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Research article

Stationary distribution of a stochastic two-species Schoener's competitive system with regime switching

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Abstract: This paper studies a stochastic two-species Schoener's competitive model with regime switching. We first investigate the sufficient conditions for the existence of a unique stationary distribution of the model. Then we prove that the convergence of transition probability to the stationary distribution is exponentially under some mild assumptions. Moreover, we also introduce several numerical simulations to validate the model against the biological significance.

Keywords: stationary distribution; Schoener's competitive model; random disturbances; ergodicity
Mathematics Subject Classification: 60H10, 60H30, 92D25

1. Introduction

Due to resource constraints, competition between several species is a common phenomenon in the natural environment, and the random disturbance of the environment has an effect on the growth of the population. In recent years, more and more scholars have done research on stochastic competitive populations, and some corresponding conclusions have been drawn (see [1–4]). The permanence and stationary distribution of the system have become significant topics to some scholars in mathematical ecology in recent years (see [5–10]). For example, in 2017, Yu and Liu ([5]) studied stationary distribution and the ergodicity of a stochastic food-chain model with Lévy jumps. In 2019, Liu ([6]) analyzed the dynamics of a stochastic regime switching predator-prey model with modified Leslie-Gower Holling-type II schemes and prey harvesting. In 2020, Wang et al. ([7]) considered the stationary distribution of a stochastic ratio-dependent predator-prey system with regime switching. Ji et al. studied permanence, extinction and periodicity for a stochastic competitive model with infinite distributed delays ([8]). In Reference [9], Rihan and Alsakaji studied the dynamics of a stochastic delay differential model for a prey-predator system with hunting cooperation in predators. For two species Lotka-Volterra competitive models, many results have been obtained in terms of the permanence and global stability of the corresponding systems (see [11]). Many scholars believe that the results are not always satisfactory for individual species (see [12]). The reason is that linearization causes many important elements to be ignored; hence, we need to introduce more complex and practical models (see [13]). In 1974, Schoener [14] proposed and investigated a competitive model, as follows:

$$\begin{cases} dx(t) = x(t)\left(-\mu_1\nu_1 - \mu_1\omega_{11}x(t) - \mu_1\omega_{12}y(t) + \frac{\mu_1\xi_1}{x(t)+\varepsilon_1}\right)dt, \\ dy(t) = y(t)\left(-\mu_2\nu_2 - \mu_2\omega_{21}x(t) - \mu_2\omega_{22}y(t) + \frac{\mu_2\xi_2}{y(t)+\varepsilon_2}\right)dt, \end{cases} \tag{1.1}$$

where $x(t)$ and $y(t)$ are the size of each species at time t, μ_1 and μ_2 stand for the spatial densities of each species and $\mu_i\nu_i(i = 1, 2)$ is its death rate. The coefficients $\mu_1\omega_{11}$ and $\mu_2\omega_{22}$ stand for the intra-specific competition rates and $\mu_1\omega_{12}$ and $\mu_2\omega_{21}$ stand for the inter-specific competition rates. In this article, the parameters $\mu_i,\ \nu_i,\ \omega_{ij},\ \xi_i$ and $\varepsilon_i(i, j = 1, 2)$ are positive constants. Now, simplifying the model (1.1), we obtain the following model:

$$\begin{cases} dx(t) = x(t)\left(-\alpha_1 - \beta_1x(t) - \gamma_1y(t) + \frac{\zeta_1}{x(t)+\delta_1}\right)dt, \\ dy(t) = y(t)\left(-\alpha_2 - \beta_2x(t) - \gamma_2y(t) + \frac{\zeta_2}{y(t)+\delta_2}\right)dt, \end{cases} \tag{1.2}$$

where $x(t)$ and $y(t)$ are the size of each species at time t, and $\alpha_i,\ \beta_i,\ \gamma_i,\ \zeta_i$ and $\delta_i(i = 1, 2)$ are also positive constants with their nature-based biological meanings. There are many scholars who have studied the Schoener model, and they reached found many important and excellent conclusions (see [14–18]). From [16], Liu et al. studied the global asymptotic stability of Schoener's competitive model with delays. In Reference [18], Zhu et al. investigated the coexistence of two species in a strongly coupled Schoener's competitive model.

In fact, population systems are often affected by environmental noise (i.e., parameters are not fixed constants in the population model). In the context of these factors, a number of authors have devoted their efforts to random population systems (see [19–25]). Nguyen and Sam ([19]) investigated the dynamics of a stochastic Lotka-Volterra model perturbed by white noise. Mao et al. obtained a significant conclusion: Even a small amount of noise can have an effect on explosions in population dynamics in Reference [20]. Wang and Liu ([23]) studied the stationary distribution of a stochastic hybrid phytoplankton-zooplankton model with toxin-producing phytoplankton. Considering the influence of the random fluctuating environment, random disturbances should be introduced into Model (1.2) to explore the effects of random disturbances on the model properties. It is assumed that random interference is white noise, and that it mainly affects growth rates and death rates, so we obtained the following stochastic model:

$$\begin{cases} dx(t) = x(t)\left[\sigma_1\dot{B}_1(t) - \alpha_1 - \beta_1x(t) - \gamma_1y(t) + \frac{\zeta_1}{x(t)+\delta_1}\right]dt, \\ dy(t) = y(t)\left[\sigma_1\dot{B}_2(t) - \alpha_2 - \beta_2x(t) - \gamma_2y(t) + \frac{\zeta_2}{y(t)+\delta_2}\right]dt, \end{cases} \tag{1.3}$$

i.e.,

$$\begin{cases} dx(t) = x(t)\left[-\alpha_1 - \beta_1x(t) - \gamma_1y(t) + \frac{\zeta_1}{x(t)+\delta_1}\right]dt + \sigma_1x(t)dB_1(t), \\ dy(t) = y(t)\left[-\alpha_2 - \beta_2x(t) - \gamma_2y(t) + \frac{\zeta_2}{y(t)+\delta_2}\right]dt + \sigma_2y(t)dB_2(t), \end{cases} \tag{1.4}$$

where σ_i^2 stands for the intensity of the white noise and $\{B_1(t), B_2(t)\}_{t\geqslant 0}$ is two-dimensional Brownian motion. Throughout this paper, the Brownian motion was defined on a complete probability space $(\Omega, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\in R_+}$ satisfying the usual condition. To the best of our knowledge, some scholars believe that parameters sometimes switch from one state to another in population models due to environmental changes. For example, the growth rates of a certain species are different at different temperatures (see [26, 27]). However, white noise does not depict these random disturbances. Therefore, according to the approach in References [28–30], we can describe the regime switching by using a continuous-time finite-state Markov chain, so we obtained the following random model with regime switching

$$\begin{cases} \mathrm{d}x(t) = x(t)\left[-\alpha_1(\lambda) - \beta_1(\lambda)x(t) - \gamma_1(\lambda)y(t) + \dfrac{\zeta_1(\lambda)}{x(t)+\delta_1(\lambda)}\right]\mathrm{d}t \\ \qquad + \sigma_1(\lambda)x(t)\mathrm{d}B_1(t), \\ \mathrm{d}y(t) = y(t)\left[-\alpha_2(\lambda) - \beta_2(\lambda)x(t) - \gamma_2(\lambda)y(t) + \dfrac{\zeta_2(\lambda)}{y(t)+\delta_2(\lambda)}\right]\mathrm{d}t \\ \qquad + \sigma_2(\lambda)y(t)\mathrm{d}B_2(t), \end{cases} \tag{1.5}$$

where $\lambda = \lambda(t)$ stands for a continuous-time Markov chain with a state space $\mathbb{S} = \{1, 2, 3, \cdots, n^*\}$. Regarding studying stochastic population models, we know that more and more attention has been paid to stationary distribution in recent years. Nevertheless, as far as we know, very little work on the stationary distribution of a stochastic two-species Schoener's competitive model with regime switching has been done. At present, the Lyapunov function method is widely being used to study the existence of a unique stationary distribution (USD) (see [31–33]).

As a matter of fact, in addition to the Lyapunov function method, we can apply the approach in Reference [34] to investigate the existence of a USD for a stochastic two-species Schoener's competitive model with regime switching.

