AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(1): 1550–1565.
DOI: 10.3934/math.2023078
Received: 21 September 2022
Revised: 12 October 2022
Accepted: 16 October 2022
Published: 21 October 2022

Research article

Sombor indices of cacti

Fan Wu, Xinhui An*and Baoyindureng Wu

Department of Mathematics, Xinjiang University, Urumqi 830046, China

* **Correspondence:** Email: xjaxh@163.com.

Abstract: For a graph G, the Sombor index $SO(G)$ of G is defined as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_G(u)^2 + d_G(v)^2},$$

where $d_G(u)$ is the degree of the vertex u in G. A cactus is a connected graph in which each block is either an edge or a cycle. Let $\mathcal{G}(n,k)$ be the set of cacti of order n and with k cycles. Obviously, $\mathcal{G}(n,0)$ is the set of all trees and $\mathcal{G}(n,1)$ is the set of all unicyclic graphs, then the cacti of order n and with $k(k \geq 2)$ cycles is a generalization of cycle number k. In this paper, we establish a sharp upper bound for the Sombor index of a cactus in $\mathcal{G}(n,k)$ and characterize the corresponding extremal graphs. In addition, for the case when $n \geq 6k-3$, we give a sharp lower bound for the Sombor index of a cactus in $\mathcal{G}(n,k)$ and characterize the corresponding extremal graphs as well. We also propose a conjecture about the minimum value of sombor index among $\mathcal{G}(n,k)$ when $n \geq 3k$.

Keywords: Sombor index; cactus; extreme value
Mathematics Subject Classification: 33C20, 33B15,11B83

1. Introduction

In this paper, we consider connected simple and finite graphs, and refer to Bondy and Murty [4] for notation and terminologies used but not defined here.

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$, $|V(G)| = n$ and $|E(G)| = m$. We denote $G-v$ and $G-uv$ the graph obtained from G by deleting a vertex $v \in V(G)$, or an edge $uv \in E(G)$, respectively. Similarly, $G+uv$ is obtained from G by adding an edge $uv \notin E(G)$, where $u, v \in V(G)$. An edge uv of a graph G is called a *cut edge* if the graph $G-uv$ is disconnected. For a vertex $u \in V(G)$, its degree $d_G(u)$ is equal to the number of vertices in G adjacent to u; the neighborhood of u is denoted by $N_G(u)$, or $N(u)$ for short. The symbols $\Delta(G)$ and $\delta(G)$ represent the maximum degree and the minimum degree of G. We use T_n, C_n, P_n and S_n to denote the tree, cycle, path and star of order n, respectively.

Gutman [13] defined a new vertex-degree-based graph invariant, called Sombor index. Precisely, for a graph G, it is denoted by $SO(G)$ and is defined as

$$SO(G) = \sum_{uv\in E(G)} \sqrt{d_G(u)^2 + d_G(v)^2}.$$

He proved that for any tree T with $n \geq 3$ vertices,

$$2\sqrt{2}(n-3) + 2\sqrt{5} \leq SO(T) \leq (n-1)\sqrt{n^2 - 2n + 2},$$

with left side of equality if and only if $T \cong P_n$, and with the right side of equality if and only if $T \cong S_n$. Chen etal. [5] determined the extremal values of the Sombor index of trees with some given parameters, including matching number, pendant vertices, diameter, segment number, branching number, etc. The corresponding extremal trees are characterized completely. Deng etal. [11] obtained a sharp upper bound for the Sombor index among all molecular trees with fixed numbers of vertices, and characterize those molecular trees achieving the extremal value. Cruz etal. [8] determined the extremal values of Sombor indices over trees with at most three branch vertices. Li etal. [19] give sharp bounds for the Sombor index of trees with a given diameter. Das and Gutman [9] present bounds on the Sombor index of trees in terms of order, independence number, and number of pendent vertices, and characterize the extremal cases. In addition, analogous results for quasi-trees are established. Sun and Du [30] present the maximum and minimum Sombor indices of trees with fixed domination number, and identified the corresponding extremal trees. Zhou etal. [36] determined the graph with minimum Sombor index among all trees with given number of vertices and maximum degree, respectively, among all unicyclic graphs with given number of vertices and maximum degree.

Cruz and Rada [7] investigate the Sombor indices of unicyclic and bicyclic graphs. Let $U(n, p, q, r) \in \mathcal{G}(n, 1)$, where $p \geq q \geq r \geq 0$ and $p + q + r = n - 3$, be a unicyclic graph obtained from 3-cycle C_3 with $V(C_3) = \{u, v, w\}$, adding p, q and r pendent vertices to the vertices u, v and w, respectively. They showed that for a unicyclic graph with $n \geq 3$ vertices,

$$2\sqrt{2}n \leq SO(G) \leq (n-3)\sqrt{(n-1)^2 + 1} + 2\sqrt{(n-1)^2 + 2^2} + 2\sqrt{2}.$$

The lower and upper bound is uniquely attained by $G \cong C_n$ and $G \cong U(n, n-3, 0, 0)$, respectively. Alidadi etal. [2] gave the minimum Sombor index for unicyclic graphs with the diameter $D \geq 2$.

Aashtab etal. [1] studied the structure of a graph with minimum Sombor index among all graphs with fixed order and fixed size. It is shown that in every graph with minimum Sombor index the difference between minimum and maximum degrees is at most 1. Cruz etal. [6] characterize the graphs extremal with respect to the Sombor index over the following families of graphs: (connected) chemical graphs, chemical trees, and hexagonal systems. Liu etal. [21] determined the minimum Sombor indices of tetracyclic (chemical) graphs. Das and Shang [10] present some lower and upper bounds on the Sombor index of graph G in terms of graph parameters (clique number, chromatic number, number of pendant vertices, etc.) and characterize the extremal graphs. For the Sombor index of a connected graph with given order, Horoldagva and Xu [16] presented sharp upper and lower bounds when its girth is fixed, a lower bound if its maximum degree is given and an upper bound in terms of given number of pendent vertices or pendent edges, respectively. In [29], Shang observe power-law and small-word effect for the simplicial networks and examine the effectiveness of the approximation method for Sombor index through computational experiments.

Relations between the Sombor index and some other well-known degree-based descriptors [14, 24, 25, 33]. A number of application of Sombor index in chemistry were reported in [3, 20, 27]. Besides, the relationship between the energy and Sombor index of a graph G is studied in [12, 15, 26, 28, 31, 32].

Some variations of Sombor index, for instance, the reduced Sombor index, average Sombor index, are investigated. Redžepović [27] examined the predictive and discriminative potentials of Sombor index, the reduced Sombor index, average Sombor index. Liu etal. [23] obtained some bounds for reduced Sombor index of graphs with given several parameters (such as maximum degree, minimum degree, matching number, chromatic number, independence number, clique number), some special graphs (such as unicyclic grahs, bipartite graphs, graphs with no triangles, graphs with no $K_r + 1$ and the Nordhaus-Gaddum-type results). A conjecture related to the chromatic number in the above paper was verified to be true by Wang and Wu [34]. Liu etal. [22] ordered the chemical trees, chemical unicyclic graphs, chemical bicyclic graphs and chemical tricyclic graphs with respect to Sombor index and reduced Sombor index. Furthermore, they determined the first fourteen minimum chemical trees, the first four minimum chemical unicyclic graphs, the first three minimum chemical bicyclic graphs, the first seven minimum chemical tricyclic graphs. Finally, the applications of reduced Sombor index to octane isomers were given. Wang and Wu [35] investigated the reduced Sombor index and the exponential reduced Sombor index of a molecular tree solving a conjecture [23] and an open problem [11].

