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Research article

On a coupled system of fractional (p, q)-differential equation with Lipschitzian matrix in generalized metric space

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Abstract: This work is concerned with the study of the existing solution for the fractional (p, q)-difference equation under first order (p, q)-difference boundary conditions in generalized metric space. To achieve the solution, we combine some contraction techniques in fixed point theory with the numerical techniques of the Lipschitz matrix and vector norms. To do this, we first associate a matrix to a desired boundary value problem. Then we present sufficient conditions for the convergence of this matrix to zero. Also, we design some algorithms to use the computer for calculate the eigenvalues of such matrices and different values of (p, q)-Gamma function. Finally, by presenting two numerical examples, we examine the performance and correctness of the proposed method. Some tables and figures are provided to better understand the issues.

Keywords: (p, q)-difference; boundary value problem; fixed point; Lipschitzian matrix; quantum calculus
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1. Introduction

As we know, to solve an array of problems in STEM fields such as mathematics, physics and engineering, we have to model the phenomena by differential systems. It is clear that, the accuracy and efficiency of the proposed model depends on several factors. For this reason, researchers have always tried to optimize their methods. One of the new methods that have recently seen dramatic growth in the study of BVPs is the use of non-integer derivatives. Perhaps the reason for this increase is the efficiency of fractional derivatives in maintaining system memory and its non-localization [1]. This high potential of fractional derivatives has led to the study of the theory of fractional operators from different perspectives and various generalizations, the most famous of which are the fractional derivatives of Riemann-Liouville and Caputo [2], and Hadamard and Caputo Fabrizio [3, 4]. In bio-mathematics, for example, some researchers have developed models for the mumps virus [5], hepatitis B [6–8], human liver [9], and COVID-19 [10, 11] using fractional calculus. In thermodynamics, models for thermostats using red the Caputo fraction derivatives, and Riemann Liouville were presented under different conditions and the stability of these models were investigated [12–14]. Some important equations in physics such as Schrödinger [15, 16], Sturm-Liouville [17–19], Pantograph [20–22], Langevin [23–26], etc. were also studied from different aspects in this field. See [27–39], for more contributions on fractional calculus.

On other hand, the history of mathematics and physics is somehow intertwined with generalization. One of these common generalizations relates to the work of the English mathematician Frank Hilton Jackson in removing the concept of limit from derivative. In 1910, he laid the foundations for the exciting world of quantum calculus with the introduction of the q-derivative [40, 41]. The concept of h-derivative was later introduced, but its growth and application were not as great as q-derivative. The basic topics related to these two types of derivatives are discussed in detail in the book "Quantum Calculus" [42]. The concepts of q-derivative, and q-integral were later developed by other researchers [43, 44]. This led to the development of quantum fractional calculus. Also, due to the possibility of using computers in discrete spaces, the fractional q-differential equations have been given special attention by researchers in the last decade. For example, in 2011, the existence of positive solutions for BVPs with fractional q-difference equation was investigated by El-Shaed, Ferreira, and Ma et al. [45–47]. Shabibi et al. studied analytical and numerical solutions for q-differential inclusion via new integral boundary conditions [48]. See [49–53], for more information.

The generalization of the derivative operator did not end with the q-derivative only, and long after the q-derivative itself was generalized. In 2004 [54], Remmel and Wachs presented (p, q)-analogues for Stirling numbers inspired by q-analogs from quantum calculus. Later in 2015 [55], Mursaleen et al. investigated (p, q)-analogs of Bernstein operators. In 2018 [56], Sadjang presented fundamental theorem of (p, q)-calculus and some (p, q)-Taylor formulas. Soontharanon and Sitthiwirattham, in 2020, reviewed some properties of fractional (p, q)-calculus [57]. After their work, some researchers investigated the boundary value problems using (p, q)-calculus. One can find more contributions about this topic in [58–63]. Promsakon et al. studied the following (p, q)-difference BVP of second order:

$$\begin{cases} D^2_{p,q}u(t) = \mathfrak{m}\left(t, u\left(p^2 t\right)\right), \quad t \in \left[0, \mathcal{T}/p^2\right], \\ \mu_1 u(0) + \mu_2 D_{p,q}u(0) = \mu_3, \quad \kappa_1 u(\mathcal{T}) + \kappa_2 D_{p,q}u(\mathcal{T}/p) = \kappa_3, \end{cases}$$

whit constants μ_j, κ_j, $j = 1, 2, 3$, and $D_{p,q}$ denote the (p, q)-difference operator, $\mathfrak{m} \in C([0, \mathcal{T}/p^2] \times \mathbb{R}, \mathbb{R})$ [59]. As mentioned at the beginning, fractional calculus is preferable to ordinary calculus due to its lower error rates in studying and modeling natural phenomena, especially in computer calculations and simulations. For this reason, we do not want to be deprived of this advantage in this research.

To the best of our knowledge, the fractional coupled system of (p, q)-difference equations in generalized metric space using the Lipschitzian matrix has not been investigated properly. Therefore, taking the idea from the above topics in the present work, we want to examine the following BVP involving the Caputo fractional (p, q)-difference operator:

$$\begin{cases} {}^{c}D_{p,q}^{\zeta_1}u(t) = \mathfrak{m}\left(t, u\left(p^{\zeta}t\right), v\left(p^{\zeta}t\right)\right), \ \ t \in \left[0, \mathcal{T}/p^{\zeta}\right], \ 1 < \zeta_1 \leq 2, \\ {}^{c}D_{p,q}^{\zeta_2}v(t) = \mathfrak{n}\left(t, u\left(p^{\zeta}t\right), v\left(p^{\zeta}t\right)\right), \ \ t \in \left[0, \mathcal{T}/p^{\zeta}\right], \ 1 < \zeta_2 \leq 2, \\ u(0) = u'(0) = 0, \quad v(0) = v'(0) = 0, \\ D_{p,q}u(\mathcal{T}/p) = \mu_1 D_{p,q}u(\eta_1), \ \ D_{p,q}v(\mathcal{T}/p) = \mu_2 D_{p,q}v(\eta_2), \end{cases} \tag{1.1}$$

where $\zeta = \max\{\zeta_1, \zeta_2\}$, $J = \left[0, \mathcal{T}/p^{\zeta}\right]$, with constants μ_j, η_j $(j = 1, 2)$, and $\mathfrak{m}, \mathfrak{n} \in C\left(J \times \mathbb{R}^2, \mathbb{R}\right)$, ${}^{c}D_{p,q}^{\zeta}$ and $D_{p,q}$ denote Caputo fractional (p, q)-derivative and first-order (p, q)-difference operator, respectively. The novelty of our method is that, at first, we associate a square matrix to the desired BVP such that its element depended on fraction order and quantum parameters (p, q). Then we will prove the existence of the solution using the fixed point theory.

2. Preliminaries

This section covers the basic concepts of quantum calculus and (p, q)-calculus that we will need to present our main results. There are also some important theorems of fixed point theory that are necessary to discuss the existence and uniqueness of the solution.

Assume that $J = [a, b] \subset \mathbb{R}$ and $p, q \in (0, 1]$. Also let $C(J, \mathbb{R})^2 := C(J, \mathbb{R}) \times C(J, \mathbb{R})$ equipped with the vector norm $\|\cdot\|$ defined by $\|z\| = (\|u\|_\infty, \|v\|_\infty)$ or norm $\|\cdot\|_X$ defined by $\|z\|_X = \left(\|u\|_\infty^2 + \|v\|_\infty^2\right)^{\frac{1}{2}}$ for $x = (u, v)$, where $\|w\|_\infty = \max_{t\in J} |w(t)|$ for $w \in C(J, \mathbb{R})$. It is obvious that $\left(C(J, \mathbb{R})^2, \|\cdot\|\right)$ or $\left(C(J, \mathbb{R})^2, \|\cdot\|_X\right)$ is a Banach space.

Definition 2.1. *[57] Let z be a real number and $0 < p, q < 1$, then the p, q-analogue of z is defined in the following manner*

$$[z]_{p,q} = \frac{p^z - q^z}{p - q}, \quad z \in \mathbb{N}. \tag{2.1}$$

Also, for the power function $(c - d)_{p,q}^{(n)}$, it's p, q-analogue with $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ reads as follow:

$$\begin{cases} (c - d)_{p,q}^{(0)} = 1, \\ (c - d)_{p,q}^{(n)} := \prod_{j=0}^{n-1}\left(cp^j - dq^j\right), \quad c, d \in \mathbb{R}. \end{cases}$$

Definition 2.2. *[57] Let $z \in \mathbb{R}$, the p, q-Gamma fuction for z is defined in the following manner*

$$\Gamma_{p,q}(z) = \frac{(p - q)_{p,q}^{(z-1)}}{(p - q)^{z-1}},$$

Note that, $\Gamma_{p,q}(z + 1) = [z]_{p,q}\Gamma_{p,q}(z)$, is valid. We presented the following Algorithm to compute the $\Gamma_{p,q}(z)$ function. Some numerical result for this function presented in Tables 1 and 2.

Table 1. Some numerical results for $\Gamma_{p,q}$ function values, with $p = 0.95$ and $z = 2.5$.

r	$q = 0.2$	$q = 0.5$	$q = 0.6$	$q = 0.7$	$q = 0.8$	$q = 0.9$
			$p = 0.95, z = 2.5$			
1	1.4766	2.6273	3.4863	5.1282	9.2364	29.7243
2	1.5263	2.9606	4.0159	6.0060	10.9431	35.4698
3	1.5368	3.1276	4.3213	6.5649	12.1048	39.5508
4	1.5390	3.2144	4.5070	6.9461	12.9633	42.7302
5	1.5395	3.2603	4.6232	7.2169	13.6317	45.3628
6	1.5396	3.2847	4.6970	7.4142	14.1704	47.6362
...	...	...	...	...	...	...
15	1.5396	3.3126	4.8270	7.9551	16.4436	61.5187
16	1.5396	3.3127	4.8279	7.9663	16.5529	62.6948
...	...	...	...	...	...	...
21	1.5396	3.3127	4.8293	7.9922	16.9100	67.9298
22	1.5396	3.3127	4.8295	7.9942	16.9543	68.8620
...	...	...	...	...	...	...
36	1.5396	3.3127	4.8295	7.9999	17.1866	78.7064
37	1.5396	3.3127	4.8295	8.0000	17.1907	79.2148
..	...	...	...	...	...	...
73	1.5396	3.3127	4.8295	8.0000	17.2131	80.1671
74	1.5396	3.3127	4.8295	8.0000	17.2132	80.6125
...	...	...	...	...	...	...
249	1.5396	3.3127	4.8295	8.0000	17.2132	89.4425
250	1.5396	3.3127	4.8295	8.0000	17.2132	89.4426

Table 2. Some numerical results for $\Gamma_{p,q}$ function values, with $p = 1$ and $z = 2.5$.