Motivated by these, in this paper, we consider the model (1.5) and establish the sufficient conditions for the existence and uniqueness of an ergodic stationary distribution. In addition, we introduce some numerical simulations and realistic scenarios to illustrate the effects of Markovian switching on the existence of stationary distribution.

2. Main results

Throughout this article, we have the following assumptions:

$\{\lambda(t)\}_{t\geqslant 0}$ is independent and irreducible; thus, $\{\lambda(t)\}_{t\geqslant 0}$ is ergodic and has a USD, which is denoted as $\pi = (\pi_1, \pi_2, \cdots, \pi_{n^*})^{\mathrm{T}}$;

$\{B_1(t), B_2(t)\}_{t\geqslant 0}$ is independent;

$\min\limits_{k\in\mathbb{S}}\{\alpha_i(k), \beta_i(k), \gamma_i(k), \zeta_i(k), \delta_i(k), i, j = 1, 2\} > 0.$

For the sake of convenience, we define some notations.

$$R^2_+ = \{m \in R^2 \mid m_i > 0, i = 1, 2\}, \overline{R}^2_+ = \{m \in R^2 \mid m_i \geqslant 0, i = 1, 2\},$$

$$\partial R^2_+ = \overline{R}^2_+ \backslash R^2_+,$$

$$z_1(j) = \frac{\zeta_1(j)}{\delta_1(j)} - \alpha_1(j) - \frac{1}{2}\sigma_1^2(j), z_2(j) = \frac{\zeta_2(j)}{\delta_2(j)} - \alpha_2(j) - \frac{1}{2}\sigma_2^2(j),$$

$$\Phi_1 = \sum_{j\in \mathbb{S}} \pi_j z_1(j), \Phi_2 = \sum_{j\in \mathbb{S}} \pi_j z_2(j).$$

$\|\cdot\|_{TV}$ represents the total variation norm (see e.g., Reference [35]).

Remark 2.1. z_1 and z_2 denote the maximum of the "stochastic" growth rate of one species' population and the other species' population in state j without the competitor, respectively. Φ_1 and Φ_2 denote the maximum of the long-term "stochastic" growth rate of a species' population and the other species' population in the hybrid system (1.5), respectively.

We state two results before giving our main result of this article. According to Theorem 4.2 in Reference [31] and (5.17) in Reference [29], we have the following lemma:

Lemma 2.1. For the logistic equation

$$d\varphi_1(t) = \varphi_1(t)\Big[\frac{\zeta_1(\lambda)}{\delta_1(\lambda)} - \alpha_1(\lambda) - \beta_1(\lambda)\varphi_1(t)\Big]dt + \sigma_1(\lambda)\varphi_1(t)dB_1(t) \tag{2.1}$$

with the initial data $(\varphi_1(0), \lambda(0)) \in R_+ \times \mathbb{S}$, if $\Phi_1 > 0$, then Eq (2.1) has a unique ergodic stationary distribution (UESD) $\eta^\varphi(\cdot\times\cdot)$ concentrated on $R_+ \times \mathbb{S}$, and

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \beta_1(\lambda(s))\varphi_1(s)ds = \sum_{j\in \mathbb{S}} \int_{R_+} \beta_1(j)z\eta^\varphi(dz, j) = \Phi_1. \tag{2.2}$$

According to a similar proof of Theorem 3.1 in Reference [30], we can show the following.

Lemma 2.2. For any initial data $(x(0), y(0), \lambda(0)) = (m, l) \in R^2_+ \times \mathbb{S}$, Model (1.5) has a unique global solution $(x(t), y(t), \lambda(t)) \in R^2_+ \times \mathbb{S}$ almost surely (a.s.).

Theorem 2.1. Consider the model (1.5), according to $\Phi_1 > 0$, we have the following: (a) if $\Phi_2 > 0$, then $(x(t), y(t), \lambda(t))$ has a UESD $\eta(\cdot\times\cdot)$ concentrated on $R^2_+ \times \mathbb{S}$ and the transition probability of $(x(t), y(t), \lambda(t))$ converges to $\eta(\cdot\times\cdot)$ exponentially under the norm of total variation. (b) If $\Phi_2 < 0$, $\lim_{t\to+\infty} y(t) = 0$ a.s., and the transition probability of $(x(t), \lambda(t))$ converges to $\eta^\varphi(\cdot\times\cdot)$. Moreover,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \beta_1(\lambda(s))x(s)ds = \sum_{j\in \mathbb{S}} \int_{R_+} \beta_1(j)z\eta^\varphi(dz, j) = \Phi_1. \tag{2.3}$$

Remark 2.2. From a biological point of view, Case (a) means that Model (1.5) is permanent; Case (b) means that Model (1.5) is collapsed.

Theorem 2.1 reveals that the permanence and collapse of Model (1.5) depend on the sign of Φ_2 under the assumption that $\Phi_1 > 0$. We can realize that the sign of Φ_2 is related to regime switching. So, we select $\mathbb{S} = \{1, 2\}$ for a better understanding. Therefore, the hybrid system (1.5) has the following two subsystems:

$$\begin{cases} dx(t) = x(t)\left[-\alpha_1(1) - \beta_1(1)x(t) - \gamma_1(1)y(t) + \dfrac{\zeta_1(1)}{x(t) + \delta_1(1)}\right]dt \\ \qquad + \sigma_1(1)x(t)dB_1(t), \\ dy(t) = y(t)\left[-\alpha_2(1) - \beta_2(1)x(t) - \gamma_2(1)y(t) + \dfrac{\zeta_2(1)}{y(t) + \delta_2(1)}\right]dt \\ \qquad + \sigma_2(1)y(t)dB_2(t), \end{cases} \tag{2.4}$$

and

$$\begin{cases} \mathrm{d}x(t) = x(t)\left[-\alpha_1(2) - \beta_1(2)x(t) - \gamma_1(2)y(t) + \dfrac{\zeta_1(2)}{x(t)+\delta_1(2)}\right]\mathrm{d}t \\ \qquad + \sigma_1(2)x(t)\mathrm{d}B_1(t), \\ \mathrm{d}y(t) = y(t)\left[-\alpha_2(2) - \beta_2(2)x(t) - \gamma_2(2)y(t) + \dfrac{\zeta_2(2)}{y(t)+\delta_2(2)}\right]\mathrm{d}t \\ \qquad + \sigma_2(2)y(t)\mathrm{d}B_2(t). \end{cases} \tag{2.5}$$

In a biological sense, there are two situations:

($\mathscr{A}$) The above two subsystems have the same behavior with respect to permanence and collapse. In this case, Theorem 2.1 reveals that the permanent and collapsing behavior of the hybrid system (1.5) does not change under the behavior of regime switching. For instance, the hybrid system (1.5) is collapsing with the regime switching under the condition of that both subsystems (2.4) and (2.5) are collapsing.

($\mathscr{B}$) The above two subsystems have different behaviors with respect to permanence and collapse. In other words, one subsystem is collapsed, and the other is permanent. The result is intriguing under the behavior of regime switching. This is the significance of this paper, i.e., that the permanence and collapse of the hybrid system (1.5) depends on the symbol Φ_2. Hybrid system (1.5) is permanent under the condition $\Phi_2 > 0$, and collapsed under the condition $\Phi_2 < 0$.

