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Research article

Novel results on fixed-point methodologies for hybrid contraction mappings in M_b-metric spaces with an application

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Abstract: By combining the results of Wardowski's cyclic contraction operators and admissible multi-valued mappings, the motif of η-cyclic (α_*, β_*)-admissible type F-contraction multivalued mappings are presented. Moreover, some novel fixed point theorems for such mappings are proved in the context of M_b-metric spaces. Also, two examples are given to clarify and strengthen our theoretical study. Finally, the existence of a solution of a pair of ordinary differential equations is discussed as an application.

Keywords: fixed point methodology; M_b-metric space; η-cyclic (α_*, β_*)-admissible F-contraction multivalued mapping
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1. Introduction

The study of fixed points (FPs) is an interesting topic because it has many applications not only in non-linear analysis but also in many aspects of engineering and physics. FP technique has gained a large number of readers due to its smoothness and ease of approach.

In the long term of studying functional analysis, the metric space (MS) is an important topic in it, which has many generalizations and extensions in different formulas. One of these generalizations is motif of b-metric space [1] which open the wide field for researchers to develop metric fixed point theory. In 1994, Matthews introduced another generalization of (MS) which is called partial metric spaces (PMSs) [2] and studied some properties of this space. In 2014, Ma et al. [3] introduced a new type of MSs which generalize the concepts of MSs and operator-valued MSs, they defined C^*-algebra-valued MSs and gave some FP results. An M-metric space (MMS) was redacted by Asadi et al. [4] in the same year of 2014, as an extension of (PMSs). Accordingly, some topological properties of said space and FP results for contraction mapping have been discussed. Altun et al. [5] presented some FP theorems for multivalued mappings of Feng-Liu type on complete MMSs. They inspected of the topological characteristics of (MMS) and asserted that the sequential topology τ_s is larger than the topology τ_m induced by open balls and the closure of a subset A of M-metric Ξ with respect to τ_s is included the closure of a subset A of M-metric Ξ with respect to (wrt) τ_m. Sahin et al. [6] generalized Feng-Liu techniques and discussed some new FP results for multivalued F-contraction mappings. Very recently, Patle et al. [7] studied Pompeiu-Hausdorff distance induced by the MMSs. Also, they established the Nadler and Kannan type FP theorems for set-valued mappings in such spaces. Monfared et al. [8, 9] applied the notion of control and ultra altering distance functions ψ and ϕ for single valued contraction mappings in an MMS. Meanwhile, Mlaiki et al. [10] introduced the concept of F_m-expanding contractive mappings and graphic FP theorems in the mentied spaces. Mlaiki et al. [11] generalized the MMS to M_b-metric space (MbMS) and proved the existence and uniqueness of a FP under suitable contraction conditions. Recently, Hu and Gu [12] derived a new concept of the probabilistic MS, which is called the Menger probabilistic S-metric space, and investigated some topological properties of this space and proved related FP theorems for λ-contraction mapping.

In 1973, Geraghty [13] introduced a fruitful generalization of Banach contraction principle and obtained FP results for a single-valued mapping. In 1989, Mizoguchi and Takahashi [14] relaxed the compactness of value of a mapping Γ to closed and bounded subsets of Ξ and they obtained FP results for multi-valued mappings of Geraghty contraction. Popescu [15] proved interesting result for α-Geraghty contraction mappings in MSs. Arshad et al. [16] extended Popescu's results to introduce the new notion of α_*-Geraghty type F-contraction multivalued mapping in b-metric like space.

On the other hand, the notion of cyclic (α, β)-admissible mapping was discussed by Alizadeh et al. [17] and several FP results under this idea were proved. Ameer et al. [18] investigated FPs of cyclic (α_*, β_*)-type-γ-FG-contractive mappings and established some FP theorems in PbMSs. For more details, see [19–28].

This manuscript is devoted to introduce the concept of η-cyclic (α_*, β_*)-admissible type F-contraction multivalued mappings. Via this idea, some common FP results are obtained in MbMSs. Finally, as an application, the existence of solution to a pair of ordinary differential equations (ODEs) are given.

2. Preliminaries

In this part, we give some elementary discussions about MMSs.

Definition 2.1. [4] Let $\Xi \neq \emptyset$. If the function $m : \Xi \times \Xi \to \mathbb{R}^+$ fulfills the stipulations below, for all $\lambda, \gamma, \kappa \in \Xi$:

(M_1) $m(\lambda, \lambda) = m(\gamma, \gamma) = m(\lambda, \gamma)$ iff $\lambda = \gamma$;

(M_2) $m_{\lambda,\gamma} \leq m(\lambda,\gamma)$;
(M_3) $m(\lambda,\gamma) = m(\gamma,\lambda)$;
(M_4) $\left(m(\lambda,\gamma) - m_{\lambda,\gamma}\right) \leq (m(\lambda,\kappa) - m_{\lambda,\kappa}) + \left(m(\kappa,\gamma) - m_{\kappa,\gamma}\right)$.

Then the pair (Ξ, m) is called an MMS.

It should be noted that the notion $m_{\lambda,\gamma}$ and $M_{\lambda,\gamma}$ are defined by Asadi et al. [4] as follows:

$$m_{\lambda,\gamma} = \min\{m(\lambda,\lambda), m(\gamma,\gamma)\},$$

and

$$M_{\lambda,\gamma} = \max\{m(\lambda,\lambda), m(\gamma,\gamma)\}.$$

Definition 2.2. [11] An MbMS on a non-empty set Ξ is a function $m_b : \Xi^2 \to R^+$ that fulfills the assumptions below, for all $\lambda, \gamma, \kappa \in \Xi$,

(Mb_1) $m_b(\lambda,\lambda) = m_b(\gamma,\gamma) = m_b(\lambda,\gamma)$ iff $\lambda = \gamma$;
(Mb_2) $m_{b_{\lambda,\gamma}} \leq m_b(\lambda,\gamma)$;
(Mb_3) $m_b(\lambda,\gamma) = m_b(\gamma,\lambda)$;
(Mb_4) There is a coefficient $s \geq 1$ so that for all $\lambda, \gamma, \kappa \in \Xi$, we have

$$m_b(\lambda,\gamma) - m_{b_{\lambda,\gamma}} \leq s\left[\left(m_b(\lambda,\kappa) - m_{b_{\lambda,\kappa}}\right) + \left(m_b(\kappa,\gamma) - m_{b_{\kappa,\gamma}}\right)\right] - m_b(\kappa,\kappa).$$

Then the pair (Ξ, m_b) is called an MbMS.

Note. Symbols $m_{b_{\lambda,\gamma}}$ and $M_{b_{\lambda,\gamma}}$ defined in [11] as follows:

$$m_{b_{\lambda,\gamma}} = \min\{m_b(\lambda,\lambda), m_b(\gamma,\gamma)\},$$

and

$$M_{b_{\lambda,\gamma}} = \max\{m_b(\lambda,\lambda), m_b(\gamma,\gamma)\}.$$

Example 2.3. [11] Let $\Xi = [0,\infty)$ and $p > 1$ be a constant. Define $m_b : \Xi^2 \longrightarrow [0,\infty)$ by

$$m_b(\lambda,\gamma) = (\max\{\lambda,\gamma\})^p + |\lambda - \gamma|^p, \ \forall \lambda,\gamma \in \Xi.$$

Then (Ξ, m_b) is an MbMS (with coefficient $s = 2^p$) and not MMS.

Example 2.4. [29] Let $\Xi = [0,1]$ and $m_b : \Xi \times \Xi \longrightarrow [0,\infty)$ be defined by

$$m_b(\lambda,\gamma) = \left(\frac{\lambda+\gamma}{2}\right)^2, \ \forall \lambda,\gamma \in \Xi.$$

Then (Ξ, m_b) is an MbMS (with coefficient $s = 2$) which is not an MMS.