A vertex of degree 1 is said to be a *pendant* vertex. Further, an edge is said to be a *pendant* edge if one of its end vertices is a pendant vertex. A connected graph that has no cut vertices is called a *block*, the blocks of G which correspond to leaves of its block tree are referred to as its *end blocks*. A *cactus* is a connected graph in which every block is either an edge or a cycle. Let $\mathcal{G}(n, k)$ be the family of all cacti with n vertices and k cycles. Clearly, $|E(G)| = n + k - 1$ for any $G \in \mathcal{G}(n, k)$. Note that $\mathcal{G}(n, 0)$ is the set of all trees and $\mathcal{G}(n, 1)$ is the set of all unicyclic graphs. Gutman [13] characterized the tree with extremal value Sombor index.

It is our main concern in this paper to study the extremal value problem of Sombor index on $\mathcal{G}(n, k)$, $k \geq 2$. In this paper, we will determine the maximum Sombor index of graphs among $G \in \mathcal{G}(n, k)$, and also characterize the corresponding extremal graphs. Later, we will determine the minimum Sombor index of graphs with given conditions among $G \in \mathcal{G}(n, k)$, and also characterize the corresponding extremal graphs.

Let $G_0(n, k) \in \mathcal{G}(n, k)$ be a bundle of k triangles with $n - 2k - 1$ pendent vertices attached to the common vertex, as illustrated in Figure 1.

By a simple computation, we have $SO(G_0(n, k)) = (n - 2k - 1)\sqrt{(n-1)^2 + 1} + 2k\sqrt{(n-1)^2 + 2^2} + 2\sqrt{2}k$. We will see that $G_0(n, k)$ has the maximum Sombor index among $\mathcal{G}(n, k)$.

Theorem 1.1. *Let $k \geq 1$ and $n \geq 3$. For any $G \in \mathcal{G}(n, k)$,*

$$SO(G) \leq (n - 2k - 1)\sqrt{(n-1)^2 + 1} + 2k\sqrt{(n-1)^2 + 2^2} + 2\sqrt{2}k,$$

with equality if and only if $G \cong G_0(n, k)$.

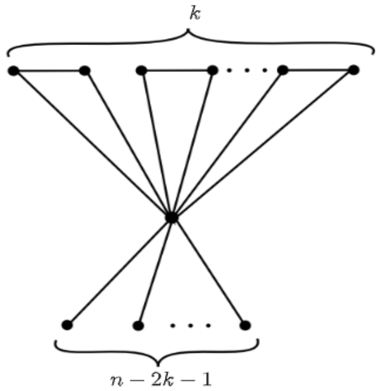

Figure 1. $G_0(n,k)$.

Let $C^*(n,k)$ denote the set of the elements G of $\mathcal{G}(n,k)$ with the following properties:

(1) $\delta(G) = 2$ and $\Delta(G) = 3$;

(2) a vertex is a cut vertex if and only if it has degree 3, and there are exactly $2k-2$ cut vertices;

(3) at least $\lceil\frac{k-3}{2}\rceil$ internal cycles with all three degrees are triangles;

(4) at most one vertex not belong to any cycle;

(5) the three degree vertices on the cycle are adjacent.

Generally speaking, if k is even, an element of $C^*(n,k)$ obtained from a tree T of order k with each vertex having degree 1 or 3 by replacing each vertex of degree 3 with a triangle and replacing each vertex of degree 1 with a cycle. If k is odd, an element of $C^*(n,k)$ obtained from a tree T of order k with exactly a vertex having degree 2 by replacing two (adjacent) vertices of degree 3 with a cycle, and other vertices having degree 1 or 3 by replacing each vertex of degree 3 with a triangle, replacing each vertex of degree 1 with a cycle or an element of $C^*(n,k)$ obtained from a tree T of order k with each vertex having degree 1 or 3 by retention one vertex of degree 3, and by replacing other vertices of degree 3 with a triangle and replacing each vertex of degree 1 with a cycle.

Three elements of $C^*(n,k)$ are shown in terms of the parity of k in Figure 2.

Theorem 1.2. *Let $k \geq 2$ and $n \geq 6k-3$. For any $G \in \mathcal{G}(n,k)$, we have*

$$SO(G) \geq 2\sqrt{2}n + 5\sqrt{2}(\lfloor\frac{k}{2}\rfloor - 2) + 2\sqrt{13}(k - \lfloor\frac{k}{2}\rfloor + 1),$$

with equality if and only if $G \in C^*(n,k)$.

The proofs of Theorems 1.1 and 1.2 are given Sections 2 and 3, respectively.

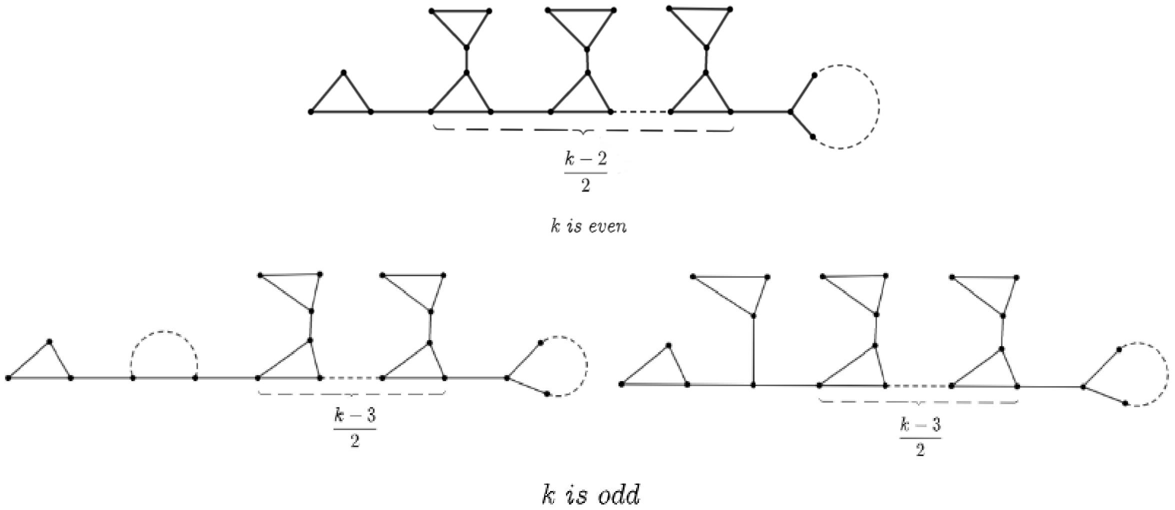

Figure 2. Three cacti in $C^*(n, k)$ when $n \geq 3k$.