r	$q = 0.2$	$q = 0.5$	$q = 0.6$	$q = 0.7$	$q = 0.8$	$q = 0.9$
			$p = 1, z = 2.5$			
1	1.3465	2.3270	3.0381	4.3529	7.4253	19.4816
2	1.3874	2.5893	3.4449	5.0035	8.6105	22.6972
3	1.3955	2.7116	3.6611	5.3813	9.3376	24.7266
4	1.3971	2.7707	3.7822	5.6157	9.8190	26.1175
5	1.3975	2.7997	3.8519	5.7672	10.1541	27.1253
...	...	...	...	...	...	...
12	1.3975	2.8282	3.9501	6.0612	10.9983	30.1220
13	1.3975	2.8284	3.9512	6.0686	11.0359	30.3085
...	...	...	...	...	...	...
18	1.3975	2.8284	3.9527	6.0829	11.1340	30.9169
19	1.3975	2.8284	3.9528	6.0838	11.1434	30.9957
...	...	...	...	...	...	...
28	1.3975	2.8284	3.9528	6.0857	11.1754	31.3959
29	1.3975	2.8284	3.9528	6.0858	11.1764	31.4195
...	...	...	...	...	...	...
44	1.3975	2.8284	3.9528	6.0858	11.1802	31.5821
45	1.3975	2.8284	3.9528	6.0858	11.1803	31.5862
...	...	...	...	...	...	...
91	1.3975	2.8284	3.9528	6.0858	11.1803	31.6224
92	1.3975	2.8284	3.9528	6.0858	11.1803	31.6225

Algorithm 1. The proposed procedure To calculate $\Gamma_{p,q}(x)$.

```
function G = gammapq(p,q,z,r)
%pq-Gamma Function
d=1;
for k=k=1:r
g=(d.*(1-(q./p).^(k+1))./(1-(q).^(z+k)))./(p-q).^(z-1);
end
end
```

Definition 2.3. *[57] Suppose that $f : [0, \mathcal{T}] \to \mathbb{R}$, then the (p, q)-derivative of f is defined by*

$$D_{p,q}f(z) = \frac{f(pz) - f(qz)}{(p-q)z}, \quad \text{for} \quad z \neq 0,$$

which $D_{p,q}f(0) = f'(0)$.

Definition 2.4. *[57] Consider $f : [0, \mathcal{T}] \to \mathbb{R}$, then the generalized quantum integral with p, q parameters is defined as following formula*

$$\int_0^x f(z)d_{p,q}z = (p-q)x \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}}x\right), \tag{2.2}$$

which the right-hand side converges. Furthermore, in Riemann-Liouville type, we have

$$\begin{aligned}\left(I_{p,q}^{\zeta}f\right)(z) &= \frac{1}{p^{\binom{\zeta}{2}}\Gamma_{p,q}(\zeta)} \int_0^z (z-qs)_{p,q}^{(\zeta-1)} f\left(\frac{s}{p^{\zeta-1}}\right) d_{p,q}s \\ &= \frac{(p-q)z}{p^{\binom{\zeta}{2}}\Gamma_{p,q}(\zeta)} \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left(z - \frac{q^{n+1}}{p^{n+1}}z\right)_{p,q}^{(\zeta-1)} f\left(\frac{q^n}{p^{\zeta+n}}z\right),\end{aligned}$$

where $z \in [0, p^{\alpha}\mathcal{T}]$.

Remark 2.5. *[57] For a continuous function f, we have:*

$$\begin{cases} \left(D_{p,q}^{\zeta}f\right)(z) = \left(D_{p,q}^{[\zeta]} I_{p,q}^{[\zeta]-\zeta} f\right)(z), \\ \left({}^{c}D_{p,q}^{\zeta}f\right)(z) = \left(I_{p,q}^{[\zeta]-\zeta} D_{p,q}^{[\zeta]} f\right)(z), \end{cases}$$

where $[\zeta]$ is the smallest integer greater than or equal to ζ. Notice that, $(D_{p,q}^{0}f)(z) = f(z)$ and ${}^{c}D_{p,q}^{0}f(z) = f(z)$.

Lemma 2.6. *[57] The following relation is established:*

$$\left(I_{p,q}^{\zeta}\, {}^{c}D_{p,q}^{\zeta}f\right)(z) = f(z) - \sum_{k=0}^{[\zeta]-1} \frac{z^k}{p^{\binom{\zeta}{2}}\Gamma_{p,q}(k+1)} \left(D_{p,q}^{k}f\right)(0),$$

where $\zeta \in (n-1, n)$. Indeed, for equation $({}^{c}D_{p,q}^{\alpha}f)(z) = 0$, it's general solution expressed by $f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_{n-1}z^{n-1}$, where $c_0, \ldots, c_{n-1} \in \mathbb{R}$.

Definition 2.7. *[64] Assume that $\mathcal{X} \neq \emptyset$, then the map $d : \mathcal{X}\times\mathcal{X} \to \mathbb{R}$, is called a vector-valued metric on $\mathcal{X}$ if the following properties met*

(i) $\forall z, w \in X$, $d(z,w) \geq 0$, *and* $d(z,w) = 0$ *if and only if* $z = w$.

(ii) $\forall z, w \in X$, $d(z,w) = d(w,z)$.

(iii) $\forall z, w, y \in X$, $d(z,w) \leq d(z,y) + d(y,w)$.

For such a space, namely $(\mathcal{X}, d)$, which is called generalized metric space, convergence and completeness are similar to those in usual metric space.

Definition 2.8. *[64] Suppose $(\mathcal{X}, d)$ be the same space defined in the above, then an operator $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is called a contraction if there exists a matrix $\mathcal{M}$ which converges to zero such that $\forall z, w \in \mathcal{X}$, we have $d(\mathcal{T}(z), \mathcal{T}(w)) \leq \mathcal{M}d(z,w)$.*

Definition 2.9. *[65] A matrix $\mathcal{M}_{n\times n}$ is called convergent to zero if $\mathcal{M}^s \to 0$, as $s \to \infty$.*

Theorem 2.10. *[65] The following proposition are equivalent:*

(i) *$\mathcal{M}_{n\times n}$ convergent to zero.*

(ii) *$\mathcal{I} - \mathcal{M}$ is nonsingular and $(\mathcal{I} - \mathcal{M})^{-1} = \sum_{s=0}^{\infty} \mathcal{M}^s$ such that $\mathcal{I}$ denotes unit matrix of the same order as $\mathcal{M}$.*

(iii) *$\forall \lambda \in \mathbb{C}$, we have $|\lambda| < 1$, such that $|\mathcal{M} - \lambda\mathcal{I}| = 0$.*

(iv) *$\mathcal{I} - \mathcal{M}$ is nonsingular and $(\mathcal{I} - \mathcal{M})^{-1}$ has nonnegative elements.*

Lemma 2.11. *[64] Let $C_{n\times n}$, $\mathcal{D}_{n\times n}$ are two matrices. If $C_{n\times n}$ converges to zero and the elements of $\mathcal{D}_{n\times n}$ are small enough, then $C_{n\times n} + \mathcal{D}_{n\times n}$ also converges to zero.*

Theorem 2.12. *[66] Let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a contractive operator with a Lipschitz matrix $\mathcal{M}$, and $(\mathcal{X}, d)$ be a complete generalized metric space. Then, T has a unique fixed point w^* and each $w_0 \in X$*

$$d\left(\mathcal{T}^k(w_0), w^*\right) \leq \mathcal{M}^k(\mathcal{I} - \mathcal{M})^{-1} d\left(w_0, T(w_0)\right), \quad \forall k \in \mathbb{N}.$$

Theorem 2.13. *[66] Let D be a nonempty closed bounded convex subset of Banach space $\mathcal{Z}$, and $\mathcal{F} : D \to D$ is a completely continuous operator. Then, $\mathcal{F}$ has at least one fixed point.*

Theorem 2.14. *[66] Let $\varepsilon > 0$, also, the following two conditions must be met at the same time:*

(i) *The operator $\mathcal{F} : \bar{\mathcal{Z}}_\varepsilon(0, \mathcal{Z}) \to \mathcal{Z}$ is a completely continuous.*

(ii) *For every solution $\mathbf{w}$, of $\mathbf{w} = \delta\mathcal{F}(\mathbf{w})$, such that $\delta \in (0,1)$, we have $\|\mathbf{w}\| < \varepsilon$.*

Then, the aforesaid operator has at least one fixed point.

Notation 2.15. *In the continuation of this section, we will introduce an important matrix.*

$$\mathcal{M}_{2\times 2} = \begin{bmatrix} a_1\Lambda_1 & b_1\Lambda_1 \\ a_2\Lambda_2 & b_2\Lambda_2 \end{bmatrix}, \tag{2.3}$$

such that $a_i, b_i > 0$, $i = 1, 2$ and

$$\begin{aligned} \Lambda_1 &= \left|\frac{T}{\Delta_1}\left(\frac{\lambda_1\eta_1^{\zeta_1-1} - T^{\zeta_1-1}}{\Gamma_{p,q}(\zeta_1)}\right)\right| - \frac{T^{\zeta_1}}{\Gamma_{p,q}(\zeta_1+1)}, \\ \Lambda_2 &= \left|\frac{T}{\Delta_2}\left(\frac{\lambda_2\eta_2^{\zeta_2-1} - T^{\zeta_2-1}}{\Gamma_{p,q}(\zeta_2)}\right)\right| - \frac{T^{\zeta_2}}{\Gamma_{p,q}(\zeta_2+1)}, \end{aligned} \tag{2.4}$$

where $\Delta_i = 1 - \mu_i \neq 0$, $i = 1, 2$.

In view of Theorem 2.10, we present some sufficient conditions for the convergence of $\mathcal{M}$.

Theorem 2.16. *Assume that one of the following three conditions hold true, then the matrix $\mathcal{M}$ which defined in* (2.3) *converges to zero*

($\mathcal{H}_1$) $4\Lambda_1\Lambda_2 a_2 b_1 + (\Lambda_2 b_2 - \Lambda_1 a_1)^2 > 0$ *and* $\left|\frac{\Lambda_2 b_2+\Lambda_1 a_1 \pm \sqrt{4\Lambda_1\Lambda_2 a_2 b_1+(\Lambda_2 b_2-\Lambda_1 a_1)^2}}{2}\right| < 1$*;*

($\mathcal{H}_2$) $4\Lambda_1\Lambda_2 a_2 b_1 + (\Lambda_2 b_2 - \Lambda_1 a_1)^2 = 0$ *and* $|\Lambda_2 b_2 + \Lambda_1 a_1| < 2$ *;*

($\mathcal{H}_3$) $4\Lambda_1\Lambda_2 a_2 b_1 + (\Lambda_2 b_2 - \Lambda_1 a_1)^2 < 0$ *and* $\Lambda_1\Lambda_2\,(a_1 b_2 - a_2 b_1) < 1$.

