left\| X(t) - \bar{X}(t) \right\| \le [M(\varphi - \bar{\varphi}) E_q(-\rho_1 t^{\rho_1})]^{\rho_2}, t \ge 0,$$

then the system (1) is Mittag-Leffler stabilization, where
$X(t) = (x_1(t), x_2(t), \cdots x_n(t))^{\mathrm{T}}$, $\bar{X}(t) = (\bar{x}_1(t), \bar{x}_2(t), \cdots \bar{x}_n(t))^{\mathrm{T}}$, $\varphi(t) = (\varphi_1(t), \varphi_2(t), \cdots \varphi_n(t))^{\mathrm{T}}$, $\bar{\varphi}(t) = (\bar{\varphi}_1(t), \bar{\varphi}_2(t), \cdots \bar{\varphi}_n(t))^{\mathrm{T}}$, $M(\varphi - \bar{\varphi}) \ge 0, M(0) = 0.$ $E_q(\cdot)$ is a Mittag-Leffler function with a parameter q.

Lemma 1. [18] $x(t)$ is a continuously differentiable function on $[0, \delta](\delta > 0)$, then $D_t^{-p} D_t^q x(t) = D_t^{-p+q} x(t)$, $0 < q < 1, n-1 \le p < n, n \in Z^+$.

Lemma 2. [19] $u(t)$ is a continuous function on $[0, +\infty)$, there exists $d_1 > 0$ and $d_2 > 0$, such that $u(t) \le -d_1 D_t^{-q} u(t) + d_2, \quad t \ge 0,$ then $u(t) \le d_2 E_q(-d_1 t^q)$, where $0 < q < 1$, $E_q(\cdot)$ is a Mittag-Leffler function with a parameter q.

Lemma 3. [18] If $r(t)$ is differentiable and $r'(t)$ is continuous, thus $\frac{1}{2} D_t^q r^2(t) \le r(t) D_t^q r(t), \ 0<q \le 1.$

Definition 5. [20] For $u(t) \in C(R)$, if $u(t+\omega) = -u(t)$ for $t \in R$, thus $u(t)$ is an anti-periodic function, where ω is a normal number.

Assumptions used in this article:

H_1: $f_i(t)$ is bounded continuous excitation function and satisfies Lipschitz conditions, there exist $l_i > 0$, $\bar{f}_i > 0$ such $\left| f_i(\xi_1) - f_i(\xi_2) \right| \le l_i \left| \xi_1 - \xi_2 \right|, \quad \left| f_i(t) \right| \le \bar{f}_i, \quad \xi_1, \xi_2 \in R, \quad i = 1, 2, \cdots n.$

H_2: Excitation function $f_i(t)$ satisfies $f_i(u) = -f_i(-u), \ u \in R, \ i = 1, 2, \cdots, n.$

H_3: Input function $I_i(t)$ satisfies $I_i(t+\omega) = -I_i(t), \ \left| I_i(t) \right| \le \bar{I}_i$, where $\omega > 0, \ \bar{I}_i \ge 0, \ i = 1, 2, \cdots, n.$

H_4: Time-varying delays function $\tau_{ij}(t)$ is bounded, and differentiable, and satisfy $0 \le \dot{\tau}_{ij}(t) \le \tau^* < 1,\ t > 0$, $i = 1, 2, \cdots, n$.

3. Main results

Theorem 1. The solution of system (1) is bounded on $[0, T](0 \le T < +\infty)$ when H_1 and H_3 hold.

Proof. There is $D_t^\alpha |g(x)| \le \mathrm{sgn}(g(x)) D_t^\alpha g(x)$ for a continuous function $g(x)$ and Definition 2. We get from (1):

$$D_t^\alpha |x_i(t)| \le -\beta_i |x_i(t)| + \sum_{j=1}^n |a_{ij}| |f_j(x_j(t))| + \sum_{j=1}^n |b_{ij}| |f_j(x_j(t - \tau_{ij}(t)))| + |I_i(t)|$$

$$\le -\beta_i |x_i(t)| + \sum_{j=1}^n (|a_{ij}| + |b_{ij}|) \bar{f}_j + \bar{I}_i. \tag{3}$$