2. The maximum Sombor index of cacti

In this section, we will determine the maximum value of the Sombor index of cacti with n vertices and k cycles, and characterize the corresponding extremal graph. We start with several known results, which will be used in the proof of Theorem 1.1.

Lemma 2.1 (Horoldagva and Xu [16])**.** *If uv is a non-pendent cut edge in a connected graph G, then $SO(G') > SO(G)$, where G' is the graph obtained by the contraction of uv onto the vertex u and adding a pendent vertex v to u.*

In 1932, Karamata proved an interesting result, which is now known as the majorization inequality or Karamata's inequailty. Let $A = (a_1, a_2, \cdots, a_n)$ and $B = (b_1, b_2, \cdots, b_n)$ be non-increasing two sequences on an interval I of real numbers such that $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$. If $a_1 + a_2 + \cdots + a_i \geq b_1 + b_2 + \cdots + b_i$ for all $1 \leq i \leq n - 1$ then we say that A majorizes B.

Lemma 2.2 (Karamata [18])**.** *Let $f : I \to \mathbb{R}$ be a strictly convex function. Let $A = (a_1, a_2, \cdots, a_n)$ and $B = (b_1, b_2, \cdots, b_n)$ be non-increasing sequences on I. If A majorizes B then $f(a_1) + f(a_2) + \cdots + f(a_n) \geq f(b_1) + f(b_2) + \cdots + f(b_n)$ with equality if and only if $a_i = b_i$ for all $1 \leq i \leq n$.*

Lemma 2.3. *Let G be a connected graph with a cycle $C_p = v_1 v_2 \cdots v_p v_1$ $(p \geq 4)$ such that $G - E(C_p)$ has exactely p components $G_1, G_2, \ldots, G_p$, where G_i is the component of $G - E(C_p)$ containing v_i for each $i \in \{1, 2, \cdots, p\}$, as shown in Figure 3. If $G' = G - \{v_{p-1}v_p, uv_p | u \in N_{G_p}(v_p)\} + \{v_1 v_{p-1}, uv_1 | u \in N_{G_p}(v_p)\}$, then $SO(G') > SO(G)$.*

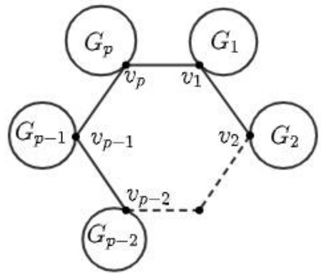

Figure 3. The graph G.

Proof. Let $N_G(v_1)\backslash\{v_2, v_p\} = \{x_1, x_2, \cdots, x_s\}$, $N_G(v_p)\backslash\{v_1, v_{p-1}\} = \{y_1, y_2, \cdots, y_t\}$, where $s = d_G(v_1) - 2$ and $t = d_G(v_p) - 2$. Hence, by the definition of $SO(G)$, we obtain

$$\begin{aligned}
& SO(G') - SO(G) \\
=& \sum_{i=1}^{s} \sqrt{d_{G'}(x_i)^2 + d_{G'}(v_1)^2} - \sum_{i=1}^{s} \sqrt{d_G(x_i)^2 + d_G(v_1)^2} + \sum_{j=1}^{t} \sqrt{d_{G'}(y_j)^2 + d_{G'}(v_1)^2} \\
& - \sum_{j=1}^{t} \sqrt{d_G(y_j)^2 + d_G(v_p)^2} + \sqrt{d_{G'}(v_1)^2 + d_{G'}(v_2)^2} - \sqrt{d_G(v_1)^2 + d_G(v_2)^2} \\
& + \sqrt{d_{G'}(v_1)^2 + d_{G'}(v_{p-1})^2} - \sqrt{d_G(v_p)^2 + d_G(v_{p-1})^2} + \sqrt{d_{G'}(v_1)^2 + 1^2} \\
& - \sqrt{d_G(v_1)^2 + d_G(v_p)^2} \\
=& \sum_{i=1}^{s} \sqrt{d_{G'}(x_i)^2 + (s+t+3)^2} - \sum_{i=1}^{s} \sqrt{d_G(x_i)^2 + (s+2)^2} + \sum_{j=1}^{t} \sqrt{d_{G'}(y_j)^2 + (s+t+3)^2} \\
& - \sum_{j=1}^{t} \sqrt{d_G(y_j)^2 + (t+2)^2} + \sqrt{(s+t+3)^2 + d_{G'}(v_2)^2} - \sqrt{(s+2)^2 + d_G(v_2)^2} \\
& + \sqrt{(s+t+3)^2 + d_{G'}(v_{p-1})^2} - \sqrt{(t+2)^2 + d_G(v_{p-1})^2} + \sqrt{(s+t+3)^2 + 1^2} \\
& - \sqrt{(s+2)^2 + (t+2)^2} \\
>& \sqrt{(s+t+3)^2 + 1^2} - \sqrt{(s+2)^2 + (t+2)^2}.
\end{aligned}$$

Since $s \geq 0, t \geq 0$, we have $[(s+t+3)^2 + 1^2] - [(s+2)^2 + (t+2)^2] = 2st + 4s + 2t + 2 > 0$, implying that $SO(G') > SO(G)$. □

Lemma 2.4. *Let n and k be two nonnegative integers with $n \geq 2k+1$. If $G \in \mathcal{G}(n,k)$ has a triangle $v_1v_2v_3v_1$ with $d_G(v_3) \geq d_G(v_2) \geq 3$ as shown in Figure 4, then $SO(G') > SO(G)$, where $G' = G - \{v_2u |\ u \in N_{G_2}(v_2)\} + \{v_3u |\ u \in N_{G_2}(v_2)\}$.*

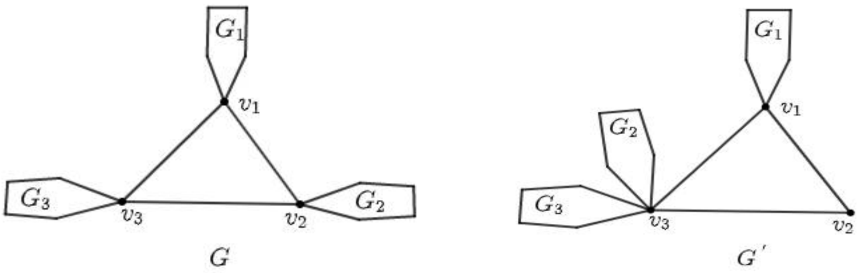

Figure 4. The graphs G and G'.