Proof. By doing a simple calculation, we get

$$|\lambda I - M_{2\times 2}| = \begin{vmatrix} \lambda - a_1\Lambda_1 & -b_1\Lambda_1 \\ -a_2\Lambda_2 & \lambda - b_2\Lambda_2 \end{vmatrix} = \lambda^2 - (a_1\Lambda_1 + b_2\Lambda_2)\,\lambda + \Lambda_1\Lambda_2\,(a_1 b_2 - a_2 b_1) = 0,$$

that will lead to:

(i) $\lambda_{1,2} = \frac{a_1\Lambda_1 + b_2\Lambda_2 \pm \sqrt{\Delta}}{2}$ when $\Delta > 0$;

(ii) $\lambda_{1,2} = \frac{a_1\Lambda_1 + b_2\Lambda_2}{2}$ when $\Delta = 0$;

(iii) $\lambda_{1,2} = \frac{a_1\Lambda_1 + b_2\Lambda_2 \pm \sqrt{-\Delta i}}{2}$ when $\Delta < 0$;

which

$$\Delta = (a_1\Lambda_1 + b_2\Lambda_2)^2 - 4\Lambda_1\Lambda_2\,(a_1 b_2 - a_2 b_1) = (a_1\Lambda_1 - b_2\Lambda_2)^2 + 4\Lambda_1\Lambda_2 a_2 b_1.$$

According to Theorem 2.10 (3), and some calculations, we get the desired result. □

We end this section with the following lemma.

Lemma 2.17. *Suppose that* $\mathfrak{m}, \mathfrak{n} \in C([0, \mathcal{T}/p^{\alpha_i}], \mathbb{R})$ *are given functions and* μ_1, μ_2 *are constants. Then a unique solution of the following BVP:*

$$\begin{cases} {}^{c}D_{p,q}^{\zeta_1} u(t) = \mathfrak{m}(t), \ t \in \left[0, \mathcal{T}/p^{\zeta_1}\right], \ 1 < \zeta_1 \leq 2, \\ {}^{c}D_{p,q}^{\zeta_2} v(t) = \mathfrak{n}(t), \ t \in \left[0, \mathcal{T}/p^{\zeta_2}\right], \ 1 < \zeta_2 \leq 2, \\ u(0) = v(0) = u^{(i)}(0) = v^{(i)}(0) = 0, \ \ i = 2, ..., n-2, \\ D_{p,q}u(T/p) = \mu_1 D_{p,q}u(\eta_1), \ \ D_{p,q}v(T/p) = \mu_2 D_{p,q}v(\eta_2), \end{cases} \tag{2.5}$$

is given by

$$
\begin{aligned}
u(t) =& \int_0^t \frac{(t-qs)_{p,q}^{(\zeta_1-1)}}{p^{\binom{\zeta_1}{2}}\Gamma_{p,q}(\zeta_1)} \mathfrak{m}\left(\frac{s}{p^{\zeta_1-1}}\right) d_{p,q}s \\
&+ \frac{t}{\Delta_1}\left\{\mu_1 \int_0^{\eta_1} \frac{(\eta_1-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)} \mathfrak{m}\left(\frac{s}{p^{\zeta_1-2}}\right) d_{p,q}s \right.\\
&\left. - \int_0^{\mathcal{T}/p} \frac{(\mathcal{T}/p-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)} \mathfrak{m}\left(\frac{s}{p^{\zeta_1-2}}\right) d_{p,q}s\right\}, \\
v(t) =& \int_0^t \frac{(t-qs)_{p,q}^{(\zeta_2-1)}}{p^{\binom{\zeta_2}{2}}\Gamma_{p,q}(\zeta_2)} \mathfrak{n}\left(\frac{s}{p^{\zeta_2-1}}\right) d_{p,q}s \\
&+ \frac{t}{\Delta_2}\left\{\mu_2 \int_0^{\eta_1} \frac{(\eta_1-qs)_{p,q}^{(\zeta_2-2)}}{p^{\binom{\zeta_2-1}{2}}\Gamma_{p,q}(\alpha-1)} \mathfrak{n}\left(\frac{s}{p^{\zeta_2-2}}\right) d_{p,q}s \right.\\
&\left. - \int_0^{\mathcal{T}/p} \frac{(\mathcal{T}/p-qs)_{p,q}^{(\zeta_2-2)}}{p^{\binom{\zeta_2-1}{2}}\Gamma_{p,q}(\zeta_2-1)} \mathfrak{n}\left(\frac{s}{p^{\zeta_2-2}}\right) d_{p,q}s\right\}.
\end{aligned}
\tag{2.6}
$$

where

$$\Delta_i = 1-\lambda_i \neq 0, \quad i = 1,2.$$

Proof. By applying the (p,q)-integral on both sides (2.5) and using Lemma 2.6, we get

$$u(t) = \int_0^t \frac{(t-qs)^{(\zeta_1-1)}}{p^{\binom{\zeta_1}{2}}\Gamma_{p,q}(\zeta_1)} \mathfrak{m}\left(\frac{s}{p^{\zeta_1-1}}\right) d_{p,q}s + c_0 + c_1 t, \quad t \in [0,\mathcal{T}] \tag{2.7}$$

where c_0, c_1 are constants. Now by using condition in (2.5), we find $c_0 = 0$, and

$$c_1 = \frac{1}{\Delta_1}\left[\mu_1 \int_0^{\eta} \frac{(\eta-qs)^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1)} \mathfrak{m}\left(\frac{s}{p^{\zeta_1-2}}\right) d_{p,q} - \int_0^{T/p} \frac{(\mathcal{T}/p-qs)^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1)} \mathfrak{m}\left(\frac{s}{p^{\zeta_1-2}}\right) d_{p,q}\right].$$

Substituting the values of c_1 in (2.7), we obtain (2.6). Proof for $v(t)$ is similar to the above. □

3. Main results

We need the following assumptions to prove our main results.

($\mathcal{L}_{1.1}$) $\mathfrak{m}, \mathfrak{n} : J \times \mathbb{R}^2 \to \mathbb{R}$ are jointly continuous functions such that satisfy Lipschitz condition $\forall u, v, \bar{u}, \bar{v} \in \mathbb{R}$, and some a_1, a_2, b_1, b_2, where

$$\begin{cases} |\mathfrak{m}(t,u,v) - \mathfrak{m}(t,\bar{u},\bar{v})| \leq a_1|u-\bar{u}| + b_1|v-\bar{v}|, \\ |\mathfrak{n}(t,u,v) - \mathfrak{n}(t,\bar{u},\bar{v})| \leq a_2|u-\bar{u}| + b_2|v-\bar{v}|. \end{cases}$$

($\mathcal{L}_{1.2}$) The Caratheodory functions $\mathfrak{m}, \mathfrak{n} : J \times \mathbb{R}^2 \to \mathbb{R}$ satisfy Lipschitz condition as following form:

$$\begin{cases} |\mathfrak{m}(t,u,v)| \leq a_1|u| + b_1|v| + c_1, \\ |\mathfrak{n}(t,u,v)| \leq a_2|u| + b_2|v| + c_2, \end{cases}$$

for all $u, v \in \mathbb{R}$, and some $a_1, a_2, b_1, b_2, c_1, c_2 > 0$.

($\mathcal{L}_{1.3}$) The jointly continuous functions $\mathfrak{m}, \mathfrak{n} : J \times \mathbb{R}^2 \to \mathbb{R}$ satisfy following inequalities:

$$\begin{cases} |\mathfrak{m}(t,u)| \leq w_1\left(t, |u|_E\right), \\ |\mathfrak{n}(t,u)| \leq w_2\left(t, |u|_E\right), \end{cases}$$

for all $u, v \in \mathbb{R}$ and $t \in J$, which $|\cdot|_E$ represent the Euclidean norm in $\mathbb{R}^2$, and w_1, w_2 are jointly continuous functions on $J \times \mathbb{R}_+$ such that nondecreasing in their second variables.

($\mathcal{L}_2$) The matrix $\mathcal{M}_{2\times 2}$ defined in (2.3) converges to zero.

($\mathcal{L}_3$) $\exists\, K_1 > 0$ which for $\sigma = (\sigma_1, \sigma_2) \in (0, +\infty)^2$, $\forall t \in J$ the following inequalities:

$$\begin{cases} \sup_{t\in J} \frac{1}{\rho_1}\Bigg[\int_0^t \frac{(t-qs)_{p,q}^{(\zeta_1-1)}}{p^{\binom{\zeta_1}{2}}\Gamma_{p,q}(\zeta_1)} w_1\left(s, |\rho|_E\right) d_{p,q}s, \\ + \frac{t}{\Delta_1}\Bigg\{\mu_1 \int_0^{\eta_1} \frac{(\eta_1-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)} w_1\left(s, |\rho|_E\right) d_{p,q}s - \int_0^{\mathcal{T}/p} \frac{(\mathcal{T}/p-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)} w_1\left(s, |\rho|_E\right) d_{p,q}s\Bigg\}\Bigg] \geq 1, \\ \sup_{t\in J} \frac{1}{\rho_2}\Bigg[\int_0^t \frac{(t-qs)_{p,q}^{(\zeta_2-1)}}{p^{\binom{\zeta_2}{2}}\Gamma_{p,q}(\zeta_2)} w_2\left(s, |\rho|_E\right) d_{p,q}s \\ + \frac{t}{\Delta_2}\Bigg\{\mu_2 \int_0^{\eta_2} \frac{(\eta_2-qs)_{p,q}^{(\zeta_2-2)}}{p^{\binom{\zeta_2-1}{2}}\Gamma_{p,q}(\zeta_2-1)} w_2\left(s, |\rho|_E\right) d_{p,q}s - \int_0^{\mathcal{T}/p} \frac{(\mathcal{T}/p-qs)_{p,q}^{(\zeta_2-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_2-1)} w_2\left(s, |\rho|_E\right) d_{p,q}s\Bigg\}\Bigg] \geq 1, \end{cases}$$

implies $|\sigma|_E \leq K_1$.