3. Proofs

Consider the following equation:

$$\mathrm{d}x(t) = f(x(t), \lambda(t))\mathrm{d}t + g(x(t), \lambda(t))\mathrm{d}B(t),$$

where $f : R^n \times \mathbb{S} \to R^n$, $g : R^n \times \mathbb{S} \to R^{n\times m}$ and $\{B(t)\}_{t\geqslant 0}$ is the m-dimensional Brownian motion. For the function $H(x, j)$, define

$$\mathcal{L}H(x, j) = H_x(x, j)f(x, j) + \frac{1}{2}trace\left[g^{\mathrm{T}}(x, j)H_{xx}(x, j)g(x, j)\right] + \sum_{k\in\mathbb{S}} q_{jk}H(x, k),$$

where $(q_{jk})_{n^*\times n^*}$ is the generator of $\lambda(t)$, and

$$H_x(x, j) = \frac{\partial H(x, j)}{\partial x},\ H_{xx}(x, j) = \frac{\partial H_x(x, j)}{\partial x}.$$

Let

$$F_1(x, y, j) = -\alpha_1(j) - \beta_1(j)x - \gamma_1(j)y + \frac{\zeta_1(j)}{x(t)+\delta_1(j)},$$

$$F_2(x, y, j) = -\alpha_2(j) - \beta_2(j)x - \gamma_2(j)y + \frac{\zeta_2(j)}{y(t)+\delta_2(j)},$$

so we obtained the following system:

$$\begin{cases} dx(t) = x(t)F_1(x,y,j)dt + \sigma_1(\lambda)x(t)dB_1(t), \\ dy(t) = y(t)F_2(x,y,j)dt + \sigma_2(\lambda)y(t)dB_2(t). \end{cases} \tag{3.1}$$

There are constants $\tilde{M} > 0$ and $L > 0$ such that

$$xF_1(x,y,j) + yF_2(x,y,j) \leqslant -\tilde{M}(1+x+y)^2$$

for $\forall (x,y,j) \in \overline{R}_+^2 \times \mathbb{S}$ with $\sqrt{x^2+y^2} \geqslant L$. This is such that there exists a constant $a \in (0,1)$ such that

$$\begin{aligned} &\frac{xF_1(x,y,j) + yF_2(x,y,j)}{1+x+y} - \frac{\sigma_1^2(j)x^2 + \sigma_2^2(j)x^2}{2(1+x+y)^2} \\ &+a\Big[3 + (\alpha_1(j) + \beta_1(j)x + \gamma_1(j)y + \frac{\zeta_1(j)}{x(t)+\delta_1(j)}) \\ &+(\alpha_2(j) + \beta_2(j)x + \gamma_2(j)y + \frac{\zeta_2(j)}{y(t)+\delta_2(j)})\Big] < 0 \end{aligned}$$

for $\forall (x,y,j) \in \overline{R}_+^2 \times \mathbb{S}$ with $\sqrt{x^2+y^2} \geqslant L$. Therefore, for an arbitrary given

$$b \in \ (0,\ \min\{\frac{a}{2}, \frac{a}{2\check{\sigma}^2}\}),$$

if $\sqrt{x^2+y^2} \geqslant L$, we have

$$\begin{aligned} M(x,y,j) := &\frac{xF_1(x,y,j) + yF_2(x,y,j)}{1+x+y} - \frac{\sigma_1^2x^2 + \sigma_2^2x^2}{2(1+x+y)^2} + a + 2b\check{\sigma}^2 + 2b \\ &+ b\Big[(\alpha_1(j) + \beta_1(j)x + \gamma_1(j)y + \frac{\zeta_1(j)}{x(t)+\delta_1(j)}) \\ &+(\alpha_2(j) + \beta_2(j)x + \gamma_2(j)y + \frac{\zeta_2(j)}{y(t)+\delta_2(j)})\Big] < 0, \forall j \in \ \mathbb{S}, \end{aligned} \tag{3.2}$$

where $\check{\sigma}^2 = \max_{i=1,\,2}\{\max_{j\in\,\mathbb{S}} \sigma_i^2(j)\}$. Thereby,

$$M_1 := \sup_{(x,y)\in\overline{R}_+^2\setminus(0,0),\, j\in\,\mathbb{S}} \{M(x,y,j)\} < +\infty. \tag{3.3}$$

For $c = (c_1, c_2) \in R_+^2$ with $\|c\| := \sqrt{c_1^2 + c_2^2} \leqslant \ b < \frac{1}{2}$, let $H(\cdot) : R_+^2 \times \mathbb{S} \to R_+$ be defined by

$$H(x,y,j) = \frac{1+x+y}{x^{c_1}y^{c_2}}.$$

We just do a direct calculation and show that $H(x,y,j) > 1$ for all $(x,y,j) \in R_+^2 \times \mathbb{S}$.

If $\Phi_1 > 0, \Phi_2 > 0$, there is a value $\tilde{c} = (\tilde{c_1}, \tilde{c_2}) \in R_+^2$ with $\|\tilde{c}\| \leqslant b$ such that $\tilde{c_1}\Phi_1 - \tilde{c_2}\Phi_2 > \ 0$. Define

$$\tilde{H}(x,y,j) = \frac{1+x+y}{x^{\tilde{c}_1}y^{\tilde{c}_2}}, (x,y) \in R_+^2, j \in \ \mathbb{S}.$$

Obviously, $\tilde{H}(x,y,j)$ is a special example of $H(x,y,j)$.

Let

$$q^* = \frac{1}{2}\min\{\tilde{c_1}\Phi_1 - \tilde{c_2}\Phi_2,\ \tilde{c_2}\Phi_2\}. \tag{3.4}$$

There exists a sufficiently large constant $M^* \in N$ such that

$$M^* a > a + M_1 + q^*. \tag{3.5}$$

Lemma 3.1. For $(x(0), y(0), \lambda(0)) = (m,\ l) \in R_+^2 \times \mathbb{S}$, the solution $(x(t), y(t), \lambda(t))$ is a Makov-Feller process, and

$$E_{m,l}[H^b(x(t), y(t), \lambda(t))] \leqslant e^{bM_1 t} H^b(m, l). \tag{3.6}$$

Proof. Notice the following equation:

$$\liminf_{n\to+\infty}\{H(x, y, j)|x > n \text{ or } \frac{1}{x} > n \text{ or } y > n \text{ or } \frac{1}{y} > n\} = +\infty. \tag{3.7}$$

By direct calculation, we have

$$\begin{aligned}\mathcal{L}H^b(x, y, j) = \quad & bH^b(x, y, j)\Bigg[\frac{xF_1(x, y, j) + yF_2(x, y, j)}{1 + x + y} + \frac{b-1}{2}\frac{\sigma_1^2(j)x^2 + \sigma_2^2(j)y^2}{(1 + x + y)^2} \\ & -c_1F_1(x, y, j) - c_2F_2(x, y, j) + \frac{c_1\sigma_1^2(j) + c_2\sigma_2^2(j)}{2} \\ & +\frac{b}{2}\left(c_1^2\sigma_1^2(j) + c_2^2\sigma_2^2(j)\right) - b\left(\frac{c_1\sigma_1^2(j)x + c_2\sigma_2^2(j)y}{1 + x + y}\right)\Bigg].\end{aligned}$$

Applying $\|c\| \leqslant b < 1$, we can see that

$$\frac{b}{2}\left(\frac{\sigma_1^2(j)x^2 + \sigma_2^2(j)y^2}{(1 + x + y)^2}\right) \leqslant \frac{b\check{\sigma}^2}{2},$$

$$\begin{aligned}-c_1F_1(x, y, j) - c_2F_2(x, y, j) \leqslant b\Bigg[&(\alpha_1(j) + \beta_1(j)x + \gamma_1(j)y + \frac{\zeta_1(j)}{x(t) + \delta_1(j)}) \\ &+(\alpha_2(j) + \beta_2(j)x + \gamma_2(j)y + \frac{\zeta_2(j)}{y(t) + \delta_2(j)})\Bigg],\end{aligned}$$

$$\frac{c_1\sigma_1^2(j) + c_2\sigma_2^2(j)}{2} + \frac{b}{2}\left(c_1^2\sigma_1^2(j) + c_2^2\sigma_2^2(j)\right) - b\left(\frac{c_1\sigma_1^2(j)x + c_2\sigma_2^2(j)y}{1 + x + y}\right) \leqslant \frac{3b\check{\sigma}^2}{2}.$$

Therefore, we have

$$\begin{aligned}\mathcal{L}H^b(x, y, j) \leqslant bH^b(x, y, j)\Bigg[&\frac{xF_1(x, y, j) + yF_2(x, y, j)}{1 + x + y} - \frac{\sigma_1^2(j)x^2 + \sigma_2^2(j)y^2}{2(1 + x + y)^2} \\ &+ 2b\check{\sigma}^2 + b\Bigg(\alpha_1(j) + \beta_1(j)x + \gamma_1(j)y + \frac{\zeta_1(j)}{x(t) + \delta_1(j)} \\ &+\alpha_2(j) + \beta_2(j)x + \gamma_2(j)y + \frac{\zeta_2(j)}{y(t) + \delta_2(j)}\Bigg).\end{aligned} \tag{3.8}$$

According to (3.2) and (3.3), we can obtain that

$$\mathcal{L}H^b(x,y,j) \leqslant bM(x,y,j)H^b(x,y,j) \leqslant bM_1(x,y,j)H^b(x,y,j). \tag{3.9}$$

According to Theorem 5.1 in Reference [36], (3.7) and (3.9), we obtain that the solution $(x(t), y(t), \lambda(t))$ is a Makov-Feller process. Moreover, we can obtain (3.6) by applying (3.9) and Gronwall's inequality.

