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Research article

New fixed point results in double controlled metric type spaces with applications

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Abstract: The concept of an $\mathcal{F}$-contraction was introduced by Wardowski, while Samet et al. introduced the class of α-admissible mappings and the concept of (α-ψ)-contractive mapping on complete metric spaces. In this paper, we study and extend two types of contraction mappings: (α-ψ)-contraction mapping and (α-$\mathcal{F}$)-contraction mapping, and establish new fixed point results on double controlled metric type spaces. Moreover, we demonstrate some examples and present an application of our result on the existence and uniqueness of the solution for an integral equation.

Keywords: fixed point; double controlled metric type spaces; $(\alpha - \psi)$-contraction; (α-$\mathcal{F}$)-contraction; b-metric spaces; Wardowski contractions
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1. Introduction

One of the most important fixed point theorems is the Banach contraction principle [1], where he established the existence and uniqueness of a fixed point for self-contractive mapping in a metric space setting. Several generalizations and extensions of his result exist in the literature, either through changing the contraction mapping or generalizing the type of metric spaces. For example, b-metric space which was introduced by Bakhtin [2], and was later extended by Czerwik [3], is in fact a fascinating generalization of metric spaces. This generalization led to the study of various fixed point results for some contractive mappings which led to wide applications in many branches of mathematics, including nonlinear analysis and its applications. An extended b-metric space was introduced by Kamran et al. [4]. Later Mlaiki et al. [5] introduced the notion of controlled metric type spaces, which was generalized into double controlled metric type spaces [6]. Then recently, several articles appeared on double controlled partial metric type spaces [7], double controlled metric-like spaces [8–11], and double controlled quasi metric-like spaces [12].

Another generalization of metric spaces was introduced by Branciari in 2000 [13], where he replaced the triangular inequality with a quadrilateral inequality, which led to the birth of rectangular metric space which is a larger class than metric space, and this resulted in the extension of the Banach contraction theorem to this new setting. In 2015, George et al. [14], generalized rectangular metric spaces to rectangular b-metric spaces. Asim et al. [15] included a control function to initiate the concept of extended rectangular b-metric space as a generalization of rectangular b-metric spaces. More recently many other spaces appeared, such as ordered partial rectangular b-metric spaces [16], rectangular M-metric spaces [17], and controlled rectangular metric-like spaces [18].

Samet et al. [19] introduced the class of α-admissible mappings on metric spaces and the concept of (α-ψ)-contractive mapping on complete metric spaces and established some fixed point theorems. Shatanawi et al. [20] introduced the concept of an (α -ψ)-contraction on extended b-metric space, thus generalizing the results of Mukheimer [21] and Mehmet and Kiziltunc [22]. Wardowski introduced another type of contraction mapping, known as an $\mathcal{F}$-contraction in 2012 [23]. Few authors have studied fixed-point theorems for an $\mathcal{F}$-contraction, and (α- $\mathcal{F}$)-contraction on some complete metric spaces [24, 25]. In [26] Vujaković et al. dealt with $\mathcal{F}$-contractions for weak α-admissible mappings in metric-like spaces; moreover, a new Wardowski-type fixed-point results were illustrated in [27].

In this paper, we study and extend two types of contraction mappings: (α-ψ)-contraction mappings and (α-$\mathcal{F}$)-contraction mappings on double controlled metric type spaces. We establish new fixed-point results on double controlled metric type spaces. Moreover, we demonstrate some examples and present an application of our result on the existence and uniqueness of the solution for an integral equation. It is worth noting that extensions of the Banach contraction principle from metric spaces to controlled metric type spaces are useful to prove the existence and uniqueness theorem for different types of integral and differential equations. This illustrates the importance of research in this area.

2. Preliminaries

In 2017, Kamran et al. [4] defined the notion of extended b-metric spaces, which led to the generalization of various results and work.

Definition 2.1. (see [4]) Consider a mapping $A : X_a \times X_a \to [1, +\infty)$, where X_a is a nonempty set. The function $d_a : X_a \times X_a \to [0, +\infty)$ is named an extended b-metric, if, for all $\hat{x}, \hat{y}, \hat{z} \in X_a$, the following holds:

(1) $d_a(\hat{x}, \hat{z}) = 0 \iff \hat{x} = \hat{z}$;
(2) $d_a(\hat{x}, \hat{z}) = d_a(\hat{z}, \hat{x})$;
(3) $d_a(\hat{x}, \hat{z}) \leq A(\hat{x}, \hat{z})[d_a(\hat{x}, \hat{y}) + d_a(\hat{y}, \hat{z})]$.

The pair (X_a, d_a) is called an extended b-metric space.

An extension of the b-metric spaces into a controlled metric type space was introduced by Mlaiki et al. [5], as follows:

Definition 2.2. (see [5]) Let $\beta : X_\beta \times X_\beta \to [1, +\infty)$ be a mapping, where X_β is a nonempty set. The function $\tilde{d}_\beta : X_\beta \times X_\beta \to [0, +\infty)$ is called a controlled metric, if, for all $\hat{x}, \hat{y}, \hat{z} \in X_\beta$, the following conditions hold:
$(d1)$ $\tilde{d}_\beta(\hat{x}, \hat{z}) = 0 \iff \hat{x} = \hat{z}$;

$(d2)$ $\tilde{d}_\beta(\hat{x},\hat{z}) = \tilde{d}_\beta(\hat{z},\hat{x})$;
$(d3)$ $\tilde{d}_\beta(\hat{x},\hat{z}) \leq \beta(\hat{x},\hat{y})\tilde{d}_\beta(\hat{x},\hat{y}) + \beta(\hat{y},\hat{z})\tilde{d}_\beta(\hat{y},\hat{z})$.
The pair $(X_\beta, \tilde{d}_\beta)$ is called a controlled metric type space.

A more general concept of b-metric type space is called a double controlled metric type space [6], which is defined below.

Definition 2.3. ([6]) Consider two non-comparable functions $\beta, \mu : Z \times Z \to [1, +\infty)$, defined on a nonempty set Z. The mapping $\tilde{d}_{\beta,\mu} : Z \times Z \to [0, +\infty)$ is called a double controlled metric type by β and μ, if, for all $z_1, z_2, z_3 \in Z$, the following conditions hold:
$(q1)$ $\tilde{d}_{\beta,\mu}(z_1, z_2) = 0 \Longleftrightarrow z_1 = z_2$;
$(q2)$ $\tilde{d}_{\beta,\mu}(z_1, z_2) = \tilde{d}_{\beta,\mu}(z_2, z_1)$;
$(q3)$ $\tilde{d}_{\beta,\mu}(z_1, z_2) \leq \beta(z_1, z_3)\tilde{d}_{\beta,\mu}(z_1, z_3) + \mu(z_3, z_2)\tilde{d}_{\beta,\mu}(z_3, z_2)$.
The pair $(Z, \tilde{d}_{\beta,\mu})$ is called a double controlled metric type space (in short we will denote it by DCMTS).

Remark 2.4. Every controlled metric type space is a DCMTS when both functions β and μ are taken to be the same function, i.e. $\beta = \mu$. The converse is not true in general, as the below example illustrates. Furthermore, in Definition 2.3, β and μ are non-comparable functions, which means none of them is greater or equal to the other.