Definition 2.5. [11] Let (Ξ, m_b) be an MbMS. Then

- A sequence $\{\lambda_n\}$ in Ξ converges to a point λ if and only if

$$\lim_{n\to\infty}\left(m_b(\lambda_n,\lambda) - m_{b_{\lambda_n,\lambda}}\right) = 0.$$

- A sequence $\{\lambda_n\}$ in Ξ is called m_b-Cauchy sequence iff

$$\lim_{n,m\to\infty}\left(m_b(\lambda_n,\lambda_m)-m_{b_{\lambda_n,\lambda_m}}\right) \text{ and } \lim_{n,m\to\infty}\left(M_{b_{\lambda_n,\lambda_m}}-m_{b_{\lambda_n,\lambda_m}}\right)$$

exist and finite.
- An MbMS is called m_b-complete if every m_b-Cauchy sequence $\{\lambda_n\}$ converges to a point λ so that

$$\lim_{n\to\infty}\left(m_b(\lambda_n,\lambda)-m_{b_{\lambda_n,\lambda}}\right)=0 \text{ and } \lim_{n\to\infty}\left(M_{b_{\lambda_n,\lambda}}-m_{b_{\lambda_n,\lambda}}\right)=0.$$

The first result concerning with the existence of FPs in the MbMS presented by Mlaiki et al. [11] as follows:

Theorem 2.6. *Let (Ξ, m_b) be an MbMS with coefficient $s \geq 1$ and Γ be a self-mapping on Ξ. If there is $k \in [0,1)$ so that*

$$m_b(\Gamma\lambda,\Gamma\gamma)\leq km_b(\lambda,\gamma),\ \forall\lambda,\gamma\in\Xi.$$

Then Γ has a unique FP ς in Ξ.

The concepts of cyclic (α,β)-admissible and cyclic (α_*,β_*)-admissible mappings are showed in the work of [17, 18] as follows:

Definition 2.7. Let $\Xi \neq \emptyset$, $\alpha,\beta:\Xi\to[0,\infty)$ be two functions. A mapping $\Gamma:\Xi\to\Xi$ is called cyclic (α,β)-admissible if for some $\lambda\in\Xi$,

$$\alpha(\lambda)\geq 1 \Rightarrow \beta(\Gamma\lambda)\geq 1,$$

and

$$\beta(\lambda)\geq 1 \Rightarrow \alpha(\Gamma\lambda)\geq 1.$$

Definition 2.8. Let $\Xi \neq \emptyset$, $\alpha,\beta:\Xi\to[0,\infty)$ be mappings and A, B be subsets of Ξ. A mapping $\Gamma:\Xi\to CB(\Xi)$ is called cyclic (α_*,β_*)-admissible if for some $\lambda\in\Xi$,

$$\alpha(\lambda)\geq 1 \Rightarrow \beta_*(\Gamma\lambda)\geq 1,$$

and

$$\beta(\lambda)\geq 1 \Rightarrow \alpha_*(\Gamma\lambda)\geq 1,$$

where $\beta_*(A)=\inf_{a\in A}\beta(a)$ and $\alpha_*(B)=\inf_{b\in B}\alpha(b)$.

Theorem 2.9. *[13] Let Ξ be a complete metric space and $\Gamma:\Xi\to\Xi$. If there is $\varphi\in\xi$ so that*

$$d(\Gamma\lambda,\Gamma\gamma)\leq\varphi(d(\lambda,\gamma))d(\lambda,\gamma),\ \forall\lambda,\gamma\in\Xi,$$

holds, where ξ is the set of all functions $\varphi:[0,\infty)\to[0,1)$ satisfying $\lim_{n\to\infty} t_n = 0$ whenever $\lim_{n\to\infty}\varphi(t_n) = 1$. Then Γ has a unique FP $\lambda^* \in \Xi$ and for each $\lambda\in\Xi$, the sequence $\{T^n\lambda\}$ converges to λ^*.

In 2012, Wardowski [30] made a great contribution to the study of new theories related to fixed points in the context of ordinary metric spaces. This contribution is called F-contraction mappings.

Definition 2.10. [30] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping fulfilling the stipulations below:

(F_1) F is strictly increasing, i.e., if $\alpha < \beta$, then $F(\alpha) < F(\beta)$, $\forall \alpha, \beta \in \mathbb{R}^+$;
(F_2) for any sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive real numbers, $\lim\limits_{n\to\infty} \alpha_n = 0$ iff $\lim\limits_{n\to\infty} F(\alpha_n) = -\infty$;
(F_3) there is $k \in (0,1)$ so that $\lim\limits_{n\to\infty} \alpha^k F(\alpha_n) = 0$.

Felhi [31] generalized the Definition 2.10 by adding the condition below to the stipulations (F_1) − (F_3):

(F_4) for any sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive real numbers so that

$$\tau + F(s\alpha_n) \leq F(\alpha_{n-1}), \ s \geq 1,$$

for all $n \in \mathbb{N}$ and some $\tau > 0$, then

$$\tau + F(s^n \alpha_n) \leq F\left(s^{n-1}\alpha_{n-1}\right), \ \forall n \in \mathbb{N}.$$

Here, F_w and F_s denote the sets of all functions F fulfilling (F_1) − (F_3) and (F_1) − (F_4), respectively.

Remark 2.11. [32] If F is right continuous and satisfies (F_1), then

$$F(\inf A) = \inf F(A) \ \forall F \subset (0,\infty) \text{ with } \inf(F) > 0.$$

Assume that (Ξ, m_b) is MbMS and$CB_{m_b}(\Xi)$ is the family of all non-empty, bounded and closed subsets of Ξ. For $\aleph, \mathbb{Q} \in CB_{m_b}(\Xi)$, define

$$H_{m_b}(\aleph, \mathbb{Q}) = \max\{\delta_{m_b}(\aleph, \mathbb{Q}), \delta_{m_b}(\mathbb{Q}, \aleph)\},$$

where $\delta_{m_b}(\aleph, \mathbb{Q}) = \sup\{m_b(p, \mathbb{Q}) : p \in \aleph\}$ and $m_b(p, \mathbb{Q}) = \inf\{m_b(p,q) : q \in \mathbb{Q}\}$.
The following results are very useful in our study. These results are taken from [4, 7].

Lemma 2.12. *Let $\aleph$ be a non-empty set in an MbMS (Ξ, m_b), then $p \in \bar{\aleph}$ iff*

$$m_b(p, \aleph) = \sup_{\lambda \in \aleph} m_{b_{p,\lambda}},$$

where $\bar{\aleph}$ denotes the closure of $\aleph$ wrt m_b.

Lemma 2.13. *Let $\aleph, \mathbb{Q}, \mathfrak{R} \in CB_{m_b}(\Xi)$, then*

(a) $\delta_{m_b}(\aleph, \aleph) = \sup\limits_{p\in\aleph}\left\{\sup\limits_{q\in\aleph} m_{b_{pq}}\right\}$,
(b) for $s \geq 1$, we have

$$\begin{aligned}&\left(\delta_{m_b}(\aleph, \mathbb{Q}) - \sup_{p\in\aleph}\sup_{q\in\mathbb{Q}} m_{b_{pq}}\right)\\ \leq\ & s\left[\left(\delta_{m_b}(\aleph, \mathfrak{R}) - \inf_{p\in\aleph}\inf_{r\in\mathfrak{R}} m_{b_{pr}}\right) + \left(\delta_{m_b}(\mathfrak{R}, \mathbb{Q}) - \inf_{r\in\mathfrak{R}}\inf_{q\in\mathbb{Q}} m_{b_{rq}}\right)\right] - \inf_{r\in\mathfrak{R}} m_b(r,r).\end{aligned}$$

Lemma 2.14. *Let $\aleph, \mathbb{Q}, \mathfrak{R} \in CB_{m_b}(\Xi)$, then*

(1)
$$H_{m_b}(\aleph,\aleph) = \delta_{m_b}(\aleph,\aleph) = \sup_{p\in\aleph}\left\{\sup_{q\in\mathbb{Q}} m_{b_{p,q}}\right\},$$

(2) $H_{m_b}(\aleph,\mathbb{Q}) = H_{m_b}(\mathbb{Q},\aleph)$,

(3) for $s \geq 1$, we get
$$\begin{aligned}&\left(H_{m_b}(\aleph,\mathbb{Q}) - \sup_{p\in\aleph}\sup_{q\in\mathbb{Q}} m_{b_{pq}}\right)\\ \leq\ & s\left[\left(H_{m_b}(\aleph,\mathfrak{R}) - \inf_{p\in\aleph}\inf_{r\in\mathfrak{R}} m_{b_{p,r}}\right) + \left(H_{m_b}(\mathfrak{R},\mathbb{Q}) - \inf_{r\in\mathfrak{R}}\inf_{q\in\mathbb{Q}} m_{b_{r,q}}\right)\right] - \inf_{r\in\mathfrak{R}} m_b(r,r).\end{aligned}$$

Lemma 2.15. *Let $\aleph, \mathbb{Q} \in CB_{m_b}(\Xi)$ and $h > 1$, then for all $p \in \aleph$, there is $q \in \mathbb{Q}$ so that*

(i) $m_b(p,q) \leq hH_{m_b}(\aleph,\mathbb{Q})$,

(ii) $m_b(p,q) \leq H_{m_b}(\aleph,\mathbb{Q}) + h$.