Now, to find the solution to our boundary value problem (1.1), we will convert it to finding a unique fixed point for an operator. We will use Lemma 2.17 to define this operator. Thus, we define $\mathcal{F} : C(J, \mathbb{R})^2 \to C(J, \mathbb{R})^2$, such that $\mathcal{F} := (\mathcal{F}_1, \mathcal{F}_2)$ which $\mathcal{F}_1, \mathcal{F}_2$ are given by

$$\begin{aligned} \mathcal{F}_1(u,v)(t) &= \int_0^t \frac{(t-qs)_{p,q}^{(\zeta_1-1)}}{p^{\binom{\zeta_1}{2}}\Gamma_{p,q}(\zeta_1)} \mathfrak{m}\left(s, u\left(p^{\zeta_1-1}s\right), v\left(p^{\zeta_2-1}s\right)\right) d_{p,q}s \\ &+ \frac{t}{\Delta_1}\Bigg\{\mu_1 \int_0^{\eta_1} \frac{(\eta_1-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)} \mathfrak{m}\left(s, u\left(p^{\zeta_1-1}s\right), v\left(p^{\zeta_2-1}s\right)\right) d_{p,q}s \\ &- \int_0^{\mathcal{T}/p} \frac{(\mathcal{T}/p-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)} \mathfrak{m}\left(s, u\left(p^{\zeta_1-1}s\right), v\left(p^{\zeta_2-1}s\right)\right) d_{p,q}s\Bigg\}, \\ \mathcal{F}_2(u,v)(t) &= \int_0^t \frac{(t-qs)_{p,q}^{(\zeta_2-1)}}{p^{\binom{\zeta_2}{2}}\Gamma_{p,q}(\zeta_2)} \mathfrak{n}\left(s, u\left(p^{\zeta_1-1}s\right), v\left(p^{\zeta_2-1}s\right)\right) d_{p,q}s \\ &+ \frac{t}{\Delta_2}\Bigg\{\mu_2 \int_0^{\eta_2} \frac{(\eta_2-qs)_{p,q}^{(\zeta_2-2)}}{p^{\binom{\zeta_2-1}{2}}\Gamma_{p,q}(\zeta_2-1)} \mathfrak{n}\left(s, u\left(p^{\zeta_1-1}s\right), v\left(p^{\zeta_2-1}s\right)\right) d_{p,q}s \\ &- \int_0^{\mathcal{T}/p} \frac{(\mathcal{T}/p-qs)_{p,q}^{(\zeta_2-2)}}{p^{\binom{\zeta_2-1}{2}}\Gamma_{p,q}(\zeta_2-1)} \mathfrak{n}\left(s, u\left(p^{\zeta_1-1}s\right), v\left(p^{\zeta_2-1}s\right)\right) d_{p,q}s\Bigg\}. \end{aligned} \tag{3.1}$$

Theorem 3.1. *Let $\mathcal{L}_{1.1}, \mathcal{L}_{1.2}, \mathcal{L}_{1.3}$, and $\mathcal{L}_2$ are hold true. Then the problem mentioned in* (1.1) *has a unique solution.*

Proof. As for assumption $\mathcal{L}_2$ and Theorem 2.10 (4), it simply follows that $I - \mathcal{M}_{2\times 2}$ is invertible and its inverse $(I - \mathcal{M}_{2\times 2})^{-1}$ has nonnegative elements. Now, we define

$$\widetilde{U} = \left\{(u, v) \in C(J, \mathbb{R})^2 : \|u\|_\infty \leq \widetilde{K}_1, \|v\|_\infty \leq \widetilde{K}_2\right\},$$

such that

$$\begin{bmatrix} \widetilde{K}_1 \\ \widetilde{K}_2 \end{bmatrix} \geq (I - \mathcal{M}_{2\times 2})^{-1} \begin{bmatrix} \tilde{\mathcal{M}}_1 \\ \tilde{\mathcal{M}}_2 \end{bmatrix}.$$

Here $\tilde{\mathcal{M}}_1 = \Lambda_1 \mathfrak{m}_{\max}$, $\tilde{\mathcal{M}}_2 = \Lambda_2 \mathfrak{n}_{\max}$ with $\mathfrak{m}_{\max} = \max_{t\in J} |\mathfrak{m}(t, 0, 0)|$, and $\mathfrak{n}_{\max} = \max_{t\in J} |\mathfrak{n}(t, 0, 0)|$. We follow the proof in two steps.

At first: we show that the operator $\mathcal{F}$ mentioned in (3.1) maps $\widetilde{U}$ into $\widetilde{U}$. For this purpose, $\forall (u, v) \in \widetilde{U}$ and $0 < t_1 < t_2 < 1$, by employing $\mathcal{L}_{1.1}$, we can write

$$\begin{aligned}
&|\mathcal{F}_1(u, v)(t_2) - \mathcal{F}_1(u, v)(t_1)| \\
&= \Bigg| \int_0^{t_2} \frac{(t_2 - qs)^{(\zeta_1-1)}}{p^{\binom{\zeta_1}{2}}\Gamma_{p,q}(\alpha)} \mathfrak{m}\left(s, u\left(p^{\zeta_1-1}s\right), v\left(p^{\zeta_2-1}s\right)\right) d_{p,q}s \\
&- \int_0^{t_1} \frac{(t_1 - qs)^{(\zeta_1-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \mathfrak{m}\left(s, u\left(p^{\zeta_1-1}s\right), v\left(p^{\zeta_2-1}s\right)\right) d_{p,q}s \Bigg| \\
&+ \frac{|t_2 - t_1|}{\Delta_1} \Bigg\{ \mu_1 \int_0^{\eta} \frac{(\eta - qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1 - 1)} \mathfrak{m}\left(s, u\left(p^{\zeta_1-1}s\right), v\left(p^{\zeta_2-1}s\right)\right) d_{p,q}s \\
&- \int_0^{T/p} \frac{(T/p - qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1 - 1)} \mathfrak{m}\left(s, u\left(p^{\zeta_1-1}s\right), v\left(p^{\zeta_2-1}s\right)\right) d_{p,q}s \Bigg\} \\
&\leq \Bigg| \int_0^{t_1} \frac{(t_2 - qs)^{(\zeta_1-1)} - (t_1 - qs)^{(\zeta_1-1)}}{p^{(\zeta_1)}2^{(\alpha)}\Gamma_{p,q}(\alpha)} \mathfrak{m}\left(s, u\left(p^{\zeta_1-1}s\right), v\left(p^{\zeta_2-1}s\right)\right) d_{p,q}s \Bigg| \\
&+ \Bigg| \int_{t_1}^{t_2} \frac{(t_2 - qs)^{(a-1)}}{p^{\binom{a}{2}}\Gamma_{p,q}(\alpha)} \mathfrak{m}\left(s, u\left(p^{\zeta_1-1}s\right), v\left(p^{\zeta_2-1}s\right)\right) d_{p,q}s \Bigg| \\
&+ \frac{|t_2 - t_1|}{\Delta_1} \Bigg\{ \mu_1 \int_0^{\eta_1} \frac{(\eta_1 - qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1 - 1)} \mathfrak{m}\left(s, u\left(p^{\zeta_1-1}s\right), v\left(p^{\zeta_2-1}s\right)\right) d_{p,q}s \\
&- \int_0^{T/p} \frac{(T/p - qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1 - 1)} \mathfrak{m}\left(s, u\left(p^{\zeta_1-1}s\right), v\left(p^{\zeta_2-1}s\right)\right) d_{p,q}s \Bigg\} \\
&\leq \Bigg[\int_0^{t_1} \frac{(t_2 - qs)^{(\zeta_1-1)} - (t_1 - qs)^{(\zeta_1-1)}}{p^{\binom{\zeta_1}{2}}\Gamma_{p,q}(\zeta_1)} \left| \mathfrak{m}\left(s, u\left(p^{\zeta_1-1}s\right), v\left(p^{\zeta_2-1}s\right)\right) \right| d_{p,q}s \\
&+ \int_{t_1}^{t_2} \frac{(t_2 - qs)^{(\zeta_1-1)}}{p^{\binom{\pi}{2}}\Gamma_{p,q}(\zeta_1)} \left| \mathfrak{m}\left(s, u\left(p^{\zeta_1-1}s\right), v\left(p^{\zeta_2-1}s\right)\right) \right| d_{p,q}s \Bigg] \\
&+ \frac{|t_2 - t_1|}{\Delta_1} \Bigg\{ \mu_1 \int_0^{\eta_1} \frac{(\eta_1 - qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1 - 1)} \left| \mathfrak{m}\left(s, u\left(p^{\zeta_1-1}s\right), v\left(p^{\zeta_2-1}s\right)\right) \right| d_{p,q}s \\
&- \int_0^{T/p} \frac{(T/p - qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1 - 1)} \left| \mathfrak{m}\left(s, u\left(p^{\zeta_1-1}s\right), v\left(p^{\zeta_2-1}s\right)\right) \right| d_{p,q}s \Bigg\},
\end{aligned} \tag{3.2}$$

which yields that

$$|\mathcal{F}_1(u,v)(t_2) - \mathcal{F}_1(u,v)(t_1)| \to 0 \quad \text{as} \quad t_1 \to t_2.$$

Therefore, $\mathcal{F}_1$ maps $\widetilde{U}$ into $C(J,\mathbb{R})^2$. Moreover, we find that

$$\begin{aligned} \left|\mathfrak{m}\left(s, u\left(p^{\zeta_1-1}s\right), v\left(p^{\zeta_2-1}s\right)\right)\right| &\leq \left|\mathfrak{m}\left(s, u\left(p^{\zeta_1-1}s\right), v\left(p^{\zeta_2-1}s\right)\right) - \mathfrak{m}(t,0,0)\right| + |\mathfrak{m}(t,0,0)| \\ &\leq a_1\tilde{K}_1 + b_1\tilde{K}_1 + \mathfrak{m}_{max}, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} |\mathcal{F}_1(u,v)(t)| \leq & \int_0^t \frac{(t-qs)_{p,q}^{(\zeta_1-1)}}{p^{\binom{\zeta_1}{2}}\Gamma_{p,q}(\zeta_1)} \left|\mathfrak{m}\left(s, u\left(p^{\zeta_1-1}s\right), v\left(p^{\zeta_2-1}s\right)\right)\right| d_{p,q}s \\ & + \frac{t}{\Delta}\left\{\mu_1 \int_0^{\eta_1} \frac{(\eta_1-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)} \left|\mathfrak{m}\left(s, u\left(p^{\zeta_1-1}s\right), v\left(p^{\zeta_2-1}s\right)\right)\right| d_{p,q}s \right. \\ & \left. - \int_0^{\mathcal{T}/p} \frac{(\mathcal{T}/p-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)} \left|\mathfrak{m}\left(s, u\left(p^{\zeta_1-1}s\right), v\left(p^{\zeta_2-1}s\right)\right)\right| d_{p,q}s \right\} \\ \leq & \left(a_1\tilde{R}_1 + b_1\tilde{R}_1 + \mathfrak{m}_{max}\right)\left[\int_0^t \frac{(t-qs)_{p,q}^{(\zeta_1-1)}}{p^{\binom{\zeta_1}{2}}\Gamma_{p,q}(\zeta_1)} d_{p,q}s \right. \\ & \left. + \frac{t}{\Delta}\left\{\mu_1 \int_0^{\eta_1} \frac{(\eta_1-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)} d_{p,q}s - \int_0^{\mathcal{T}/p} \frac{(\mathcal{T}/p-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)} d_{p,q}s \right\}\right]. \end{aligned} \tag{3.4}$$

Thus,

$$\begin{aligned} |\mathcal{F}_1(u,v)(t)| &\leq \Delta_1\left(a_1\tilde{K}_1 + b_1\tilde{K}_1 + \mathfrak{m}_{max}\right) \\ &\leq \tilde{K}_1. \end{aligned} \tag{3.5}$$

It can also be proved in a similar way that $\mathcal{F}_2$ maps $\widetilde{U}$ into $C(J,\mathbb{R})^2$, and

$$\begin{aligned} |\mathcal{F}_2(u,v)(t)| &\leq \Delta_2\left(a_2\tilde{K}_2 + b_2\tilde{K}_2 + \mathfrak{n}_{max}\right) \\ &\leq \tilde{K}_2. \end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6), we get

$$\begin{bmatrix} \|\mathcal{F}_1(u,v)\|_\infty \\ \|\mathcal{F}_2(u,v)\|_\infty \end{bmatrix} \leq \begin{bmatrix} \widetilde{K}_1 \\ \widetilde{K}_2 \end{bmatrix},$$

that is, we proved that $\mathcal{F}(\widetilde{U}) \subset \widetilde{U}$. Hence, $\mathcal{F}$ maps $\widetilde{U}$ into $\widetilde{U}$.