Lemma 3.2. If $\Phi_1 > 0$, then

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s)\mathrm{d}s \geqslant \frac{\Phi_1}{\check{\beta}_1}, \tag{3.10}$$

where $\check{\beta}_1 = \max_{j\in\mathbb{S}}\{\beta_1(j)\}$.

Proof. By definition of $\check{\beta}_1$, we get

$$\frac{1}{t}\int_0^t x(s)\check{\beta}_1 ds \geq \frac{1}{t}\int_0^t x(s)\beta_1(\lambda(s))ds.$$

We take the infimum for the left-hand side, and then we take the limit of both sides:

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s)\check{\beta}_1 ds \geq \lim_{t\to\infty}\frac{1}{t}\int_0^t x(s)\beta_1(\lambda(s))ds = \Phi_1.$$

So,

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \geq \frac{\Phi_1}{\check{\beta}_1}.$$

According to (3.10), there is a t_1 such that $x(t_1) \geqslant \frac{\Phi_1}{2\check{\beta}_1}$. That is to say, without loss of generality, we make an assumption; suppose that

$$x_1 \geqslant \frac{\Phi_1}{2\check{\beta}_1}.$$

Then, define that

$$N = \left\{m = (m_1, m_2) \in \overline{R}_+^2 \mid m_1 > \frac{\Phi_1}{2\check{\beta}_1}, \|m\| \leqslant L\right\}.$$

Lemma 3.3. If $\Phi_1 > 0$ and $\Phi_2 > 0$, then there is a $T^* > 0$ such that for all $t \geqslant T^*$ and $(x(0), y(0), \lambda(0)) = (\tilde{m}, l) \in (\partial R_+^2 \bigcap N) \times \mathbb{S}$,

$$\frac{1}{t}\int_0^t E_{\tilde{m},l}\left[F_3(x(s), y(s), \lambda(s))\right]\mathrm{d}s \leqslant -q^*, \tag{3.11}$$

where q^* is given in (3.4), and

$$\begin{aligned} F_3(x,y,j) = &\frac{xF_1(x,y,j) + yF_2(x,y,j)}{1+x+y} - \frac{\sigma_1^2(j)x^2 + \sigma_2^2(j)y^2}{2(1+x+y)^2} \\ &- \tilde{c}_1\left[\frac{\zeta_1(j)}{x+\delta_1(j)} - \frac{1}{2}\sigma_1^2 - \alpha_1(j) - \beta_1(j)x - \gamma_1(j)y\right] \\ &- \tilde{c}_2\left[\frac{\zeta_2(j)}{y+\delta_2(j)} - \frac{1}{2}\sigma_2^2 - \alpha_2(j) - \beta_2(j)x - \gamma_2(j)y\right]. \end{aligned}$$

Proof. We argue by contradiction. Suppose that the conclusion of this lemma is not true. Then, we can find that $(x_k, y_k, k) \in (\partial R_+^2 \bigcap N) \times \mathbb{S}$, and that $t_k > 0$ and $\lim_{k\to\infty} t_k = \infty$ such that

$$\frac{1}{t_k}\int_0^{t_k} E_{\tilde{m}_k,k} F_3(x(s), y(s), \lambda(s))ds > -q^*.$$

Note that

$$\Pi_t^{\tilde{m}_k,k}(d\mathbf{y}) := \frac{1}{t}\int_0^t P_{\tilde{m}_k,k}\{(x(s), y(s), \lambda(s)) \in d\mathbf{y}\}ds.$$

By Tonelli's theorem, we get that

$$\begin{aligned}\int_{(\partial R_+^2 \bigcap N)\times\mathbb{S}} (1 + \mathbf{c}^T\mathbf{y})^b \Pi_t^{\tilde{m}_k,k}(d\mathbf{y}) &= \int_{(\partial R_+^2 \bigcap N)\times\mathbb{S}} (1 + \mathbf{c}^T\mathbf{y})^b \frac{1}{t}\int_0^t P_{\tilde{m}_k,k}\{(x(s), y(s), \lambda(s)) \in d\mathbf{y}\}ds \\ &= \frac{1}{t}\int_0^t E_{\tilde{m}_k,k}(1 + \mathbf{c}^T(x(s), y(s), \lambda(s)))^b ds.\end{aligned}$$

Applying Lemma 3.2 in Reference [34],

$$\begin{aligned}\sup_{k\in N, t\geq 0}\int_{(\partial R_+^2 \bigcap N)\times\mathbb{S}} (1 + \mathbf{c}^T\mathbf{y})^b \Pi_t^{\tilde{m}_k,k}(d\mathbf{y}) &= \sup_{k\in N, t\geq 0}\frac{1}{t}\int_0^t E_{\tilde{m}_k,k}(1 + \mathbf{c}^T(x(s), y(s), \lambda(s)))^b ds \\ &\leq \sup_{\|\mathbf{x}\|\leq L, t\geq 0}\frac{1}{t}\int_0^t (\tilde{M}_1 + (1 + \mathbf{c}^T\mathbf{y})^b e^{-bas})ds \\ &< \infty,\end{aligned}$$

where$\tilde{M}_1 = \frac{1}{a}M_1 \sup_{\|\mathbf{x}\|\leq L}(1 + \mathbf{c}^T\mathbf{x})^b$.

This implies that the family $(\Pi_{t_k}^{\tilde{m}_k,k})_{k\in N}$ is tight in R_+^2. As a result, $(\Pi_{t_k}^{\tilde{m}_k,k})_{k\in N}$ has a convergent subsequence in the weak* topology. Without loss of generality, we can suppose that $\{\Pi_{t_k}^{\tilde{m}_k,k} : k \in N\}$ is a convergent sequence in the weak* topology. It can be shown that its limit is an invariant probability measure μ of $(x(t), y(t), \lambda(t))$. As a consequence of Lemma 3.4 in Reference [34],

$$\lim_{k\to\infty}\frac{1}{t_k}\int_0^{t_k} E_{\tilde{m}_k,k} F_3(x(t), y(t), \lambda(t))dt = \int_{(\partial R_+^2 \bigcap N)\times\mathbb{S}} F_3(\mathbf{x})\mu(d\mathbf{x}).$$

By the definition of q^*, we get

$$\lim_{k\to\infty}\frac{1}{t_k}\int_0^{t_k} E_{\tilde{m}_k,k} F_3(x(t), y(t), \lambda(t))dt \leq -q^*,$$

which is a contradiction of the assumptions.

Lemma 3.4. If $\Phi_1 > 0$ and $\Phi_2 > 0$, then there are two positive constants $\tau \in (0, \frac{b}{2})$ and M_τ such that, for $\forall t \in (T^*, M^*T^*)$ and $(x(0), y(0), \lambda(0)) = (m, l) \in (R_+^2 \bigcap N) \times \mathbb{S}$,

$$E_{m,l}\left[H^\tau(x(t), y(t), \lambda(t))\right] \leqslant H^\tau(m, l)e^{-\frac{\tau q^* t}{4}} + M_\tau, \tag{3.12}$$

where M^* is given in (3.5) and T^* is given in Lemma 3.3.