Example 2.5. ([6]) Let $\mathcal{Z} = [0, +\infty)$; define the mapping $\tilde{d}_{\beta,\mu} : \mathcal{Z} \times \mathcal{Z} \to [0, +\infty)$ by

$$\tilde{d}_{\beta,\mu}(x,y) = \begin{cases} 0 & \text{iff } x = y, \\ \frac{1}{x} & \text{if } x \geq 1 \text{ and } y \in [0,1), \\ \frac{1}{y} & \text{if } y \geq 1 \text{ and } x \in [0,1), \\ 1 & \text{otherwise.} \end{cases}$$

Let $\beta, \mu : \mathcal{Z} \times \mathcal{Z} \to [1, +\infty)$ be two functions defined by

$$\beta(x,y) = \begin{cases} x & \text{if } x, y \geq 1, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$\mu(x,y) = \begin{cases} 1 & \text{if } x, y < 1, \\ max\{x,y\} & \text{otherwise.} \end{cases}$$

Then $(\mathcal{Z}, \tilde{d}_{\beta,\mu})$ is a double controlled metric type space. First, note that Conditions $(q1)$ and $(q2)$ hold. To show that Condition $(q3)$ is satisfied, we note that if either $z = x$ or $z = y$, then $(q3)$ holds. Thus, suppose $x \neq y$, which means $x \neq y \neq z$. We consider the following cases:

Case 1: If $x \geq 1$ and $y \in [0,1)$, or $y \geq 1$ and $x \in [0,1)$, then for any z, clearly Condition $(q3)$ is satisfied.

Case 2: If x, $y > 1$, and $z \geq 1$, then one can easily observe that $\tilde{d}_{\beta,\mu}(x,y) = 1 \leq x(1) + max\{y,z\}(1)$. While, if $z \in [0,1)$, then we obtain

$$\tilde{d}_{\beta,\mu}(x,y) = 1 \leq \frac{1}{x} + y\frac{1}{y}; \text{ hence, } (q3) \text{ is satisfied.}$$

Case 3: If $x, y < 1$ and $z \geq 1$, we obtain

$$\tilde{d}_{\beta,\mu}(x, y) = 1 \leq \frac{1}{z} + z\frac{1}{z}.$$

While, if $z \in [0, 1)$, then easily Condition ($q3$) holds. Therefore, $(\mathcal{Z}, \tilde{d}_{\beta,\mu})$ is a double controlled metric type space, which is not a controlled metric type space; by taking $\beta = \mu$, we observe

$$\tilde{d}_{\beta}(0, \frac{1}{2}) = 1 > \frac{1}{2} = \frac{1}{4} + \frac{1}{4} = \beta(0, 4)\tilde{d}_{\beta}(0, 4) + \beta(4, \frac{1}{2})\tilde{d}_{\beta}(4, \frac{1}{2}).$$

Definition 2.6. Let $(Z, \tilde{d}_{\beta,\mu})$ be a DCMTS, and let $\{z_n\}_{n\geq 0}$ be any sequence in Z. Then

(1) For any $z \in Z$ and $\varepsilon > 0$, the open ball $B(z, \varepsilon)$ is defined as

$$B(z, \varepsilon) = \{w \in Z, \tilde{d}_{\beta,\mu}(z, w) < \varepsilon\}.$$

(2) A sequence $\{z_n\}$ is said to converge to some w in Z if, for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$, such that $\tilde{d}_{\beta,\mu}(z_n, w) < \varepsilon$ for all $n \geq N$. In this case, we write $\lim_{n\to+\infty} z_n = w$.

(3) We say that $\{z_n\}$ is a Cauchy sequence if, for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $\tilde{d}_{\beta,\mu}(z_m, z_n) < \varepsilon$ for all $m, n \geq N$.

(4) The space $(Z, \tilde{d}_{\beta,\mu})$ is called complete if every Cauchy sequence in Z is convergent.

(5) The mapping $T : Z \to Z$ is said to be continuous at $z \in Z$ if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $T(B(z, \delta)) \subseteq B(Tz, \varepsilon)$. The continuity of the mapping T at w can also be defined in terms of convergent sequences; thus, if $z_n \to w$, then $Tz_n \to Tw$ as $n \to +\infty$.

To see how the open ball looks, consider Example 2.5, taking $x = 3$ and $\varepsilon = 1/2 > 0$; then,

$$B(3, 1/2) = \{w \in Z, \tilde{d}_{\beta,\mu}(3, w) < 1/2\} = \{3, w \in [0, 1)\}.$$

3. (α-ψ)-contraction and fixed point theorems

Let Ψ denote the set of all functions $\psi : [0, +\infty) \to [0, +\infty)$ such that

(1) ψ is non-decreasing,
(2) $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for all $t > 0$, where ψ^n is the n-th iterate of ψ.

Now, we recall the following lemma; consult [19, 20].

Lemma 3.1. *For each $\psi \in \Psi$, the following holds:*

(1) $\lim_{n\to+\infty} \psi^n(t) = 0$*, for all* $t > 0$*;*
(2) $\psi(t) < t$ *for all* $t > 0$*;*
(3) $\psi(0) = 0$.

We start by presenting a definition of the set of double controlled comparison functions, $\Psi_{\beta,\mu}$, which will be defined as an extension of the b-comparison function of Berinde, and the set of all controlled comparison functions [28].

Definition 3.2. Consider two non-comparable mappings $\beta, \mu : Z \times Z \to [1, + + \infty)$, where Z is a nonempty set. A function $\psi \in \Psi$ is said to be a double controlled comparison function if ψ satisfies the following conditions:

(1) ψ is non-decreasing;
(2) $\sum_{n=1}^{+\infty}\psi^n(t)\prod_{i=1}^{n}\mu(z_i,z_m)\beta(z_n,z_{n+1}) < +\infty$ and $\lim_{n\to+\infty}\psi^n(t)\beta(z_n,z_{n+1}) < +\infty$ for any sequence $\{z_n\}_{n=1}^{+\infty}$ in Z, for all $t>0$ and for the non-negative integer m, where ψ^n is the n-th iterate of ψ.

The set of all double controlled comparison functions will be denoted by $\Psi_{\beta,\mu}$.

Observe that when $\beta=\mu$, then the double controlled comparsion function $\Psi_{\beta,\mu}$ becomes the controlled comparison function as defined in [28].

Note that if $\psi\in\Psi_{\beta,\mu}$, then we have $\sum_{n=1}^{+\infty}\psi^n(t)<+\infty$, since $\psi^n(t)\prod_{i=1}^{n}\mu(z_i,z_m)\geq\psi^n(t)$, for all $t>0$. Also,

$$\psi^n(t)\prod_{i=1}^{n}\mu(z_i,z_m)\beta(z_n,z_{n+1})\geq\psi^n(t).$$

Hence, by Lemma 3.1, we have $\psi(t)<t$.

The family $\Psi_{\beta,\mu}$ is a non-empty set, as the below example illustrates.

Example 3.3. Let $Z=\{0,1,2\}$. Define two symmetric functions $\beta,\mu: Z\times Z\to[1,+\infty)$ by

$$\beta(0,0)=\beta(2,2)=\beta(1,1)=\beta(0,2)=1,\quad \beta(1,2)=\frac{5}{8},\quad \beta(0,1)=\frac{11}{10},$$

and

$$\mu(0,0)=\mu(2,2)=\mu(1,1)=1,\quad \mu(0,2)=\frac{3}{2},\quad \mu(1,2)=\frac{5}{4},\quad \mu(0,1)=\frac{11}{10}.$$

The mapping $\tilde{d}_{\beta,\mu}$ is given by

$$\tilde{d}_{\beta,\mu}(0,0)=\tilde{d}_{\beta,\mu}(1,1)=\tilde{d}_{\beta,\mu}(2,2)=0,$$

and

$$\tilde{d}_{\beta,\mu}(0,1)=\tilde{d}_{\beta,\mu}(1,0)=1,\quad \tilde{d}_{\beta,\mu}(0,2)=\tilde{d}_{\beta,\mu}(2,0)=\frac{1}{2},\quad \tilde{d}_{\beta,\mu}(1,2)=\tilde{d}_{\beta,\mu}(2,1)=\frac{2}{5}.$$

One can easily show that $(Z,\tilde{d}_{\beta,\mu})$ is a DCMTS. Let $0<a<1$; define the function $\psi(t)=\frac{at}{3}$. Note that $\beta(z_1,z_2)\leq 3$ and $\mu(z_1,z_2)\leq 3$. Then we have

$$\psi^n(t)\prod_{i=1}^{n}\mu(z_i,z_m)\beta(z_n,z_{n+1})\leq\frac{a^n t}{3^n}.3^{n+1}=3a^n t.$$

Thus,

$$\sum_{n=1}^{+\infty}\psi^n(t)\prod_{i=1}^{n}\mu(z_i,z_m)\beta(z_n,z_{n+1})\leq\sum_{n=1}^{+\infty}3a^n t<+\infty.$$

Similarly, one can easily show that $\lim_{n\to+\infty}\psi^n(t)\beta(z_n,z_{n+1})<+\infty$. Therefore, $\psi\in\Psi_{\beta,\mu}$.