Proof. (i) Suppose that there exists an $p \in \aleph$ such that
$$m_b(p,q) > hH_{m_b}(\aleph,\mathbb{Q}),$$
for all $q \in \mathbb{Q}$. This implies that
$$\inf\{m_b(p,q) : q \in \mathbb{Q}\}.$$
Now
$$m_b(\aleph,\mathbb{Q}) \geq \delta_{m_b}(\aleph,\mathbb{Q}) = \sup\{m_b(p,\mathbb{Q}) : p \in \aleph\} \geq m_b(p,\mathbb{Q}) \geq H_{m_b}(\aleph,\mathbb{Q}),$$
this is a contradiction since $H_{m_b}(\aleph,\mathbb{Q}) \neq 0$ and $h > 1$. Hence,
$$m_b(p,q) \leq hH_{m_b}(\aleph,\mathbb{Q}).$$

(ii) Suppose that there exists $p \in \aleph$ such that $m_b(p,q) > H_{m_b}(\aleph,\mathbb{Q}) + h$ for all $q \in \mathbb{Q}$, then we have
$$m_b(\aleph,\mathbb{Q}) + h \leq m_b(p,q) \leq \delta_{m_b}(\aleph,\mathbb{Q}) \leq H_{m_b}(\aleph,\mathbb{Q}) + h,$$
a contradiction again. Since $H_{m_b}(\aleph,\mathbb{Q}) \neq 0$ and $h > 1$. Thus,
$$m_b(p,q) \leq H_{m_b}(\aleph,\mathbb{Q}) + h.$$

□

Remark 2.16. For all λ, γ, κ in an MbMS (Ξ, m_b), then

(1) $M_{b_{\lambda,\gamma}} + m_{b_{\lambda,\gamma}} = m_b(\lambda,\lambda) + m_b(\gamma,\gamma)$,

(2) $M_{b_{\lambda,\gamma}} - m_{b_{\lambda,\gamma}} = |m_b(\lambda,\lambda) - m_b(\gamma,\gamma)|$,

(3) For $s \geq 1$, we have
$$M_{b_{\lambda,\gamma}} - m_{b_{\lambda,\gamma}} \leq s\left[\left(M_{b_{\lambda,\kappa}} - m_{b_{\lambda,\kappa}}\right) + \left(M_{b_{\kappa,\gamma}} - m_{b_{\kappa,\gamma}}\right)\right].$$

Notice that:
If $s = 1$, then we get Remark 1.1 in [4].

3. The results

In this part, the following are established:

- Definition of η-cyclic (α_*,β_*)-admissible mappings,
- The notion of Geraghty contraction type mappings,
- Some new common FP theorem for a pair of generalized (α_*,β_*)-Geraghty F-contraction multivalued mapping in an MbMS.

Definition 3.1. Let $\Xi \neq \emptyset$, $\alpha,\beta,\eta : \Xi \to [0,\infty)$ be mappings and A, B be subsets of Ξ. A mapping $\Gamma : \Xi \to CB_{m_b}(\Xi)$ is called η-cyclic (α_*,β_*)-admissible if for some $\lambda \in \Xi$,

$$\alpha(\lambda) \geq \eta(\lambda) \Rightarrow \beta_*(\Gamma\lambda) \geq \eta_*(\Gamma\lambda),$$

and

$$\beta(\lambda) \geq \eta(\lambda) \Rightarrow \alpha_*(\Gamma\lambda) \geq \eta_*(\Gamma\lambda)$$

where $\beta_*(A) = \inf_{a\in A}\beta(a)$ and $\alpha_*(B) = \inf_{b\in B}\alpha(b)$.

Definition 3.2. Let $\Xi \neq \emptyset$, $\alpha,\beta,\eta : \Xi \to [0,\infty)$ be mappings and A, B be subsets of Ξ. Two mappings $\Im, \Gamma : \Xi \to CB_{m_b}(\Xi)$ are called η-cyclic (α_*,β_*)-admissible if for some $\lambda \in \Xi$,

$$\alpha(\lambda) \geq \eta(\lambda) \Rightarrow \beta_*(\Im\lambda) \geq \eta_*(\Im\lambda),$$

and

$$\beta(\lambda) \geq \eta(\lambda) \Rightarrow \alpha_*(\Gamma\lambda) \geq \eta_*(\Gamma\lambda).$$

Notice that:

- If $\eta = \eta_* = 1$ and $\Im = \Gamma$, then we get Definition 2.2 in [18].
- Definition 3.2 reduces to Definition 3.1, if we put $\Im = \Gamma$.

Example 3.3. Let $\Xi = [0,\infty)$. Define the mappings $\Im, \Gamma : \Xi \to CB_{m_b}(\Xi)$ and $\alpha,\beta,\eta : \Xi \to [0,\infty)$ by $\Im\lambda = \{3\lambda\}$, $\Gamma\lambda = \left\{\lambda^2\right\}$, $\eta(\lambda) = \lambda$, $\forall\lambda \in \Xi$,

$$\alpha(\lambda) = \begin{cases} e^{3\lambda^2}, & \text{if } \lambda > 0, \\ 1, & \text{otherwise,} \end{cases} \quad \text{and } \beta(\lambda) = \begin{cases} 5^{2\lambda}, & \text{if } \lambda > 0, \\ 1, & \text{otherwise.} \end{cases}$$

For all $\lambda > 0$, we get

$$\alpha(\lambda) = e^{3\lambda^2} \geq \lambda = \eta(\lambda) \Rightarrow \beta_*(\Im\lambda) = \beta_*(3\lambda) = 5^{6\lambda} \geq 3\lambda = \eta_*(\Im\lambda).$$

Similarly,

$$\beta(\lambda) = 5^{2\lambda} \geq \lambda = \eta(\lambda).$$

Otherwise, for $\lambda = 0$ the conditions of definition are satisfied. Then the pair $(\Im, \Gamma)$ is η-cyclic (α_*,β_*)-admissible mappings.