Proof. Applying Itô's formula, we have

$$\ln H(x, y, \lambda) = \ln H(m, l) + r(t),$$

where

$$r(t) = \int_0^t F_3(x(s), y(s), \lambda(s))\mathrm{d}s + \int_0^t \frac{\sigma_1(\lambda(s))x(s)}{1 + x(s) + y(s)}\mathrm{d}B_1(s) + \int_0^t \frac{\sigma_2(\lambda(s))y(s)}{1 + x(s) + y(s)}\mathrm{d}B_2(s) - \sum_{i=1}^{2} \int_0^t c_i\sigma_i(\lambda(s))\mathrm{d}B_i(s).$$

Thereby, we obtain

$$b \ln H(x, y, \lambda) = b \ln H(m, l) + br(t),$$

$$H^b(x, y, \lambda) = e^{br(t)}H^b(m, l),$$

i.e.,

$$E_{m,l}\left[H^b(x, y, \lambda)\right] = H^b(m, l)E_{m,l}\left[e^{br(t)}\right].$$

In light of (3.6), one can see that

$$E_{m,l}(e^{br(t)}) = \frac{E_{m,l}\left[H^b(x, y, \lambda)\right]}{H^b(m, l)} \leqslant e^{bM_1t}. \tag{3.13}$$

Define

$$H_1(x, y, j) = (1 + x + y)x^{c_1}y^{c_2}, (x, y, j) \in R_+^2 \times \mathbb{S}.$$

Using Itô's formula again, we obtain

$$\frac{E_{m,l}\left[H_1^b(x, y, \lambda)\right]}{H_1^b(m, l)} \leqslant e^{bM_1t}. \tag{3.14}$$

Notice that

$$H^{-b}(x, y, j) = (1 + x + y)^{-2b}H_1^b(x, y, j) \leqslant H_1^b(x, y, j).$$

Thus,

$$E_{m,l}\left[e^{-br(t)}\right] = \frac{E_{m,l}\left[H^{-b}(x, y, \lambda)\right]}{H^{-b}(m, l)} \leqslant \frac{E_{m,l}\left[H_1^b(x, y, \lambda)\right]}{H^{-b}(m, l)} = E_{m,l}\left[H_1^b(x, y, \lambda)\right]\frac{(1 + m_1 + m_2)^{2b}}{H_1^b(m, l)},$$

so we get

$$E_{m,l}\left[e^{-br(t)}\right] \leqslant (1 + m_1 + m_2)^{2b}\frac{E_{m,l}\left[H_1^b(x, y, \lambda)\right]}{H_1^b(m, l)}. \tag{3.15}$$

Substituting (3.14) into (3.15) results in

$$E_{m,l}\left[e^{-br(t)}\right] \leqslant (1 + m_1 + m_2)^{2b}e^{bM_1t}. \tag{3.16}$$

Therefore, we have

$$E_{m,l}\left[e^{br(t)}\right]+E_{m,l}\left[e^{-br(t)}\right]\leqslant\left[1+(1+m_1+m_2)^{2b}\right]e^{bM_1t}$$
$$\leqslant\left[1+(1+m_1+m_2)^{2b}\right]e^{bM_1M^*T^*}=:M_2. \tag{3.17}$$

Applying Lemma 3.5 in Reference [34], we can see that $H_{m,l,t}(\tau):=\ln E_{m,l}\left[e^{\tau r(t)}\right]$ is twice differentiable on $[0,\frac{b}{2})$, and that

$$\frac{\mathrm{d}H_{m,l,t}(\tau)}{\mathrm{d}\tau}=E_{m,l}(r(t)),0\leqslant\frac{\mathrm{d}^2H_{m,l,t}(\tau)}{\mathrm{d}\tau^2}\leqslant M_3,\forall\tau\in[0,\frac{b}{2}),t\in[T^*,T^*M^*], \tag{3.18}$$

where M_3 is a constant that depends on M_2.

Obviously, $F_3(x(s),y(s),\lambda(s))$ is a continuous function, so $\int_0^t F_3(x(s),y(s),\lambda(s))ds$ is also a continuous function. By the Feller property ([22]) of $(x(t),y(t),\lambda(t))$, we get that the mapping

$$(x(s),y(s),s)\to E_{m,l}\int_0^t\left[F_3(x(s),y(s),\lambda(s))\right]\mathrm{d}s$$

is continuous.

If $0<dist(m,\partial R_+^2)<M_4$, we have

$$E_{m,l}\int_0^t\left[F_3(x(s),y(s),\lambda(s))\right]\mathrm{d}s=\int_0^t E_{m,l}\left[F_3(x(s),y(s),\lambda(s))\right]\mathrm{d}s\leqslant -q^*t,$$

where $M_4\in(0,\frac{\Phi_1}{3\breve{\beta}_1})$, $m\in N$.

In fact, take a point $(\tilde{m},l)\in(\partial R_+^2\cap N)\times\mathbb{S}$; for any $\varepsilon>0$, when $(m,l)\in(U(\tilde{m},M_4)\cap N)\times\mathbb{S}$,

$$\left|\int_0^t E_{m,l}\left[F_3(x(s),y(s),\lambda(s))\right]\mathrm{d}s-\int_0^t E_{\tilde{m},l}\left[F_3(x(s),y(s),\lambda(s))\right]\mathrm{d}s\right|<\varepsilon.$$

So,

$$\int_0^t E_{m,l}\left[F_3(x(s),y(s),\lambda(s))\right]\mathrm{d}s<\int_0^t E_{\tilde{m},l}\left[F_3(x(s),y(s),\lambda(s))\right]\mathrm{d}s+\varepsilon<-q^*t+\varepsilon;$$

because ε is arbitrary, we have

$$\int_0^t E_{m,l}\left[F_3(x(s),y(s),\lambda(s))\right]\mathrm{d}s\leq -q^*t.$$

Then, we have

$$E_{m,l}\left[r(t)\right]=E_{m,l}\int_0^t F_3(x(s),y(s),\lambda(s))\mathrm{d}s+E_{m,l}\int_0^t\frac{\sigma_1(\lambda(s))x(s)}{1+x(s)+y(s)}\mathrm{d}B_1(s)$$
$$+E_{m,l}\int_0^t\frac{\sigma_2(\lambda(s))y(s)}{1+x(s)+y(s)}\mathrm{d}B_2(s)-E_{m,l}\sum_{i=1}^2\int_0^t c_i\sigma_i(\lambda(s))\mathrm{d}B_i(s)$$
$$=\int_0^t E_{m,l}\left[F_3(x(s),y(s),\lambda(s))\right]\mathrm{d}s+0\leqslant -q^*t\leqslant -\frac{q^*t}{2},t\in[T^*,T^*M^*]. \tag{3.19}$$

For

$$t \in [T^*, T^*M^*], 0 < dist(m, \partial R_+^2) < M_4, m \in N, \tau \in [0, \frac{b}{2}),$$

expanding $H_{m,l,t}(\tau)$ around 0, and according to (3.18) and (3.19), we can know that, for a sufficiently small τ,

$$H_{m,l,t}(\tau) \leqslant -\frac{q^* t\tau}{2} + M_3\tau^2 \leqslant -\frac{q^* t\tau}{4}.$$

By (3.13), for such a τ and

$$0 < dist(m, \partial R_+^2) < M_4, m \in N, t \in [T^*, T^*M^*],$$

we have

$$\frac{E_{m,l}\left[H^\tau(x(t), y(t), \lambda(t))\right]}{H^\tau(m, l)} = E_{m,l}\left[e^{\tau r(t)}\right] = e^{H_{m,l,t}(\tau)} \leqslant e^{-\frac{q^*\tau t}{4}}. \tag{3.20}$$

If $dist(m, \partial R_+^2) \geqslant M_4$, for $m \in N$ and $t \in [T^*, T^*M^*]$, applying (3.6), we have

$$E_{m,l}\left[H^\tau(x(t), y(t), \lambda(t))\right] \leqslant e^{\tau M_1 t} H^\tau(m, l) \leqslant e^{\tau M_1 \mathrm{T^*M^*}} \max_{m \in N, l \in \mathbb{S}} [H^\tau(m, l)] := M_\tau. \tag{3.21}$$

Thus (3.20) and (3.21) give the desired conclusion (3.12).