Samet et al. [19] introduced the class of α-admissible mappings; see also [29].

Definition 3.4. Let $(Z,\tilde{d}_{\beta,\mu})$ be a DCMTS, where Z is a nonempty set, and let $\alpha: Z\times Z\to[0,+\infty)$. A mapping $T: Z\longrightarrow Z$ is said to be α-admissible if, whenever $\alpha(\hat{x},\hat{y})\geq 1$, it implies $\alpha(T\hat{x},T\hat{y})\geq 1$, for all $\hat{x},\hat{y}\in Z$.

Example 3.5. Let $Z = (0, +\infty)$. Define $\alpha : Z \times Z \to [0, +\infty)$ and $T : Z \longrightarrow Z$ by

$$\alpha(x, y) = \begin{cases} 5 & \text{if } x \geq y, \\ 0 & \text{otherwise,} \end{cases}$$

and $T(x) = ln(x)$ for all $x \in Z$. Then, T is α-admissible, consult [19] for more examples.

Definition 3.6. Let $T : Z \to Z$ be a mapping on a DCMTS $(Z, \tilde{d}_{\beta,\mu})$, where Z is a nonempty set, and let $\Psi_{\beta,\mu}$ be a double controlled comparison function, as in Definition 3.2. Then T is said to be an $(\alpha$-$\psi)$-double contractive mapping if there exist functions $\alpha : Z \times Z \to [0, +\infty)$ and $\psi \in \Psi_{\beta,\mu}$ such that for all $x, y \in Z$, the following holds,

$$\alpha(x, y)\tilde{d}_{\beta,\mu}(Tx, Ty) \leq \psi(\tilde{d}_{\beta,\mu}(x, y)). \tag{3.1}$$

Our first main result on the fixed point theorem is as follows.

Theorem 3.7. *Let $(Z, \tilde{d}_{\beta,\mu})$ be a complete DCMTS, and let $T : Z \to Z$ be an $(\alpha$-$\psi)$-double contractive mapping for some $\psi \in \Psi_{\beta,\mu}$. Assume the following holds:*

(1) T is α-admissible;
(2) There exists $z_0 \in Z$ such that $\alpha(z_0, Tz_0) \geq 1$;
(3) T is continuous.

Then, T has a fixed point. Moreover, if for any two fixed points of T in Z say, a and b with $\alpha(a, b) \geq 1$, then T has a unique fixed point in Z.

Proof. Let $z_0 \in Z$ such that $\alpha(z_0, Tz_0) \geq 1$. Define the sequence $\{z_n\}_{n\geq 0}$ in Z by $Tz_0 = z_1$ and $Tz_1 = z_2$; thus, $T^n z_0 = z_n$ for all $n \in \mathbb{N}$.

Observe that if $z_n = z_{n+1}$ for some $n \in \mathbb{N}$, then we are done, and z_n is the fixed point of T. Therefore, we may assume that $z_n \neq z_{n+1}$ for all $n \in \mathbb{N}$.

From the hypothesis, we know that $1 \leq \alpha(z_0, Tz_0) = \alpha(z_0, z_1)$, and using the fact that T is α-admissible, we can easily deduce that for all $n \in \mathbb{N}$, we have $\alpha(z_n, z_{n+1}) \geq 1$.

As T is an $(\alpha$-$\psi)$-double contractive mapping, we obtain

$$\begin{aligned} \tilde{d}_{\beta,\mu}(z_n, z_{n+1}) &\leq \alpha(z_{n-1}, z_n)\tilde{d}_{\beta,\mu}(z_n, z_{n+1}) \\ &= \alpha(z_{n-1}, z_n)\tilde{d}_{\beta,\mu}(Tz_{n-1}, Tz_n) \\ &\leq \psi(\tilde{d}_{\beta,\mu}(z_{n-1}, z_n)) \leq .... \leq \psi^n(\tilde{d}_{\beta,\mu}(z_0, z_1)). \end{aligned} \tag{3.2}$$

Therefore, let $m, n \in \mathbb{N}$, with $m > n$, and by utilizing the triangular inequality of the double controlled metric type, we get

$$\begin{aligned} \tilde{d}_{\beta,\mu}(z_n, z_m) &\leq \beta(z_n, z_{n+1})\tilde{d}_{\beta,\mu}(z_n, z_{n+1}) + \mu(z_{n+1}, z_m)\tilde{d}_{\beta,\mu}(z_{n+1}, z_m). \quad (3.3) \\ &\leq \beta(z_n, z_{n+1})\tilde{d}_{\beta,\mu}(z_n, z_{n+1}) + \mu(z_{n+1}, z_m)[\beta(z_{n+1}, z_{n+2})\tilde{d}_{\beta,\mu}(z_{n+1}, z_{n+2}) \\ &\quad + \mu(z_{n+2}, z_m)\tilde{d}_{\beta,\mu}(z_{n+2}, z_m)]. \\ \tilde{d}_{\beta,\mu}(z_n, z_m) &\leq \beta(z_n, z_{n+1})\tilde{d}_{\beta,\mu}(z_n, z_{n+1}) + \mu(z_{n+1}, z_m)\beta(z_{n+1}, z_{n+2})\tilde{d}_{\beta,\mu}(z_{n+1}, z_{n+2}) \\ &\quad + \mu(z_{n+1}, z_m)\mu(z_{n+2}, z_m)\tilde{d}_{\beta,\mu}(z_{n+2}, z_m). \quad (3.4) \end{aligned}$$

$$\begin{aligned}\tilde{d}_{\beta,\mu}(z_n,z_m) &\le \beta(z_n,z_{n+1})\tilde{d}_{\beta,\mu}(z_n,z_{n+1}) + \mu(z_{n+1},z_m)\beta(z_{n+1},z_{n+2})\tilde{d}_{\beta,\mu}(z_{n+1},z_{n+2})\\ &\quad + \mu(z_{n+1},z_m)\mu(z_{n+2},z_m)\beta(z_{n+2},z_{n+3})\tilde{d}_{\beta,\mu}(z_{n+2},z_{n+3})\\ &\quad + \mu(z_{n+1},z_m)\mu(z_{n+2},z_m)\mu(z_{n+3},z_m)\tilde{d}_{\beta,\mu}(z_{n+3},z_m).\\ &\le \beta(z_n,z_{n+1})\psi^n(\tilde{d}_{\beta,\mu}(z_0,z_1))\\ &\quad + \mu(z_{n+1},z_m)\beta(z_{n+1},z_{n+2})\psi^{n+1}(\tilde{d}_{\beta,\mu}(z_0,z_1))\\ &\quad + \mu(z_{n+1},z_m)\mu(z_{n+2},z_m)\beta(z_{n+2},z_{n+3})\psi^{n+2}(\tilde{d}_{\beta,\mu}(z_0,z_1))\\ &\quad \vdots\\ &\quad + \mu(z_{n+1},z_m)\mu(z_{n+2},z_m)\mu(z_{n+3},z_m)......\beta(z_{m-2},z_{m-1})\psi^{m-2}(\tilde{d}_{\beta,\mu}(z_0,z_1))\\ &\quad + \mu(z_{n+1},z_m)\mu(z_{n+2},z_m)\mu(z_{n+3},z_m)......\mu(z_{m-1},z_m)\psi^{m-1}(\tilde{d}_{\beta,\mu}(z_0,z_1)).\end{aligned} \tag{3.5}$$