Proof. Let $N_G(v_3)\backslash\{v_2\} = \{v_1, x_1, x_2, \cdots, x_s\}$, $N_G(v_2)\backslash\{v_3\} = \{v_1, y_1, y_2, \cdots, y_t\}$, then $s = d_G(v_3) - 2,\ t = d_G(v_2) - 2$. Defined according to G', $d_{G'}(v_3) = s + t + 2$. Assume that $d_{G'}(v_1) = d_G(v_1) = p$, $\sqrt{(s+t+2)^2 + 2^2} + \sqrt{(s+2)^2 + (t+2)^2} = q$.

$$
\begin{aligned}
& SO(G') - SO(G) \\
=& \sum_{i=1}^{s}\Big(\sqrt{d_{G'}(v_3)^2 + d_{G'}(x_i)^2} - \sqrt{d_G(v_3)^2 + d_G(x_i)^2}\Big) + \sum_{j=1}^{t}\Big(\sqrt{d_{G'}(v_3)^2 + d_{G'}(y_j)^2} \\
& - \sqrt{d_G(v_2)^2 + d_G(y_j)^2}\Big) + \Big(\sqrt{d_{G'}(v_3)^2 + d_{G'}(v_1)^2} - \sqrt{d_G(v_3)^2 + d_G(v_1)^2}\Big) \\
& + \Big(\sqrt{d_{G'}(v_2)^2 + d_{G'}(v_3)^2} - \sqrt{d_G(v_2)^2 + d_G(v_3)^2}\Big) + \Big(\sqrt{d_{G'}(v_1)^2 + d_{G'}(v_2)^2} \\
& - \sqrt{d_G(v_1)^2 + d_G(v_2)^2}\Big) \\
=& \sum_{i=1}^{s}\sqrt{(s+t+2)^2 + d_{G'}(x_i)^2} - \sum_{i=1}^{s}\sqrt{(s+2)^2 + d_G(x_i)^2} \\
& + \sum_{j=1}^{t}\sqrt{(s+t+2)^2 + d_{G'}(y_j)^2} - \sum_{j=1}^{t}\sqrt{(t+2)^2 + d_G(y_j)^2} \\
& + \sqrt{(s+t+2)^2 + d_{G'}(v_1)^2} - \sqrt{(s+2)^2 + d_G(v_1)^2} + \sqrt{(s+t+2)^2 + 2^2} \\
& - \sqrt{(s+2)^2 + (t+2)^2} + \sqrt{d_{G'}(v_1)^2 + 2^2} - \sqrt{d_G(v_1)^2 + (t+2)^2} \\
>& \sqrt{p^2 + (s+t+2)^2} + \sqrt{p^2 + 2^2} - \sqrt{p^2 + (s+2)^2} \\
& - \sqrt{p^2 + (t+2)^2} + \sqrt{(s+t+2)^2 + 2^2} - \sqrt{(s+2)^2 + (t+2)^2} \\
=& p\Big(\sqrt{1 + (\frac{s+t+2}{p})^2} + \sqrt{1 + (\frac{2}{p})^2} - \sqrt{1 + (\frac{s+2}{p})^2} - \sqrt{1 + (\frac{t+2}{p})^2}\Big) + \frac{2st}{q}.
\end{aligned}
\tag{2.1}
$$

Let us consider a function $f(x) = \sqrt{1 + x^2}$ and it is easy to see that this function is strictly convex for $x \in [0, +\infty)$. Since $A = \{\frac{s+t+2}{p}, \frac{2}{p}\}$ majorizes $B = \{\frac{s+2}{p}, \frac{t+2}{p}\}$, By Karamata's inequality, $f(\frac{s+t+2}{p}) + f(\frac{2}{p}) > f(\frac{s+2}{p}) + f(\frac{t+2}{p})$. Combining this with (2.1), it follows that $SO(G') > SO(G)$. □

Now, we are ready to present the proof of Theorem 1.1.

Proof of Theorem 1.1:

Let G be a cactus with the maximum Sombor index value among $\mathcal{G}(n,k)$. By Lemma 2.1, each cut edge of G is pendent. By Lemma 2.3, every cycles of G is a triangle. Furthermore, by Lemma 2.4, $G \cong G_0(n,k)$. Thus, the maximum Sombor index of cacti among $\mathcal{G}(n,k)$ is

$$SO(G) = (n-2k-1)\sqrt{(n-1)^2+1} + 2k\sqrt{(n-1)^2+2^2} + 2\sqrt{2}k.$$

3. The minimum Sombor index of cacti

In this section, we determine the minimum Sombor index of graphs in $\mathcal{G}(n,k)$, and characterize the corresponding extremal graphs.

First, we introduce some additional notations. For a graph G, $V_i(G) = \{v \in V(G) | \ d(v) = i\}$, $n_i = |V_i(G)|$, $E_{i,j}(G) = \{uv \in E(G) | \ d(u) = i, \ d(v) = j\}$ and $e_{i,j}(G) = |E_{i,j}(G)|$. Obviously, $e_{i,j}(G) = e_{j,i}(G)$. If there is no confusion, $e_{i,j}(G)$ is simply denoted by $e_{i,j}$. For any simple graph G of order n, we have

$$n = n_1 + n_2 + \cdots + n_{n-1}, \tag{3.1}$$

and

$$\begin{cases} 2e_{1,1} + e_{1,2} + \cdots + e_{1,n-1} &= n_1 \\ e_{2,1} + 2e_{2,2} + \cdots + e_{2,n-1} &= 2n_2 \\ \vdots & \\ e_{n-1,1} + e_{n-1,2} + \cdots + 2e_{n-1,n-1} &= (n-1)n_{n-1} \end{cases} \tag{3.2}$$

Let

$$L_{n,n} = \{(i,j) | \ i,j \in \mathbb{N}, 1 \le i \le j \le n-1\}.$$

it follows easily from (3.1) and (3.2) that

$$n = \sum_{(i,j)\in L_{n,n}} \frac{i+j}{ij} e_{i,j}, \tag{3.3}$$

Let $G \in \mathcal{G}(n,k)$ with $n \ge 2k+1$ and $k \ge 1$. It implies that $e_{1,1}(G) = 0$ and $e_{i,j}(G) = 0$ for any $1 \le i \le j \le n-1$ with $i+j > n+k$. Let $L^k_{n,n} = \{(i,j) \in L_{n,n} : i+j \le n+k\}$, $L^{k'}_{n,n} = L^k_{n,n} - \{(2,2),(2,3),(3,3)\}$. By a simple calculation we obtain the following result.