In the setting of the MbMS, we define a generalized (α_*,β_*)-Geraghty F-contraction mappings as follows:

Definition 3.4. Let (Ξ, m_b) be an MbMS, $\alpha, \beta, \eta : \Xi \to [0, \infty)$ be functions. Two multivalued mappings $\Im, \Gamma : \Xi \to CB_{m_b}(\Xi)$ is called a pair of generalized (α_*, β_*)-Geraghty F-contraction mappings if there exist $\varphi \in \xi$ and $F \in F_s$ so that for all $\lambda, \gamma \in \Xi$, $s \geq 1$ and $\tau \in \mathbb{R}_+$ with $H_{m_b}(\Im\lambda, \Gamma\gamma) > 0$,

$$\begin{aligned} \beta_*(\Im\lambda)\,\alpha_*(\Gamma\gamma) \quad &\geq \quad \eta_*(\Im\lambda)\,\eta_*(\Gamma\gamma) \\ &\Longrightarrow \quad \tau + F\left(sH_{m_b}(\Im\lambda, \Gamma\gamma)\right) \leq F\left(\varphi\left(M_{m_b}(\lambda, \gamma)\right) M_{m_b}(\lambda, \gamma)\right), \end{aligned} \tag{3.1}$$

where

$$M_{m_b}(\lambda, \gamma) = \max\{m_b(\lambda, \gamma), m_b(\lambda, \Im\gamma), m_b(\gamma, \Gamma\gamma), \frac{m_b(\lambda, \Im\lambda) m_b(\gamma, \Gamma\gamma)}{s + m_b(\lambda, \gamma)}\}. \tag{3.2}$$

Theorem 3.5. *Let (Ξ, m_b) be a complete MbMS, $\alpha, \beta, \eta : \Xi \to [0, \infty)$ be a given functions, and $\Im, \Gamma : \Xi \to CB_{m_b}(\Xi)$ be two multivalued mappings satisfy the postulates below:*

(1) the pair $(\Im, \Gamma)$ is generalized (α_*, β_*)-Geraghty F-contraction;
(2) the pair $(\Im, \Gamma)$ is η-cyclic (α_*, β_*)-admissible;
(3) either there is $\lambda_0 \in \Xi$ so that $\alpha_*(\Gamma\lambda_0) \geq \eta_*(\Gamma\lambda_0)$ or $\gamma_0 \in \Xi$ so that $\beta_*(\Im\gamma_0) \geq \eta_*(\Im\gamma_0)$.

Then $\Im$ and Γ have a common FP $\lambda^* \in \Xi$.

Proof. Let $\lambda_0 \in \Xi$ so that $\alpha(\lambda_0) \geq \eta(\lambda_0)$, by axiom (2) $\exists \lambda_1 \in \Im\lambda_0$ and $\lambda_2 \in \Gamma\lambda_1$ so that

$$\alpha(\lambda_0) \geq \eta(\lambda_0) \Rightarrow \beta(\lambda_1) \geq \beta_*(\Im\lambda_0) \geq \eta_*(\Im\lambda_0)$$

and

$$\alpha(\lambda_2) \geq \alpha_*(\Gamma\lambda_1) \geq \eta_*(\Gamma\lambda_1).$$

Therefore

$$\alpha_*(\Gamma\lambda_1)\beta_*(\Im\lambda_0) \geq \eta_*(\Gamma\lambda_1)\,\eta_*(\Im\lambda_0),$$

Since F is right continuous, then from Remark 2.11, we have

$$F(sm_b(\lambda_1, \Gamma\lambda_1)) = \inf_{\gamma \in \Gamma\lambda_1} F(sm_b(\lambda_1, \gamma)).$$

Thus, there is $\gamma = \lambda_2 \in \Gamma\lambda_1$, so that

$$\begin{aligned} F(sm_b(\lambda_1, \lambda_2)) &\leq F\left(sH_{m_b}(\Im\lambda_0, \Gamma\lambda_1)\right) \\ &\leq F\left(\varphi\left(M_{m_b}(\lambda_0, \lambda_1)\right) M_{m_b}(\lambda_0, \lambda_1)\right) - \tau, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} M_{m_b}(\lambda_0, \lambda_1) &= \max\left\{ \begin{array}{c} m_b(\lambda_0, \lambda_1), m_b(\lambda_0, \Im\lambda_0), m_b(\lambda_1, \Gamma\lambda_1), \\ \frac{m_b(\lambda_0, \Im\lambda_0) m_b(\lambda_1, \Gamma\lambda_1)}{s + m_b(\lambda_0, \lambda_1)} \end{array} \right\} \\ &= \max\left\{ m_b(\lambda_0, \lambda_1), m_b(\lambda_1, \lambda_2), \frac{m_b(\lambda_0, \lambda_1) m_b(\lambda_1, \lambda_2)}{s + m_b(\lambda_0, \lambda_1)} \right\} \\ &\leq \max\left\{ m_b(\lambda_0, \lambda_1), m_b(\lambda_1, \lambda_2), \frac{m_b(\lambda_0, \lambda_1) m_b(\lambda_1, \lambda_2)}{m_b(\lambda_0, \lambda_1)} \right\} \\ &= \max\left\{ m_b(\lambda_0, \lambda_1), m_b(\lambda_1, \lambda_2) \right\}. \end{aligned}$$

If $M_{m_b}(\lambda_0,\lambda_1) \leq m_b(\lambda_1,\lambda_2)$, then from (3.3), we can write

$$\begin{aligned} F(sm_b(\lambda_1,\lambda_2)) &\leq F(\varphi(m_b(\lambda_1,\lambda_2))\, m_b(\lambda_1,\lambda_2)) - \tau \\ &< F(m_b(\lambda_1,\lambda_2)). \end{aligned}$$

Applying (F_1), we have

$$sm_b(\lambda_1,\lambda_2) < m_b(\lambda_1,\lambda_2),$$

a contradiction. If $M_{m_b}(\lambda_0,\lambda_1) \leq m_b(\lambda_0,\lambda_1)$, then by (3.3), we get

$$\begin{aligned} F(sm_b(\lambda_1,\lambda_2)) &\leq F(\varphi(m_b(\lambda_0,\lambda_1))\, m_b(\lambda_0,\lambda_1)) - \tau \\ &< F(m_b(\lambda_0,\lambda_1)). \end{aligned}$$

Again, from (F_1), we obtain

$$sm_b(\lambda_1,\lambda_2) < m_b(\lambda_0,\lambda_1),$$

Analogous to (3.3), there is $\lambda_3 \in \Im\lambda_2$ so that

$$\begin{aligned} F(sm_b(\lambda_2,\lambda_3)) &\leq F(\varphi(M_{m_b}(\lambda_1,\lambda_2)))\, M_{m_b}(\lambda_1,\lambda_2)) - \tau \\ &\leq F(\varphi(m_b(\lambda_1,\lambda_2)))\, m_b(\lambda_1,\lambda_2)) - \tau. \end{aligned}$$

Continuing with the same scenario, we construct a sequence $\{\lambda_n\}$ in Ξ so that $\lambda_{2n+1} \in \Im\lambda_{2n}$ and $\lambda_{2n+2} \in \Gamma\lambda_{2n+1}$. Since the pair $(\Im,\Gamma)$ is η-cyclic (α_*,β_*)-admissible, we have

$$\beta_*(\Im\lambda_{2n})\alpha_*(\Gamma\lambda_{2n+1}) \geq \eta_*(\Im\lambda_{2n})\eta_*(\Gamma\lambda_{2n+1}),\ \forall n \geq 0.$$

Subsequently, by (3.1), we get

$$\begin{aligned} \tau + F(sm_b(\lambda_{2n+1},\lambda_{2n+2})) &\leq \tau + F(sH_{m_b}(\Im\lambda_{2n},\Gamma\lambda_{2n+1})) \\ &\leq F(\varphi(M_{m_b}(\lambda_{2n},\lambda_{2n+1}))\, M_{m_b}(\lambda_{2n},\lambda_{2n+1})) \\ &\leq F(\varphi(m_b(\lambda_{2n},\lambda_{2n+1}))\, m_b(\lambda_{2n},\lambda_{2n+1})) \\ &\leq F(m_b(\lambda_{2n},\lambda_{2n+1}), \end{aligned} \tag{3.4}$$

therefore (3.4) implies that

$$\tau + F(sm_b(\lambda_{2n+1},\lambda_{2n+2})) \leq F(m_b(\lambda_{2n},\lambda_{2n+1}).$$

Set $\rho_{2n+1} = m_b(\lambda_{2n+1},\lambda_{2n+2})$ and $\mu_{2n+1} = s^{2n+1}\rho_{2n+1}$, $\forall n \geq 0$, then, we can write

$$\tau + F(s\rho_{2n+1}) \leq F(\rho_{2n}),\ \forall n \geq 0.$$

By (F_4), one can obtain

$$\tau + F(\mu_{2n+1}) \leq F(\mu_{2n}),\ \forall n \geq 0. \tag{3.5}$$

Repeating the inequality (3.5), we obtain

$$F(\mu_{2n+1}) \leq F(\mu_{2n}) - \tau \leq ... \leq F(\mu_0) - (2n+1)\tau,\ \forall n \geq 0. \tag{3.6}$$