Secondly: We shall show that the operator $\mathcal{F}$ mentioned in (3.1) is a generalized contraction. For achieve it, $\forall (u,v),(\bar{u},\bar{v})\in \widetilde{U}$, let $\bar{\mathfrak{m}}=\mathfrak{m}(\cdot,\bar{u},\bar{v})$, using $\mathcal{L}_{1.1}$, we have

$$\begin{aligned}
&\Big|\mathcal{F}_1(u,v)(t)-\mathcal{F}_1(\bar{u},\bar{v})(t)\Big|\\
&\le \int_0^t \frac{(t-qs)_{p,q}^{(\zeta_1-1)}}{p^{\binom{\zeta_1}{2}}\Gamma_{p,q}(\zeta_1)}\Big|\mathfrak{m}\left(s,u\left(p^{\zeta_1-1}s\right),v\left(p^{\zeta_2-1}s\right)\right)-\bar{\mathfrak{m}}\left(s,\bar{u}\left(p^{\zeta_1-1}s\right),\bar{v}\left(p^{\zeta_2-1}s\right)\right)\Big|d_{p,q}s\\
&+\frac{t}{\Delta_1}\bigg\{\mu_1\int_0^{\eta_1}\frac{(\eta_1-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)}\Big|\mathfrak{m}\left(s,u\left(p^{\zeta_1-1}s\right),v\left(p^{\zeta_2-1}s\right)\right)-\bar{\mathfrak{m}}\left(s,\bar{u}\left(p^{\zeta_1-1}s\right),\bar{v}\left(p^{\zeta_2-1}s\right)\right)\Big|d_{p,q}s\\
&-\int_0^{T/p}\frac{(T/p-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)}\Big|\mathfrak{m}\left(s,u\left(p^{\zeta_1-1}s\right),v\left(p^{\zeta_2-1}s\right)\right)-\bar{\mathfrak{m}}\left(s,\bar{u}\left(p^{\zeta_1-1}s\right),\bar{v}\left(p^{\zeta_2-1}s\right)\right)\Big|d_{p,q}s\bigg\}\\
&\le \int_0^t \frac{(t-qs)_{p,q}^{(\zeta_1-1)}}{p^{\binom{\zeta_1}{2}}\Gamma_{p,q}(\zeta_1)}\Big(a_1|u-\bar{u}|+b_1|v-\bar{v}|\Big)d_{p,q}s \qquad (3.7)\\
&+\frac{t}{\Delta_1}\bigg\{\mu_1\int_0^{\eta_1}\frac{(\eta_1-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)}\Big(a_1|u-\bar{u}|+b_1|v-\bar{v}|\Big)d_{p,q}s\\
&-\int_0^{\mathcal{T}/p}\frac{(\mathcal{T}/p-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)}\Big(a_1|u-\bar{u}|+b_1|v-\bar{v}|\Big)d_{p,q}s\bigg\}\\
&\le \Big(a_1|u-\bar{u}|+b_1|v-\bar{v}|\Big)\bigg[\int_0^t \frac{(t-qs)_{p,q}^{(\zeta_1-1)}}{p^{\binom{\zeta_1}{2}}\Gamma_{p,q}(\zeta_1)}d_{p,q}s\\
&+\frac{t}{\Delta_1}\bigg\{\mu_1\int_0^{\eta_1}\frac{(\eta_1-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)}d_{p,q}s-\int_0^{\mathcal{T}/p}\frac{(\mathcal{T}/p-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)}d_{p,q}s\bigg\}\bigg],
\end{aligned}$$

which yields that

$$\Big\|\mathcal{F}_1(u,v)(t)-\mathcal{F}_1(\bar{u},\bar{v})(t)\Big\| \le \Delta_1\Big(a_1\|u-\bar{u}\|+b_1\|v-\bar{v}\|\Big). \qquad (3.8)$$

Similarly, we can obtain

$$\Big\|\mathcal{F}_2(u,v)(t)-\mathcal{F}_2(\bar{u},\bar{v})(t)\Big\| \le \Delta_2\Big(a_2\|u-\bar{u}\|+b_2\|v-\bar{v}\|\Big). \qquad (3.9)$$

We can then put (3.8) and (3.9) together and rewrite as

$$\begin{bmatrix} \|\mathcal{F}_1(u,v)-\mathcal{F}_1(\bar{u},\bar{v})\|_\infty \\ \|\mathcal{F}_2(u,v)-\mathcal{F}_2(\bar{u},\bar{v})\|_\infty \end{bmatrix} \le \mathcal{M}_{2\times 2}\begin{bmatrix} \|u-\bar{u}\|_\infty \\ \|v-\bar{v}\|_\infty \end{bmatrix}.$$

Now, according to $\mathcal{L}_2$, one can apply Theorem 2.12 (Perov's fixed point theorem) to achieve what is intended. □

Theorem 3.2. *Let $\mathcal{L}_{1.2}$ and $\mathcal{L}_2$ are satisfied. Then the problem* (1.1) *has at least one solution.*

Proof. Let

$$\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \ge (I-\mathcal{M}_{2\times 2})^{-1}\begin{bmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{bmatrix},$$

where $\tilde{c}_1 = c_1\Delta_1$, and $\tilde{c}_2 = c_2\Delta_2$. Define

$$U = \left\{(u, v) \in C(J, \mathbb{R})^2 : \|u\|_\infty \le K_1, \|v\|_\infty \le K_2\right\}.$$

Obviously, $U \neq \emptyset$ is a closed, bounded and convex subset of $C(J, \mathbb{R})^2$. We follow the proof in three steps.

Step 1: At first, we prove that $\mathcal{F}(U) \subset U$. For this purpose, $\forall u, v \in C(J, \mathbb{R})$ and $\|u\|_\infty \le R_1, \|v\|_\infty \le R_2$, by employing $\mathcal{L}_{1.2}$, we can write

$$\begin{aligned}\left\|\mathcal{F}_1(u, v)(t)\right\| &\le \Delta_1\Big(a_1\|u\| + b_1\|v\| + c_1\Big)\\ &\le \Delta_1 a_1 K_1 + \Delta_1 b_1 K_2 + \tilde{c}_1\\ &\le R.\end{aligned} \tag{3.10}$$

Similarly, we can obtain

$$\begin{aligned}\left\|\mathcal{F}_2(u, v)(t)\right\| &\le \Delta_2\Big(a_2\|u\| + b_2\|v\| + c_2\Big)\\ &\le \Delta_2 a_2 R_1 + \Delta_2 b_2 R_2 + \tilde{c}_2\\ &\le R.\end{aligned} \tag{3.11}$$

We can then put (3.10) and (3.11) together and rewrite as

$$\begin{bmatrix} \|\mathcal{F}_1(u, v)\|_\infty \\ \|\mathcal{F}_2(u, v)\|_\infty \end{bmatrix} \le \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}.$$

Thus, we conclude that $\mathcal{F}(U) \subset U$.

Step 2: In this step we will show the operator $\mathcal{F}$ is continuous. Suppose that (u_n, v_n) be a sequence which $(u_n, v_n) \to (u, v)$ in U. For convenience put $\mathfrak{m}_n(\cdot) = \mathfrak{m}(\cdot, u_n(\cdot), v_n(\cdot))$ and $\mathfrak{m}(\cdot) = \mathfrak{m}(\cdot, u(\cdot), v(\cdot))$. Then $\forall t \in J$, we find

$$\begin{aligned}&\Big|\mathcal{F}_1(u_n, v_n)(t) - \mathcal{F}_1(u, v)(t)\Big|\\ &\le \int_0^t \frac{(t - qs)_{p,q}^{(\zeta_1-1)}}{p^{\binom{\zeta_1}{2}}\Gamma_{p,q}(\zeta_1)}\Big|\mathfrak{m}\left(s, u_n\left(p^{\zeta_1-1}s\right), v_n\left(p^{\zeta_2-1}s\right)\right) - \mathfrak{m}\left(s, u\left(p^{\zeta_1-1}s\right), v\left(p^{\zeta_2-1}s\right)\right)\Big| d_{p,q}s\\ &+ \frac{t}{\Delta_1}\Bigg\{\mu_1 \int_0^{\eta_1} \frac{(\eta_1 - qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1 - 1)}\Big|\mathfrak{m}\left(s, u_n\left(p^{\zeta_1-1}s\right), v_n\left(p^{\zeta_2-1}s\right)\right) - \mathfrak{m}\left(s, u\left(p^{\zeta_1-1}s\right), v\left(p^{\zeta_2-1}s\right)\right)\Big| d_{p,q}s\\ &- \int_0^{T/p} \frac{(T/p - qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1 - 1)}\Big|\mathfrak{m}\left(s, u_n\left(p^{\zeta_1-1}s\right), v_n\left(p^{\zeta_2-1}s\right)\right) - \mathfrak{m}\left(s, u\left(p^{\zeta_1-1}s\right), v\left(p^{\zeta_2-1}s\right)\right)\Big| d_{p,q}s\Bigg\}\\ &\le \Delta_1\Big\|\mathfrak{m}(\cdot, u_n(\cdot), v_n(\cdot)) - \mathfrak{m}(s, u(\cdot), v(\cdot))\Big\|.\end{aligned}$$

Thus, $\mathcal{F}_1$ is continuous. As same way we arrive that $\mathcal{F}_2$ is continuous. Hence, $\mathcal{F}$ is continuous.