Combined with Lemma 1, it can be deduced from (3):

$$|x_i(t)| \le -\beta_i D_t^{-\alpha} |x_i(t)| + D_t^{-\alpha} [\sum_{j=1}^n (|a_{ij}| + |b_{ij}|) \bar{f}_j + \bar{I}_i]$$

$$= -\beta_i D_t^{-\alpha} |x_i(t)| + [\sum_{j=1}^n (|a_{ij}| + |b_{ij}|) \bar{f}_j + \bar{I}_i] \frac{t^\alpha}{\Gamma(\alpha + 1)}$$

$$\le -\beta_i D_t^{-\alpha} |x_i(t)| + [\sum_{j=1}^n (|a_{ij}| + |b_{ij}|) \bar{f}_j + \bar{I}_i] \frac{T^\alpha}{\Gamma(\alpha + 1)}.$$

From Lemma 2:

$$|x_i(t)| \le \frac{[\sum_{j=1}^n (|a_{ij}| + |b_{ij}|) \bar{f}_j + \bar{I}_i] T^\alpha}{\Gamma(\alpha + 1)} E_\alpha(-\beta_i t^\alpha),\ t \ge 0,\ i = 1, 2, \cdots, n.$$

That is the solution $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^{\mathrm{T}}$ is bounded on $0 \le t \le T < +\infty$, where $E_\alpha(\cdot)$ is a Mittag-Leffler function with a parameter α.

Theorem 2. The solution of system (1) is Mittag-Leffler stabilization on $[0, T](T < +\infty)$, if

$$\eta = \min_{1 \le i \le n} \{2\beta_i - \sum_{j=1}^n (|a_{ij}| + |b_{ij}|)\, l_j - \sum_{j=1}^n (|a_{ji}| + \frac{|b_{ji}|}{1 - \tau^*})\, l_i\} > 0,$$

when H_1 and H_3 hold.

Proof. Suppose $x^*(t) = (x_1^*(t), x_2^*(t), \cdots, x_n^*(t))^{\mathrm{T}}$ and $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^{\mathrm{T}}$ are the solutions of $x_i^*(s) = \varphi_i^*(s),\ D_t^\alpha x_i^*(s) = \psi_i^*(s)$ and $x_i(s) = \varphi_i(s),\ D_t^\alpha x_i(s) = \psi_i(s)$. Let $y_i(t) = x_i(t) - x_i^*(t)$, combined formula (1):

$$D_t^\alpha y_i(t) = -\beta_i y_i(t) + \sum_{j=1}^n a_{ij} [f_j(x_j(t)) - f_j(x_j^*(t))] + \sum_{j=1}^n b_{ij} [f_j(x_j(t - \tau_{ij}(t))) - f_j(x_j^*(t - \tau_{ij}(t)))]. \tag{4}$$

We get $D_t^\alpha y_i^2(t) \le 2 y_i(t) D_t^\alpha y_i(t)$ from Lemma 3, and from (4):

$$
\begin{aligned}
D_t^{\alpha} y_i^2(t) &\le 2y_i(t)\{-\beta_i y_i(t)+\sum_{j=1}^{n} a_{ij}[f_j(x_j(t))-f_j(x_j^*(t))]+\sum_{j=1}^{n} b_{ij}[f_j(x_j(t-\tau_{ij}(t)))-f_j(x_j^*(t-\tau_{ij}(t)))]\} \\
&\le -2\beta_i y_i^2(t)+2\left|y_i(t)\right|[\sum_{j=1}^{n}\left|a_{ij}\right| l_j \left|y_j(t)\right|+\sum_{j=1}^{n}\left|b_{ij}\right| l_j \left|y_j(t-\tau_{ij}(t))\right|] \\
&\le -2\beta_i y_i^2(t)+\sum_{j=1}^{n}\left|a_{ij}\right| l_j \left|y_i^2(t)+y_j^2(t)\right|+\sum_{j=1}^{n}\left|b_{ij}\right| l_j \left|y_i^2(t)+y_j^2(t-\tau_{ij}(t))\right| \\
&=[-2\beta_i+\sum_{j=1}^{n}(\left|a_{ij}\right|+\left|b_{ij}\right|)l_j]y_i^2(t)+\sum_{j=1}^{n}\left|a_{ij}\right| l_j y_j^2(t)+\sum_{j=1}^{n}\left|b_{ij}\right| l_j y_j^2(t-\tau_{ij}(t)).
\end{aligned} \tag{5}
$$