$$\begin{aligned}\tilde{d}_{\beta,\mu}(z_n,z_m) &\le \beta(z_n,z_{n+1})\psi^n(\tilde{d}_{\beta,\mu}(z_0,z_1)) + \sum_{j=n+1}^{m-2}\psi^j(\tilde{d}_{\beta,\mu}(z_0,z_1))\prod_{i=n+1}^{j}\mu(z_i,z_m)\beta(z_j,z_{j+1})\\ &\quad + \prod_{i=n+1}^{m-1}\mu(z_i,z_m)\psi^{m-1}(\tilde{d}_{\beta,\mu}(z_0,z_1)).\end{aligned} \tag{3.6}$$

$$\begin{aligned}\tilde{d}_{\beta,\mu}(z_n,z_m) &\le \beta(z_n,z_{n+1})\psi^n(\tilde{d}_{\beta,\mu}(z_0,z_1)) + \sum_{j=n+1}^{m-1}\psi^j(\tilde{d}_{\beta,\mu}(z_0,z_1))\left(\prod_{i=n+1}^{j}\mu(z_i,z_m)\right)\beta(z_j,z_{j+1}).\\ &\le \beta(z_n,z_{n+1})\psi^n(\tilde{d}_{\beta,\mu}(z_0,z_1)) + (\Omega_{m-1}-\Omega_{n-1}),\end{aligned} \tag{3.7}$$

where

$$\Omega_p = \sum_{j=1}^{p}\psi^j(\tilde{d}_{\beta,\mu}(z_0,z_1))\prod_{i=1}^{j}\mu(z_i,z_m)\beta(z_j,z_{j+1}). \tag{3.8}$$

By the properties of the double controlled comparison functions $\psi \in \Psi_{\beta,\mu}$, we have that $\lim_{n\to+\infty}\psi^n(t)\beta(z_n,z_{n+1})$ is finite and converges. Moreover, using the ratio test on Ω_p, one can easily deduce that $\lim_{n,m\to+\infty}[\Omega_{m-1}-\Omega_{n-1}] = 0$. Therefore, $\{z_n\}_{n\ge 0}$ is a Cauchy sequence. The completeness of the DCMTS $(Z,\tilde{d}_{\beta,\mu})$ implies that $\{z_n\}$ converges to some $z \in Z$, i.e. $\lim_{n\to\infty}\tilde{d}_{\beta,\mu}(z_n,z) = 0$. By $(q3)$ of Definition 2.3, we obtain

$$\begin{aligned}\tilde{d}_{\beta,\mu}(z,Tz) &\le \beta(z,z_{n+1})\tilde{d}_{\beta,\mu}(z,z_{n+1}) + \mu(z_{n+1},Tz)\tilde{d}_{\beta,\mu}(z_{n+1},Tz).\\ &= \beta(z,z_{n+1})\tilde{d}_{\beta,\mu}(z,z_{n+1}) + \mu(z_{n+1},Tz)\tilde{d}_{\beta,\mu}(Tz_n,Tz).\end{aligned}$$

Taking the limit as n tends to infinity in the above inequality, and the fact that T is continuous, we have that $Tz_n \to Tz$, i.e. $\lim_{n\to\infty}\tilde{d}_{\beta,\mu}(Tz_n,Tz) = \tilde{d}_{\beta,\mu}(Tz,Tz) = 0$, so we obtain $\tilde{d}_{\beta,\mu}(z,Tz) = 0$; that is, $Tz = z$. Thus, T has a fixed point as desired.

To prove the uniqueness of the fixed point, assume T has two fixed points, say, a, b, i.e., $T(a) = a$ and $T(b) = b$, such that $\alpha(a,b) \ge 1$. Using the fact that T is an $(\alpha$-$\psi)$-double contractive mapping which is α-admissible, we obtain

$$\begin{aligned}\tilde{d}_{\beta,\mu}(a,b) &= \tilde{d}_{\beta,\mu}(Ta,Tb).\\ &\le \alpha(a,b)\tilde{d}_{\beta,\mu}(Ta,Tb).\\ &\le \psi(\tilde{d}_{\beta,\mu}(a,b))\end{aligned}$$

$$\vdots$$
$$\leq \psi^n(\tilde{d}_{\beta,\mu}(a,b)).$$

Since $\psi \in \Psi_{\beta,\mu}$, taking the limit as n tends to infinity in the above inequalities, we deduce by Lemma 3.1 that $\tilde{d}_{\beta,\mu}(a,b) = 0$, which implies that $a = b$. Thereby, T has a unique fixed point.

□

Example 3.8. Let $Z = [0,+\infty)$, and consider the mapping $\tilde{d}_{\beta,\mu} : Z \times Z \to [0,+\infty)$ defined by $\tilde{d}_{\beta,\mu}(x,y) = |x-y|$. Then $(Z,\tilde{d}_{\beta,\mu})$ is a complete DCMTS controlled by two functions $\beta,\mu : Z\times Z \to [1,+\infty)$, where

$$\beta(x,y) = max\{x,y\} + 1, \text{ and } \mu(x,y) = max\{x,y\} + 2.$$

Let the mapping $T : Z \to Z$ be defined by

$$T(x) = \begin{cases} 2x - \frac{17}{9} & \text{if } x > 1, \\ \frac{x}{9} & \text{if } 0 \leq x \leq 1. \end{cases}$$

We define two functions, $\alpha : Z \times Z \to (-\infty,+\infty)$ and $\psi(t) : [0,+\infty) \to [0,+\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x,y \in [0,1], \\ 0 & \text{otherwise,} \end{cases}$$

and $\psi(t) = \frac{1}{9}t$. Let $x_0 = 1$; then, $x_1 = T(1) = \frac{1}{9}$. Hence, $x_n = T^n(1) = \frac{1}{9^n}$ for all $n \geq 1$. Note that $\beta(x_n,x_{n+1}) \leq 2$, $\mu(x_n,x_{n+1}) \leq 3$, and $\psi^n(t) = \frac{1}{9^n}t$. One can easily show that

$$\sum_{n=1}^{+\infty} \psi^n(t) \prod_{i=1}^{n} \mu(z_i,z_m)\beta(z_n,z_{n+1}) \leq \sum_{n=1}^{+\infty} t(\frac{6}{9})^n < +\infty.$$

Similarly, one can show that $\lim_{n\to+\infty} \psi^n(t)\beta(z_n,z_{n+1}) < +\infty$. Therefore, $\psi \in \Psi_{\beta,\mu}$.

We will show the mapping T satisfies the conditions of Theorem 3.7. Clearly, T is continuous, and it is α-admissible since if $\alpha(x,y) \geq 1$, $x,y \in [0,1]$, so $Tx = \frac{x}{9} \in [0,1]$, and $Ty = \frac{y}{9} \in [0,1]$; consequently, $\alpha(Tx,Ty) \geq 1$. Furthermore, with $x_0 = 1$, we have that $\alpha(1,T(1)) = \alpha(1,\frac{1}{9}) = 1 \geq 1$. To verify that T is an (α -ψ)-double contractive mapping, consider any $x,y \in Z$; the case if $x > 1$ or $y > 1$ is trivial; thus, assume $x,y \in [0,1]$; we have

$$\alpha(x,y)\tilde{d}_{\beta,\mu}(Tx,Ty) = |\frac{x}{9} - \frac{y}{9}| = \frac{1}{9}|x-y| \leq \psi(\tilde{d}_{\beta,\mu}(x,y)).$$

Therefore, T satisfies all of the conditions of Theorem 3.7. Consequently, T has a fixed point, which is $x = 0$.

The following is an immediate consequence of Theorem 3.7.

Corollary 3.9. *Let $(Z,\tilde{d}_{\beta,\mu})$ be a complete DCMTS, and let $T : Z \to Z$ be an (α-ψ)-double contractive mapping satisfying the following conditions:*

(1) T is continuous.
(2) There exists $\psi \in \Psi_{\beta,\mu}$, such that $\tilde{d}_{\beta,\mu}(Tz,Tw) \leq \psi(\tilde{d}_{\beta,\mu}(z,w))$ for all $z,w \in Z$.