Step 3: Finally in this step, we prove that $\mathcal{F}(U)$ is relatively compact. In view of $\mathcal{F}(U) \subset U$, we find $\mathcal{F}(U)$ is uniformly bounded. So the only thing left is to show $\mathcal{F}$ is an equi-continuous operator. For achieve this, $\forall(u,v)\in U$ and $t_1,t_2\in J$ such that $t_1<t_2$, we have

$$\begin{aligned}
&|\mathcal{F}_1(u,v)(t_2)-\mathcal{F}_1(u,v)(t_1)|\\
&\leq\left(a_1K_1+b_1K_2+c_1\right)\left[\int_0^{t_1}\frac{(t_2-qs)^{(\zeta_1-1)}-(t_1-qs)^{(\zeta_1-1)}}{p^{\binom{\zeta_1}{2}}\Gamma_{p,q}(\zeta_1)}d_{p,q}s+\int_{t_1}^{t_2}\frac{(t_2-qs)^{(\zeta_1-1)}}{p^{\binom{\pi}{2}}\Gamma_{p,q}(\zeta_1)}d_{p,q}s\right]\\
&+\frac{|t_2-t_1|}{\Delta}\left\{\mu_1\int_0^{\eta_1}\frac{(\eta_1-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)}d_{p,q}s-\int_0^{\mathcal{T}/p}\frac{(\mathcal{T}/p-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)}d_{p,q}s\right\}\\
&\to 0 \text{ as } t_2\to t_1.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&|\mathcal{F}_2(u,v)(t_2)-\mathcal{F}_2(u,v)(t_1)|\\
&\leq\left(a_2K_1+b_2K_2+c_2\right)\left[\int_0^{t_1}\frac{(t_2-qs)^{(\zeta_2-1)}-(t_1-qs)^{(\zeta_2-1)}}{p^{\binom{\zeta_2}{2}}\Gamma_{p,q}(\zeta_2)}d_{p,q}s+\int_{t_1}^{t_2}\frac{(t_2-qs)^{(\zeta_2-1)}}{p^{\binom{\pi}{2}}\Gamma_{p,q}(\zeta_2)}d_{p,q}s\right]\\
&+\frac{|t_2-t_1|}{\Delta}\left\{\mu_1\int_0^{\eta_1}\frac{(\eta_2-qs)_{p,q}^{(\zeta_2-2)}}{p^{\binom{\zeta_2-1}{2}}\Gamma_{p,q}(\zeta_2-1)}d_{p,q}s-\int_0^{\mathcal{T}/p}\frac{(\mathcal{T}/p-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_2-1}{2}}\Gamma_{p,q}(\zeta_2-1)}d_{p,q}s\right\}\\
&\to 0 \text{ as } t_2\to t_1.
\end{aligned}$$

Thus, we conclude that $\mathcal{F}(U)$ is an equi-continuous and this yields $\mathcal{F}(U)$ is relatively compact.

Hence, by utilize Theorem 2.13, we conclude that the problem mentioned in (1.1) has a solution in U. □

Theorem 3.3. *Let assumptions $\mathcal{L}_{1.3}$ and $\mathcal{L}_3$ are satisfied. Then the problem formulated in (1.1) has at least one solution.*

Proof. As mentioned in aforesaid our Banach space is $X=C(J,\mathbb{R})^2$ equipped with the norm $\|u\|_X$.

Suppose that $K>K_1$ and define the map $\mathcal{F}:\bar{B}_K\to C(J,\mathbb{R})^2$ which $\mathcal{F}$ is formulated in (3.1), and $\bar{B}_K:=\bar{B}_K\left(0,C(J,\mathbb{R})^2\right)=\left\{u\in C(J,\mathbb{R})^2:\|u\|\leq K\right\}$. We follow the proof in two steps.

Step 1: In this step, we shall prove that $\mathcal{F}$ is a completely continuous operator. According to Theorem 3.2, we have $\mathcal{F}$ is continuous. Therefore, we shall prove that $\mathcal{F}\left(\bar{B}_R\right)$ is relatively compact set. For achieve this, at first, we prove that $\mathcal{F}\left(\bar{B}_R\right)$ is uniformly bounded.

Thus, $\forall(u,v)\in\bar{B}_K, t\in J$, we find

$$\begin{aligned}
|\mathcal{F}_1(u,v)(t)|\leq&\int_0^t\frac{(t-qs)_{p,q}^{(\zeta_1-1)}}{p^{\binom{\zeta_1}{2}}\Gamma_{p,q}(\zeta_1)}w_1(s,|u|_E)\,d_{p,q}s\\
&+\frac{t}{\Delta}\left\{\mu_1\int_0^{\eta_1}\frac{(\eta_1-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)}w_1(s,|u|_E)\,d_{p,q}s\right.\\
&\left.-\int_0^{\mathcal{T}/p}\frac{(\mathcal{T}/p-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)}w_1(s,|u|_E)\,d_{p,q}s\right\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^t \frac{(t-qs)_{p,q}^{(\zeta_1-1)}}{p^{\binom{\zeta_1}{2}}\Gamma_{p,q}(\zeta_1)} w_1\left(s,\ \sqrt{2}K\right) d_{p,q}s \\
&+ \frac{t}{\Delta}\left\{\mu_1 \int_0^{\eta_1} \frac{(\eta_1-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)} w_1\left(s,\ \sqrt{2}K\right) d_{p,q}s \right.\\
&\left. - \int_0^{\mathcal{T}/p} \frac{(\mathcal{T}/p-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)} w_1\left(s,\ \sqrt{2}K\right) d_{p,q}s\right\} \\
&\leq \max_{t\in J}\left\{w_1(t,\ \sqrt{2}K)\right\}\left[\int_0^t \frac{(t-qs)_{p,q}^{(\zeta_1-1)}}{p^{\binom{\zeta_1}{2}}\Gamma_{p,q}(\zeta_1)} d_{p,q}s \right.\\
&\left. + \frac{t}{\Delta}\left\{\mu_1 \int_0^{\eta_1} \frac{(\eta_1-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)} d_{p,q}s - \int_0^{\mathcal{T}/p} \frac{(\mathcal{T}/p-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)} d_{p,q}s\right\}\right],
\end{aligned}
$$

which yields $\mathcal{F}_1(B_R)$ is uniformly bounded. As same way, it is easy to check that $\mathcal{F}_2(B_R)$ is also uniformly bounded. Hence, we conclude that $\mathcal{F}\left(\bar{B}_R\right)$ is uniformly bounded. Now, we prove that $\mathcal{F}\left(\bar{B}_R\right)$ is an equi-continuous set. For do this, $\forall (u,v) \in \bar{B}_R$ and $t_1, t_2 \in J$ such that $t_1 < t_2$, we can write

$$
\begin{aligned}
&|\mathcal{F}_1(u,v)(t_2) - \mathcal{F}_1(u,v)(t_1)| \\
&\leq \max_{t\in J}\left\{w_1(t,\ \sqrt{2}K)\right\}\left[\int_0^{t_1} \frac{(t_2-qs)^{(\zeta_1-1)} - (t_1-qs)^{(\zeta_1-1)}}{p^{\binom{\zeta_1}{2}}\Gamma_{p,q}(\zeta_1)} d_{p,q}s + \int_{t_1}^{t_2} \frac{(t_2-qs)^{(\zeta_1-1)}}{p^{\binom{\pi}{2}}\Gamma_{p,q}(\zeta_1)} d_{p,q}s\right] \\
&+ \frac{|t_2-t_1|}{\Delta}\left\{\mu_1 \int_0^{\eta_1} \frac{(\eta_1-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)} d_{p,q}s - \int_0^{\mathcal{T}/p} \frac{(\mathcal{T}/p-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)} d_{p,q}s\right\} \\
&\to 0 \text{ as } t_2 \to t_1.
\end{aligned}
$$

Similarly, note that

$$
\begin{aligned}
&|\mathcal{F}_2(u,v)(t_2) - \mathcal{F}_2(u,v)(t_1)| \\
&\leq \max_{t\in J}\left\{w_2(t,\ \sqrt{2}K)\right\}\left[\int_0^{t_1} \frac{(t_2-qs)^{(\zeta_2-1)} - (t_1-qs)^{(\zeta_2-1)}}{p^{\binom{\zeta_2}{2}}\Gamma_{p,q}(\zeta_2)} d_{p,q}s + \int_{t_1}^{t_2} \frac{(t_2-qs)^{(\zeta_2-1)}}{p^{\binom{\pi}{2}}\Gamma_{p,q}(\zeta_2)} d_{p,q}s\right] \\
&+ \frac{|t_2-t_1|}{\Delta}\left\{\mu_1 \int_0^{\eta_1} \frac{(\eta_2-qs)_{p,q}^{(\zeta_2-2)}}{p^{\binom{\zeta_2-1}{2}}\Gamma_{p,q}(\zeta_2-1)} d_{p,q}s - \int_0^{\mathcal{T}/p} \frac{(\mathcal{T}/p-qs)_{p,q}^{(\zeta_2-2)}}{p^{\binom{\zeta_2-1}{2}}\Gamma_{p,q}(\zeta_2-1)} d_{p,q}s\right\} \\
&\to 0 \text{ as } t_2 \to t_1.
\end{aligned}
$$

Thus, we arrive at $\mathcal{F}\left(\bar{B}_R\right)$ is an equi-continuous set. Hence, $\mathcal{F}\left(\bar{B}_R\right)$ is relatively compact set.

Step 2: In this step, we show that the set $\mathcal{Z} = \{z : z = \mu\mathcal{F}(z), \text{ for } \mu \in J\}$ is bounded, such that $z = (u,v)$. Note that, for $t \in J$, we have

$$
\begin{aligned}
|u(t)| &= |\mu\mathcal{F}_1(u,v)(t)| \\
&= \mu\left|\int_0^t \frac{(t-qs)_{p,q}^{(\zeta_1-1)}}{p^{\binom{\zeta_1}{2}}\Gamma_{p,q}(\zeta_1)} \mathfrak{m}\left(s, u\left(p^{\zeta_1-1}s\right), v\left(p^{\zeta_2-1}s\right)\right) d_{p,q}s\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{t}{\Delta}\Bigg\{\mu_1\int_0^{\eta_1}\frac{(\eta_1-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)}\mathfrak{m}\left(s,u\left(p^{\zeta_1-1}s\right),v\left(p^{\zeta_2-1}s\right)\right)d_{p,q}s\\
&-\int_0^{T/p}\frac{(T/p-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)}\mathfrak{m}\left(s,u\left(p^{\zeta_1-1}s\right),v\left(p^{\zeta_2-1}s\right)\right)d_{p,q}s\Bigg\}\Bigg|\\
&\le\int_0^{t}\frac{(t-qs)_{p,q}^{(\zeta_1-1)}}{p^{\binom{\zeta_1}{2}}\Gamma_{p,q}(\zeta_1)}\left|\mathfrak{m}\left(s,u\left(p^{\zeta_1-1}s\right),v\left(p^{\zeta_2-1}s\right)\right)\right|d_{p,q}s\\
&+\frac{t}{\Delta}\Bigg\{\mu_1\int_0^{\eta_1}\frac{(\eta_1-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)}\left|\mathfrak{m}\left(s,u\left(p^{\zeta_1-1}s\right),v\left(p^{\zeta_2-1}s\right)\right)\right|d_{p,q}s\\
&-\int_0^{T/p}\frac{(T/p-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)}\left|\mathfrak{m}\left(s,u\left(p^{\zeta_1-1}s\right),v\left(p^{\zeta_2-1}s\right)\right)\right|d_{p,q}s\Bigg\}\\
&\le\sup_{t\in J}\Bigg[\int_0^{t}\frac{(t-qs)_{p,q}^{(\zeta_1-1)}}{p^{\binom{\zeta_1}{2}}\Gamma_{p,q}(\zeta_1)}w_1\left(s,|x\left(p^{\alpha-1}s\right)|_E\right)d_{p,q}s\\
&+\frac{t}{\Delta}\Bigg\{\mu_1\int_0^{\eta_1}\frac{(\eta_1-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)}w_1\left(s,|x\left(p^{\alpha-1}s\right)|_E\right)d_{p,q}s \qquad (3.12)\\
&-\int_0^{T/p}\frac{(T/p-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)}w_1\left(s,|x\left(p^{\alpha-1}s\right)|_E\right)d_{p,q}s\Bigg\}\Bigg]. \qquad (3.13)
\end{aligned}
$$