Lemma 3.5. For $\forall T > 0$, $\{x(nT), y(nT), \lambda(nT)\}_{n\in N}$ is irreducible and aperiodic. In addition, $E \times \{l\}$ is petite, where $E \in R_+^2$ is an arbitrary compact set and $l \in \mathbb{S}$ is arbitrary.

Proof. Let $W \subset R_+^2$ be an open set with a smooth boundary ∂W such that $E \subset W$ and $(x(0), y(0), \lambda(0)) = (m, l) \in E \times \{l\}$. For $\forall A \in E$, $j \in \mathbb{S}$ and $t \geqslant 0$, define

$$P_{m,l}^W(t, A \times \{j\}) = \mathrm{P}\Big[((x(t), y(t), \lambda(t)) \in A \times \{j\}) \bigcap ((x(0), y(0), \lambda(0)) \in E \times \{l\}) \bigcap ((x(s), y(s), \lambda(s)) \in W \times \mathbb{S}, 0 < s < t)\Big].$$

We can see that the density function $p_{m,l}^W(t, m', j)$ of $P_{m,l}^W$ is positive by using Lemma 3.8 in Reference [37], and that it is jointly continuous in t, m and m'.

For $\forall\ m' \in W$, we define that

$$p^E(t, m', j) = \min_{l \in \mathbb{S}} \left(\inf_{m \in E} \{p_{m,l}^W(t, m', j)\}\right).$$

For $m' \notin W$, define $p^E(t, m', j) = 0$. Let π^E is the corresponding measure of $p^E(T, m', j)$. Then, we have

$$P_{m,l}(T, A \times \{j\}) \geqslant P_{m,l}^W(T, A \times \{j\}) \geqslant \pi^E(A \times \{j\}).$$

That is to say, $E \times \{l\}$ is petite for $\{x(nT), y(nT), \lambda(nT)\}_{n\in N}$. Moreover, if $\pi^E(A \times \{j\}) > 0$, then

$$P_{m,l}(T, A \times \{j\}) \geqslant \pi^E(A \times \{j\}) > 0. \tag{3.22}$$

So, $\{x(nT), y(nT), \lambda(nT)\}_{n\in N}$ is irreducible.

The above have proved that $\{x(nT), y(nT), \lambda(nT)\}_{n\in N}$ is irreducible; next, we prove that $\{x(nT), y(nT), \lambda(nT)\}_{n\in N}$ is aperiodic. If the argument is not true, Theorem 2.2 in

Reference [38] (page 21) means that there exist disjoint $A_0 \times \{l_0\}, \cdots, A_{n-1} \times \{l_{n-1}\} \subset R^2_+ \times \mathbb{S}$ with $n \geqslant 2$ such that, for $\forall (m,l) \in A_i \times \{l_i\}$,

$$P_{m,l}(T, A_{i+1} \times \{l_{i+1}\}) = 1,\ i = 0,\ \cdots,\ n-1 \ (\mathrm{mod}\ n).$$

Therefore, $P_{m,l}(T, A_i \times \{l_i\}) = 0$. By (3.22), it is a contradiction.

Remark 3.1. For the definitions of aperiodicity, petiteness and irreducibility, one can refer to References [35, 38].

Proof of Theorem 2.1

(a). Define

$$\mu = \inf\Big\{t \geqslant 0 | x^2(t) + y^2(t) \leqslant L^2\Big\}.$$

According to (3.8), we can see that

$$\begin{aligned}\mathcal{L}H^\tau(x,y,j) \leqslant \tau H^\tau(x,y,j)\left[M(x,y,j) - a - 2b\right] \leqslant \tau H^\tau(x,y,j)\left[M(x,y,j) - a\right] \\ = \tau M(x,y,j) H^\tau(x,y,j) \\ - \tau a H^\tau(x,y,j).\end{aligned}$$

Applying (3.2), we have

$$\mathcal{L}H^\tau(x,y,j) \leqslant -\tau a H^\tau(x,y,j), x^2 + y^2 \geqslant L^2.$$

Then, following form Dynkin's formula (e.g., [22]), we obtain that

$$\begin{aligned}&E_{m,l}\Big[e^{\tau a(\mu \wedge M^* T^*)} H^\tau\big(x(\mu \wedge M^* T^*), y(\mu \wedge M^* T^*), \lambda(\mu \wedge M^* T^*)\big)\Big] \\ \leqslant& H^\tau(m,l) + E_{m,l}\int_0^{\mu \wedge M^* T^*} e^{\tau a s}\Big[\mathcal{L}H^\tau(x(s), y(s), \lambda(s)) + \tau a H^\tau(x(s), y(s), \lambda(s))\Big] \mathrm{d}s \\ \leqslant& H^\tau(m,l).\end{aligned}$$

Thus,

$$\begin{aligned}&H^\tau(m,l) \\ \geqslant& E_{m,l}\Big[e^{\tau a(\mu \wedge M^* T^*)} H^\tau\big(x(\mu \wedge M^* T^*), y(\mu \wedge M^* T^*), \lambda(\mu \wedge M^* T^*)\big)\Big] \\ =& E_{m,l}\Big[1_{\{\mu \leqslant (M^*-1)T^*\}} e^{\tau a(\mu \wedge M^* T^*)} H^\tau\big(x(\mu \wedge M^* T^*), y(\mu \wedge M^* T^*), \lambda(\mu \wedge M^* T^*)\big)\Big] \\ &+ E_{m,l}\Big[1_{\{(M^*-1)T^* < \mu < M^* T^*\}} e^{\tau a(\mu \wedge M^* T^*)} H^\tau\big(x(\mu \wedge M^* T^*), y(\mu \wedge M^* T^*), \lambda(\mu \wedge M^* T^*)\big)\Big] \\ &+ E_{m,l}\Big[1_{\{\mu \geqslant M^* T^*\}} e^{\tau a(\mu \wedge M^* T^*)} H^\tau\big(x(\mu \wedge M^* T^*), y(\mu \wedge M^* T^*), \lambda(\mu \wedge M^* T^*)\big)\Big] \\ =& E_{m,l}\Big[1_{\{\mu \leqslant (M^*-1)T^*\}} e^{\tau a \mu} H^\tau\big(x(\mu), y(\mu), \lambda(\mu)\big)\Big] \\ &+ E_{m,l}\Big[1_{\{(M^*-1)T^* < \mu < M^* T^*\}} e^{\tau a \mu} H^\tau\big(x(\mu), y(\mu), \lambda(\mu)\big)\Big] \\ &+ E_{m,l}\Big[1_{\{\mu \geqslant M^* T^*\}} e^{\tau a M^* T^*} H^\tau\big(x(M^* T^*), y(M^* T^*), \lambda(M^* T^*)\big)\Big] \\ \geqslant& E_{m,l}\Big[1_{\{\mu \leqslant (M^*-1)T^*\}} H^\tau\big(x(\mu), y(\mu), \lambda(\mu)\big)\Big] \\ &+ e^{\tau a(M^*-1)T^*} E_{m,l}\Big[1_{\{(M^*-1)T^* < \mu < M^* T^*\}} H^\tau\big(x(\mu), y(\mu), \lambda(\mu)\big)\Big] \\ &+ e^{\tau a M^* T^*} E_{m,l}\Big[1_{\{\mu \geqslant M^* T^*\}} H^\tau\big(x(M^* T^*), y(M^* T^*), \lambda(M^* T^*)\big)\Big].\end{aligned} \tag{3.23}$$

According to (3.12) and the Markov property of $(x(t), y(t), \lambda(t))$, we have

$$\begin{aligned}
&E_{m,l}\Big[1_{\{\mu\leqslant(M^*-1)T^*\}}H^\tau\big(x(M^*T^*), y(M^*T^*), \lambda(M^*T^*)\big)\Big]\\
\leqslant&E_{m,l}\Big[1_{\{\mu\leqslant(M^*-1)T^*\}}\Big(M_\tau + e^{-\frac{\tau q^*(M^*T^*-\mu)}{4}}H^\tau\big(x(\mu), y(\mu), \lambda(\mu)\big)\Big)\Big]\\
\leqslant&M_\tau + e^{-\frac{\tau q^*T^*}{4}}E_{m,l}\Big[1_{\{\mu\leqslant(M^*-1)T^*\}}H^\tau\big(x(\mu), y(\mu), \lambda(\mu)\big)\Big].
\end{aligned} \tag{3.24}$$