Then, T has a unique fixed point.

Proof. Define the function $\alpha : Z\times Z \to [0, +\infty)$ via $\alpha(z, w) = 1$. Then, T is α-admissible. Furthermore, T satisfies all of the conditions of Theorem 3.7, so T has a unique fixed point. □

Corollary 3.10. *Let $(Z, \tilde{d}_{\beta,\mu})$ be a complete DCMTS, and let $T : Z \to Z$ be an α-admissible map satisfying the following conditions:*

(1) There exists $z_0 \in Z$ such that $\alpha(z_0, Tz_0) \geq 1$.
(2) T is continuous.
(3) There exists $r \in [0, 1)$ such that for any sequence $\{z_n\}_{n=1}^{+\infty}$ in Z,

$$\sum_{n=1}^{+\infty}(rt)^n \prod_{i=1}^{n} \mu(z_i, z_m)\beta(z_n, z_{n+1}) < +\infty,$$

and $\lim_{n\to+\infty}(rt)^n\beta(z_n, z_{n+1}) < +\infty$ for all $t > 0$.
(4) For any $z, w \in Z$, the mapping T satisfies $\alpha(z, w)\tilde{d}_{\beta,\mu}(Tz, Tw) \leq r(\tilde{d}_{\beta,\mu}(z, w))$.

Then, T has a fixed point.

Proof. Define a map $\psi : [0, +\infty) \to [0, +\infty)$ via $\psi(t) = rt$. Then T is an (α-ψ) double contractive mapping. Moreover, T satisfies all of the conditions of Theorem 3.7; therefore, T has a fixed point. □

In the following theorem, the hypothesis of continuity is replaced with a weaker condition.

Theorem 3.11. *Let $(Z, \tilde{d}_{\beta,\mu})$ be a complete DCMTS, and let $T : Z \to Z$ be an (α-ψ)-double contractive mapping for some $\psi \in \Psi_{\beta,\mu}$. Suppose that the following conditions hold:*

(1) T is α-admissible.
(2) There exists $z_0 \in Z$ such that $\alpha(z_0, Tz_0) \geq 1$.
(3) If $\{z_n\}$ is a sequence in Z such that $\alpha(z_n, z_{n+1}) \geq 1$ and $z_n \to z$ as $n \to +\infty$, then $\alpha(z_n, z) \geq 1$ for all n.

Then, T has a fixed point. Moreover, if we assume that for all $x, y \in Z$, there exists $w \in Z$ such that $\alpha(x, w) \geq 1$ and $\alpha(y, w) \geq 1$, then T has a unique fixed point.

Proof. To prove this result, we repeat the same steps as in the proof of Theorem 3.7; thus, we construct a sequence $\{z_n\}$ that converges to a point $z \in Z$. The constructed sequence has the property $\alpha(z_n, z_{n+1}) \geq 1$, for all $n \in \mathbb{N}$. The last assumption of the result implies that $\alpha(z_n, z) \geq 1$. Finally, we prove that z is a fixed point for T. The double controlled triangle inequality implies that

$$\tilde{d}_{\beta,\mu}(z, Tz) \leq \beta(z, z_{n+1})\tilde{d}_{\beta,\mu}(z, z_{n+1}) + \mu(z_{n+1}, Tz)\tilde{d}_{\beta,\mu}(z_{n+1}, Tz). \tag{3.9}$$

Note that the first term on the right-hand side of (3.9), $\beta(z, z_{n+1})\tilde{d}_{\beta,\mu}(z, z_{n+1})$ converges to 0 since the sequence $\{z_n\}$ converges to 0. As for the second term, using the fact that $\alpha(z_n, z) \geq 1$, and by (3.2) of Theorem 3.7, we obtain

$$\begin{aligned}\mu(z_{n+1}, Tz)\tilde{d}_{\beta,\mu}(z_{n+1}, Tz) &\leq \mu(z_{n+1}, Tz)\tilde{d}_{\beta,\mu}(Tz_n, Tz)\alpha(z_n, z),\\ &\leq \mu(z_{n+1}, Tz)\psi(\tilde{d}_{\beta,\mu}(z_n, z)), \leq \mu(z_{n+1}, Tz)\tilde{d}_{\beta,\mu}(z_n, z).\end{aligned}$$

Hence, $\mu(z_{n+1}, Tz)\tilde{d}_{\beta,\mu}(z_{n+1}, Tz)$ converges to 0, as n tends to infinity. Therefore, $\tilde{d}_{\beta,\mu}(z, Tz) = 0$, and z is a fixed point for T.

To prove the uniqueness of the fixed point, suppose a, b are two fixed points of T, i.e. $T(a) = a$, and $T(b) = b$. Hence, there exists $w \in Z$ such that $\alpha(a, w) \geq 1$ and $\alpha(b, w) \geq 1$. Since T is α-admissible, we obtain $\alpha(a, T^n w) \geq 1$, and $\alpha(b, T^n w) \geq 1$, for all $n \in \mathbb{N}$. Thus we have

$$\begin{aligned}
\tilde{d}_{\beta,\mu}(a, T^n w) &= \tilde{d}_{\beta,\mu}(Ta, T^n w). \\
&\leq \alpha(a, T^{n-1}w)\tilde{d}_{\beta,\mu}(Ta, T^n w)). \\
&\leq \psi(\tilde{d}_{\beta,\mu}(a, T^{n-1}w)). \\
&\vdots \\
&\leq \psi^n(\tilde{d}_{\beta,\mu}(a, w)).
\end{aligned}$$

Since $\psi \in \Psi_{\beta,\mu}$, taking the limit as n tends to infinity in the above inequalities we deduce that $\lim_{n\to+\infty} T^n w = a$. Next, repeating the same process with the other fixed point, we obtain $\lim_{n\to+\infty} T^n w = b$. From the uniqueness of the limit, we deduce that $a = b$. Hence, T has a unique fixed point. □

4. Wardowski and (α-$\mathcal{F}$)-contractions and fixed-point theorems

A new type of contractions known as $\mathcal{F}$-contractions was introduced by Wardowski in 2012 [23], which resulted in establishing new fixed point theorems in complete metric spaces.

Definition 4.1. Let $\mathcal{F}$ be the family of all functions $F : (0, +\infty) \to (-\infty, +\infty)$ satisfying the following:
($F1$) F is a strictly increasing function.
($F2$) For each sequence $\{s_n\}$ of postive real numbers, the following holds:

$$\lim_{n\to+\infty} s_n = 0 \iff \lim_{n\to+\infty} F(s_n) = -\infty.$$

($F3$) There exists $k \in (0, 1) \ni \lim_{s\to 0^+} s^k F(s) = 0$.

The family $\mathcal{F}$ is not empty, as the below examples illustrate.

Example 4.2. Consider the following functions: $G(s) = ln(s)$, $H(s) = \frac{-1}{\sqrt{s}}$ and $L(s) = ln(s) + s$ for $s > 0$. Clearly each one of these functions satisfies the conditions $(F1), (F2)$ and $(F3)$; thus, they belong to $\mathcal{F}$. For more details, consult [23].

Definition 4.3. A self mapping $T : Z \to Z$ defined on a DCMTS $(Z, \tilde{d}_{\beta,\mu})$, where Z is a nonempty set, is said to be an $\mathcal{F}$-contraction if there exist a function $F \in \mathcal{F}$ and a constant $\tau > 0$ such that the following holds:

$$\tilde{d}_{\beta,\mu}(Tx, Ty) > 0 \implies \tau + F(\tilde{d}_{\beta,\mu}(Tx, Ty)) \leq F(\tilde{d}_{\beta,\mu}(x, y)), \text{ for all } x, y \in Z. \quad (4.1)$$

Next, we define the (α- $\mathcal{F}$)-contraction mappings on a DCMTS.