Letting $n \to \infty$ in (3.6), we have

$$\lim_{n\to\infty} F(\mu_{2n+1}) = -\infty.$$

It follows from (F_2) that

$$\lim_{n\to\infty} \mu_{2n+1} = 0.$$

By (F_3), there is $k \in (0,1)$ so that

$$\lim_{n\to\infty} \mu_{2n+1}^k F(\mu_{2n+1}) = 0.$$

From (3.6), we get

$$\begin{aligned} \mu_{2n+1}^k F(\mu_{2n+1}) - \mu_{2n+1}^k F(\mu_0) &\leq \left(\mu_{2n+1}^k (F(\mu_0) - (2n+1)\tau) - \mu_{2n+1}^k F(\mu_0)\right) \\ &= -\mu_{2n+1}^k (2n+1)\tau \leq 0, \ \forall n \geq 0. \end{aligned}$$

Taking the limit as $n \to \infty$ and since $\tau > 0$, we obtain

$$\lim_{n\to\infty} \mu_{2n+1}^k (2n+1) = 0.$$

Thus, there is $n_1 \in \mathbb{N}$ so that

$$\mu_{2n+1}^k (2n+1) \leq 1 \Rightarrow \mu_{2n+1} \leq \frac{1}{(2n+1)^{\frac{1}{k}}} \ \forall n \geq n_1.$$

This leads to the series $\sum_n \mu_{2n+1}$ is convergent.

Now, we prove that $\{\lambda_n\}$ is an m_b-Cauchy sequence in Ξ. Using (Mb_4), we get

$$\begin{aligned} & m_b(\lambda_{2n+1}, \lambda_{2n+3}) - m_{b_{\lambda_{2n+1},\lambda_{2n+3}}} \\ \leq\ & s\left[m_b(\lambda_{2n+1}, \lambda_{2n+2}) - m_{b_{\lambda_{2n+1},\lambda_{2n+2}}} + m_b(\lambda_{2n+2}, \lambda_{2n+3}) - m_{b_{\lambda_{2n+2},\lambda_{2n+3}}}\right] \\ & -m_b(\lambda_{2n+2}, \lambda_{2n+2}) \\ \leq\ & s\left[m_b(\lambda_{2n+1}, \lambda_{2n+2}) - m_{b_{\lambda_{2n+1},\lambda_{2n+2}}} + m_b(\lambda_{2n+2}, \lambda_{2n+3}) - m_{b_{\lambda_{2n+2},\lambda_{2n+3}}}\right] \\ \leq\ & s m_b(\lambda_{2n+1}, \lambda_{2n+2}) + s^2 m_b(\lambda_{2n+2}, \lambda_{2n+3}). \end{aligned}$$

Similarly

$$\begin{aligned} & m_b(\lambda_{2n+1}, \lambda_{2n+4}) - m_{b_{\lambda_{2n+1},\lambda_{2n+4}}} \\ \leq\ & s\left[m_b(\lambda_{2n+1}, \lambda_{2n+2}) - m_{b_{\lambda_{2n+1},\lambda_{2n+2}}} + m_b(\lambda_{2n+2}, \lambda_{2n+4}) - m_{b_{\lambda_{2n+2},\lambda_{2n+4}}}\right] \\ & -m_b(\lambda_{2n+2}, \lambda_{2n+2}) \\ \leq\ & s m_b(\lambda_{2n+1}, \lambda_{2n+2}) + s^2 m_b(\lambda_{2n+2}, \lambda_{2n+3}) + s^3 m_b(\lambda_{2n+3}, \lambda_{2n+4}). \end{aligned}$$

In general, for all $q > p > n_1$ with $p = 2n+1$, we obtain

$$m_b\left(\lambda_p, \lambda_q\right) - m_{b_{\lambda_p,\lambda_q}} \leq \sum_{i=p}^{q-1} s^{i-p+1} m_b(\lambda_i, \lambda_{i+1}) \leq \sum_{i=p}^{q-1} s^i m_b(\lambda_i, \lambda_{i+1}) \leq \sum_{i=p}^{\infty} \mu_i.$$

The convergence of the series $\sum_{i=p}^{\infty} \mu_i$ leads to

$$\lim_{p,q\to\infty} \left(m_b \left(\lambda_p, \lambda_q \right) - m_{b_{\lambda_p,\lambda_q}} \right) = 0.$$

By the same way and from Remark 2.16, we obtain

$$\begin{aligned} & M_{b_{\lambda_{2n+1},\lambda_{2n+4}}} - m_{b_{\lambda_{2n+1},\lambda_{2n+4}}} \\ \leq \ & s \left(M_{b_{\lambda_{2n+1},\lambda_{2n+2}}} - m_{b_{\lambda_{2n+1},\lambda_{2n+2}}} \right) + s^2 \left(M_{b_{\lambda_{2n+2},\lambda_{2n+3}}} - m_{b_{\lambda_{2n+2},\lambda_{2n+3}}} \right) \\ & + s^3 \left(M_{b_{\lambda_{2n+3},\lambda_{2n+4}}} - m_{b_{\lambda_{2n+3},\lambda_{2n+4}}} \right). \end{aligned}$$

In general, for all $q > p > n_1$ with $p = 2n + 1$, we obtain

$$\begin{aligned} M_{b_{\lambda_p,\lambda_q}} - m_{b_{\lambda_p,\lambda_q}} & \leq \sum_{i=p}^{q-1} s^{i-p+1} \left(M_{b_{\lambda_i,\lambda_{i+1}}} - m_{b_{\lambda_i,\lambda_{i+1}}} \right) \leq \sum_{i=p}^{q-1} s^{i-p+1} M_{b_{\lambda_i,\lambda_{i+1}}} \\ & \leq \sum_{i=p}^{q-1} s^{i-p+1} m_b \left(\lambda_i, \lambda_i \right) \leq \sum_{i=p}^{q-1} s^{i-p+1} m_b \left(\lambda_i, \lambda_{i+1} \right) \\ & \leq \sum_{i=p}^{q-1} s^i m_b \left(\lambda_i, \lambda_{i+1} \right) \leq \sum_{i=p}^{\infty} \mu_i. \end{aligned}$$

The convergence of the series $\sum_{i=p}^{\infty} \mu_i$ leads to

$$\lim_{p,q\to\infty} \left(M_{b_{\lambda_p,\lambda_q}} - m_{b_{\lambda_p,\lambda_q}} \right) = 0.$$

Therefore, $\{\lambda_n\}$ is an m_b-Cauchy sequence in Ξ. Since Ξ is m_b-complete, there exists $\lambda^* \in \Xi$ so $\lambda_n \longrightarrow \lambda^*$ as $n \longrightarrow \infty$, implies $\lambda_{2n+1} \to \lambda^*$ and $\lambda_{2n+2} \to \lambda^*$ as $n \to \infty$. Thus, we have

$$\lim_{n\to\infty} \left(m_b \left(\lambda_{2n+1}, \lambda^* \right) - m_{b_{\lambda_{2n+1},\lambda^*}} \right) = 0. \tag{3.7}$$

Since $\lim_{n\to\infty} m_b \left(\lambda_{2n+1}, \lambda_{2n+1} \right) = 0$, then by (3.7), we get

$$\lim_{n\to\infty} m_b \left(\lambda_{2n+1}, \lambda^* \right) = 0. \tag{3.8}$$

It follows from (3.1), (3.4) and (3.8) that

$$\lim_{n\to\infty} H_{m_b} \left(\Im \lambda_{2n}, \Gamma \lambda^* \right) = 0. \tag{3.9}$$