Similarly, one can obtain

$$
\begin{aligned}
|v(t)|\le\sup_{t\in J}\Bigg[&\int_0^{t}\frac{(t-qs)_{p,q}^{(\zeta_2-1)}}{p^{\binom{\zeta_2}{2}}\Gamma_{p,q}(\zeta_2)}w_2\left(s,|x\left(p^{\alpha-1}s\right)|_E\right)d_{p,q}s\\
&+\frac{t}{\Delta_2}\Bigg\{\mu_2\int_0^{\eta_2}\frac{(\eta_2-qs)_{p,q}^{(\zeta_2-2)}}{p^{\binom{\zeta_2-1}{2}}\Gamma_{p,q}(\zeta_2-1)}w_2\left(s,|x\left(p^{\alpha-1}s\right)|_E\right)d_{p,q}s\\
&-\int_0^{T/p}\frac{(T/p-qs)_{p,q}^{(\zeta_2-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_2-1)}w_2\left(s,|x\left(p^{\alpha-1}s\right)|_E\right)d_{p,q}s\Bigg\}\Bigg].
\end{aligned}
\tag{3.14}
$$

Assume that $\sigma_1=\|u\|_\infty,\sigma_2=\|v\|_\infty$. It follows from (3.12) and (3.14), we have

$$
\begin{cases}
\rho_1\le\sup_{t\in J}\Bigg[\int_0^{t}\frac{(t-qs)_{p,q}^{(\zeta_1-1)}}{p^{\binom{\zeta_1}{2}}\Gamma_{p,q}(\zeta_1)}w_1\left(s,|\rho|_E\right)d_{p,q}s\\
+\frac{t}{\Delta_1}\Bigg\{\mu_1\int_0^{\eta_1}\frac{(\eta_1-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)}w_1\left(s,|\rho|_E\right)d_{p,q}s-\int_0^{T/p}\frac{(T/p-qs)_{p,q}^{(\zeta_1-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_1-1)}w_1\left(s,|\rho|_E\right)d_{p,q}s\Bigg\}\Bigg],\\
\rho_2\le\sup_{t\in J}\Bigg[\int_0^{t}\frac{(t-qs)_{p,q}^{(\zeta_2-1)}}{p^{\binom{\zeta_2}{2}}\Gamma_{p,q}(\zeta_2)}w_2\left(s,|\rho|_E\right)d_{p,q}s\\
+\frac{t}{\Delta_2}\Bigg\{\mu_2\int_0^{\eta_2}\frac{(\eta_2-qs)_{p,q}^{(\zeta_2-2)}}{p^{\binom{\zeta_2-1}{2}}\Gamma_{p,q}(\zeta_2-1)}w_2\left(s,|\rho|_E\right)d_{p,q}s-\int_0^{T/p}\frac{(T/p-qs)_{p,q}^{(\zeta_2-2)}}{p^{\binom{\zeta_1-1}{2}}\Gamma_{p,q}(\zeta_2-1)}w_2\left(s,|\rho|_E\right)d_{p,q}s\Bigg\}\Bigg].
\end{cases}
$$

In view of $\mathcal{L}_3$ we deduce that $|\sigma|_E\le K_1$. Since $|\sigma|_E=\|z\|_x$ and $K_1<K$, one has $\|z\|_x<K$.

Thanks to Theorem 2.14 to obtain the existence result. □

4. Examples

Example 4.1. *Consider the following fractional (p,q)-boundary value problem*

$$\begin{cases} {}^cD^{\frac{3}{2}}_{\frac{1}{4},\frac{1}{5}}u(t) = 0.5 + \dfrac{1}{8} \times \dfrac{u^2(\frac{t}{8})}{1+u^2(t)} cos(4v(\frac{t}{4\sqrt{2}})), \quad t \in [0,8], \\ {}^cD^{\frac{5}{4}}_{\frac{1}{4},\frac{1}{5}}v(t) = 0.5 + \dfrac{1}{10} \times \dfrac{u^2(\frac{t}{8})}{1+u^2(t)} sin(8v(\frac{t}{4\sqrt{2}})), \quad t \in [0,8], \\ u(0) = u'(0) = 0, \quad v(0) = v'(0) = 0, \\ D_{p,q}u(4) = 2D_{p,q}u(1), \ \ D_{p,q}v(4) = 3D_{p,q}v(1). \end{cases} \tag{4.1}$$

In this case we take $\zeta_1 = \frac{3}{2}$, $\zeta_2 = \frac{5}{4}$, $T = 1$, $p = \frac{1}{4}$, $q = \frac{1}{5}$, $\eta_1 = \eta_2 = 1$, and $\mu_1 = 2$, $\mu_2 = 3$. It is easy to check that:

$$\Lambda_1 = \frac{1}{3\Gamma_{\frac{1}{4},\frac{1}{5}}(\frac{3}{2})} = 0.7454, \quad and \quad \Lambda_2 = \frac{1}{5\Gamma_{\frac{1}{4},\frac{1}{5}}(\frac{5}{4})} = 0.2991.$$

Further, note that

$$\sup_{u.v\in\mathbb{R}} \left|\frac{\partial \mathfrak{m}(u,v)}{\partial v}\right| \le \frac{1}{2} := a_1, \quad \sup_{u.v\in\mathbb{R}} \left|\frac{\partial \mathfrak{m}(u,v)}{\partial u}\right| \le \frac{3\sqrt{3}}{64} := b_1,$$

$$\sup_{u.v\in\mathbb{R}} \left|\frac{\partial \mathfrak{n}(u,v)}{\partial v}\right| \le \frac{4}{5} := a_2, \quad \sup_{u.v\in\mathbb{R}} \left|\frac{\partial \mathfrak{n}(u,v)}{\partial u}\right| \le \frac{3\sqrt{3}}{80} := b_2.$$

So, we arrive at

$$\mathcal{M}_{2\times 2} = \begin{bmatrix} 0.3727 & 0.0605 \\ 0.2393 & 0.0194 \end{bmatrix},$$

and

$$(\mathcal{I} - \mathcal{M}_{2\times 2})^{-1} = \begin{bmatrix} 1.6326 & 0.1008 \\ 0.3984 & 1.0444 \end{bmatrix}.$$

This matrix has two eigenvalues $\lambda_1 = 0.4098$, and $\lambda_2 = -0.0177$, which in both case, we have $|\lambda_1| < 1$ and $|\lambda_2| < 1$. Also rank$(\mathcal{I} - \mathcal{M}_{2\times 2}) = 2$, and all member of $(\mathcal{I} - \mathcal{M}_{2\times 2})^{-1}$, are nonnegative. Thus, $\mathcal{M}_{2\times 2} \to 0$. Hence all conditions of Theorem 3.1 are valid and the problem (4.1) *has a unique solution. Moreover, the data in Table 3, show that convergence of $\mathcal{M}_{2\times 2}$ is independent of quantum parameters (p,q). Also, to better understand this example, the graphs of functions $\mathfrak{m}, \mathfrak{n}$, and heatmap of Table 3 are presented in Figures 1–3.*

Table 3. Eignvalues of $\mathcal{M}_{2\times 2}$ with different value of p, q.

p	q	$\Gamma_{p,q}(1.5)$	$\Gamma_{p,q}(1.25)$	Λ_1	Λ_2	$\lvert\lambda_1\rvert$	$\lvert\lambda_2\rvert$
0.25	0.2	2.2361	1.4953	0.7454	0.2991	0.4098	0.0177
0.3	0.1	1.2247	1.1067	0.4082	0.2213	0.2314	0.0128
0.47	0.18	1.2731	1.1283	0.4103	0.2257	0.2327	0.0129
0.7	0.59	2.5226	1.5883	0.8409	0.3177	0.4599	0.0189
0.91	0.81	3.0166	1.7368	1.0055	0.3474	0.5461	0.0208

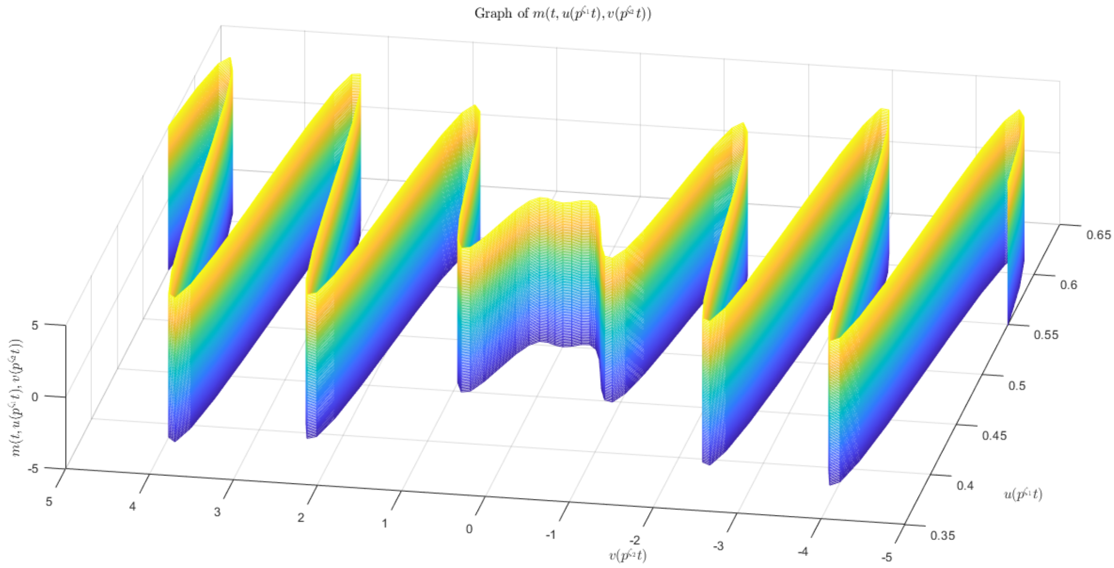

Figure 1. The graph of $\mathfrak{m}(t, u(p^{\zeta_1} t), v(p^{\zeta_2} t))$ in Example 4.1.