According to (3.6) and the Markov property of $(x(t), y(t), \lambda(t))$, we have

$$\begin{aligned}
&E_{m,l}\Big[1_{\{(M^*-1)T^*<\mu<M^*T^*\}}H^\tau\big(x(M^*T^*), y(M^*T^*), \lambda(M^*T^*)\big)\Big]\\
\leqslant&E_{m,l}\Big[1_{\{(M^*-1)T^*<\mu<M^*T^*\}}e^{\tau M_1(M^*T^*-\mu)}H^\tau\big(x(\mu), y(\mu), \lambda(\mu)\big)\Big]\\
\leqslant&e^{\tau M_1T^*}E_{m,l}\Big[1_{\{(M^*-1)T^*<\mu<M^*T^*\}}H^\tau\big(x(\mu), y(\mu), \lambda(\mu)\big)\Big].
\end{aligned} \tag{3.25}$$

Substituting (3.24) and (3.25) into (3.23) results in

$$\begin{aligned}
&H^\tau(m, l)\\
\geqslant&e^{\frac{\tau q^*T^*}{4}}E_{m,l}\Big[1_{\{\mu\leqslant(M^*-1)T^*\}}H^\tau\big(x(M^*T^*), y(M^*T^*), \lambda(M^*T^*)\big)\Big] - M_\tau e^{\frac{\tau q^*T^*}{4}}\\
&+ e^{\tau a(M^*-1)T^*}e^{-\tau M_1T^*}E_{m,l}\Big[1_{\{(M^*-1)T^*<\mu<M^*T^*\}}H^\tau\big(x(M^*T^*), y(M^*T^*), \lambda(M^*T^*)\big)\Big]\\
&+ e^{\tau aM^*T^*}E_{m,l}\Big[1_{\{\mu\geqslant M^*T^*\}}H^\tau\big(x(M^*T^*), y(M^*T^*), \lambda(M^*T^*)\big)\Big]\\
\geqslant&e^{\tau M_5T^*}E_{m,l}\Big[H^\tau\big(x(M^*T^*), y(M^*T^*), \lambda(M^*T^*)\big)\Big] - M_\tau e^{\frac{\tau q^*T^*}{4}},
\end{aligned} \tag{3.26}$$

where

$$M_5 = \min\Big\{\frac{q^*}{4}, aM^*, a(M^*-1)\Big\} = \frac{q^*}{4}.$$

Therefore,

$$E_{m,l}\Big[H^\tau\big(x(M^*T^*), y(M^*T^*), \lambda(M^*T^*)\big)\Big] \leqslant e^{-\frac{\tau q^*T^*}{4}}H^\tau(m, l) + M_\tau. \tag{3.27}$$

According to Lemma 3.5, (3.27) and Geometric Ergodic Theorem of Reference [35], $\{x(nM^*T^*), y(nM^*T^*), \lambda(nM^*T^*)\}_{n\in N}$ has positive Harris recurrence, and there exists an invariant measure $\eta(\cdot\times\cdot)$ on $R^2_+\times\mathbb{S}$ such that, for some $\varepsilon\in(0, 1)$ and $M_\tau > 0$,

$$\|P_{m,l}(nM^*T^*, \cdot\times\cdot) - \eta(\cdot\times\cdot)\|_{TV} \leqslant M_\tau\varepsilon^n. \tag{3.28}$$

Since $\{(x(nM^*T^*), y(nM^*T^*), \lambda(nM^*T^*))\}_{n\in N}$ is positive Harris recurrent, $\{x(t), y(t), \lambda(t)\}$ is positive recurrent. According to Theorems 4.3 and 4.4 of Reference [39], there is a UESD to $\{(x(t), y(t), \lambda(t))\}$. According to (3.28), we can see that the UESD is η.

According to the virtue of Theorem 5 of Reference [40], $\|P_{m,l}(t, \cdot\times\cdot) - \eta(\cdot\times\cdot)\|_{TV}$ is decreasing with respect to t. Therefore, the last conclusion of Theorem 2.1(a) follows from (3.28).

(b). The proof is routine, so we just give the outline. Applying Itô's formula to the second equation in Model (1.5), we have

$$\begin{aligned}
\mathrm{d}\ln y(t) =&\Big[-\alpha_2(\lambda) + \frac{\zeta_2(\lambda)}{y(t)+\delta_2(\lambda)} - \frac{1}{2}\sigma_2^2(\lambda) - \beta_2(\lambda)x(t) - \gamma_2(\lambda)y(t)\Big]\mathrm{d}t\\
&+ \sigma_2(\lambda)\mathrm{d}B_2(t)\\
\leqslant&\Big[-\alpha_2(\lambda) + \frac{\zeta_2(\lambda)}{y(t)+\delta_2(\lambda)} - \frac{1}{2}\sigma_2^2(\lambda)\Big]\mathrm{d}t + \sigma_2(\lambda)\mathrm{d}B_2(t).
\end{aligned}$$

That is to say,

$$\begin{aligned}\ln y(t) \leqslant & \ln y(0) + \int_0^t \left[-\alpha_2(\lambda(s)) + \frac{\zeta_2(\lambda(s))}{y(s)+\delta_2(\lambda(s))} - \frac{1}{2}\sigma_2^2(\lambda(s))\right] \mathrm{d}s \\ & + \int_0^t \sigma_2(\lambda(s)) \mathrm{d}B_2(s) \\ \leqslant & \ln y(0) + \int_0^t \left[-\alpha_2(\lambda(s)) + \frac{\zeta_2(\lambda(s))}{\delta_2(\lambda(s))} - \frac{1}{2}\sigma_2^2(\lambda(s))\right] \mathrm{d}s \\ & + \int_0^t \sigma_2(\lambda(s)) \mathrm{d}B_2(s).\end{aligned}$$

Therefore, we have

$$\frac{1}{t}\ln y(t) \leqslant \frac{1}{t}\ln y(0) + \frac{1}{t}\int_0^t [-z_2]\,\mathrm{d}s + \frac{1}{t}\int_0^t \sigma_2(\lambda(s))\mathrm{d}B_2(s).$$

Note that

$$\lim_{t\to+\infty}\left[\frac{1}{t}\ln y(0) + \frac{1}{t}\int_0^t (-z_2)\mathrm{d}s + \frac{1}{t}\int_0^t \sigma_2(\lambda(s))\mathrm{d}B_2(s)\right] = \Phi_2 < 0;$$

hence, $\lim_{t\to+\infty} y(t) = 0$. Therefore the transition probability of $(x(t), \lambda(t))$ converges weakly to $\eta(\cdot \times \cdot)$. Furthermore, Lemma 2.1 means that (2.3) holds.

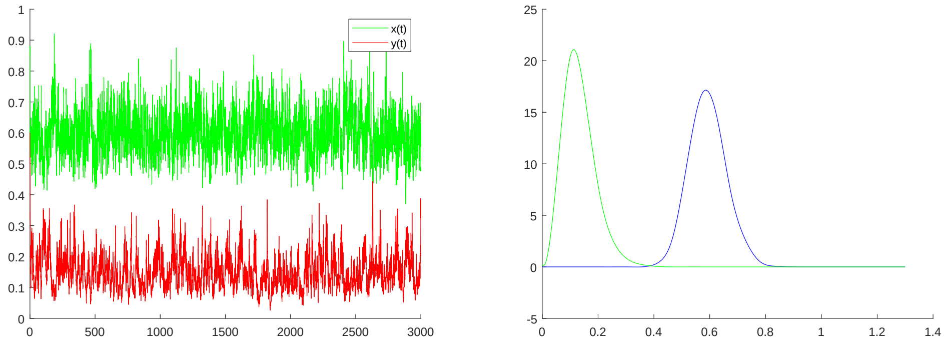
4. Example

To see two situations in Section 2 more clearly, let us use several simulations to illustrate the impacts. Here, we just present the situation ($\mathcal{B}$) by letting the stationary distribution of the Markov chain change (i.e., let π change). In the following example, the values of the parameters are hypothesized.