Since $\lambda_{2n+1} \in \Im \lambda_{2n}$ and

$$m_b \left(\lambda_{2n+1}, \Gamma \lambda^* \right) \leq H_{m_b} \left(\Im \lambda_{2n}, \Gamma \lambda^* \right).$$

Then after taking the limit as $n \to \infty$, we obtain that

$$\lim_{n\to\infty} m_b \left(\lambda_{2n+1}, \Gamma \lambda^* \right) = 0. \tag{3.10}$$

By (Mb_2), one can write

$$m_{b_{\lambda_{2n+1},\Gamma\lambda^*}} \leq m_b\left(\lambda_{2n+1},\Gamma\lambda^*\right),$$

that is

$$\lim_{n\to\infty} m_{b_{\lambda_{2n+1},\Gamma\lambda^*}} = 0. \tag{3.11}$$

Now, utilizing (Mb_4), we have

$$\begin{aligned}
& m_b\left(\lambda^*,\Gamma\lambda^*\right) - \sup_{\gamma\in\Gamma\lambda^*} m_{b_{\lambda^*,\gamma}} \\
\leq\ & m_b\left(\lambda^*,\Gamma\lambda^*\right) - m_{b_{\lambda^*,\Gamma\lambda^*}} \\
\leq\ & s\left[m_b\left(\lambda^*,\lambda_{2n+1}\right) - m_{b_{\lambda^*,\lambda_{2n+1}}} + m_b\left(\lambda_{2n+1},\Gamma\lambda^*\right) - m_{b_{\lambda_{2n+1},\Gamma\lambda^*}}\right] \\
& - m_b\left(\lambda_{2n+1},\lambda_{2n+1}\right).
\end{aligned} \tag{3.12}$$

Letting $n\to\infty$ in (3.12) and from (3.8), (3.10) and (3.11), we conclude that

$$m_b\left(\lambda^*,\Gamma\lambda^*\right) \leq \sup_{\gamma\in\Gamma\lambda^*} m_{b_{\lambda^*,\gamma}}. \tag{3.13}$$

Using (Mb_2), for all $\gamma\in\Gamma\lambda^*$, we get

$$m_{b_{\lambda^*,\gamma}} \leq m_b\left(\lambda^*,\gamma\right),$$

yields

$$m_{b_{\lambda^*,\gamma}} - m_b\left(\lambda^*,\gamma\right) \leq 0.$$

Thus

$$\sup\left\{m_{b_{\lambda^*,\gamma}} - m_b\left(\lambda^*,\gamma\right) : \gamma\in\Gamma\lambda^*\right\} \leq 0,$$

this implies that

$$\sup_{\gamma\in\Gamma\lambda^*} m_{b_{\lambda^*,\gamma}} - \sup_{\gamma\in\Gamma\lambda^*} m_b\left(\lambda^*,\gamma\right) \leq 0.$$

Therefore

$$\sup_{\gamma\in\Gamma\lambda^*} m_{b_{\lambda^*,\gamma}} \leq m_b\left(\lambda^*,\Gamma\lambda^*\right). \tag{3.14}$$

From (3.13) and (3.14), we obtain

$$m_b\left(\lambda^*,\Gamma\lambda^*\right) = \sup_{\gamma\in\Gamma\lambda^*} m_{b_{\lambda^*,\gamma}}.$$

Hence by Lemma 2.12, we get $\lambda^*\in\overline{\Gamma\lambda^*}=\Gamma\lambda^*$. Similarly, we can easily conclude that $\lambda^*\in\Im\lambda^*$. Therefore λ^* is a common FP of $\Im$ and Γ. □

Remark 3.6. Theorem 3.5 still valid if we consider the following:

- If we put $s=1$ in Definition 3.4, then generalized (α_*,β_*)-Geraghty F-contraction mappings take the form: $H_m(\Im\lambda,\Gamma\gamma)>0$,

$$\begin{aligned}
\alpha_*(\Im\lambda)\beta_*(\Gamma\gamma) \quad &\geq \quad \eta_*\left(\Im\lambda\right)\eta_*\left(\Gamma\gamma\right) \\
&\Longrightarrow \quad \tau + F\left(H_m(\Im\lambda,\Gamma\gamma)\right) \leq F\left(\varphi\left(M(\lambda,\gamma)\right)M(\lambda,\gamma)\right)
\end{aligned}$$

where $\varphi \in \xi, F \in F_s,\ \tau \in \mathbb{R}_+$ and

$$M(\lambda,\gamma) = \max\{m(\lambda,\gamma), m(\lambda,\Im\lambda), m(\gamma,\Gamma\gamma), \frac{m(\lambda,\Im\lambda)m(\gamma,\Gamma\gamma)}{1+m(\lambda,\gamma)}\}.$$

Moreover, under the same conditions (1)–(3) of Theorem 3.5, $\Im$ and Γ have a common FP in a complete MMS (Ξ, m).

- If we take $M_{m_b}(\lambda,\gamma) = m_b(\lambda,\gamma)$ in Definition 3.4, then we have a common FP of $\Im$ and Γ in complete MbMS, provided that the stipulations (1)–(3) of Theorem 3.5 hold.
- If we consider $\Im = \Gamma$ in Definition 3.4, then the result is given quickly in the same manner as the proof of Theorem 3.5.

The example below supports Theorem 3.5.

Example 3.7. Let $\Xi = [0,\infty)$ and $m_b : \Xi \times \Xi \longrightarrow [0,\infty)$ defined by

$$m_b(\lambda,\gamma) = \max\{\lambda,\gamma\}^p + |\lambda-\gamma|^p,\ \forall \lambda,\gamma \in \Xi.$$

Clearly, (Ξ, m_b) is an MbMS with $p > 1$ and $s = 2^p$.

If we take $\lambda = 5$, $\gamma = 1$ and $\kappa = 4$, we obtain that

$$m_b(\lambda,\gamma) - m_{b_{\lambda,\gamma}} > m_b(\lambda,\kappa) - m_{b_{\lambda,\kappa}} - m_b(\kappa,\gamma) - m_{b_{\kappa,\gamma}}.$$

This means (Ξ, m_b) is not MMS. Define $\Im, \Gamma : \Xi \to CB_{m_b}(\Xi)$ by

$$\Im\lambda = \begin{cases} \left\{\frac{\lambda}{64}\right\}, & \text{if } \lambda \in (0,1], \\ \left\{0, \frac{1}{16}\right\}, & \text{otherwise,} \end{cases} \quad \text{and } \Gamma\lambda = \begin{cases} \left\{0, \frac{\lambda}{48}\right\}, & \text{if } \lambda \in (0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Describe the functions $\alpha, \beta, \eta : \Xi \to [0,\infty)$ as $\eta(\lambda) = \lambda + 1$,

$$\alpha(\lambda) = \begin{cases} 3e^{2\lambda^2,} & \text{if } \lambda > 0, \\ 3, & \lambda = 0, \end{cases},\ \beta(\lambda) = \begin{cases} 5\lambda + 1, & \lambda > 0, \\ 1, & \lambda = 0. \end{cases}$$

for all $\lambda \in \Xi$. Now, for $\lambda \in (0,1]$, we have $\alpha(\lambda) \geq \eta(\lambda)$ implies

$$\beta_*(\Im\lambda) = \beta_*\left(\frac{\lambda}{64}\right) = \frac{5\lambda}{64} + 1 \geq \frac{\lambda}{64} + 1 = \eta_*\left(\left\{\frac{\lambda}{64}\right\}\right) = \eta_*(\Im\lambda).$$

When $\Gamma\lambda = 0$, then $\beta(\lambda) \geq \eta(\lambda)$ implies

$$\alpha_*(\Gamma\lambda) = \alpha_*(0) = 3 > 1 = \eta_*(0) = \eta_*(\Gamma\lambda),$$

if $\Gamma\lambda = \frac{\lambda}{48}$, then, we get $\beta(\lambda) \geq \eta(\lambda)$ implies

$$\alpha_*(\Gamma\lambda) = \alpha_*\left(\frac{\lambda}{48}\right) = 3e^{2\left(\frac{\lambda}{48}\right)^2} \geq \frac{\lambda}{48} + 1 = \eta_*\left(\frac{\lambda}{48}\right) = \eta_*(\Gamma\lambda).$$