Definition 4.4. Let $(Z, \tilde{d}_{\beta,\mu})$ be a DCMTS, where Z is a nonempty set. A self-mapping $T : Z \to Z$ is said to be an (α- $\mathcal{F}$)-contraction mapping, if there exists a mapping $\alpha : Z \times Z \to [0, +\infty)$, $F \in \mathcal{F}$ and a constant $\tau > 0$, such that the following holds

$$\tau + \alpha(x, y)F(\tilde{d}_{\beta,\mu}(Tx, Ty)) \leq F(\tilde{d}_{\beta,\mu}(x, y)), \quad (4.2)$$

for all $x, y \in Z$, with $\tilde{d}_{\beta,\mu}(Tx, Ty) > 0$.

Next, we state our main fixed point theorem.

Theorem 4.5. *Let $(Z, \tilde{d}_{\beta,\mu})$ be a complete DCMTS, and let $T : Z \to Z$ be (α- $\mathcal{F}$)-contractive mapping, such that the following holds:*

(1) T is α-admissible.
(2) There exists $z_0 \in Z$ such that $\alpha(z_0, Tz_0) \geq 1$.
(3) T is continuous.
(4) For $z_0 \in Z$, define the seqeunce $\{z_n\}$ by $z_n = T^n z_0$, and assume

$$\sup_{m\geq 1} \lim_{i\to+\infty} \frac{\beta(z_{i+1}, z_{i+2})}{\beta(z_i, z_{i+1})}\mu(z_{i+1}, z_m) < 1. \tag{4.3}$$

In addition, for each $z \in Z$

$$\lim_{n\to+\infty} \beta(z, z_n) \text{ and } \lim_{n\to+\infty} \mu(z_n, z) \text{ exist and are finite.} \tag{4.4}$$

Then, T has a fixed point. Moreover, if for any two fixed points of T in Z, say ξ, and η with $\alpha(\xi, \eta) \geq 1$, then T has a unique fixed point in Z.

Proof. Let z_0 be chosen as in Condition (2), where $\alpha(z_0, Tz_0) \geq 1$. Define the sequence $\{z_n\}$ as follows, let $Tz_0 = z_1, T^2z_0 = Tz_1 = z_2$; thus, for any $n \in \mathbb{N}$, we obtain

$$T^n z_0 = T^{n-1} z_1 = \cdots = Tz_{n-1} = z_n.$$

Note that if there exists n, such that $z_n = z_{n+1}$, then we are done and z_n is the fixed point of T. Therefore, we may assume that $z_n \neq z_{n+1}$ for all $n \geq 0$.

As T is α-admissible, for all $n \geq 0$, $\alpha(z_n, z_{n+1}) \geq 1$. Now, using (4.2), we have

$$\begin{aligned} \tau + F(\tilde{d}_{\beta,\mu}(z_n, z_{n+1})) &= \tau + F(\tilde{d}_{\beta,\mu}(Tz_{n-1}, z_{n+1})). \\ &\leq \tau + \alpha(z_n, z_{n+1})F(\tilde{d}_{\beta,\mu}(Tz_{n-1}, Tz_n)). \\ &\leq F(\tilde{d}_{\beta,\mu}(z_{n-1}, z_n)), \end{aligned}$$

which gives $F(\tilde{d}_{\beta,\mu}(z_n, z_{n+1})) \leq F(\tilde{d}_{\beta,\mu}(z_{n-1}, z_n)) - \tau$. Repeating the processing several times, we obtain

$$\begin{aligned} F(\tilde{d}_{\beta,\mu}(z_n, z_{n+1})) &\leq F(\tilde{d}_{\beta,\mu}(z_{n-1}, z_n)) - \tau. \\ &\leq F(\tilde{d}_{\beta,\mu}(z_{n-2}, z_{n-1})) - 2\tau. \\ &\leq \cdots \leq F(\tilde{d}_{\beta,\mu}(z_0, z_1)) - n\tau. \end{aligned} \tag{4.5}$$

Letting $n \to +\infty$ in (4.5), and since $\tau > 0$, we have

$$\lim_{n\to+\infty} F(\tilde{d}_{\beta,\mu}(z_n, z_{n+1})) = -\infty. \tag{4.6}$$

Since $F \in \mathcal{F}$, by (F2), it follows that $\lim_{n\to+\infty} \tilde{d}_{\beta,\mu}(z_n, z_{n+1}) = 0$. By (F3), there exists $k \in (0, 1)$, such that

$$\lim_{n\to+\infty} (\tilde{d}_{\beta,\mu}(z_n, z_{n+1}))^k F(\tilde{d}_{\beta,\mu}(z_n, z_{n+1})) = 0. \tag{4.7}$$

From (4.5), we get

$$F(\tilde{d}_{\beta,\mu}(z_n, z_{n+1})) - F(\tilde{d}_{\beta,\mu}(z_0, z_1)) \leq -n\tau.$$

Which implies

$$\begin{aligned}(\tilde{d}_{\beta,\mu}(z_n, z_{n+1}))^k F(\tilde{d}_{\beta,\mu}(z_n, z_{n+1})) - (\tilde{d}_{\beta,\mu}(z_n, z_{n+1}))^k F(\tilde{d}_{\beta,\mu}(z_0, z_1)) \\ \leq -n\tau(\tilde{d}_{\beta,\mu}(z_n, z_{n+1}))^k \leq 0. \end{aligned} \tag{4.8}$$

Taking the limit as n tends to infinity in (4.8), we have

$$\lim_{n\to+\infty} n((\tilde{d}_{\beta,\mu}(z_n, z_{n+1}))^k = 0. \tag{4.9}$$

Therefore, $\lim_{n\to+\infty} n^{1/k}(\tilde{d}_{\beta,\mu}(z_n, z_{n+1})) = 0$, so there exists some $n_0 \in \mathbb{N}$, such that

$$\tilde{d}_{\beta,\mu}(z_n, z_{n+1}) \leq \frac{1}{n^{1/k}}, \ for\ all\ n \geq n_0. \tag{4.10}$$

For all $m, n \in \mathbb{N}$ with $m > n$, and using the triangular inequality as in (3.3), we deduce:

$$\begin{aligned}
\tilde{d}_{\beta,\mu}(z_n, z_m) &\leq \beta(z_n, z_{n+1})\tilde{d}_{\beta,\mu}(z_n, z_{n+1}) + \mu(z_{n+1}, z_m)\tilde{d}_{\beta,\mu}(z_{n+1}, z_m). \\
&\leq \beta(z_n, z_{n+1})\tilde{d}_{\beta,\mu}(z_n, z_{n+1}) + \mu(z_{n+1}, z_m)[\beta(z_{n+1}, z_{n+2})\tilde{d}_{\beta,\mu}(z_{n+1}, z_{n+2}) \\
&\quad + \mu(z_{n+2}, z_m)\tilde{d}_{\beta,\mu}(z_{n+2}, z_m)]. \\
&= \beta(z_n, z_{n+1})\tilde{d}_{\beta,\mu}(z_n, z_{n+1}) + \mu(z_{n+1}, z_m)\beta(z_{n+1}, z_{n+2})\tilde{d}_{\beta,\mu}(z_{n+1}, z_{n+2}) \\
&\quad + \mu(z_{n+1}, z_m)\mu(z_{n+2}, z_m)\tilde{d}_{\beta,\mu}(z_{n+2}, z_m). \\
&\leq \beta(z_n, z_{n+1})\tilde{d}_{\beta,\mu}(z_n, z_{n+1}) + \mu(z_{n+1}, z_m)\beta(z_{n+1}, z_{n+2})\tilde{d}_{\beta,\mu}(z_{n+1}, z_{n+2}) \\
&\quad + \mu(z_{n+1}, z_m)\mu(z_{n+2}, z_m)[\beta(z_{n+2}, z_{n+3})\tilde{d}_{\beta,\mu}(z_{n+2}, z_{n+3}) \\
&\quad + \mu(z_{n+3}, z_m)\tilde{d}_{\beta,\mu}(z_{n+3}, z_m)]. \\
&\ \vdots \\
\tilde{d}_{\beta,\mu}(z_n, z_m) &\leq \beta(z_n, z_{n+1})\tilde{d}_{\beta,\mu}(z_n, z_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\mu(z_j, z_m)\right)\beta(z_i, z_{i+1})\tilde{d}_{\beta,\mu}(z_i, z_{i+1}) \\
&\quad + \prod_{k=n+1}^{m-1}\mu(z_k, z_m)\tilde{d}_{\beta,\mu}(z_{m-1}, z_m). \\
&\leq \beta(z_n, z_{n+1})\tilde{d}_{\beta,\mu}(z_n, z_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\mu(z_j, z_m)\right)\beta(z_i, z_{i+1})\tilde{d}_{\beta,\mu}(z_i, z_{i+1}) \\
&\quad + \prod_{k=n+1}^{m-1}\mu(z_k, z_m)\beta(z_{m-1}, z_m)\tilde{d}_{\beta,\mu}(z_{m-1}, z_m). \\
\tilde{d}_{\beta,\mu}(z_n, z_m) &\leq \beta(z_n, z_{n+1})\tilde{d}_{\beta,\mu}(z_n, z_{n+1}) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\mu(z_j, z_m)\right)\beta(z_i, z_{i+1})\tilde{d}_{\beta,\mu}(z_i, z_{i+1}). \\
&\leq \beta(z_n, z_{n+1})\tilde{d}_{\beta,\mu}(z_n, z_{n+1}) + \sum_{i=n+1}^{m-1}\left(\prod_{j=0}^{i}\mu(z_j, z_m)\right)\beta(z_i, z_{i+1})\tilde{d}_{\beta,\mu}(z_i, z_{i+1}).
\end{aligned}$$

Applying (4.10) in the last inequality implies $\tilde{d}_{\beta,\mu}(z_i, z_{i+1}) \leq \frac{1}{i^{1/k}}$; we obtain

$$\tilde{d}_{\beta,\mu}(z_n, z_m) \leq \beta(z_n, z_{n+1})(\frac{1}{n^{1/k}}) + \sum_{i=n+1}^{m-1}\left(\prod_{j=0}^{i}\mu(z_j, z_m)\right)\beta(z_i, z_{i+1})(\frac{1}{i^{1/k}}).$$
$$= \beta(z_n, z_{n+1})(\frac{1}{n^{1/k}}) + [\Delta_{m-1} - \Delta_n], \tag{4.11}$$

where

$$\Delta_p = \sum_{i=1}^{p}\left(\prod_{j=0}^{i}\mu(z_j, z_m)\right)\beta(z_i, z_{i+1})(\frac{1}{i^{1/k}}). \tag{4.12}$$

By using the ratio test and applying (4.3), and the fact that $(\frac{i}{i+1})^{1/k} < 1$ as $k \in (0, 1)$, we obtain $\lim_{n,m\to+\infty}[\Delta_{m-1} - \Delta_n] = 0$. Furthermore, by (4.4) we deduce that $\lim_{n\to+\infty}\beta(z_n, z_{n+1})(\frac{1}{n^{1/k}}) = 0$. Hence

$$\lim_{n,m\to+\infty}\tilde{d}_{\beta,\mu}(z_n, z_m) = 0. \tag{4.13}$$

Thus, $\{z_n\}$ is a Cauchy sequence in a complete DCMTS $(Z, \tilde{d}_{\beta,\mu})$; therefore, it converges to some $\xi \in Z$.

Next, we show that ξ is a fixed point of T, i.e. $T\xi = \xi$. Since T is continuous and $\lim_{n\to+\infty}\tilde{d}_{\beta,\mu}(z_n, \xi) = 0$, we have that $\lim_{n\to+\infty}\tilde{d}_{\beta,\mu}(Tz_n, T\xi) = 0$. This gives us

$$\tilde{d}_{\beta,\mu}(\xi, T\xi) = \lim_{n\to+\infty}\tilde{d}_{\beta,\mu}(z_{n+1}, T\xi) = \lim_{n\to+\infty}\tilde{d}_{\beta,\mu}(Tz_n, T\xi) = 0; \tag{4.14}$$

hence $T\xi = \xi$.

To prove the uniqueness of the fixed point, assume there exist two fixed points, ξ and η, such that $\xi \neq \eta$ and $\alpha(\xi, \eta) \geq 1$. Hence $T\xi = \xi \neq \eta = T\eta$ implies that $\tilde{d}_{\beta,\mu}(T\xi, T\eta) > 0$. For $\tau > 0$, Eq (4.2) gives

$$\begin{aligned}\tau + F(\tilde{d}_{\beta,\mu}(T\xi, T\eta)) &\leq \tau + \alpha(\xi, \eta)F(\tilde{d}_{\beta,\mu}(T\xi, T\eta)).\\ &\leq F(\tilde{d}_{\beta,\mu}(\xi, \eta)) = F(\tilde{d}_{\beta,\mu}(T\xi, T\eta)),\end{aligned}$$

which indicates that $\tau \leq 0$, thus reaching a contradiction. Hence $\xi = \eta$ and the fixed point is unique. □

Next, we present a supporting example for Theorem 4.5.

Example 4.6. Let $Z = [0, +\infty)$ and consider the mapping $\tilde{d}_{\beta,\mu} : Z\times Z \to [0, +\infty)$ defined by $\tilde{d}_{\beta,\mu}(x, y) = |x - y|$. Then $(Z, \tilde{d}_{\beta,\mu})$ is a complete DCMTS, where $\beta, \mu : Z \times Z \to [1, +\infty)$ are defined by $\beta(x, y) = max\{x, y\} + 1$, and

$$\mu(x, y) = \begin{cases} x + y & \text{if } x \in [0, 1],\\ 1 & \text{if } x > 1.\end{cases}$$

Let the mapping $T : Z \to Z$ be

$$T(x) = \begin{cases} \frac{x}{3} & \text{if } x \in [0, 1],\\ x - \frac{2}{3} & \text{if } x > 1.\end{cases}$$

Let $\alpha : Z \times Z \to (-\infty, +\infty)$, and $F : (0, +\infty) \to (-\infty, +\infty)$ be defined by,

$$\alpha(x,y)=\begin{cases}1 & \text{if } x,y\in[0,1],\\ 0 & \text{otherwise,}\end{cases}$$

and $F(t)=ln(t)$. Clearly, T is continuous, and it is α-admissible since if $\alpha(x,y)\geq 1$, $x,y\in[0,1]$; consequently, $\alpha(Tx,Ty)=\alpha(\frac{x}{3},\frac{y}{3})\geq 1$. To show that T is an $(\alpha\text{-}\mathcal{F})$-contraction mapping, we only need to look at the case when $x,y\in[0,1]$. Note that

$$\tilde{d}_{\beta,\mu}(Tx,Ty)=|Tx-Ty|=\frac{1}{3}|x-y|=\frac{1}{3}\tilde{d}_{\beta,\mu}(x,y)<\frac{2}{3}\tilde{d}_{\beta,\mu}(x,y).$$

Hence

$$ln(\frac{3}{2})+\alpha(x,y)ln(\tilde{d}_{\beta,\mu}(Tx,Ty))\leq ln(\frac{3}{2})+ln(\tilde{d}_{\beta,\mu}(Tx,Ty))\leq ln(\tilde{d}_{\beta,\mu}(x,y)). \tag{4.15}$$

By taking $\tau=ln(\frac{3}{2})>0$ in (4.15), we have

$$\tau+\alpha(x,y)F(\tilde{d}_{\beta,\mu}(Tx,Ty))\leq F(\tilde{d}_{\beta,\mu}(x,y)).$$