From (5):

$$
\sum_{i=1}^{n} y_i^2(t) \le \sum_{i=1}^{n}[-2\beta_i+\sum_{j=1}^{n}(\left|a_{ij}\right|+\left|b_{ij}\right|)l_j+\sum_{j=1}^{n}\left|a_{ji}\right| l_i]\mathrm{D}_t^{-\alpha} y_i^2(t)+\sum_{i=1}^{n}\sum_{j=1}^{n}\left|b_{ji}\right| l_i \mathrm{D}_t^{-\alpha} y_i^2(t-\tau_{ij}(t)). \tag{6}
$$

$t-\tau_{ij}(t)\in[-\tau_{ij}(t),0]$ when $t\in[0,\tau_{ij}(t)]$. Let $u=s-\tau_{ij}(s)$, then

$$
\begin{aligned}
D_t^{-\alpha} y_i^2(t-\tau_{ij}(t)) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y_i^2(s-\tau_{ij}(s))\mathrm{d}s \\
&= \frac{1}{\Gamma(\alpha)}\int_{-\tau_{ij}(0)}^{t-\tau_{ij}(t)} \frac{(t-u-\tau_{ij}(s))^{\alpha-1} y_i^2(u)}{1-\dot{\tau}_{ij}(s)}\mathrm{du} \\
&\le \frac{\varphi_i^*}{(1-\tau^*)\Gamma(\alpha)}\int_{-\tau_{ij}(0)}^{t-\tau_{ij}(t)} (t-u-\tau_{ij}(t))^{\alpha-1}\mathrm{du} \\
&= \frac{\varphi_i^*(t+\tau_{ij}(0)-\tau_{ij}(t))^{\alpha}}{(1-\tau^*)\Gamma(\alpha)\alpha} \\
&\le \frac{\varphi_i^* T^{\alpha}}{(1-\tau^*)\Gamma(\alpha+1)},
\end{aligned} \tag{7}
$$

where $\varphi_i^* = \sup_{-\tau\le s\le 0}\{(\varphi_i^*(s)-\varphi_i(s))^2\},\ i=1,2,\cdots,n.$

$t-\tau_{ij}(t)\in[0,+\infty)$ when $t\in[\tau_{ij}(t),+\infty)$. Let $u=s-\tau_{ij}(s)$, then

$$
\begin{aligned}
D_t^{-\alpha} y_i^2(t-\tau_{ij}(t)) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y_i^2(s-\tau_{ij}(s))\mathrm{d}s \\
&= \frac{1}{\Gamma(\alpha)}\int_{-\tau_{ij}(0)}^{t-\tau_{ij}(t)} \frac{(t-u-\tau_{ij}(s))^{\alpha-1} y_i^2(u)}{1-\dot{\tau}_{ij}(s)}\mathrm{du} \\
&= \frac{1}{\Gamma(\alpha)}[\int_{-\tau_{ij}(0)}^{0} \frac{(t-u-\tau_{ij}(s))^{\alpha-1} y_i^2(u)}{1-\dot{\tau}_{ij}(s)}\mathrm{du}+\int_0^{t-\tau_{ij}(t)} \frac{(t-u-\tau_{ij}(s))^{\alpha-1} y_i^2(u)}{1-\dot{\tau}_{ij}(s)}\mathrm{du}] \\
&\le \frac{1}{(1-\tau^*)\Gamma(\alpha)}[\int_{-\tau_{ij}(0)}^{0} (-u)^{\alpha-1} y_i^2(u)\mathrm{du}+\int_0^{t-\tau_{ij}(t)} (t-u-\tau_{ij}(t))^{\alpha-1} y_i^2(u)\mathrm{du}] \\
&\le \frac{1}{1-\tau^*}[\frac{\varphi_i^* T^{\alpha}}{\Gamma(\alpha+1)}+D_i^{-\alpha} y_i^2(t)],
\end{aligned} \tag{8}
$$

where $\varphi_i^* = \sup_{-\tau\le s\le 0}\{(\varphi_i^*(s)-\varphi_i(s))^2\},\ i=1,2,\cdots,n.$

We obtain from (7) and (8):

$$D_t^{-\alpha} y_i^2(t-\tau_{ij}(t)) \le \frac{1}{1-\tau^*}[\frac{\varphi_i^* T^\alpha}{\Gamma(\alpha+1)} + D_i^{-\alpha} y_i^2(t)],\ i=1,2,\cdots,n. \quad (9)$$