Lemma 3.1. *For any graph $G \in \mathcal{G}(n,k)$ ($k \ge 1$),*

$$SO(G) = 2\sqrt{2}n + (6\sqrt{13} - 10\sqrt{2})(k-1) + (5\sqrt{2} - 2\sqrt{13})e_{3,3} + \sum_{(i,j)\in L^{k'}_{n,n}} g(i,j)e_{i,j},$$

where

$$g(i,j) = \sqrt{i^2+j^2} - (12\sqrt{2} - 6\sqrt{13})\frac{i+j}{ij} + (10\sqrt{2} - 6\sqrt{13}). \tag{3.4}$$

Proof. For any $G \in \mathcal{G}(n,k)$,

$$n = \sum_{(i,j)\in L^k_{n,n}} \frac{i+j}{ij} e_{i,j}, \tag{3.5}$$

$$n+k-1=\sum_{(i,j)\in L_{n,n}^{k}} e_{i,j}. \tag{3.6}$$

Relations (3.5) and (3.6) can be rewritten as

$$5e_{2,3}+4e_{3,3}=6n-6e_{2,2}-6\sum_{(i,j)\in L_{n,n}^{k'}}\frac{i+j}{ij}e_{i,j},$$

$$e_{2,3}+e_{3,3}=n+k-1-e_{2,2}-\sum_{(i,j)\in L_{n,n}^{k'}} e_{i,j}.$$

Combining the above, we have

$$e_{2,3}=6k-6-2e_{3,3}+\sum_{(i,j)\in L_{n,n}^{k'}}(6\frac{i+j}{ij}-6)e_{i,j}, \tag{3.7}$$

$$e_{2,2}=n-5k+5+e_{3,3}-\sum_{(i,j)\in L_{n,n}^{k'}}(6\frac{i+j}{ij}-5)e_{i,j}. \tag{3.8}$$

$g(2,2)=g(2,3)=0$, $g(3,3)=(5\sqrt{2}-2\sqrt{13})<0$.

Thus,

$$\begin{aligned} SO(G)=&\sqrt{13}e_{2,3}+3\sqrt{2}e_{3,3}+2\sqrt{2}e_{2,2}+\sum_{(i,j)\in L_{n,n}^{k'}}\sqrt{i^2+j^2}e_{i,j}\\ =&2\sqrt{2}n+(6\sqrt{13}-10\sqrt{2})(k-1)+(5\sqrt{2}-2\sqrt{13})e_{3,3}+\sum_{(i,j)\in L_{n,n}^{k'}}g(i,j)e_{i,j}. \end{aligned} \tag{3.9}$$

□

Lemma 3.2 (Chen, Li, Wang [5]). *Let* $f(x,y)=\sqrt{x^2+y^2}$ *and* $h(x,y)=f(x,y)-f(x-1,y)$, *where* $x,y\geq 1$. *If* $x,y\geq 1$, *then* $h(x,y)$ *strictly decreases with* y *for fixed* x *and increases with* x *for fixed* y.

Since $f(x+k,y)-f(x,y)=\sum_{i=1}^{k}[f(x+i,y)-f(x+i-1,y)]=\sum_{i=1}^{k}h(x+i,y)$ for any $k\in\mathbb{Z}^+$, we have the following corollary.

Corollary 3.1. *If* $x,y\geq 1$ *and* $k\in\mathbb{Z}^+$, *then* $f(x+k,y)-f(x,y)$ *strictly decreases with* y *for fixed* x *and increases with* x *for fixed* y.

Let $P_l=u_0u_1\cdots u_l$, $l\geq 1$ be a path of G with $d(u_0)\geq 3$, $d(u_i)=2$ for $1\leq i\leq l-1$ when $l>1$. We call P_l an internal path if $d(u_l)\geq 3$, and a pendent path if $d(u_l)=1$.

Lemma 3.3. *Let* G *be a cactus graph of order* $n\geq 4$. *If there exists two edges* $uu', v_1v_2\in E(G)$ *such that* $d(u)=1$ *and* $\min\{d_G(v_1),\ d_G(v_2)\}\geq 2$. *Let* $G'=G-uu'-v_1v_2+uv_1+uv_2$, *then* $SO(G')<SO(G)$.

Proof. Let $N_G(u')=\{u,w_1,w_2,\cdots,w_{t-1}\}$, where $t=d(u')(t\geq 2)$. By the assumption, $d_{G'}(u)=2$ and $d_{G'}(u')=t-1$. Since G is a cactus graph, then $d(w_i)\leq n-t+1(i=1,\cdots,t-1)$, $d(v_j)\leq n-t+1(j=1,2)$, $t<n-1$. Thus,

$$
\begin{aligned}
&SO(G) - SO(G') \\
=&[\sum_{i=1}^{t-1} f(t, d(w_i)) + f(t,1) + f(d(v_1), d(v_2))] - [\sum_{i=1}^{t-1} f(t-1, d(w_i)) + f(2, d(v_1)) + f(2, d(v_2))] \\
=&\sum_{i=1}^{t-1} h(t, d(w_i)) + f(t,1) - f(2, d(v_1)) + [f(d(v_1), d(v_2)) - f(2, d(v_2))] \\
\geq&(t-1)h(t, n-t+1) + f(t,1) - f(2, d(v_1)) + f(d(v_1), n-t+1) - f(2, n-t+1) \\
\geq&(t-1)h(t, n-t+1) + f(t,1) + f(n-t+1, n-t+1) - 2f(2, n-t+1) \\
\geq&h(2, n-t+1) + f(2,1) + f(n-t+1, n-t+1) - 2f(2, n-t+1) \\
=&\sqrt{2^2 + (n-t+1)^2} - \sqrt{1^2 + (n-t+1)^2} + \sqrt{5} + (n-t+1)\sqrt{2} - 2\sqrt{2^2 + (n-t+1)^2} \\
=&\sqrt{5} + (n-t+1)(\sqrt{2} - \sqrt{1^2 + (\frac{1}{n-t+1})^2} - \sqrt{(\frac{2}{n-t+1})^2 + 1^2}) \\
>&\sqrt{5} + 2(\sqrt{2} - \sqrt{1^2 + (\frac{1}{2})^2} - \sqrt{(\frac{2}{2})^2 + 1^2}) = 0.
\end{aligned}
$$

□

A repeated application of the above lemma result in the following consequence.

Corollary 3.2. *If G is a cactus has a pendent path $P_l = u_0u_1 \cdots u_l$ with $d(u_l) = 1$ and $v_1v_2 \in E(G)$, $\min\{d_G(v_1),\ d_G(v_2)\} \geq 2$, then $SO(G') < SO(G)$, where $G' = G - u_0u_1 - v_1v_2 + u_1v_1 + u_lv_2$.*

The following result is immediate from the above corollary.

Corollary 3.3. *Let $k \geq 1$. If G is a cactus has the minimum Sombor index among $\mathcal{G}(n,k)$, then $\delta(G) \geq 2$.*

The following result is due to Jiang and Lu [17], which is a key lemma in the proof of Theorem 1.2.

Lemma 3.4 (Jiang and Lu [17])**.** *Let k and n be two integers with $k \geq 2$ and $n \geq 6k-4$. If $G \in \mathcal{G}(n,k)$ with $\delta(G) \geq 2$, then there exists a path $x_1x_2x_3x_4$ of length 3 in G such that $d(x_2) = d(x_3) = 2$ and $x_1 \neq x_4$.*

Lemma 3.5. *Let k and n be two integers with $k \geq 2$ and $n \geq 6k-3$. If G has the minimum Sombor index among $\mathcal{G}(n,k)$, then $\Delta(G) \leq 3$.*

Proof. By contradiction, suppose that $\Delta(G) \geq 4$. Let $v \in V(G)$ with $d_G(v) = \Delta(G)$. Since $n \geq 6k-3$, by Lemma 3.4, there exists a path $x_1x_2x_3x_4$ in G such that $d_G(x_2) = d_G(x_3) = 2$ and $x_1 \neq x_4$. Let $G_1 = G - x_1x_2 - x_2x_3 + x_1x_3$. Clearly, $G_1 \in \mathcal{G}(n-1,k)$ and $d_{G_1}(u) = d_G(u)$ for all $u \in V(G_1)$.