Hence the pair $(\Im, \Gamma)$ is η-cyclic (α_*, β_*)-admissible mappings. Consider $\varphi(t) = \frac{1}{6^p}$, so for $\lambda, \gamma \in (0,1]$, one can write

$$F(sH_{m_b}(\Im\lambda,\Gamma\gamma)) \;=\; F(2^p(\max\left\{\sup_{a\in\Im\lambda} m_b(a,\Gamma\gamma), \sup_{b\in\Gamma\gamma} m_b(\Im\lambda, b)\right\}))$$

$$
\begin{aligned}
&= F\left(2^p \max\left(m_b\left(\frac{\lambda}{64},\left\{0,\frac{\gamma}{48}\right\}\right), m_b\left(\frac{\lambda}{64},\frac{\gamma}{48}\right)\right)\right)\\
&= F\left(2^p \max\left(m_b\left(\frac{\lambda}{64},0\right), m_b\left(\frac{\lambda}{64},\frac{\gamma}{48}\right)\right)\right)\\
&= F\left(2^p m_b\left(\frac{\lambda}{64},\frac{\gamma}{48}\right)\right) = F\left(\frac{2^p}{16^p} m_b\left(\frac{\lambda}{4},\frac{\gamma}{3}\right)\right)\\
&= F\left(\left(\frac{1}{8}\right)^p m_b\left(\frac{\lambda}{4},\frac{\gamma}{3}\right)\right) = F\left(\frac{1}{2^{3p}} m_b\left(\frac{\lambda}{4},\frac{\gamma}{3}\right)\right)\\
&\leq F\left(\left(\frac{16}{8\times 12}\right)^p m_b(\lambda,\gamma)\right) = F\left(\frac{1}{6^p} m_b(\lambda,\gamma)\right).
\end{aligned}
$$

By taking $F(\lambda) = \ln\lambda$, we have

$$
\ln(sH_{m_b}(\Im\lambda,\Gamma\gamma)) \leq \ln\left(\frac{1}{6^p} m_b(\lambda,\gamma)\right) = -p\ln(6) + \ln(m_b(\lambda,\gamma)),
$$

which implies that

$$
\ln(sH_{m_b}(\Im\lambda,\Gamma\gamma)) \leq \ln(m_b(\lambda,\gamma)) - \tau.
$$

Since $m_b(\lambda,\gamma) \leq M_b(\lambda,\gamma)$, and using the definition of φ, then we obtain

$$
F(sH_{m_b}(\Im\lambda,\Gamma\gamma)) \leq F(\varphi(M_b(\lambda,\gamma))) - \tau \leq F(\varphi(M_{mb}(\lambda,\gamma))\, M_{mb}(\lambda,\gamma)) - \tau.
$$

Otherwise, the inequality below holds

$$
F(sH_{m_b}(\Im\lambda,\Gamma\gamma) \leq F(\varphi(M_{mb}(\lambda,\gamma))\, M_{mb}(\lambda,\gamma)) - \tau.
$$

Analogously, for each $\lambda,\gamma \in \Xi$, we can find some $\tau > 0$ satisfy the above inequality. Hence, all hypotheses of Theorem 3.5 are fulfilled with $\tau = p\ln(6)$ and $\lambda^* = 0$ is a common FP of $\Im$ and Γ.

4. Some consequences

This part is a reduction of the previous part by taking $\Im$ and Γ are single-valued mappings.

Definition 4.1. Let $\Xi \neq \emptyset$ and $\alpha,\beta,\eta : \Xi \to [0,\infty)$ be given functions. The mapping $\Gamma : \Xi \to \Xi$ is called η-cyclic (α,β)-admissible if for some $\lambda \in \Xi$,

$$
\alpha(\lambda) \geq \eta(\lambda) \Rightarrow \beta(\Gamma\lambda) \geq \eta(\Gamma\lambda),
$$

and

$$
\beta(\lambda) \geq \eta(\lambda) \Rightarrow \alpha(\Gamma\lambda) \geq \eta(\Gamma\lambda).
$$

Definition 4.2. Let $\Xi \neq \emptyset$ and $\alpha,\beta,\eta : \Xi \to [0,\infty)$ be given functions. The mappings $\Im,\Gamma : \Xi \to \Xi$ are called η-cyclic (α,β)-admissible if for some $\lambda \in \Xi$,

$$
\alpha(\lambda) \geq \eta(\lambda) \Rightarrow \beta(\Im\lambda) \geq \eta(\Im\lambda),
$$

and

$$
\beta(\lambda) \geq \eta(\lambda) \Rightarrow \alpha(\Gamma\lambda) \geq \eta(\Gamma\lambda).
$$

Now, we present some results related to the existence of FPs which can be proven in a similar way to Theorem 3.5.

Corollary 4.3. *Let* (Ξ, m_b) *be a complete MbMS and* $\alpha, \beta, \eta : \Xi \to [0, \infty)$ *be given functions. Assume that the mapping* $\Gamma : \Xi \to \Xi$ *satisfies the following condition: There are* $\varphi \in \xi$ *and* $F \in F_s$ *so that for all* $\lambda, \gamma \in \Xi$, $s \geq 1$ *and* $\tau > 1$,

$$\alpha(\lambda)\beta(\gamma) \geq \eta(\lambda)\eta(\gamma) \Longrightarrow \tau + F(sm_b(\Gamma\lambda, \Gamma\gamma)) \leq F(\varphi(M_{m_b}(\lambda, \gamma)) M_{m_b}(\lambda, \gamma)),$$

where $M_{m_b}(\lambda, \gamma)$ *is defined in (3.2). Assume also that the following hypotheses are satisfied:*

(i) Γ *is an* η*-cyclic* (α, β)*-admissible;*
(ii) there is $\lambda_0 \in \Xi$ *so that* $\alpha(\lambda_0) \geq \eta(\lambda_0)$ *or* $\beta(\lambda_0) \geq \eta(\lambda_0)$.

Then Γ *has a FP* $\lambda^* \in \Xi$.

Corollary 4.4. *Let* (Ξ, m_b) *be a complete MbMS and* $\alpha, \beta, \eta : \Xi \to [0, \infty)$ *be given functions. Consider the mappings* $\Im, \Gamma : \Xi \to \Xi$ *satisfy the assumption below: There are* $\varphi \in \xi$ *and* $F \in F_s$ *so that for all* $\lambda, \gamma \in \Xi$, $s \geq 1$ *and* $\tau > 1$,

$$\alpha(\lambda)\beta(\gamma) \geq \eta(\lambda)\eta(\gamma) \Longrightarrow \tau + F(sm_b(\Im\lambda, \Gamma\gamma)) \leq F(\varphi(M_{m_b}(\lambda, \gamma)) M_{m_b}(\lambda, \gamma)),$$

where $M_{m_b}(\lambda, \gamma)$ *is defined in (3.2). Suppose also the following two conditions hold:*

(i) $(\Im, \Gamma)$ *is a pair of* η*-cyclic* (α, β)*-admissible;*
(ii) there is $\lambda_0 \in \Xi$ *so that* $\alpha(\lambda_0) \geq \eta(\lambda_0)$ *or* $\gamma_0 \in \Xi$ *so that* $\beta(\gamma_0) \geq \eta(\gamma_0)$.

Then $\Im$ *and* Γ *have a common FP* $\lambda^* \in \Xi$.

If we set $\alpha(\lambda) = \beta(\gamma) = \eta(\lambda) = \eta(\gamma) = 1$ in Corollary 4.4, we have the following result.

Corollary 4.5. *Let* (Ξ, m_b) *be a complete MbMS,* $\Im$ *and* Γ *be self-mappings defined on* Ξ*. If there are* $\varphi \in \xi$ *and* $F \in F_s$ *so that for all* $\lambda, \gamma \in \Xi$, $s \geq 1$ *and* $\tau > 1$,

$$\tau + F(sm_b(\Im\lambda, \Gamma\gamma)) \leq F(\varphi(M_{m_b}(\lambda, \gamma)) M_{m_b}(\lambda, \gamma)), \tag{4.1}$$

where $M_{m_b}(\lambda, \gamma)$ *is described as (3.2). Then* $\Im$ *and* Γ *have a common FP* $\lambda^* \in \Xi$.