Substitute the result of (9) into (6):

$$\begin{aligned}\sum_{i=1}^n y_i^2(t) &\le \sum_{i=1}^n[-2\beta_i + \sum_{j=1}^n(|a_{ij}|+|b_{ij}|)l_j + \sum_{j=1}^n(|a_{ji}|+\frac{|b_{ji}|}{1-\tau^*})l_i]D_t^{-\alpha} y_i^2(t) + \sum_{i=1}^n\sum_{j=1}^n |b_{ji}| l_i \frac{\varphi_i^* T^\alpha}{(1-\tau^*)\Gamma(\alpha+1)} \\ &\le -\min_{1\le i\le n}[2\beta_i - \sum_{j=1}^n(|a_{ij}|+|b_{ij}|)l_j - \sum_{j=1}^n(|a_{ji}|+\frac{|b_{ji}|}{1-\tau^*})l_i]\sum_{i=1}^n D_t^{-\alpha} y_i^2(t) + \sum_{i=1}^n\sum_{j=1}^n |b_{ji}| l_i \frac{\varphi_i^* T^\alpha}{(1-\tau^*)\Gamma(\alpha+1)}. \quad (10)\end{aligned}$$

Combined with Lemma 2, it can be deduced from (10):

$$\|x-x^*\| = \sum_{i=1}^n (x-x^*)^2 \le M(\varphi-\varphi^*)E_\alpha(-\eta t^\alpha),\ t>0, \quad (11)$$

where $M(\varphi-\varphi^*) = \sum_{i=1}^n\sum_{j=1}^n |b_{ji}| l_i \frac{\varphi_i^* T^\alpha}{(1-\tau^*)\Gamma(\alpha+1)}$,

$\eta = \min_{1\le i\le n}\{2\beta_i - \sum_{j=1}^n(|a_{ij}|+|b_{ij}|)l_j - \sum_{j=1}^n(|a_{ji}|+\frac{|b_{ji}|}{1-\tau^*})l_i\} > 0.$ Obviously $M(\varphi-\varphi^*)\ge 0$, and $M(0)=0$, thus the solution of system (1) is Mittag-Leffler stabilization from Definition 4.

Theorem 3. System (1) has an anti-periodic solution when the Theorem 2 and H_2 hold, and the solution is Mittag-Leffler stabilization.

Proof. For a positive integer k and a normal number ω from H_2 and H_3, we obtain from (1):

$$\begin{aligned} D_t^\alpha[(-1)^{k+1}x_i(t+(k+1)\omega)] &= (-1)^{k+1}[-\beta_i x_i(t+(k+1)\omega) + \sum_{j=1}^n a_{ij} f_j(x_j(t+(k+1)\omega)) \\ &\quad + \sum_{j=1}^n b_{ij} f_j(x_j(t+(k+1)\omega-\tau_{ij}(t))) + I_i(t+(k+1)\omega)] \\ &= -\beta_i(-1)^{k+1}x_i(t+(k+1)\omega) + \sum_{j=1}^n a_{ij} f_j((-1)^{k+1}x_j(t+(k+1)\omega)) \\ &\quad + \sum_{j=1}^n b_{ij} f_j((-1)^{k+1}x_j(t+(k+1)\omega-\tau_{ij}(t))) + I_i(t),\ i=1,2,\cdots,n. \end{aligned} \quad (12)$$

So $(-1)^{k+1}x_i(t+(k+1)\omega)$ is the solution of system (1) for a positive integer k. $x(t)$ is bounded from Theorem 1, then there exists a positive constant N such that:

$$|(-1)^{k+1}x_i(t+(k+1)\omega)| \le NE_\alpha[-\eta(t+(k+1)\omega)^\alpha],\ i=1,2,\cdots,n.$$

Because $0\le E_\alpha(-(\lambda t)^\alpha)\le 1, \lambda>0$, so the sequence $\{(-1)^{k+1}x_i(t+k\omega)\}$ is equicontinuous and bounded uniformly. Reapplication Arzela-Ascoli theorem $\{(-1)^k x_i(t+k\omega)\}_{k\in N}$ converges to a continuous function $x_i^*(t)$ uniformly on any compact set in $[0,+\infty]$ by selecting a subsequence $\{k\omega\}_{k\in N}$, that is