Choose $\alpha_1(j) = 0.4$, $\beta_1(j) = 0.3$, $\gamma_1(j) = 0.4$, $\zeta_1 = 0.8$, $\delta_1 = 0.65$, $\alpha_2(j) = 0.2$, $\beta_2(j) = 0.6$, $\gamma_2(j) = 0.2$, $\zeta_2 = 0.3$, $\delta_2 = 0.55$ and $\sigma_1(j) = 0.1$, $j = 1, 2$, so we have $z_1 = 0.825$ and $\Phi_1 = \Phi_1(1) = \Phi_1(2) = 0.825 > 0$.

In Regime 1, choose $\sigma_2(1) = 0.2$; thus, $z_2(1) = 0.32$ and $\Phi_2(1) = 0.32 > 0$. According to (*a*) in Theorem 2.1, there is a UESD $\eta_1(\cdot)$ concentrated on R_2^+ to Subsystem (2.4). Namely, Subsystem (2.4) is permanent; see Figure 1.

In Regime 2, choose $\sigma_2(2) = 0.9$; therefore, $z_2(1) = -0.11$ and $\Phi_2(2) = -0.11 < 0$. According to (*b*) in Theorem 2.1, Subsystem (2.5) is collapsed: one species is permanent, and the other is collapsing; see Figure 2.



(a) fig.1a

(b) fig.1b

Figure 1. (a) Sample trajectory; (b) probability density function of the solution at t=3000. They all show that Subsystem (2.4) is permanent.

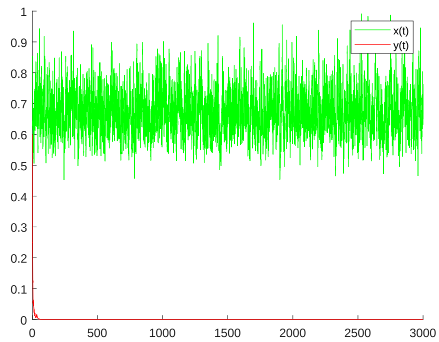


Figure 2. illustration showing that Subsystem (2.5) is collapsed.

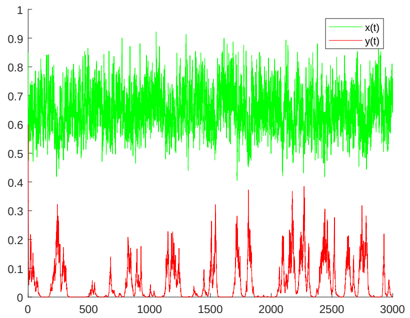
Then, we are going to choose different values of π.

Case 1. $\pi = (0.5, 0.5)^{\mathrm{T}}$. Thus, $\Phi_2 = 0.5 \times 0.32 - 0.5 \times 0.11 = 0.105 > 0$. By applying (a) of Theorem 2.1, $(x(t), y(t), \lambda(t))$ has a UESD $\eta_1(\cdot \times \cdot)$ concentrated on $R^2_+ \times \mathbb{S}$ in the hybrid model (1.5). Therefore, the hybrid system (1.5) is permanent under the behavior of regime switching, See Figure 3a and 3b.

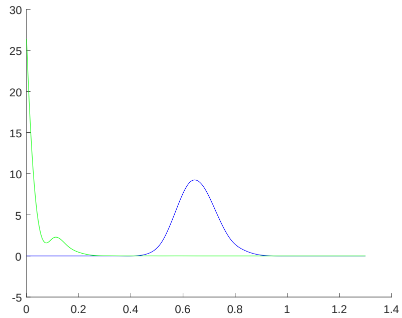
Case 2. $\pi = (0.1, 0.9)^{\mathrm{T}}$. Then, $\Phi_2 = 0.1 \times 0.32 - 0.9 \times 0.11 = -0.067 < 0$. Applying (b) of Theorem 2.1, the $y(t)$ population dies out, $(x(t), \lambda(t))$ has a UESD $\eta_2(\cdot \times \cdot)$ and $\eta_2(\cdot \times \cdot)$ is weakly concentrated on $R^2_+ \times \mathbb{S}$; in addition,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \varphi_1(s)\mathrm{d}s = \sum_{j\in\,\mathbb{S}} \int_{R_+} z\eta^{\varphi}(\mathrm{d}z, j) = 0.9167.$$

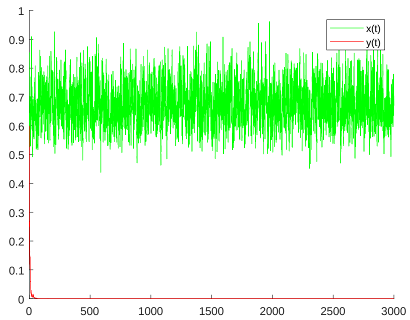
Therefore, the hybrid system (1.5) is collapsed under the behavior of regime switching. See Figure 3c.



(a) fig.3a



(b) fig.3b



(c) fig.3c

Figure 3. (a) and (b): $\pi = (0.5, 0.5)^{\mathrm{T}}$, where (a) is a sample trajectory and (b) is the probability density function of the solution at t=3000; they all show that the hybrid system (1.5) is permanent; results with $\pi = (0.1, 0.9)^{\mathrm{T}}$, showing that the hybrid system (1.5) is collapsed.

After the above numerical simulation, there are similar examples in reality. In 2008, Banks et al. [41] studied Neanderthal extinction by competitive exclusion. Despite a long history of investigation, considerable debate revolves around whether Neanderthals became extinct because of climate change or competition with anatomically modern humans (AMHs). The southerly contraction of Neanderthal range in southwestern Europe during Greenland Interstadial 8 was not due to climate change or a change in adaptation; rather, concurrent AMHs geographic expansion appears to have produced competition that led to Neanderthal extinction. In 2012, Sarwardi et al. [42] analyzed a competitive prey-predator system with prey refuge. Further, in a field survey of the Sundarban mangrove ecosystem, two very commercially viable detritivorus fishes, viz., Liza parsia and Liza tade, as well as another commercially important predator fish, viz., Lates calcarifer, are usually found in this ecosystem. Lates calcarifer depends on the predation of these two, and these fish are in competition by grazing upon detritus as food source from the supralittoral zone of the estuary during high tide ([43]). From the field surveys and studies in the Sundarban mangrove ecosystem, it has been observed that two detritivorous fish (prey population), by using refuges, coexist in nature in the presence of the predator fish population Lates calcarifer ([44]).

5. Conclusions

More and more people have begun to pay attention to the stationary distribution of random population models (see [31–33]). At present, the most common method to study stationary distribution is to construct a Lyapunov function, but, in the paper, we applied the approach in Reference [34] to investigate the existence of a unique stationary distribution. The difference between our model and the model in Reference [34] is that our model has switching.

In this article, the stationary distribution of a stochastic two-species Schoener's competitive model with regime switching was explored. We have the following conclusion: under the condition of $\Phi_1 > 0$, if $\Phi_2 > 0$, we proved that the UESD of the model is concentrated on $R_+^2 \times \mathbb{S}$. In other words, the model is permanent and the convergence rate of the transition probability is exponentially fast about the UESD; if $\Phi_2 < 0$, the model does not have stationary distribution. That is to say, when one species becomes extinct while another continues to exist, it would be a biological collapse.

As far as we know, this paper is the first on the UESD of a stochastic two-species Schoener's competitive model with regime switching. Our model, i.e., Model (1.5), is more reasonable. It takes the effect of a randomly fluctuating environment into account. The results in this paper show several key effects of regime switching on the collapse and permanence.

There are several questions that deserve further consideration. This work involved Markov switching, so it is significant to consider semi-Markov switching. Other random disturbances can also be considered, such as Lévy jumps (see [45–47]). These questions need to be further explored.

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Conflict of interest

The authors declare no conflicts of interest.

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