Hence, T is an $(\alpha\text{-}\mathcal{F})$-contraction mapping. Let $x_0=1$; then, $\alpha(x_0,Tx_0)\geq 1$. We form a sequence by $x_1=T(x_0)=T(1)=\frac{1}{3}$; hence, $x_n=T^n(x_0)=T^n(1)=\frac{1}{3^n}$ for all $n\geq 1$. Finally to show that (4.3) and (4.4) hold, observe that

$$\sup_{m\geq 1}\lim_{i\to+\infty}\frac{\beta(x_{i+1},x_{i+2})}{\beta(x_i,x_{i+1})}\mu(x_{i+1},x_m)=\sup_{m\geq 1}\lim_{i\to+\infty}(\frac{max\{\frac{1}{3^{i+1}},\frac{1}{3^{i+2}}\}+1)}{max\{\frac{1}{3^i},\frac{1}{3^{i+1}}\}+1)})(\frac{1}{3^{i+1}}+\frac{1}{3^m})<1. \tag{4.16}$$

Moreover for any $x\in Z$, both $\lim_{n\to+\infty}\beta(x,x_n)$ and $\lim_{n\to+\infty}\mu(x_n,x)$ exist and are finite. Therefore, T satisfies all of the conditions of Theorem 4.5. Consequently, T has a fixed point, which is $x=0$.

The following is an immediate consequence of Theorem 4.5.

Corollary 4.7. *Let $(Z,\tilde{d}_{\beta,\mu})$ be a complete DCMTS, and let $T:Z\to Z$ be a continuous mapping satisfying the following:*

$$\tau+F(\tilde{d}_{\beta,\mu}(T\xi,T\eta))\leq F(\tilde{d}_{\beta,\mu}(\xi,\eta)),\ \textit{for all}\ \xi,\eta\in Z. \tag{4.17}$$

Let $\xi_0\in Z$ and consider the sequence $\xi_n=T^n\xi_0$. Suppose

$$\sup_{m\geq 1}\lim_{i\to+\infty}\frac{\beta(\xi_{i+1},\xi_{i+2})}{\beta(\xi_i,\xi_{i+1})}\mu(\xi_{i+1},\xi_m)<1. \tag{4.18}$$

Also, assume the following holds

$$\lim_{n\to+\infty}\beta(\xi,\xi_n)\ \textit{and}\ \lim_{n\to+\infty}\mu(\xi_n,\xi)\ \textit{exist and are finite for every}\ \xi\in Z. \tag{4.19}$$

Then, T has a unique fixed point.

Proof. Let the map $\alpha:Z\times Z\to[0,+\infty)$ be defined by $\alpha(\xi,\eta)=1$ for all $\xi,\eta\in Z$. Repeat the proof of Theorem 4.4 by considering the defined α. □

Letting $\beta=\mu$, in Theorem 4.5, we obtain the following corollary; for comparison, see [25].

Corollary 4.8. *Let $(Z, \tilde{d}_\beta)$ be a complete DCMTS, and let $T : Z \to Z$ be an $(\alpha$-$\mathcal{F})$-contractive mapping, such that the following holds:*

(1) T is α-admissible.
(2) There exists $z_0 \in Z$ such that $\alpha(z_0, Tz_0) \geq 1$.
(3) T is continuous.
(4) For $z_0 \in Z$, define the sequence $\{z_n\}$ by $z_n = T^n z_0$, and assume

$$\sup_{m\geq 1} \lim_{i\to+\infty} \frac{\beta(z_{i+1}, z_{i+2})}{\beta(z_i, z_{i+1})} \beta(z_{i+1}, z_m) < 1. \tag{4.20}$$

In addition, for each $z \in Z$

$$\lim_{n\to+\infty} \beta(z, z_n) \text{ and } \lim_{n\to+\infty} \beta(z_n, z) \text{ exist and are finite}. \tag{4.21}$$

Then, T has a fixed point.

Proof. In Theorem 4.5, let $\mu = \beta$ and repeat the proof. □

5. Application to Fredholm type integral equation

Let $Z = C([0,1], \mathbb{R})$ be the space of all continuous real valued functions defined on the interval $[0, 1]$. Let the mapping $\tilde{d}_{\beta,\mu} : Z \times Z \longrightarrow [0, +\infty)$ be defined by

$$\tilde{d}_{\beta,\mu}(f, g) = max_{t\in[0,1]} |\frac{f(t) - g(t)}{2}|.$$

The two controlled functions $\beta, \mu : Z \times Z \to [1, +\infty)$ are defined by $\beta(f, g) = 1$ and $\mu(f, g) = 2, respectively$. It is not difficult to see that $(Z, \tilde{d}_{\beta,\mu})$ is a complete DCMTS.

The Fredholm type integral equation is defined as

$$f(t) = \int_0^1 \chi(t, \xi, f(\xi)) d\xi \text{ for } t, \xi \in [0, 1], \tag{5.1}$$

where $\chi(t, \xi, f(\xi)) : [0, 1]^2 \longrightarrow \mathbb{R}$ is a continuous function.

Theorem 5.1. *Consider $(Z, \tilde{d}_{\beta,\mu})$ a complete DCMTS, as defined above, and assume for any $f, g \in Z$ the following condition holds:*

$$|\chi(t, \xi, f(\xi)) - \chi(t, \xi, g(\xi))| \leq \lambda |f(\xi) - g(\xi)|, \textit{ for some } \lambda \in (0, 1).$$

Then, the integral equation (5.1) has a unique solution.

Proof. Let the mapping $T : Z \longrightarrow Z$ be defined as a Fredholm type integral equation:

$$Tf(t) = \int_0^1 \chi(t, \xi, f(\xi)) d\xi, \text{ for } t, \xi \in [0, 1]. \tag{5.2}$$

Define the mapping $\alpha : Z \times Z \to [0, +\infty)$ by

$$\alpha(f,g)=\begin{cases}1/2 & \text{if } f=g,\\ 1/4 & \text{if } f=0, \text{or } g=0,\\ |g|/|f| & \text{if } f>g,\\ 1 & \text{if } f<g.\end{cases}$$

Let $\psi(t):[0,+\infty)\to[0,+\infty)$ be defined by $\psi(t)=\lambda t$. For any $t\in[0,1]$, we have

$$\begin{aligned}\Big|\frac{Tf(t)-Tg(t)}{2}\Big| &= \frac{1}{2}\Big|\int_0^1 \chi(t,\xi,f(\xi))d\xi-\int_0^1 \chi(t,\xi,g(\xi))d\xi\Big|\\ &= \frac{1}{2}\Big|\int_0^1 \Big[\chi(t,\xi,f(\xi))-\chi(t,\xi,g(\xi))\Big]d\xi\Big|\\ &\le \frac{1}{2}\int_0^1 \Big|\chi(t,\xi,f(\xi))-\chi(t,\xi,g(\xi))\Big|d\xi\\ &\le \frac{1}{2}\int_0^1 \lambda|f(\xi)-g(\xi)|d\xi\\ &\le \lambda\,\tilde{d}_{\beta,\mu}(f,g)\\ &= \psi(\tilde{d}_{\beta,\mu}(f,g)),\end{aligned}$$

which gives

$$\tilde{d}_{\beta,\mu}(Tf,Tg)=max_{t\in[0,1]}\Big|\frac{Tf(t)-Tg(t)}{2}\Big|\le\psi(\tilde{d}_{\beta,\mu}(f,g)).$$

Hence,

$$\alpha(f,g)\tilde{d}_{\beta,\mu}(Tf,Tg)\le\psi(\tilde{d}_{\beta,\mu}(f,g)).$$

Therefore, T satisfies the hypothesis of Theorem 3.7, which implies the integral equation (5.1) has a unique solution as desired. □

6. Conclusions

In this article, we dealt with the notion of DCMTS, which was introduced by Mlaiki et al. [5]. Thus, we explored two types of contraction mappings in the complete DCMTS and established some fixed-point theorems in this setting. Furthermore, we presented some examples and an application of our result.

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Conflict of interest

The author declares no conflict of interest.

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