$\lim_{k\to+\infty}(-1)^k x_i(t+k\omega)=x_i^*(t),\ i=1,2,\cdots,n.$

On the other hand, owing to

$x_i^*(t+\omega)=\lim_{k\to+\infty}(-1)^k x_i(t+\omega+k\omega)=-\lim_{k\to+\infty}(-1)^{k+1}x_i(t+(k+1)\omega)=-x_i^*(t),\ i=1,2,\cdots,n.$

So $x^*(t)$ is ω-anti-periodic function. Owing $(-1)^k x_i(t+k\omega)$ is the solution of system (1) for any $k\in N$, we obtain from (1):

$$D_t^\alpha[(-1)^k x_i(t+k\omega)]=-\beta_i(-1)^k x_i(t+k\omega)+\sum_{j=1}^n a_{ij}f_j((-1)^k x_j(t+k\omega))$$
$$+\sum_{j=1}^n b_{ij}f_j((-1)^k x_j(t+k\omega-\tau_{ij}(t)))+I_i(t),\quad i=1,2,\cdots,n.$$

We can continue to get when $f_i(\cdot)$ is continuous, then

$$\lim_{k\to+\infty}D_t^\alpha[(-1)^k x_i(t+k\omega)]=-\beta_i x_i^*(t)+\sum_{j=1}^n a_{ij}f_j(x_j^*(t))+\sum_{j=1}^n b_{ij}f_j(x_j^*(t+k\omega-\tau_{ij}(t)))+I_i(t),\quad i=1,2,\cdots,n.$$

So $x^*(t)$ is an anti-periodic solution of system (1). For any $x(t)$, the inequality holds from (11):

$$\left\|x(t)-x^*(t)\right\|=\sum_{i=1}^n\left|x_i(t)-x_i^*(t)\right|\le M(\varphi-\varphi^*)E_\alpha(-\eta t^\alpha),\ t>0,$$

so $x^*(t)$ is an anti-periodic solution and Mittag-Leffler stabilization.

Remark: The stabilization and existence of anti-periodic solutions of nonlinear differential equations are of great significance in dynamic behavior, which plays a key role in physical phenomena [1–5]. The model of integer- order neural network system is a nonlinear differential equation, and fractional-order neural network system is a generalization of integer-order neural network system, so fractional-order neural network system is also a model of nonlinear differential equation generalization. From previous data, there are only discussions on the boundedness and asymptotic stabilization of almost periodic solution and ω-periodic solution for fractional-order neural networks (e.g., [11–15]), we have not seen the results of authors exploring the dynamic behavior of the anti-periodic solution of a system. In the article, we mainly give the sufficient conditions for the existence of anti-periodic solutions and Mittag-Leffler stabilization of fractional order neural network systems. The results are new. This provides a new basis to further explore the dynamic properties of a system in theoretical research and practical application.

4. Numerical simulation

We consider fractional-order neural networks with time-varying delays:

$$\mathrm{D}_t^\alpha x_i(t)=-\beta_i x_i(t)+\sum_{j=1}^2 a_{ij}f_j(x_j(t))+\sum_{j=1}^2 b_{ij}f_j(x_j(t-\tau_{ij}(t)))+I_i(t),\ i=1,2. \tag{13}$$

We get $\alpha=0.85$, $\beta_1=1.85$, $\beta_2=1.9$, $a_{11}=\frac{1}{16}$, $a_{12}=\frac{1}{32}$, $a_{21}=-\frac{1}{16}$, $a_{22}=-\frac{1}{32}$, $b_{11}=-\frac{1}{16}$, $b_{12}=\frac{1}{32}$, $b_{21}=\frac{1}{16}$, $b_{22}=-\frac{1}{32}$, $f_i(x)=\frac{|x_i+1|-|x_i-1|}{50}$, $I_i(t)=\frac{\cos(8t)}{150}$, $\tau_{ij}(t)=\frac{2-e^{-t}}{3}$, therefore

$$I_i(t+\frac{\pi}{8})=-I_i(t), \quad f_i(-x)=-\frac{|x_i+1|-|x_i-1|}{50}=-f_i(x), \ i=1,2.$$

Let $l_i=\frac{1}{25}$, $\bar{f}_i=\frac{1}{25}$, $\bar{I}_i=\frac{1}{150}$, $\tau^*=\frac{1}{3}$, $\omega=\frac{\pi}{8}$.