Note. The pair $(\Im, \Gamma)$ that satisfy (4.1) is called generalized Geraghty F-contraction mappings.

5. An application

In this part, we apply Corollary 4.5 to discuss the existence of solution to the pair of ODEs. Consider the following pair of ODEs:

$$\begin{cases} -\frac{d^2\lambda}{dt^2} = f(t, \lambda(t)), & t \in [0,1] \\ \lambda(0) = \lambda(1) = 0, \end{cases} \quad \text{and} \quad \begin{cases} -\frac{d^2\gamma}{dt^2} = g(t, \gamma(t)), & t \in [0,1], \\ \gamma(0) = \gamma(1) = 0. \end{cases} \tag{5.1}$$

where $f, g : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions. So, the pair of ODEs (5.1) is equivalent to the following integral equations:

$$\lambda(t) = \int_0^1 G(t,s) f(s, \lambda(s)) ds \text{ and } \gamma(t) = \int_0^1 G(t,s) g(s, \gamma(s)) ds. \tag{5.2}$$

The Green's function $G : [0,1] \times [0,1] \to \mathbb{R}$ associated with (5.2) is described as

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases}$$

Let $\Xi = C([0,1], \mathbb{R})$ be the set of all continuous functions defined on $[0,1]$. Define a function $m : \Xi \times \Xi \to \mathbb{R}^+$ by

$$m_b(\lambda, \gamma) = \max_{t \in I} \left(\left| \frac{\lambda(t) + \gamma(t)}{2} \right| \right)^2, \ \forall \lambda, \gamma \in \Xi.$$

Obviously, (Ξ, m_b) is a complete MbMS with a constant $s = 2$.

The ODEs (5.1) will be considered under the two postulates below:

(1) there is a function $\omega : \mathbb{R} \longrightarrow (0,1)$ so that for all $z_1, z_2 \in \mathbb{R}$, we have

$$|f(t, z_1)| + |g(t, z_2)| \le \sqrt{\omega(t) M_{m_b}(z_1, z_2)}, \ \forall t \in [0,1],$$

where

$$M_{m_b}(z_1, z_2) = \max \left\{ \begin{array}{c} \left|\frac{z_1 + z_2}{2}\right|^2, \left|\frac{z_1 + \Im z_1}{2}\right|^2, \left|\frac{z_2 + \Gamma z_2}{2}\right|^2, \\ \frac{\left|\frac{z_1 + \Im z_1}{2}\right|^2 \left|\frac{z_2 + \Gamma z_2}{2}\right|^2}{s + \left|\frac{z_1 + z_2}{2}\right|^2} \end{array} \right\};$$

(2) there is $s \ge 1$ so that $\int_0^1 G(t,r)\, dr \le \sqrt{\frac{12e^{-\tau}}{7s}}$, for some $\tau > 0$.

Now, we present our main theorem in this part.

Theorem 5.1. *Under the postulates (1) and (2), ODEs (5.1) has at least one solution $\lambda^* \in \Xi$.*

Proof. Describe the operators $\Im, \Gamma : \Xi \longrightarrow \Xi$ as

$$\Im\lambda(t) = \int_0^1 G(t,s) f(s, \lambda(s)) ds \text{ and } \Gamma\gamma(t) = \int_0^1 G(t,s) g(s, \gamma(s)) ds,$$

for all $t \in [0,1]$. Clearly, the solution of the integral equations (5.2) is equivalent to find a common FP of the operators $\Im$ and Γ. Let $\lambda, \gamma \in \Xi$, by our assumption, for all $t \in [0,1]$, we get

$$\begin{aligned} \left[\left| \Im\lambda(t) \right| + |\Gamma\gamma(t)| \right]^2 &= \left[\left| \int_0^1 G(t,s) f(s, \lambda(s)) ds \right| + \left| \int_0^1 G(t,s) g(s, \gamma(s)) ds \right| \right]^2 \\ &\le \left[\int_0^1 \left[|G(t,s) f(s, \lambda(s))| + |G(t,s) g(s, \gamma(s))| \right] ds \right]^2 \\ &\le \left[\int_0^1 G(t,s) \left(|f(s, \lambda(s))| + |g(s, \gamma(s))| \right) ds \right]^2 \end{aligned}$$

$$
\begin{aligned}
&\leq \left[\int_0^1 G(t,s)\sqrt{\omega(t)\, M_{m_b}(\lambda,\gamma)}ds\right]^2 \\
&\leq \left[\int_0^1 G(t,s)\sqrt{\omega(t)\, M_{m_b}(\lambda,\gamma)}ds\right]^2 \\
&= \left[\sqrt{\omega(t)\, M_{m_b}(\lambda,\gamma)}\right]^2 \left[\int_0^1 G(t,s)ds\right]^2 \\
&\leq \left[\sqrt{\omega(t)\, M_{m_b}(\lambda,\gamma)}\right]^2 \left[\sqrt{\frac{12e^{-\tau}}{7s}}\right]^2 \\
&= \omega(t)\,\frac{12e^{-\tau}}{7s} M_{m_b}(\lambda,\gamma).
\end{aligned}
$$

Consequently, we get

$$
\begin{aligned}
sm_b(\Im\lambda,\Gamma\gamma) &\leq \frac{3\omega(t)}{7} e^{-\tau} M_{m_b}(\lambda,\gamma) \\
&\leq e^{-\tau}\varphi(M_{m_b}(\lambda,\gamma))\, M_{m_b}(\lambda,\gamma),
\end{aligned}
$$

which implies that

$$
\tau + \ln(sm_b(\Im\lambda,\Gamma\gamma)) \leq \ln[\varphi(M_{m_b}(\lambda,\gamma))\, M_{m_b}(\lambda,\gamma)],
$$

where $F(\lambda) = \ln\lambda \in F_s$ and $\varphi(t) = \frac{3\omega(t)}{7}$, for all $t \in [0,1]$. Thus, all stipulations of Corollary 4.5 are fulfilled. Therefore, the operators $\Im$ and Γ have a common FP, which is a solution to the ODEs (5.1).

□

Remark 5.2. It should be noted that under the same conditions, we cannot obtain the solution of the ODEs (5.1) by the classical FP theorem because of the definition of the function $m : \Xi \times \Xi \to \mathbb{R}^+$. It is defined as

$$
m_b(\lambda,\gamma) = \max_{t\in I}\left(\left|\frac{\lambda(t)+\gamma(t)}{2}\right|\right)^2, \ \forall\lambda,\gamma \in \Xi.
$$

On a complete metric space, the classical theorem holds true, but the first metric space requirement is not met as follows:

$$
\text{for } \lambda,\gamma \in \Xi,\ if\ \lambda = \gamma, \text{ then } m_b(\lambda,\lambda) = \max_{t\in I}\left(\left|\frac{\lambda(t)+\lambda(t)}{2}\right|\right)^2 = \max_{t\in I}(|\lambda(t)|)^2 > 0.
$$

So not equal 0. Hence, (Ξ, m_b) is a complete MbMS with a constant $s = 2$ and not a complete metric space.

6. Conclusions

After the large number of papers published in the field of fixed point, we can assert that this technique is the backbone of non-linear analysis due to its smoothness and pivotality in many life disciplines. Therefore, in our manuscript, a new type of contraction was defined, called η-cyclic (α_*,β_*)-admissible type F-contraction multivalued mappings. Under this contraction, some results concerned with FPs have been proven in the context of MbMSs. Also, our new results generalize and unify many papers in this regard. Moreover, some examples have been discussed to clarify the obtained results. Finally, we applied our main result to study the existence of a solution to a pair of ODEs.

Conflict of interest

The authors declare that they have no competing interests concerning the publication of this article.

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