Since $n-1 \geq 6k-4$, by Lemma 3.4, there exists a path $y_1y_2y_3y_4$ in G_1 such that $d_{G_1}(y_2) = d_{G_1}(y_3) = 2$ and $y_1 \neq y_4$. Let $G_2 = G_1 - y_1y_2 - y_2y_3 + y_1y_3$. Then $G_2 \in \mathcal{G}(n-2,k)$ and $d_{G_2}(u) = d_{G_1}(u)$ for all $u \in V(G_2)$. Since $d_G(v) \geq 4$, $v \notin \{x_2, y_2\}$ and $d_G(v) = d_{G_1}(v) = d_{G_2}(v)$.

For convenience, let $t = \Delta(G)$. Let $N_{G_2}(v) = \{w_1, \cdots, w_t\}$. Assume that w_1, w_2, v are in the same block if v is contained in a cycle in G_2. Let $G' = G_2 - vw_1 - vw_2 + vx_2 + x_2y_2 + y_2w_1 + y_2w_2$. Then $G' \in \mathcal{G}(n,k)$, $d_{G'}(v) = t-1$, $d_{G'}(x_2) = 2$, $d_{G'}(y_2) = 3$ and $d_{G'}(u) = d_{G_2}(u) = d_G(u) \geq 2$ for all $u \in V(G_2) \setminus \{v\}$. Next, by showing $SO(G') < SO(G)$, we arrive at a contradiction.

By the construction above, $SO(G_1) = SO(G) - \sqrt{2^2+2^2} = SO(G) - 2\sqrt{2}$ and $SO(G_2) = SO(G_1) - \sqrt{2^2+2^2} = SO(G) - 4\sqrt{2}$. Thus,

$$
\begin{aligned}
&SO(G) - SO(G') = SO(G_2) - SO(G') + 4\sqrt{2}\\
&= \sum_{i=1}^{2}[f(d_G(v), d_G(w_i)) - f(d_{G'}(y_2), d_{G'}(w_i))] + \sum_{i=3}^{t}[f(d_G(v), d_G(w_i)) - f(d_{G'}(v), d_{G'}(w_i))] + 4\sqrt{2}\\
&\quad - f(d_{G'}(v), d_{G'}(x_2)) - f(d_{G'}(x_2), d_{G'}(y_2))\\
&= \sum_{i=1}^{2}[f(t, d_{G'}(w_i)) - f(3, d_{G'}(w_i))] + \sum_{i=3}^{t}[f(t, d_{G'}(w_i)) - f(t-1, d_{G'}(w_i))] + 4\sqrt{2} - f(t-1,2) - f(2,3)\\
&\geq 2[f(t,t) - f(3,t)] + (t-2)[f(t,t) - f(t-1,t)] + 4\sqrt{2} - f(t-1,2) - f(2,3)\\
&= \sqrt{2}t^2 - 2\sqrt{3^2+t^2} - (t-2)\sqrt{(t-1)^2+t^2} - \sqrt{(t-1)^2+2^2} + 4\sqrt{2} - \sqrt{13}.
\end{aligned}
$$

Hence, to show $SO(G') < SO(G)$, it suffices to show that $f(t) > 0$ for $t \geq 4$, where $f(t) = \sqrt{2}t^2 - 2\sqrt{3^2+t^2} - (t-2)\sqrt{(t-1)^2+t^2} - \sqrt{(t-1)^2+2^2} + 4\sqrt{2} - \sqrt{13}$. One can see that for any $t \geq 4$,

$$
\begin{aligned}
f'(t) =& t\Big(2\sqrt{2} - \frac{2}{\sqrt{t^2+3^2}} - \frac{1}{\sqrt{(t-1)^2+2^2}} - \frac{2}{\sqrt{1+(1-\frac{1}{t})^2}}\Big) + \frac{5t}{\sqrt{t^2+(t-1)^2}} - \sqrt{t^2+(t-1)^2}\\
&+ \frac{1}{\sqrt{(t-1)^2+2^2}} - \frac{2}{\sqrt{t^2+(t-1)^2}}\\
\geq& t\Big(2\sqrt{2} - \frac{2}{5} - \frac{1}{\sqrt{13}} - \frac{8}{5}\Big) + \frac{5t}{\sqrt{t^2+(t-1)^2}} - \sqrt{t^2+(t-1)^2} + \frac{1}{\sqrt{(t-1)^2+2^2}} - \frac{2}{\sqrt{t^2+(t-1)^2}}\\
=& \Big(2\sqrt{2} - 2 - \frac{1}{\sqrt{13}}\Big)t + \frac{5t-5}{\sqrt{t^2+(t-1)^2}} + \frac{3}{\sqrt{t^2+(t-1)^2}} - \sqrt{t^2+(t-1)^2} + \frac{1}{\sqrt{(t-1)^2+2^2}}\\
=& \Big(2\sqrt{2} - 2 - \frac{1}{\sqrt{13}}\Big)t + \frac{5}{\sqrt{1+(1+\frac{1}{t-1})^2}} - (t-1)\sqrt{1+\Big(1+\frac{1}{t-1}\Big)^2} + \frac{3}{\sqrt{t^2+(t-1)^2}}\\
&+ \frac{1}{\sqrt{(t-1)^2+2^2}} > 4\Big(2\sqrt{2} - 2 - \frac{1}{\sqrt{13}}\Big) + 3 - 5 > 0.
\end{aligned}
$$

Hence, $f(t)$ is an increasing function with respect to $t \in [4, n-1]$, implying $f(t) \geq f(4) = 20\sqrt{2} - 4\sqrt{13} - 10 > 0$. This contradicts the minimality of G. □

Lemma 3.6. *Let $k \geq 2$ and $n \geq 6k-3$. If G has the minimum Sombor index in $\mathcal{G}(n,k)$, then it does not exist a path $v_1v_2\cdots v_l (l \geq 3)$ in G such that $d_G(v_1) = d_G(v_l) = 3$ and $d_G(v_i) = 2$ $(i = 2, \cdots, l-1)$, where v_1 and v_l are not adjacent. Thus,*

(1) if a cycle C is not an end block of G, then all vertices of it have degree three in G, or it contains exactly two (adjacent) vertices of degree three.

(2) any vertices of degree two lie on a cycle.