By calculating we have: $\eta=\min_{1\le i\le 2}\{2\beta_i-\sum_{j=1}^{2}(|a_{ij}|+|b_{ij}|)\,l_j-\sum_{j=1}^{2}(|a_{ji}|+\frac{|b_{ji}|}{1-\tau^*})\,l_i\}=2.8884375>0$, so Theorem 3 holds, the system (13) has a $\frac{\pi}{8}$-anti-periodic solution with Mittag-Leffler stabilization.

On the other hand, giving the transient change of $(x_1(t),y_1(t))$ and $(x_2(t),y_2(t))$ for system (13) by numerical simulation, as shown in the figures below (see Figures 1 and 2).

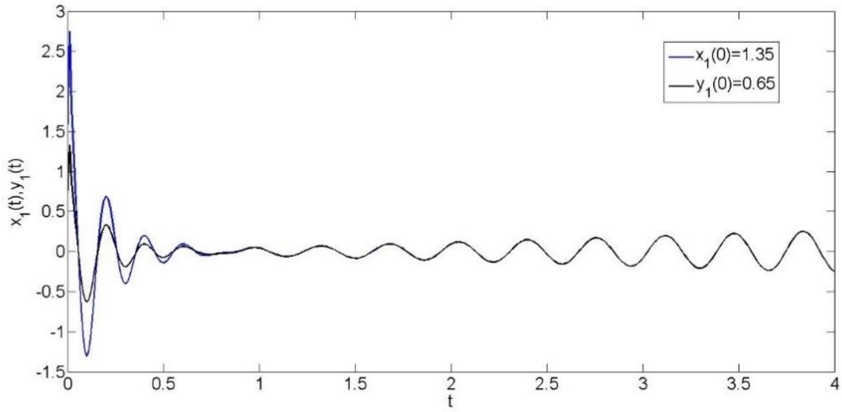

Figure 1. Transient change of $(x_1(t),y_1(t))$.

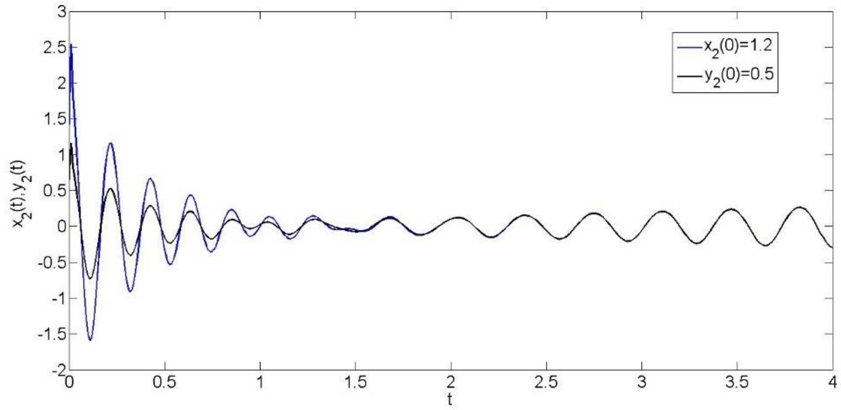

Figure 2. Transient change of $(x_2(t),y_2(t))$.

We gain a $\frac{\pi}{8}$-anti-periodic solution from the figures, it is consistent with the conclusion of theorems.

5. Conclusions

We study the dynamic behavior of fractional-order neural networks with time-varying delays in the article. Frist deriving the relationship between fractional-order integrals of state functions with and without time delay through the division of time interval and the properties of fractional-order calculus, the research method is innovative. Moreover, constructing the sequence solution of the system function which converges to a continuous function uniformly with the Arzela-Asoli theorem. In addition, giving the sufficient conditions the Mittag-Leffler stabilization, boundedness, and the existence of anti-periodic solutions for systems. Finally, the conclusion is feasible by a numerical simulation. Similarly, we can use the theoretical basis of this article to study the Mittag-Leffler stabilization of anti-periodic solutions of fractional-order Cohen-Grossberg neural networks and inertial Cohen-Grossberg neural networks, and so on.

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Conflict of interest

The authors declare no conflict of interest.

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