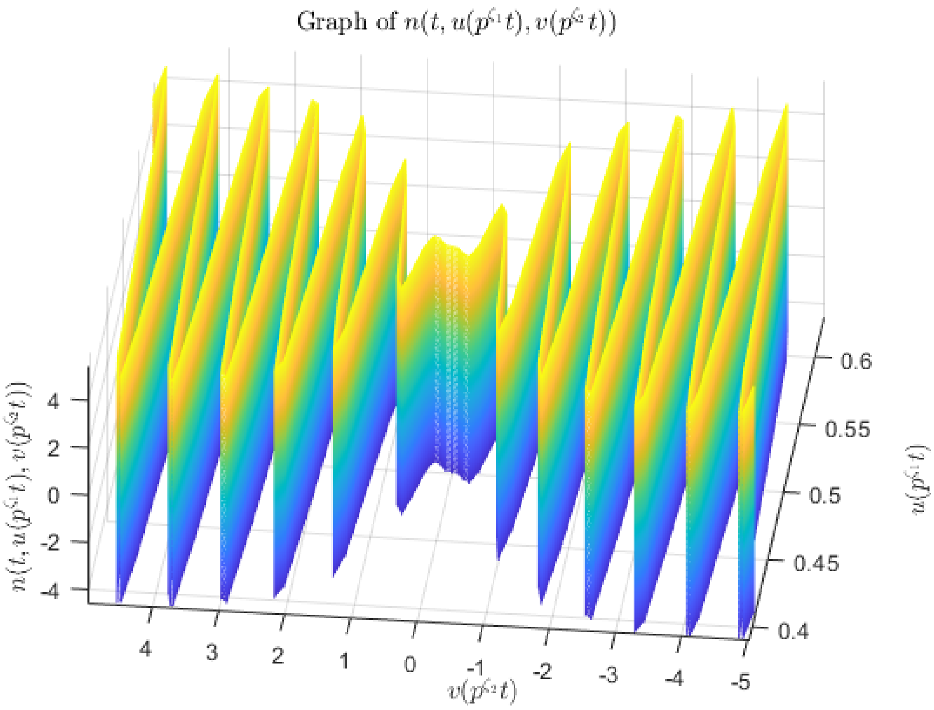

Figure 2. The graph of $\mathfrak{n}(t, u(p^{\zeta_1} t), v(p^{\zeta_2} t))$ in Example 4.1.

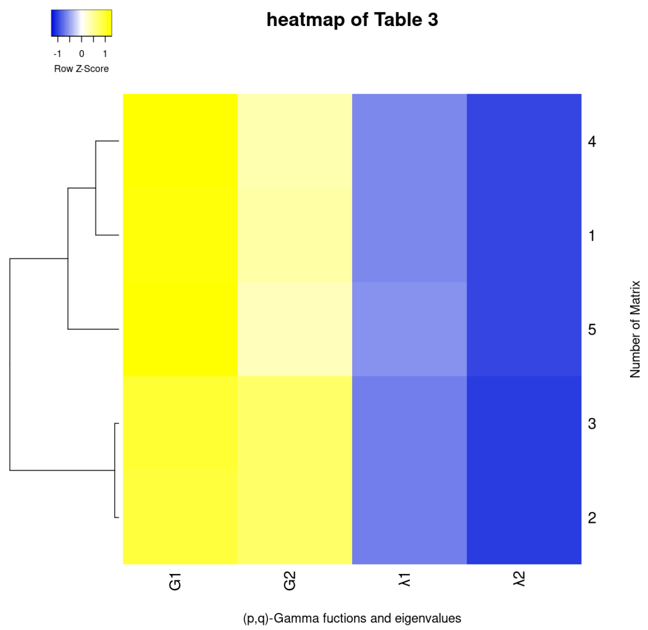

Figure 3. The heatmap of Table 3.

Example 4.2. *Consider the following fractional (p, q)-boundary value problem*

$$\begin{cases} {}^cD^{\frac{3}{2}}_{\frac{1}{4},\frac{1}{5}}u(t) = -\dfrac{3}{10}u(\frac{t}{8}) - \dfrac{2u(\frac{t}{8})v(\frac{t}{4\sqrt{2}})}{3+u^2(\frac{t}{8})} + 11, \quad t \in [0,8], \\ {}^cD^{\frac{5}{4}}_{\frac{1}{4},\frac{1}{5}}v(t) = -\dfrac{7}{13}v(\frac{t}{4\sqrt{2}}) - \dfrac{2u(\frac{t}{8})v(\frac{t}{4\sqrt{2}})}{3+u^2(\frac{t}{8})} + 13, \quad t \in [0,8], \\ u(0) = u'(0) = 0, \quad v(0) = v'(0) = 0, \\ D_{p,q}u(4) = 2D_{p,q}u(1), \quad D_{p,q}v(4) = 3D_{p,q}v(1). \end{cases} \tag{4.2}$$

In this case we take $\zeta_1 = \frac{3}{2}$, $\zeta_2 = \frac{5}{4}$, $T = 1$, $p = \frac{1}{4}$, $q = \frac{1}{5}$, $\eta_1 = \eta_2 = 1$, *and* $\mu_1 = 2$, $\mu_2 = 3$. *With a simple computation, we obtain*

$$a_1 = 0.3, \quad a_2 = 0, \quad b_1 = \frac{\sqrt{3}}{3}, \quad b_2 = \frac{7}{13} + \frac{\sqrt{3}}{6}.$$

Then, we define

$$\mathcal{M}_{2\times 2} = \begin{bmatrix} 0.2236 & 0.4304 \\ 0 & 0.2474 \end{bmatrix},$$

which this yields

$$\lambda_1 = 0.2236, \quad \lambda_2 = 0.2474,$$

and

$$(\mathcal{I} - \mathcal{M}_{2\times 2})^{-1} = \begin{bmatrix} 1.2880 & 0.7365 \\ 0 & 1.3287 \end{bmatrix},$$

also

$$rank(\mathcal{I} - \mathcal{M}_{2\times 2}) = 2.$$

From the above facts it can be concluded that, $\mathcal{M}_{2\times 2}$ convergence to zero. Thus, all assumption of Theorem 3.2 are hold and so the problem (4.2) *has at least one solution. Moreover, the data in Table 4, show that convergence of $\mathcal{M}_{2\times 2}$ is independent of quantum parameters* (p, q)*. Also, to better understand this example, the graph of the function* $\mathfrak{m}$ *and heatmap of Table 4 are presented in Figures 4 and 5.*

Table 4. Eignvalues of $\mathcal{M}_{2\times 2}$ with different value of p, q.

p	q	$\Gamma_{p,q}(1.5)$	$\Gamma_{p,q}(1.25)$	Λ_1	Λ_2	$\lvert\lambda_1\rvert$	$\lvert\lambda_2\rvert$
0.25	0.2	2.2361	1.4953	0.7454	0.2991	0.2236	0.2474
0.3	0.1	1.2247	1.1067	0.4082	0.2213	0.1225	0.1830
0.47	0.18	1.2731	1.1283	0.4103	0.2257	0.1231	0.1867
0.7	0.59	2.5226	1.5883	0.8409	0.3177	0.2523	0.2628
0.91	0.81	3.0166	1.7368	1.0055	0.3474	0.3017	0.2873

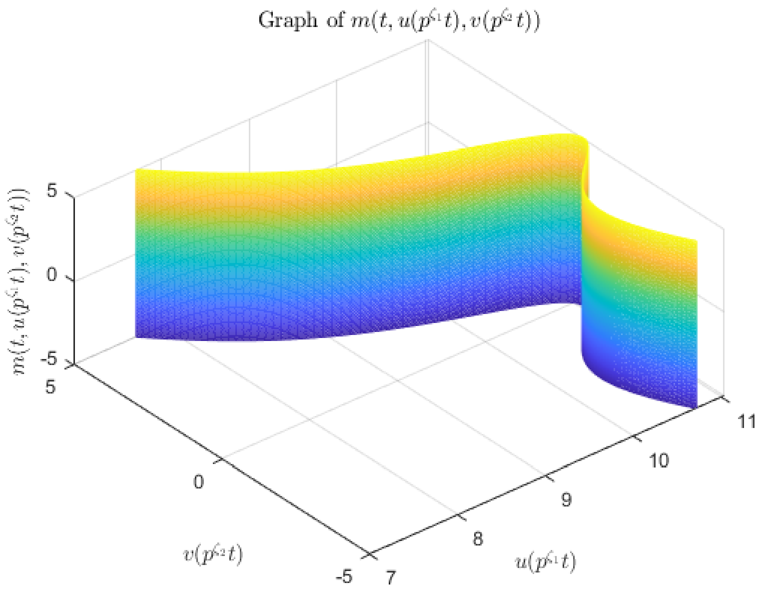

Figure 4. The graph of $\mathfrak{m}(t, u(p^{\zeta_1}t), v(p^{\zeta_2}t))$ in Example 4.2.

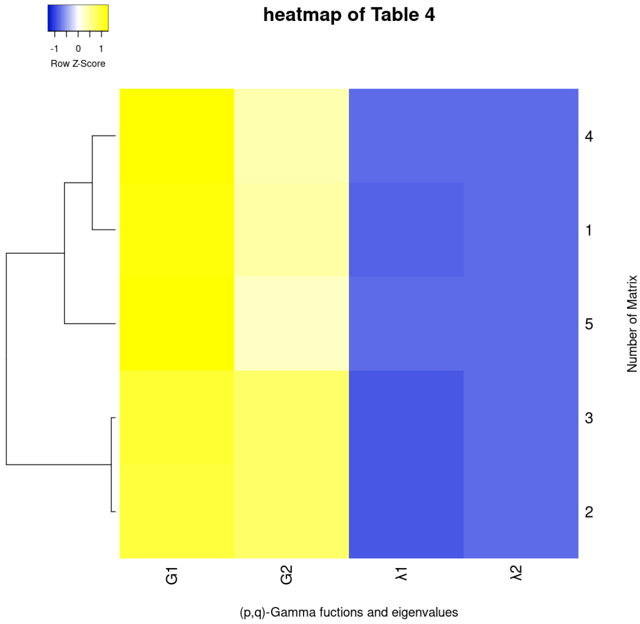

Figure 5. The heatmap of Table 4.

5. Conclusions

In this work, we investigate the fractional (p, q)-difference equation under non-local boundary conditions with a new method. We introduce the Lipchitzian matrix for our problem such that elements of this matrix depend on the fractional order ζ and the quantum Gamma function $\Gamma_{p,q}(\zeta)$. Then, using the fixed point theory and providing sufficient conditions for convergence to the zero of the mentioned matrix, we will follow the theory of existence. Finally, we go to the numerical analysis of our introduced technique to confirm its accuracy and validity. The data from the presented examples indicate the independence of our method from the p and q quantum parameters. This paper, and the methods presented in it, can provide the basis for further study of generalized quantum differential equations and the use of numerical techniques in providing sufficient conditions for the existence of the solution.

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Conflict of interest

The authors declare no conflicts of interest.

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