Proof. Suppose that there exist a path $P_l = v_1v_2\cdots v_l \in G$ as given in the assumption of the lemma. From Corollary 3.3 and Lemmas 3.5, we have $2 \leq d(v) \leq 3$ of any vertex v in G. By Lemma 3.1, in equation (3.9), $L_{n,n}^{k'} = \emptyset$,

$$SO(G) = 2\sqrt{2}n + (6\sqrt{13} - 10\sqrt{2})(k-1) + (5\sqrt{2} - 2\sqrt{13})e_{3,3}(G).$$

Let C_s be an end block of G, $w_1, w_2 \in V(C_s)$, $w_1w_2 \in E(C_s)$, $d_G(w_1) = d_G(w_2) = 2$. Let $G' = G - v_1v_2 - v_{l-1}v_l + v_1v_l - w_1w_2 + w_1v_2 + w_2v_{l-1}$. Clearly,

$$SO(G') = 2\sqrt{2}n + (6\sqrt{13} - 10\sqrt{2})(k-1) + (5\sqrt{2} - 2\sqrt{13})(e_{3,3}(G) + 1).$$

Thus, $SO(G') < SO(G)$, a contradiction.

Thus, (1) and (2) is immediate.

□

Lemma 3.7. *Let $k \geq 2$ and $n \geq 6k-3$. Let $G \in \mathcal{G}(n,k)$ has the minimum Sombor index, then $e_{3,3}(G) \leq \lfloor \frac{5k}{2} \rfloor - 4 = 2k + \lfloor \frac{k}{2} \rfloor - 4$, the equality holds if and only if $G \in C^*(n,k)$.*

Proof. By Corollary 3.3 and Lemma 3.5, we have $2 \leq d(v) \leq 3$ of any vertex v in G.

Let n_3 be the number of vertices with degree 3 not belongs to a cycle, c_1 the number of end blocks, and c_2 the number of cycles with exactly two (adjacent) vertices of degree three in G. Clearly $n_3 \geq 0$ and $c_2 \geq 0$. By Lemma 3.6 (1), there are $k - c_1 - c_2$ remaining cycles, denoted by $C_1, \ldots, C_{k-c_1-c_2}$, all vertices of which have degree three. Let d_i be the length of the cycle C_i.

Let T_{k+n_3} be the tree obtained from contracting each cycle of G into a vertex. By the hand-shaking lemma, we have

$$3n_3 + d_1 + d_2 + \cdots + d_{k-c_1-c_2} + 2c_2 + c_1 = 2(k + n_3 - 1) \tag{3.10}$$

Since $d_i \geq 3$ for each $i \in \{1, \ldots, k - c_1 - c_2\}$, by (3.10), we have

$$2k - n_3 - c_1 - 2c_2 - 2 = d_1 + d_2 + \cdots + d_{k-c_1-c_2} \geq 3(k - c_1 - c_2), \tag{3.11}$$

relation (3.11) can be rewritten as

$$(k - c_1 - c_2) + k - n_3 - c_2 - 2 \geq 3(k - c_1 - c_2), \tag{3.12}$$

implying that

$$k - c_1 - c_2 \leq \frac{k-2}{2} - \frac{n_3 + c_2}{2}. \tag{3.13}$$

On the other hand, by Lemma 3.6,

$$e_{3,3}(G) = (k + n_3 - 1) + c_2 + (d_1 + d_2 + \cdots + d_{k-c_1-c_2}). \tag{3.14}$$

It follows from (3.10) and (3.14) that

$$e_{3,3}(G) = 2k + (k - c_1 - c_2) - 3. \tag{3.15}$$

Combining (3.13) and (3.15), it yields

$$e_{3,3}(G) \leq (2k-3) + (\frac{k-2}{2} - \frac{n_3 + c_2}{2}) = \frac{5k}{2} - 4 - \frac{n_3 + c_2}{2}.$$

Since $n_3 \geq 0$, $c_2 \geq 0$,

$$e_{3,3}(G) \leq \lfloor\frac{5k}{2}\rfloor - 4. \tag{3.16}$$

If k is even,

$$e_{3,3}(G) \leq \frac{5k}{2} - 4,$$

the equality holds if and only if $n_3 = c_2 = 0$, $d_1 = d_2 = \cdots = d_{k-c_1-c_2} = 3$, that is, $G \in C^*(n,k)$.

If k is odd,

$$e_{3,3}(G) \leq \frac{5k}{2} - \frac{9}{2},$$

the equality holds if and only if $n_3 + c_2 = 1$, $d_1 = d_2 = \cdots = d_{k-c_1-c_2} = 3$, then $n_3 = 0, c_2 = 1$ or $n_3 = 1, c_2 = 0$, that is, $G \in C^*(n,k)$.

□

Proof of Theorem 1.2: Assume that under which $k \geq 2$ and $n \geq 6k - 3$ condition G has the minimum Sombor index in $\mathcal{G}(n,k)$. By Corollary 3.3 and Lemma 3.5, $2 \leq d_G(v) \leq 3$ for any vertex v in G. By Lemma 3.1,

$$SO(G) = 2\sqrt{2}n + (6\sqrt{13} - 10\sqrt{2})(k-1) + (5\sqrt{2} - 2\sqrt{13})e_{3,3}(G). \tag{3.17}$$

Thus, by Lemma 3.7,

$$SO(G) \geq 2\sqrt{2}n + 5\sqrt{2}(\lfloor\frac{k}{2}\rfloor - 2) + 2\sqrt{13}(k - \lfloor\frac{k}{2}\rfloor + 1),$$

with equality if and only if $G \in C^*(n,k)$.

4. Conclusions

Recall that $\mathcal{G}(n,k)$ denotes the set of cacti of order n and with k cycles. In this paper, we establish a sharp upper bound for the Sombor index of a cactus in $\mathcal{G}(n,k)$ and characterize the corresponding extremal graphs. In addition, for the case when $n \geq 6k-3$, we give a sharp lower bound for the Sombor index of a cactus in $\mathcal{G}(n,k)$ and characterize the corresponding extremal graphs as well. We believe that Theorem 1.2 is true for the case when $3k \leq n \leq 6k-4$.

Conjucture 4.1. *Let k and n be two integers with $n \geq 3k$ and $k \geq 2$. For any graph $G \in \mathcal{G}(n,k)$,*

$$SO(G) \geq 2\sqrt{2}n + 5\sqrt{2}(k - \lceil\frac{k}{2}\rceil - 2) + 2\sqrt{13}(\lceil\frac{k}{2}\rceil + 1),$$

with equality if and only if $G \in C^*(n,k)$.

Acknowledgments

The research of cactus graph is supported by the National Natural Science Foundation of China(11801487,12061073).

Conflict of interest

The authors declare no conflict of interest.

References

1. A. Aashtab, S. Akbari, S. Madadinia, M. Noei, F. Salehi, On the graphs with minimum Sombor index, *MATCH Commun. Math. Co.,* **88** (2022), 553–559. https://doi.org/10.46793/match.88-3.553A
2. A. Alidadi, A. Parsian, H. Arianpoor, The minimum Sombor index for unicyclic graphs with fixed diameter, *MATCH Commun. Math. Co.*, **88** (2022), 561–572. https://doi.org/10.46793/match.88-3.561A
3. S. Alikhani, N. Ghanbari, Sombor index of polymers, *MATCH Commun. Math. Co.*, **86** (2021), 715–728. https://doi.org/10.48550/arXiv.2103.13663
4. J. A. Bondy, U. S. R. Murty, *Graph Theory*, Springer, New York, (2008).
5. H. Chen, W. Li, J. Wang, Extremal values on the Sombor index of trees, *MATCH Commun. Math. Co.*, **87** (2022), 23–49. https://doi.org/10.46793/match.87-1.023C
6. R. Cruz, I. Gutman, J. Rada, Sombor index of chemical graphs, *Appl. Math. Comput.*, **399** (2021), 126018. https://doi.org/10.1016/j.amc.2021.126018
7. R. Cruz, J. Rada, Extremal values of the Sombor index in unicyclic and bicyclic graphs, *J. Math. Chem.*, **59** (2021), 1098–1116. https://doi.org/10.1007/s10910-021-01232-8
8. R. Cruz, J. Rada, J. M. Sigarreta, Sombor index of trees with at most three branch vertices, *Appl. Math. Comput.*, **409** (2021), 126414. https://doi.org/10.1016/j.amc.2021.126414
9. K. C. Das, I. Gutman, On Sombor index of trees, *Appl. Math. Comput.*, **412** (2022) 126575. https://doi.org/10.1016/j.amc.2021.126575
10. K. C. Das, Y. Shang, Some extremal graphs with respect to Sombor index, *Mathematics*, **9** (2021), 1202. https://doi.org/10.3390/math9111202
11. H. Deng, Z. Tang, R. Wu, Molecular trees with extremal values of Sombor indices, *Int. J. Quantum Chem.*, **121** (2021), e26622. https://doi.org/10.1002/qua.26622
12. K. J. Gowtham, N. N. Swamy, On Sombor energy of graphs, *Nanosystems: Phys. Chem. Math.*, **12** (2021), 411–417. https://doi.org/ 10.17586/2220-8054-2021-12-4-411-417
13. I. Gutman, Geometric approach to degree-based topological indices: Sombor indices, *MATCH Commun. Math. Co.*, **86** (2021), 11–16.
14. I. Gutman, Some basic properties of Sombor indices, *Open J. Discret. Appl. Math.*, **4** (2021), 1–3. https://doi.org/10.30538/psrp-odam2021.0047
15. I. Gutman, Spectrum and energy of the Sombor matrix, *Milirary Technical Courier*, **69** (2021), 551–561. https://doi.org/10.5937/vojtehg69-31995
16. B. Horoldagva, C. Xu, On Sombor index of graphs, *MATCH Commun. Math. Co.*, **86** (2021), 703–713. https://doi.org/10.47443/cm.2021.0006

17. Y. Jiang, M. Lu, A note on the minimum inverse sum indeg index of cacti, *Discrete Appl. Math.*, **302** (2021), 123–128. https://doi.org/10.1016/j.dam.2021.06.011

18. J. Karamata, Sur une inégalité relative aux fonctions convexes, *Publ. Inst. Math.*, **1** (1932), 145–147.

19. S. Li, Z. Wang, M. Zhang, On the extremal Sombor index of trees with a given diameter, *Appl. Math. Comput.*, **416** (2022), 126731. https://doi.org/10.1016/j.amc.2021.126731

20. H. Liu, H. Chen, Q. Xiao, X. Fang, Z. Tang, More on Sombor indices of chemical graphs and their applications to the boiling point of benzenoid hydrocarbons, *Int. J. Quantum Chem.*, **121** (2021), e26689. https://doi.org/10.1002/qua.26689

21. H. Liu, L. You, Y. Huang, Extremal Sombor indices of tetracyclic (chemical) graphs, *MATCH Commun. Math. Co.*, **88** (2022), 573–581. https://doi.org/10.46793/match.88-3.573L

22. H. Liu, L. You, Y. Huang, Ordering chemical graphs by Sombor indices and its applications, *MATCH Commun. Math. Comput. Chem.*, **87** (2022), 5–22. https://doi.org/10.48550/arXiv.2103.05995

23. H. C. Liu, L. H. You, Z. K. Tang, J. B. Liu, On the reduced Sombor index and its applications, *MATCH Commun. Math. Co.*, **86** (2021), 729–753.

24. I. Milovanovic, E. Milovanovic, M. Matejic, On some mathematical properties of Sombor indices, *Bull. Int. Math. Virtual Inst.*, **11** (2021), 341–353. https://doi.org/10.7251/BIMVI2102341M

25. J. Rada, J. M. Rodríguez, J. M. Sigarreta, General properties on Sombor indices, *Discr. Appl. Math.*, **299** (2021), 87–97. https://doi.org/10.1016/j.dam.2021.04.014

26. B. A. Rather, M. Imran, Sharp bounds on the Sombor energy of graphs, *MATCH Commun. Math. Co.,* **88** (2022), 605–624. https://doi.org/10.46793/match.88-3.605R

27. I. Redžepović, Chemical applicability of Sombor indices, *J. Serb. Chem. Soc.*, **86** (2021), 445–457. http://dx.doi.org/10.2298/JSC201215006R

28. I. Redžepović, I. Gutman, Comparing energy and Sombor Energy-An empirical study, *MATCH Commun. Math. Co.*, **88** (2022), 133–140. http://dx.doi.org/10.46793/match.88-1.133R

29. Y. Shang, Sombor index and degree-related properties of simplicial networks, *Appl. Math. Comput.*, **419** (2022), 126881. https://doi.org/10.1016/j.amc.2021.126881

30. X. Sun, J. Du, On Sombor index of trees with fixed domination number, *Appl. Math. Comput.*, **421** (2022), 126946. https://doi.org/10.1016/j.amc.2022.126946

31. A. Ülker, A. Gürsoy, N. K. Gürsoy, The energy and Sombor index of graphs, *MATCH. Commun. Math. Co.*, **87** (2022), 51–58. https://doi.org/10.46793/match.87-1.051U

32. A. Ülker, A. Gürsoy, N. K. Gürsoy, I. Gutman, Relating graph energy and Sombor index, *Discr. Math. Lett.*, **8** (2022), 6–9. https://doi.org/10.47443/dml.2021.0085

33. Z. Wang, Y. Mao, Y. Li, B. Furtula, On relations between Sombor and other degree-based indices, *J. Appl. Math. Comput.*, **68** (2022), 1–17. https://doi.org/10.1007/s12190-021-01516-x

34. F. Wang, B. Wu, The proof of a conjecture on the reduced Sombor index, *MATCH Commun. Math. Co.*, **88** (2022), 583–591. https://doi.org/10.46793/match.88-3.583W

35. F. Wang, B. Wu, The reduced Sombor index and the exponential reduced Sombor index of a molecular tree, *J. Math. Anal. Appl.*, (2022), 126442. https://doi.org/10.1016/j.jmaa.2022.126442

36. T. Zhou, Z. Lin, L. Miao, The Sombor index of trees and unicyclic graphs with given maximum degree, *Discrete Math. Lett.*, **7** (2021), 24–29. https://doi.org/10.48550/arXiv.2103.07947

AIMS Press